Foundations of Module and Ring Theory

A Handbook for Study and Research

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Contents

Preface

On the one hand this book intends to provide an introduction to module theory and the related part of ring theory. Starting from a basic understanding of linear algebra the theory is presented with complete proofs. From the beginning the approach is categorical.

On the other hand the presentation includes most recent results and includes new ones. In this way the book will prove stimulating to those doing research and serve as a useful work of reference.

Since the appearance of Cartan-Eilenberg's *Homological Algebra* in the 1950s module theory has become a most important part of the theory of associative rings with unit. The category *R-MOD* of unital modules over a ring R also served as a pattern for the investigation of more general *Grothendieck categories* which are presented comprehensively in Gabriel's work of 1962 (*Bull.Soc.Math.France*).

Whereas ring theory and category theory initially followed different directions it turned out in the 1970s – e.g. in the work of Auslander – that the study of functor categories also reveals new aspects for module theory. In our presentation many of the results obtained this way are achieved by purely module theoretic methods avoiding the detour via abstract category theory (Chapter 10). The necessary extension of usual module theory to get this is gained by an artifice.

From the very beginning the central point of our considerations is not the entire category *R-MOD* but a full subcategory of it: for an R-module M we construct the 'smallest' subcategory of *R-MOD* which contains M and is a *Grothendieck category*. This is the subcategory $\sigma[M]$ which is *subgenerated by* M, i.e. its objects are submodules of M-generated modules.

The elaboration of module theoretic theorems in $\sigma[M]$ is not more tedious than in *R-MOD*. However, the higher generality gained this way without effort yields significant advantages.

The correlation of (internal) properties of the module M with properties of the category $\sigma[M]$ enables a *homological classification* of modules. Among other things, the *Density Theorem* has a new interpretation (in 15.8). All in all the approach chosen here leads to a clear refinement of the customary module theory and, for $M = R$, we obtain well-known results for the entire module category over a ring with unit.

In addition the more general assertions also apply to rings without units and comprise the module theory for s-unital rings and rings with local units. This will be especially helpful for our investigations of functor rings.

For example, a new proof is obtained for the fact that a ring of *left finite (representation) type* is also of *right finite type* (see 54.3). For *serial rings* and *artinian principal ideal rings* we derive interesting characterizations involving properties of the functor rings (see 55.15, 56.10).

Another special feature we could mention is the definition of *linearly compact* modules through an exactness condition on the inverse limit (in 29.7). This permits more transparent proofs in studying dualities between module categories (in section 47).

Let us indicate some more applications of our methods which are not covered in the book. Categories of the type $\sigma[M]$ are the starting point for a rich module theory over non-associative rings A. For this, A is considered as module over the (associative) multiplication algebra $M(A)$ and the category $\sigma[A]$ is investigated. Also torsion modules over a topological ring and graded modules over a graded ring form categories of the type $\sigma[M]$.

For orientation, at the beginning of every section the titles of the paragraphs occurring in it are listed. At the end of the sections exercises are included to give further insight into the topics covered and to draw attention to related results in the literature. References can be found at the end of the paragraphs. Only those articles are cited which appeared after 1970. In citations of monographs the name of the author is printed in capital letters.

This book has evolved from lectures given at the Universities of Nantes and Düsseldorf from 1978 onwards. The printing was made possible through the technical assistance of the Rechenzentrum of the University of Düsseldorf.

I wish to express my sincere thanks to all who helped to prepare and complete the book.

Düsseldorf, Summer 1988

Besides several minor changes and improvements this English edition contains a number of new results. In 48.16 cogenerator modules with commutative endomorphism rings are characterized. In 51.13 we prove that a category $\sigma[M]$ which has a generator with right perfect endomorphism ring also has a projective generator. In 52.7 and 52.8 the functor rings of regular and semisimple modules are described. Three more theorems are added in section 54.

Also a number of additional exercises as well as references are included. I am very indebted to Patrick Smith and Toma Albu for their help in correcting the text.

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Robert Wisbauer

Symbols

$N(R)$	nil radical of R	*11*
$Np(R)$	sum of the nilpotent ideals in R	*11*
$P(R)$	prime radical of R	*11*
RG	semigroup ring	*32*
$An_R(M)$	annihilator of an R-module M	*42*
ENS	category of sets	*44*
GRP	category of groups	*45*
AB	category of abelian groups	*45*
$R\text{-}MOD$	category of left R-modules	*45*
$R\text{-}mod$	category of finitely generated left R-modules	*46*
$T_{A,B}$	morphism map for the functor T	*81*
$Tr(\mathcal{U}, L)$	trace of $\mathcal{U}$ in L	*107*
$Re(L,\mathcal{U})$	reject of $\mathcal{U}$ in L	*113*
$\sigma[M]$	subcategory of $R\text{-}MOD$ subgenerated by M	*118*
$\prod_{\Lambda}^{M} N_\lambda$	product of N_λ in $\sigma[M]$	*118*
$t(M)$	torsion submodule of a $\mathbb{Z}$-module M	*124*
$p(M)$	p-component of a $\mathbb{Z}$-module M	*124*
$\mathbb{Z}_{p^\infty}$	Prüfer group	*125*
$K \trianglelefteq M$	K is an essential submodule of M	*137*
$\widehat{N}$	M-injective hull of N	*141*
$K \ll M$	K is a superfluous submodule of M	*159*
$Soc\, M$	socle of M	*174*
$Rad\, M$	radical of M	*176*
$Jac\, R$	Jacobson radical of R	*178*
$\varinjlim M_i$	direct limit of modules M_i	*197*
$An(K)$	annihilator of a submodule $K \subset M$	*230*
$Ke(X)$	annihilator of a submodule $X \subset Hom_R(N, M)$	*230*
$\varprojlim N_i$	inverse limit of modules N_i	*239*
$lg(M)$	length of M	*267*
L^*, L^{**}	U-dual and U-double dual module of L	*411*
$\sigma_f[M]$	submodules of finitely M-generated modules	*426*
$\widehat{H}om(V, N)$	morphisms which are zero almost everywhere	*485*
$\widehat{E}nd(V)$	endomorphisms which are zero almost everywhere	*485*

Chapter 1

Elementary properties of rings

Before we deal with deeper results on the structure of rings with the help of module theory we want to provide elementary definitions and constructions in this chapter.

1 Basic notions

A *ring* is defined as a non-empty set R with two compositions $+\,,\cdot\,: R \times R \to R$ with the properties:

(i) $(R, +)$ is an abelian group (zero element 0);

(ii) $(R,\,\cdot\,)$ is a semigroup;

(iii) for all $a, b, c \in R$ the distributivity laws are valid:
$(a + b)c = ac + bc,\ a(b + c) = ab + ac.$

The ring R is called *commutative* if $(R, \cdot)$ is a commutative semigroup, i.e. if $ab = ba$ for all $a, b \in R$. In case the composition $\cdot$ is not necessarily associative we will talk about a *non-associative* ring.

An element $e \in R$ is a *left unit* if $ea = a$ for all $a \in R$. Similarly a *right unit* is defined. An element which is both a left and right unit is called a *unit* (also *unity, identity*) of R.

In the sequel R will always denote a ring. In this chapter we will not generally demand the existence of a unit in R but assume $R \neq \{0\}$.

The symbol 0 will also denote the subset $\{0\} \subset R$.

1.1 For non-empty subsets $A, B \subset R$ we define:

$$A + B := \{a + b \mid a \in A,\ b \in B\} \subset R,$$
$$AB := \{\sum_{i \leq k} a_i b_i \mid a_i \in A,\ b_i \in B, k \in I\!N\} \subset R.$$

With these definitions we are also able to form the sum and product of finitely many non-empty subsets $A, B, C, \ldots$ of R. The following rules are easy to verify:

$$(A + B) + C = A + (B + C),\ (AB)C = A(BC).$$

It should be pointed out that $(A + B)C = AC + BC$ is not always true. However, equality holds if $0 \in A \cap B$. For an arbitrary collection $\{A_\lambda\}_\Lambda$ of subsets $A_\lambda \subset R$ with $0 \in A_\lambda$, Λ an index set, we can form a 'sum':

$$\sum_{\lambda \in \Lambda} A_\lambda := \{\sum a_\lambda \,|\, a_\lambda \in A_\lambda,\ a_\lambda \neq 0 \text{ for only finitely many } \lambda \in \Lambda\}.$$

A subgroup I of $(R, +)$ is called a *left ideal* of R if $RI \subset I$, and a *right ideal* if $IR \subset I$. I is an *ideal* if it is both a left and right ideal.

I is a *subring* if $II \subset I$. Of course, every left or right ideal in R is also a subring of R. The intersection of (arbitrary many) (left, right) ideals is again a (left, right) ideal.

The following assertions for subsets A, B, C of R are easily verified:

If A is a left ideal, then AB is a left ideal.

If A is a left ideal and B is a right ideal, then AB is an ideal and $BA \subset A \cap B$.

If A, B are (left, right) ideals, then $A + B$ is a (left, right) ideal and $(A + B)C = AC + BC$ and $C(A + B) = CA + CB$.

1.2 A map between rings $f : R \to S$ is called a *(ring) homomorphism* if for all $a, b \in R$

$$(a + b)f = (a)f + (b)f,\ (ab)f = (a)f\,(b)f.$$

In case the rings R and S have units e_R, e_S, then we demand in addition $(e_R)f = e_S$. For surjective homomorphisms this condition is automatically satisfied.

Maps are usually written on the right side of the argument. Sometimes we write af instead of $(a)f$. For the composition of $f : R \to S,\ g : S \to T$ we have

$(a)fg = ((a)f)g \quad$ for all $a \in R$.

The *kernel* of a homomorphism $f : R \to S$ is defined as

$$Ke\, f := \{a \in R \mid (a)f = 0\}.$$

Obviously $Ke\, f$ is an ideal in R. On the other side every ideal $I \subset R$ is also the kernel of a homomorphism $g : R \to S$, e.g. of the canonical projection $p_I : R \to R/I$, where R/I is the ring of cosets $\{a + I \mid a \in R\}$ with the compositions

$$(a+I) + (b+I) := (a+b) + I, \quad (a+I)(b+I) := ab + I, \quad a, b \in R.$$

Since I is an ideal these definitions make sense and imply a ring structure on the cosets for which p_I is a (ring) homomorphism. The ring R/I is called the *factor ring* of R modulo I. The importance of factor rings is seen in:

Homomorphism Theorem. *If $f : R \to S$ is a ring homomorphism and I an ideal of R with $I \subset Ke\, f$, then there is exactly one homomorphism $\bar{f} : R/I \to S$ with $f = p_I \bar{f}$, i.e. the following diagram is commutative:*

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ {\scriptstyle p_I} \searrow & & \nearrow {\scriptstyle \bar{f}} \\ & R/I & \end{array}$$

If $I = Ke\, f$, then $\bar{f}$ is injective. If f is surjective, then $\bar{f}$ is also surjective.

From this we deduce:

Isomorphism Theorems. *Let I, J be ideals in the ring R. Then:*

(1) If $I \subset J$, then J/I is an ideal in R/I and there is a ring isomorphism

$$(R/I)\,/\,(J/I) \simeq R/J.$$

(2) There is a ring isomorphism

$$(I + J)\,/J \simeq I\,/\,(I \cap J).$$

Proof: (1) For the surjective homomorphism $f : R/I \to R/J$, $a + I \mapsto a + J$, we have $Ke\, f = \{a + I \mid a \in J\} = J/I$.

(2) Consider the surjective ring homomorphism $g : I \to I + J\,/J$, $a \mapsto a + J$, with $Ke\, g = I \cap J$.

1.3 If A is a subset of the ring R, then the smallest (left, right) ideal of R, which contains A, is called the *(left, right) ideal generated by* A and we denote it by ${}_R(A)$, $(A)_R$ resp. (A).

This is just the intersection of all (left, right) ideals of R containing A, for example:

$$ {}_R(A) = \bigcap \{I \subset R \mid I \text{ a left ideal}, A \subset I\}. $$

A possible representation is:

$$ {}_R(A) = \{\sum_{i=1}^{n} k_i a_i + \sum_{i=1}^{n} r_i a_i \mid n \in I\!N,\ k_i \in Z\!\!Z,\ r_i \in R,\ a_i \in A\}. $$

In case R has a unit this simplifies to

$$ {}_R(A) = \{\sum_{i=1}^{n} r_i a_i \mid n \in I\!N,\ r_i \in R,\ a_i \in A\} = RA. $$

If a (left, right) ideal I of R is generated by a *finite* subset $A \subset R$, then I is called *finitely generated*. If I is generated by a single element $a \in R$, then it is called a *(left, right) principal ideal*, e.g.

$$ {}_R(a) \;=\; \{ka + ra \mid k \in Z\!\!Z,\ r \in R\} \;=\; Z\!\!Za + Ra. $$

1.4 R is said to be a *direct sum* of (left, right) ideals A, $B \subset R$ if $R = A + B$ and $A \cap B = 0$. Then A and B are called *direct summands*.

Notation: $R = A \oplus B$. In this case every $r \in R$ can be *uniquely* written as $r = a + b$ with $a \in A$, $b \in B$.

If R is a direct sum of two *ideals* A, B, then every ideal in the ring A is also an ideal in R and we have $AB = BA \subset A \cap B = 0$.

In this case $R = A \oplus B$ can also be considered as the *cartesian product* of the two rings A and B: For $r_1 = a_1 + b_1$, $r_2 = a_2 + b_2$, $a_i \in A$, $b_i \in B$ we obtain from the observation above that

$r_1 r_2 = (a_1 + b_1)(a_2 + b_2) = a_1 a_2 + b_1 b_2$,

i.e. the product in R is just the product in $A \times B$ with the canonical ring structure.

In an analogous way we obtain a representation of R as (ring) product of ideals $A_1, \ldots, A_n$ if

$$ R = \sum_{i=1}^{n} A_i \text{ and } A_k \cap \sum_{i \neq k} A_i \;=\; 0 \text{ for } 1 \leq k \leq n. $$

A *ring* R is called (left, right) *indecomposable* if it cannot be represented as a direct sum of non-zero (left, right) ideals.

A (left, right) *ideal* is called *indecomposable* if it is not a direct sum of non-zero (left, right) ideals (def. as for rings).

1.5 In the definition of a ring R we did not demand the existence of a unit. As a consequence, for example, every ideal in R can be considered as ring. Certainly there need not be a unit in every ring. However, every ring is a subring of a ring with unit:

Dorroh's Theorem.
For every ring R there exists a ring R^* with unit and the properties:
(i) R is isomorphic to a subring of R^*;
(ii) the image of R is an ideal in R^*;
(iii) if R is commutative, then R^* is also commutative.
R^* as constructed below is called the Dorroh overring of R.

Proof: As a set we take $R^* = \mathbb{Z} \times R$ and for pairs (m, a), $(n, b) \in \mathbb{Z} \times R$ we define the compositions

$$(m, a) + (n, b) := (m + n, a + b), \quad (m, a) \cdot (n, b) := (mn, ab + na + mb).$$

It is easy to verify that $(R^*, +)$ is an abelian group with zero element $(0, 0)$, and $(R^*, \cdot)$ is a semigroup with unit element $(1, 0)$ (commutative if $(R, \cdot)$ is commutative). Also the distributivity laws are valid and hence $(R^*, +, \cdot)$ is a ring with unit.

The map $\varepsilon : \; R \to R^*, \quad r \mapsto (0, r)$ obviously is injective and a ring homomorphism, and $R \simeq (R)\varepsilon \subset R^*$. Since

$(n, b)(0, r) = (0, br + nr) \in (R)\varepsilon$ and $(0, r)(n, b) = (0, rb + nr) \in (R)\varepsilon$, $(R)\varepsilon$ is an ideal in R^*.

The ring R^* constructed this way certainly is not the only ring with unit containing R (as an ideal). For example, R may already have a unit and R^* is always a proper extension. In this case we have:

If e is the unit in R, then R^* is the direct sum of the ideals $R^*(0, e) \simeq R$ and $R^*(1, -e)$.

Proof: First we see $R^* = R^*(0, e) + R^*(1, -e)$ since for every $s \in R^*$: $s = s(1, 0) = s(0, e) + s(1, -e)$. For $z \in R^*(0, e) \cap R^*(1, -e)$ we have: $z = (n, b)(0, e) = (m, a)(1, -e)$ for suitable $(n, b), (m, a) \in R^*$. This means $(0, be + ne) = (m, -ae + 1a - me)$, so that $m = 0$ and hence $z = 0$.

Remark. Most of the assertions in this paragraph do not use the associativity of multiplication in R. They are also true for non-associative rings (Exceptions: The assertions at the end of 1.1 and the representation of the ideals generated by a subset in 1.3).

1.6 Exercises. *Show for a ring R :*

(1) The following assertions are equivalent:
(a) R has a unit;
(b) if R is an ideal in a ring S, then R is a direct summand of S;
(c) if R is an ideal in a ring S, then R is a homomorphic image of S.

(2) If R has a unit, then there is a ring homomorphism $\mathbb{Z} \to R$.

(3) For a left ideal $I \subset R$ and $k \in \mathbb{N}$ we have $(I + IR)^k \subset I^k + I^k R$.

(4) If R has a unit and A, B are left (or right) ideals in R with $A+B = R$, then $A \cap B \subset AB + BA$.

2 Special elements and ideals in rings

1.Properties of elements. 2.Annihilators. 3.Direct decomposition and idempotents. 4.Peirce decomposition. 5.Properties of left ideals. 6.Existence of maximal ideals. 7.Properties of minimal left ideals. 8.Maximal ideals are prime. 9.Ideals generated by regular elements. 10.Sum of nilpotent ideals. 11.Nil radical. 12.Sum of all nilpotent ideals. 13.Characterization of the prime radical. 14.Exercises.

After considering special properties of elements of a ring we will turn to special properties of ideals.

2.1 Properties of elements. Definitions.

An element a of the ring R is called a
left zero divisor if $ab = 0$ for some non-zero $b \in R$;
right zero divisor if $ba = 0$ for some non-zero $b \in R$;
zero divisor if it is a left or right zero divisor;
idempotent if $a^2 = a$;
nilpotent if $a^k = 0$ for some $k \in \mathbb{N}$;
regular if there is a $b \in R$ with $aba = a$;
left (right) invertible if R has a unit 1 and there is a $b \in R$ with $ba = 1$ (resp. $ab = 1$) ;
invertible if it is left and right invertible;
central if $ab - ba = 0$ for all $b \in R$.

Two idempotents $e, f \in R$ are called *orthogonal* if $ef = fe = 0$.

An idempotent in R is called *primitive* if it cannot be written as a sum of two non-zero orthogonal idempotents.

Corollaries. (1) An idempotent $e \in R$ which is not a (right) unit is a right zero divisor: For some $a \in R$ we have $a - ae \neq 0$ and $(a - ae)e = 0$.

(2) Every nilpotent element is a zero divisor.

(3) Every left (right) invertible element $a \in R$ is regular: From $ba = 1$ we get $aba = a$, $b \in R$.

(4) Every idempotent is regular.

(5) If $a \in R$ is regular and for $b \in R$ $aba = a$, then ab and ba are idempotent: $(ab)^2 = (aba)b = ab$, $(ba)^2 = b(aba) = ba$.

(6) If zero is the only nilpotent element in R, then every idempotent $e \in R$ is central: since $(e(a - ae))^2 = e(a - ae)e(a - ae) = 0$ we get $ea = eae$ and similarly $ae = eae$ for all $a \in R$.

(7) The central elements of R form a subring, the *centre* $Z(R)$ of R. If R has a unit 1, then of course $1 \in Z(R)$.

2.2 Annihilators. For a non empty subset $A \subset R$ denote by
$An^l_R(A) := \{b \in R \,|\, ba = 0 \text{ for all } a \in A\}$, the *left annihilator* of A,
$An^r_R(A) := \{b \in R \,|\, ab = 0 \text{ for all } a \in A\}$, the *right annihilator* of A,
$An_R(A) := An^l_R(A) \cap An^r_R(A)$, the *annihilator* of A in R.

The following notations are also in use:
$An^l_R(A) = An^l(A) = l(A)$; $\quad An^r_R(A) = An^r(A) = r(A)$.

Properties: Let A be a non empty subset of R. Then:

(1) $An^l(A)$ is a left ideal, $An^r(A)$ a right ideal in R.

(2) If $A \subset Z(R)$, then $An^l(A) = An^r(A)$ is an ideal in R.

(3) If A is a left ideal (right ideal), then $An^l(A)$ (resp. $An^r(A)$) is an ideal in R.

(4) $A \subset An_R(An_R(A))$.

2.3 Direct decomposition and idempotents. *Let R be a ring.*

(1) If a left ideal $A \subset R$ is generated by the idempotent $e \in R$, i.e. $A =_R (e)$, then $R = A \oplus An^l(e)$ is a decomposition in left ideals.

(2) If an ideal $B \subset R$ is generated by a central idempotent $f \in R$, then $R = B \oplus An_R(f)$ is a decomposition in ideals.

(3) If R has a unit 1, then every (left) ideal which is a direct summand is generated by an idempotent $f \in Z(R)$ (resp. $e \in R$).

In this case $An_R(f) = R(1-f)$ (resp. $An^l(e) = R(1-e)$).

Proof: (1) For every $a \in R$ we have $a = ae + a - ae$ with $ae \in Re \subset A$ and $(a - ae) \in An^l(e)$. If $b \in_R (e) \cap An^l(e)$, then $b = re + ne$ for some $r \in R$, $n \in \mathbb{Z}$, and $b = be = 0$, i.e. the sum is direct.

(2) If f is central, then $An^l(f) = An_R(f)$ is a two-sided ideal and the assertion follows from (1) .

(3) Let $R = I \oplus J$ be a decomposition of R in left ideals and $1 = i + j$ with $i \in I$, $j \in J$. Then $i = i^2 + ij$ and $ij = i - i^2 \in I \cap J = 0$. For every $a \in I$ we get $a = ai + aj$ and hence $aj = a - ai \in I \cap J = 0$, i.e. $a = ai$.

If I and J are ideals, then for $b \in R$ we have $bi + bj = b1 = 1b = ib + jb$. Since the representation is unique this implies $bi = ib$, i.e. $i \in Z(R)$.

Obviously $R(1-e) \subset An^l(e)$. On the other hand, if $a \in An^l(e)$, then $a(1-e) = a - ae = a$, i.e. $An^l(e) \subset R(1-e)$.

2.4 Peirce decomposition. *Let e be an idempotent in R. Then*

$$R = eRe \,+\, eAn^l(e) \,+\, An^r(e)e \,+\, An^l(e) \cap An^r(e)$$

is a decomposition of R as a sum of rings.

If $e \in Z(R)$, then this is a decomposition in ideals: $R = Re \oplus An_R(e)$.

Proof: It is easy to check that every summand is a ring. For every $a \in R$ we have

$a = eae + e(a - ae) + (a - ea)e + (a - ea - ae + eae)$,

i.e. R is the sum of the given rings.

The decomposition is unique: Assume $0 = a_1 + a_2 + a_3 + a_4$ with a_i in the corresponding ring. We have to show that all $a_i = 0$. Multiplying with e from one or both sides we get step by step

$0 = e0e = ea_1e = a_1, \ \ 0 = e0 = ea_2 = a_2, \ \ 0 = 0e = a_3e = a_3$

and hence also $a_4 = 0$.

2.5 Properties of left ideals. Definitions.

A left ideal I of R is called

minimal if $I \neq 0$ and it does not properly contain any non-zero left ideal of R;

maximal if $I \neq R$ and it is not properly contained in any left ideal $\neq R$;

nil ideal if every element in I is nilpotent;

nilpotent if there is a $k \in I\!N$ with $I^k = 0$;

idempotent if $I^2 = I$.

In a similar way minimal, maximal, nilpotent, idempotent and nil *right ideals* and *ideals* are defined.

A proper ideal $I \subset R$ is called

prime if for ideals $A, B \subset R$ the relation $AB \subset I$ implies $A \subset I$ or $B \subset I$,

semiprime if it is an intersection of prime ideals.

For a ring R with unit 1 and left ideal $K \neq R$ consider the set

$$\{I \subset R \mid I \text{ left ideal in } R \text{ with } K \subset I \text{ and } 1 \notin I\}.$$

This is an inductive ordered set and hence – by *Zorn's Lemma* – contains maximal elements. These are obviously maximal left ideals containing K.

In the same way we see the existence of maximal ideals and maximal right ideals in rings with units, i.e.:

2.6 Existence of maximal ideals.

In a ring with unit every proper (left, right) ideal is contained in a maximal (left, right) ideal.

By contrast, in arbitrary rings with unit there need not be *minimal* (left) ideals. In case there are any they have the following properties:

2.7 Properties of minimal left ideals.

Let A be a minimal left ideal in the ring R. Then either $A^2 = 0$ or $A = Re$ for some idempotent $e \in A$, i.e. A is either nilpotent or generated by an idempotent.

Proof: If $A^2 \neq 0$, then there is an $a \in A$ with $Aa \neq 0$ and hence $Aa = A$ since A is a minimal left ideal. For (at least) one $e \in A$ we get $ea = a$ and $(e^2 - e)a = 0$. The intersection $An^l(a) \cap A$ is a left ideal of R contained in A and hence zero since $Aa = A \neq 0$. This implies $e^2 = e$. Since $(Ae)a = A(ea) = A \neq 0$ also $Re \neq 0$ is a left ideal contained in A, i.e. $Re = A$.

2.8 Maximal ideals are prime.

In a ring R with unit every maximal ideal is a prime ideal.

Proof: Let A, B and M be ideals, M maximal, and $AB \subset M$. Assume $A \not\subset M$. Then $R = M + A$ and hence
$B = (M + A)B = MB + AB \subset M$.

2.9 Ideals generated by regular elements.

In a ring R with unit every (left) ideal generated by a regular element is idempotent.

Proof: For a regular element $a \in R$ we get from $a \in aRa$ that $Ra \subset RaRa \subset Ra$, i.e. $(Ra)^2 = Ra$, and

$$RaR \subset RaRaR \subset RaR, \text{ i.e. } (RaR)^2 = RaR.$$

While every nilpotent (left) ideal is a nil (left) ideal, the converse need not be true. Important for these ideals is the following observation:

2.10 Sum of nilpotent ideals.

In any ring R we have:

(1) The sum of finitely many nilpotent (left) ideals is nilpotent.

(2) The sum of (arbitrary many) nil ideals is a nil ideal.

Proof: (1) Let I, J be left ideals in R with $I^m = 0$ and $J^n = 0$ for $m, n \in I\!N$. Then $(I + J)^{m+n}$ is a sum of expressions which contain $m + n$ factors from I or J. Any of these expressions contains at least m factors from I or n factors of J and hence is zero. Therefore also $(I + J)^{m+n} = 0$.

(2) By the definition of the sum of ideals in R it suffices to show that the sum of two nil ideals is again a nil ideal: Let I, J be nil ideals and $a + b = z \in I + J$. We have $a^k = 0$ for some $k \in I\!N$. Hence $z^k = a^k + c = c$

for a suitable $c \in J$. Since J is a nil ideal we get $c^r = 0$ for an $r \in I\!N$ and $z^{kr} = c^r = 0$.

Observe that (2) is only shown for two-sided ideals.

2.11 Nil radical. The sum of all nil ideals of a ring R is called the *nil radical N(R)* of R. According to 2.10 $N(R)$ is a nil ideal in R and, by construction, all nil ideals of R are contained in $N(R)$.

Observe that in general $N(R)$ does not contain all nilpotent elements of R. In case R is commutative, then $N(R)$ is equal to the set of all nilpotent elements.

The nil radical of the factor ring $R/N(R)$ is zero, $N(R/N(R)) = \bar{0}$: Assume $I \subset R$ is an ideal in R such that $\overline{I} = (I + N(R))/N(R)$ is a nilideal in $R/N(R)$. Then for every $a \in I$ we have $a^k \in N(R)$ for some $k \in I\!N$, and we find an $r \in I\!N$ with $(a^k)^r = a^{kr} = 0$. Hence I is a nil ideal and $I \subset N(R)$.

For every left ideal $I \in R$ and $k \in I\!N$ we have $(I + IR)^k \subset I^k + I^k R$ (see Exercise 1.6,(3)). If I is a nilpotent left ideal this implies that the (two-sided) ideal $I + IR$ is also nilpotent. Hence every nilpotent left ideal is contained in a nilpotent ideal. Therefore we get:

2.12 Sum of all nilpotent ideals. *In any ring R we have:*
Np(R) := sum of all nilpotent left ideals
= sum of all nilpotent right ideals
= sum of all nilpotent ideals.

Np(R) obviously is an ideal not necessarily nilpotent but nil, i.e. contained in $N(R)$. If $N(R)$ is nilpotent then *N(R)=Np(R)*. In case *Np(R)* is not nilpotent the factor *R/Np(R)* may have nilpotent ideals $\neq 0$ and the following question arises:

How can we get the smallest ideal I for which the factor ring R/I has no nilpotent ideals? We will see from 3.13 that this is the following ideal:

Definition. The *prime radical P(R)* of R is defined to be the intersection of all prime ideals in R.

To describe this intersection we use the following variant of the notion of nilpotency: An element $a \in R$ is called *strongly nilpotent* if every sequence $a_0, a_1, a_2, \ldots$ in R with

$$a_0 = a, \quad a_{n+1} \in a_n R a_n \text{ for all } n \in I\!N,$$

becomes zero after a finite number of steps.

2.13 Characterization of the prime radical.
The prime radical $P(R)$ of a ring R with unit is exactly the set of all strongly nilpotent elements and $Np\,(R) \subset P(R) \subset N(R)$.

Proof: Suppose $a \notin P(R)$. Then there is a prime ideal $P \subset R$ with $a_0 = a \notin P$ and $a_0Ra_0 \not\subset P$. Hence there is an $a_1 \in a_0Ra_0$ with $a_1 \notin P$ and applying this argument repeatedly we obtain an infinite sequence $\{a_i\}_{i\in I\!N}$ with $0 \neq a_{n+1} \in a_nRa_n$, i.e. a is not strongly nilpotent.

Now assume a not to be strongly nilpotent. Then there exists a sequence $\{a_i\}_{i\in I\!N}$ with $a_0 = a$ and $0 \neq a_{n+1} \in a_nRa_n$ for all $n \in I\!N$. Consider the set $S = \{a_i \,|\, i \in I\!N\}$. We have $0 \notin S$ and there is a maximal element P in the set of ideals $I \subset R$ with $I \cap S = \emptyset$.

This P is a prime ideal: Let A, B be ideals in R and $AB \subset P$. Suppose $A \not\subset P$ and $B \not\subset P$. By the choice of P there are $j, k \in I\!N$ with $a_j \in P + A$ and $a_k \in P + B$. Assume $j \leq k$ then $a_k \in a_jRa_j \subset a_jR \subset P + A$ and

$$a_{k+1} \in a_kRa_k \subset (a_jR)(a_kR) \subset (P + A)(P + B) \subset P.$$

This contradicts the choice of P. Hence P is a prime ideal.

If Rb is a nilpotent left ideal, $b \in R$, and Q a prime ideal in R, then for some $k \in I\!N$ $(Rb)^k = 0 \subset Q$ and hence $Rb \subset Q$. Therefore $Rb \subset P(R)$ and also $Np\,(R) \subset P(R)$.

2.14 Exercises. *Show for a ring R:*

(1) An idempotent $e \in R$ is central if and only if it commutes with all idempotents in R.

(2) Let R be a ring with unit and a direct sum of ideals $R_1, \ldots, R_n$.
(i) The centre of R is direct sum of the centres of the rings $R_1, \ldots, R_n$.
(ii) Any ideal $I \subset R$ is a direct sum of ideals in $R_1, \ldots, R_n$.

(3) Let A be an ideal in R which has no zero divisors as a ring.
(i) If, for $r \in R$, $ra = 0$ for some non-zero $a \in A$, then $rA = Ar = 0$;
(ii) $B = \{r \in R \,|\, rA = 0\}$ is an ideal in R and the ring R/B has no zero divisors;
(iii) the ideal $\bar{A} = \{a + B \,|\, a \in A\}$ in R/B is isomorphic (as a ring) to A.

(4) If R has no zero divisors, then there exists an overring of R with unit and without zero divisors. (Hint: Exercise (3).)

(5) Let R be a ring with unit e. Assume that the element $a \in R$ has more than one left inverse. Then there are infinitely many left inverses of a in R. (Hint: With $a_0a = e$, $a_1a = e$ form $aa_1 + a_0 - e$.)

(6) Let R be a commutative ring with unit. If $a \in R$ is invertible and $b \in R$ nilpotent, then $a + b$ is invertible.

(7) A left ideal L in a ring R with unit is maximal if and only if for every $r \in R \setminus L$ there is an $s \in R$ with $1 - sr \in L$.

(8) Let $f : R \to S$ be a surjective ring homomorphism.
(i) For a prime ideal $P \subset R$ with $Ke\, f \subset P$ the image $(P)f$ is a prime ideal in S;
(ii) for a prime ideal $Q \subset S$ the preimage $(Q)f^{-1}$ is a prime ideal in R (with $Ke\, f \subset (Q)f^{-1}$);
(iii) there is a bijection between the prime ideals in R containing $Ke\, f$ and the prime ideals in S.

(9) For an ideal $P \subset R$ consider the following properties:
(a) P is a prime ideal;
(b) for left ideals $A, B \subset R$ with $AB \subset P$ we get $A \subset P$ or $B \subset P$;
(c) for $a, b \in R$ we have: if $aRb \subset P$ then $a \in P$ or $b \in P$;
(d) for a left ideal $A \subset R$ and a right ideal $B \subset R$ with $AB \subset P$ we get $A \subset P$ or $B \subset P$;
(e) for $a, b \in R$ we have: if $ab \in P$ then $a \in P$ or $b \in P$.
Which of the given properties are (always) equivalent? What can be said if R has a unit? What if R is commutative?

(10) Let R be a commutative ring with unit, I an ideal, and S a non-empty subset of R closed under multiplication with $I \cap S = \emptyset$. If an ideal $P \subset R$ is maximal with respect to the properties $I \subset P$ and $P \cap S = \emptyset$, then P is a prime ideal.

(11) Set $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$.
(i) Find all direct summands of ${}_RR$ and R_R with the corresponding idempotents.
(ii) Find idempotents $e, f \in R$ with $Re = Rf$, $eR \neq fR$.

3 Special rings

1.Properties of rings. 2.Left simple rings. 3.Simple rings. 4.Left semisimple rings. 5.Semisimple rings. 6.Structure of left semisimple rings. 7.Minimal idempotent left ideals. 8.Simple rings with a minimal left ideal. 9.Endomorphism ring of a vector space. 10.Regular rings. 11.Strongly regular rings. 12.Subdirect product of rings. 13.Semiprime rings. 14.Commutative semiprime rings. 15.Left fully idempotent rings. 16.Centre of a fully idempotent ring. 17.Fully idempotent rings. 18.Exercises.

There are classes of rings characterized by special properties of their elements or ideals. Some of them we shall consider and describe in this section.

3.1 Properties of rings. Definitions.
A ring R is called
left simple if $R^2 \neq 0$ and there are no non-trivial left ideals in R;
simple if $R^2 \neq 0$ and there are no non-trivial ideals in R;
left semisimple if R is a direct sum of minimal left ideals;
semisimple if R is a direct sum of minimal ideals;
regular if every element $a \in R$ is regular;
strongly regular if for every $a \in R$ there is a $b \in R$ with $a = a^2b$ (or $a = ba^2$);
(left) fully idempotent if every (left) ideal in R is idempotent;
prime if 0 is a prime ideal;
semiprime if 0 is a semiprime ideal.

Next we want to characterize these rings separately and elaborate relations between them:

3.2 Left simple rings. Characterizations.
For a non-trivial ring R the following properties are equivalent :
(a) R is left simple;
(b) R has a unit and every non-zero $a \in R$ has a left inverse;
(c) R is a division ring (skew field) ;
(d) $R^2 \neq 0$ and 0 is a maximal left ideal;
(e) there is an idempotent $e \in R$ which generates every left ideal $\neq 0$ of R.

Proof: $(a) \Rightarrow (b)$ Set $I = \{a \in R \,|\, Ra = 0\}$. I is a left ideal in R and, by (a), we have $I = 0$ or $I = R$. $I = R$ would imply $R^2 = 0$. Hence $I = 0$. For $0 \neq b \in R$ the left ideal $An^l(b) = 0$ since $Rb = 0$ would imply $b \in I = 0$. This means $Rb = R$, i.e. there is a $c \in R$ with $cb = b$. But $(c^2 - c)b = 0$ and hence $c^2 = c$. Since there are no non-zero nilpotent elements in R we get

$c \in Z(R)$ (see 2.1, (6)). For any $d \in R = Rc$ there is an $r \in R$ with $d = rc$ and hence $dc = rc^2 = rc = d$, i.e. c is unit in R. Since $R = Rd$ for $d \neq 0$ there exists a left inverse for d.

$(b) \Rightarrow (c)$ We have to show that there is a (two-sided) inverse for every $0 \neq a \in R$. By assumption there exists $b \in R$ with $ba = 1$ and $c \in R$ with $cb = 1$. Hence $a = (cb)a = c(ba) = c$, i.e. b is a left and right inverse of a.

$(c) \Rightarrow (d)$ Every left ideal $I \neq 0$ contains an element $0 \neq a \in R$ and for some $b \in R$ we get $ba = 1 \in I$ and $R \cdot 1 \subset I$, i.e. $I = R$. Hence 0 is a maximal left ideal.

$(d) \Rightarrow (a)$ Under the given conditions R is (a minimal and) the only non-zero left ideal.

$(c) \Rightarrow (e) \Rightarrow (a)$ are trivial.

Obviously every left simple ring is also simple. On the other hand, a simple ring may have non-trivial left ideals.

As elementary characterizations of simple rings with non-trivial centre we state:

3.3 Simple rings. Characterizations.

For a ring R the following properties are equivalent:

(a) R is simple and $Z(R) \neq 0$;

(b) R has a unit and 0 is a maximal ideal ;

(c) $Z(R)$ is a field and for every ideal $I \neq 0$ of R we have $I \cap Z(R) \neq 0$;

(d) there is a central idempotent ($\neq 0$) in R which generates every non-zero ideal of R.

Proof: $(a) \Rightarrow (b)$ Similar to the proof of 3.2, $(a) \Rightarrow (b)$, it can be seen that there are no elements $0 \neq c \in Z(R)$ with $Rc = 0$. Hence $An(c) = 0$ and $Rc = R$ for every $0 \neq c \in Z(R)$ and we find an $e \in R$ with $ec = c$ so that $(e^2 - e)c = 0$ and $e^2 = e$. For any $b \in R$ we have $(be - eb)c = bec - ecb = bc - cb = 0$ which implies $be = eb$ and $e \in Z(R)$. Therefore $Re = eR = R$, i.e. e is the unit of R.

$(b) \Rightarrow (a)$ is evident.

$(a) \Rightarrow (c)$ We have seen above that R has a unit e. Since $Rc = R$ for every $0 \neq c \in Z(R)$, we find an element $a \in R$ with $ac = e$, and for every $b \in R$ we get $(ba - ab)c = b(ac) - (ac)b = 0$, i.e. $ba = ab$ and $a \in Z(R)$. Hence $Z(R)$ is a field.

$(c) \Rightarrow (a)$ Let e be the unit of $Z(R)$. Then $(An(e) \cap Z(R))e = 0$ and hence $An(e) \cap Z(R) = 0$ which implies $An(e) = 0$ (by (c)). Now for all $a \in R$ we have $(ae - a)e = 0$, i.e. $ae = a$ and e is the unit of R. By (c), we see that e is contained in every non-zero ideal I of R and hence $I = R$.

$(a) \Leftrightarrow (d)$ is obvious.

Of special importance is the class of rings formed by the left semisimple rings with unit which can be described in the following way:

3.4 Left semisimple rings. Characterizations.

For a ring R with unit, the following properties are equivalent:

(a) R is left semisimple;

(b) R is a (finite) sum of minimal left ideals;

(c) every left ideal is a direct summand of R;

(d) every left ideal of R is generated by an idempotent.

Proof: $(a) \Rightarrow (b)$ Let R be a (direct) sum of minimal left ideals $\{U_\lambda\}_\Lambda$: $R = \sum_{\lambda\in\Lambda} U_\lambda$. Then there are elements $u_1, \cdots, u_k$ with $u_i \in U_{\lambda_i},\ \lambda_i \in \Lambda,\ i \leq k$, and $1 = u_1 + \cdots + u_k$. For any $a \in R$ this implies

$a = a1 = au_1 + \cdots + au_k$, i.e. $a \in \sum_{i\leq k} U_{\lambda_i}$ and $R = \sum_{i\leq k} U_{\lambda_i}$.

$(b) \Rightarrow (a)$ Let $U_1, \cdots, U_r$ be minimal left ideals with $R = \sum_{i\leq r} U_i$. If $U_1 \cap \sum_{2\leq i\leq r} U_i \neq 0$, then $U_1 \subset \sum_{2\leq i\leq r} U_i$ and U_1 is superfluous in the above representation of R. If the intersection is zero then U_1 is a direct summand. Then consider U_2 in the remaining sum. Deleting all superfluous summands we obtain a representation of R as (finite) direct sum.

$(c) \Leftrightarrow (d)$ follows (for rings with unit) from 2.3.

$(a) \Rightarrow (c)$ Assume $R = U_1 \oplus \cdots \oplus U_r$ for minimal left ideals $U_i \subset R$ and let $K \subset R$ be a left ideal. Then $U_i \cap K = 0$ or $U_i \cap K = U_i$, i.e. $U_i \subset K$. Without loss of generality suppose $U_1 \cap K = 0$. Then we look for further summands $(:= U_2)$ such that $(U_1 \oplus U_2) \cap K = 0$. Eventually we find (by suitably numbering) $U_1, \cdots, U_s,\ s \leq r$, such that $(U_1 \oplus \cdots \oplus U_s) \cap K = 0$ but every longer partial sum of the U_i's has non-trivial intersection with K. Setting $L = (\oplus_{i\leq s} U_i) \oplus K \subset R$ we want to show that $L = R$:
Certainly $U_1, \cdots, U_s \subset L$. Assume $U_k \cap L = 0$ for some $s < k \leq r$. Then $(U_1 \oplus \cdots \oplus U_s \oplus U_k) \cap K = 0$ contradicting the choice of s. Hence for all $k \leq r$ we have $U_k \cap L \neq 0$ and $U_k \subset L$, i.e. $R = \bigoplus_{i\leq r} U_i \subset L$.

$(c) \Rightarrow (b)$ First we want to show that for every non-zero idempotent $e \in R$ the left ideal $L = Re$ contains a minimal left ideal: Consider the set of left ideals $I \subset R$ with $I \subset L$ and $e \notin I$ and let K be a maximal element in this set (*Zorn's Lemma*). By assumption $K = Rf$ for some idempotent $f \in K$ and obviously $L = Rf \oplus R(e - ef)$. Assume $R(e - ef)$ properly contains an ideal U of R, $0 \neq U \subset R(e - ef)$ and $e - ef \notin U$. Then $e \notin Rf + U \subset L$, contradicting the choice of $K = Rf$.

Now let M denote the sum of all minimal left ideals in R. By assumption there is a left ideal $C \subset R$ with $R = M \oplus C$. Since C does not contain a minimal left ideal of R it has to be zero by the preceding considerations.

Replacing left ideals by ideals and idempotents by central idempotents essentially the same proof yields a characterization of semisimple rings. Similar arguments we shall apply to *semisimple modules* (in 20.2).

3.5 Semisimple rings. Characterizations.

For a ring R with unit, the following properties are equivalent:

(a) R is semisimple;

(b) R is a (finite) sum of minimal ideals;

(c) every ideal is a direct summand of R;

(d) every ideal of R is generated by a central idempotent;

(e) R is a finite product of simple rings (with unit).

Proof: The equivalence of (e) and (b) follows from interpreting the representation of R as sum of ideals as a cartesian product (see 1.4) and the observation that an ideal in a direct summand is also an ideal in R.

Observe that, in general, left semisimple rings do not allow a (ring direct) decomposition in *left simple* rings. However we get:

3.6 Structure of left semisimple rings.

For a left semisimple ring R with unit we have:

(1) R is a direct sum of minimal ideals;

(2) every of this summands is a simple, left semisimple ring.

Proof: (1) We prove that every ideal I is generated by a central idempotent. By assumption there is an idempotent $e \in R$ with $I = Re$. $An^r_R(e) = An^r_R(I)$ is an ideal and $(An^r_R(I) \cap I)^2 \subset I \cdot An^r_R(I) = 0$. Since there are no non-zero nilpotent (left) ideals in R (every ideal is idempotent) this implies $An^r_R(I) \cap I = 0$. For all $t \in I$ we have $e(et - t) = 0$, i.e. $et - t \in An^r_R(e) \cap I = 0$ and $et = t$. For any $r \in R$ obviously $er, re \in I$ and we obtain $re = e(re) = (er)e = er$, i.e. $e \in Z(R)$.

(2) Let T be a minimal ideal in R, $T = Re$ for $e^2 = e \in Z(R)$. A left ideal $J \subset T$ is also a left ideal in R, i.e. $J = Rf$ for some $f^2 = f \in J$. Then $T = Rf \oplus R(e - f)$ is a direct decomposition of T, i.e. J is a direct summand in T.

Now one may ask for the structure of simple, left semisimple rings. To prepare for the answer we first show:

3.7 Minimal idempotent left ideals.

Let R be a ring with unit and $N(R) = 0$. For an idempotent $e \in R$ the following assertions are equivalent:

(a) Re is a minimal left ideal;
(b) eRe is a division ring;
(c) eR is a minimal right ideal.

Proof: eRe is a subring of R with unit e.

$(a) \Rightarrow (b)$ Let Re be a minimal left ideal and $a \in eRe$ with $a \neq 0$. Then $Ra = Re$ and there is an $x \in R$ with $xa = e$. Hence $e = exa = (ex)(ea) = (exe)a$, i.e. a has a left inverse in eRe and eRe is a division ring.

$(b) \Rightarrow (a)$ Assume eRe to be a division ring and I a left ideal in R with $I \subset Re$. Then eI is a left ideal in eRe, i.e. $eI = 0$ or $eI = eRe$. The first equation implies $I^2 \subset ReI = 0$ and $I \subset N(R) = 0$. From $eI = eRe$ we get $e \in eI \subset I$ and $I = Re$. Hence Re is a minimal left ideal.

$(b) \Leftrightarrow (c)$ is seen in a similar way.

Since a simple ring has no non-zero nilpotent left ideals, the minimal left ideals are idempotent (see 2.7) and we get:

3.8 Structure of simple rings with a minimal left ideal.

Let R be a simple ring with unit and I a minimal left ideal in R, i.e. $I = Re$ with $e^2 = e \in R$. Then I is a finite dimensional right vector space over the division ring $D = eRe$ and

$$R \simeq End(I_D) \simeq D^{(k,k)}, \;\; k \in I\!N,$$

where $D^{(k,k)}$ denotes the (k,k)-matrix ring over D.

Proof: By the preceding lemma, $D = eRe$ is a division ring and $I = Re$ is a right vector space over D. For every $a \in R$ we define a map

$f_a : I \to I, \;\; f_a(i) = ai$ for $i \in I$.

It is easy to see that f is a D-vector space homomorphism. (Since I_D is a *right* vector space we write f_a to the *left*.) It is also readily checked that the map

$f : R \to End(I_D), \;\; a \mapsto f_a$ for $a \in R$,

is a ring homomorphism. The kernel of f is an ideal in R and hence zero since it certainly does not contain the unit of R. Now, for every $h \in End(I_D)$ and $b \in I$, we have

$$f_{h(b)}(i) = h(b)\, i = h(b)\, ei = h(bei) = h(bi) = h(f_b(i)) \text{ for all } i \in I,$$

i.e. $f_{h(b)} = h \circ f_b$. This implies that $f(I)$ is a left ideal in $End(I_D)$. From $IR = R$ we derive (since f is a ring homomorphism) $f(R) = f(I)f(R)$ and hence $f(R)$ is also a left ideal. Because $id_I = f_1 \in f(R)$, this means $f(R) = End(I_D)$. Therefore f is a ring isomorphism and, R being simple,

$End(I_D)$ also has to be a simple ring. We will see in our next theorem that this is only possible if I_D has finite dimension.

We know from Linear Algebra that endomorphism rings of k-dimensional D-vector spaces are isomorphic to the (k,k)-matrix rings over D.

Recall that the *rank* of a vector space homomorphism is defined as the dimension of the image space.

3.9 Structure of the endomorphism ring of a vector space.
Let V be a vector space over the division ring K and $S = End(_KV)$.
(1) For every $f \in S$, there exists $g \in S$ with $fgf = f$ (S is regular);
(2) every left ideal containing an epimorphism $f \in S$ is equal to S;
(3) every right ideal containing a monomorphism $g \in S$ is equal to S;
(4) if $dim(_KV) = n \in I\!N$, then $End(_KV)$ is a simple ring;
(5) if $dim(_KV)$ is infinite, then, for $I = \{f \in S \,|\, rank\, f \text{ is finite}\}$, we have:
(i) I is a minimal ideal in S and $I^2 = I$;
(ii) I is not generated by a central idempotent;
(iii) I is not finitely generated as a left ideal;
(iv) if $_KV$ has a countable basis, then I is the only non-trivial ideal in S.
(6) If $dim(_KV)$ is infinite, then, for every infinite cardinal number $\kappa \leq dim(_KV)$, the set $I_\kappa = \{f \in S \mid rank\, f < \kappa\}$ is an ideal in S and every ideal in S is of this form.

Proof: For two vector spaces V, W and a linear map $f : V \to W$ we get from the basis extension theorem (Linear Algebra):

(α) If f is injective, then there is a homomorphism $h : W \to V$ with $fh = id_V$.

(β) If f is surjective, then there is a homomorphism $k : W \to V$ with $kf = id_W$.

(1) Now let $f \in S$ and $\overline{V} = V/Ke\, f$. With the canonical projection $p : V \to \overline{V}$ we have the commutative diagramm

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "p"'] & & V \\ & \overline{V} \arrow[ur, "\bar{f}"'] & \end{tikzcd} .$$

p being surjective, there is a $q : \overline{V} \to V$ with $qp = id_{\overline{V}}$. $\bar{f}$ being injective, we find a $g : V \to \overline{V}$ with $\bar{f}g = id_{\overline{V}}$ and we get

$f = p\bar{f} = (pqp)\bar{f} = p(qf) = (p\bar{f}g)qf = f(gq)f$, with $gq \in S$.

(2) If I is a left ideal and $f \in I$ surjective then there exists $h \in S$ with $hf = id_V \in I$, i.e. $I = S$.

(3) If I is a right ideal and $g \in I$ injective then there exists $k \in S$ with $gk = id_V \in I$, i.e. $I = S$.

(4) Let I be an ideal, $f \in I$ and $v_1, \cdots, v_k$ a basis of V. If $(x)f \neq 0$ for some $x \in V$ then we extend $(x)f =: v$ to a basis by $x_2, \cdots, x_k$ and consider the linear maps:

$g_1 : V \to V,\ \ (v_1)g_1 = x,\ (v_i)g_1 = 0$, and

$h_1 : V \to V,\ \ (v)h_1 = v_1,\ (x_i)h_1 = 0.$
Then $g_1fh_1 \in I$ is a map with $v_1 \in Im\, g_1fh_1$. Constructing $g_i, h_i \in S$ with $v_i \in Im\, g_ifh_i$ in a similar way, we get a surjective map $\tilde{f} = \sum_{i=1}^{k} g_ifh_i \in I$ and hence $I = S$.

(5) Obviously I is an ideal. From the proof of (4) we see that for every non-zero $f \in I$ the ideal generated by f has elements with arbitrary high (finite) rank. This implies (i).

(ii),(iii): Assume $I = Se$, $e^2 = e$ and $rank\, e = k_o \in I\!N$. Then, for all $f \in I$, we have $rank\, f \leq rank\, e = k_o$. However, in I we may find maps with arbitrary high rank.

(iv) If $f \notin I$, then $(V)f \simeq V$, i.e. there is an element $g \in S$ for which fg is surjective. By (2), the ideal generated by f is equal to S.

(6) Using basic facts about cardinal numbers this can be shown in a similar way to the preceding assertions.

We have seen in the above theorem that the endomorphism ring of any vector space is regular. The regular rings were introduced by *John von Neumann* in 1936 in connection with investigations of axiomatic foundations of Geometry (*Continuous Geometry*) and hence are also called *von Neumann regular*.

3.10 Regular rings. Characterizations.

For a ring R with unit, the following properties are equivalent:

(a) R is regular (every $a \in R$ is regular);

(b) every left principal ideal is generated by an idempotent;

(c) every left principal ideal is a direct summand in R;

(d) every finitely generated left ideal is a direct summand in R.

(b), (c) and (d) are also true for right ideals.

Proof: $(a) \Rightarrow (b)$ Let $a \in R$ and $b \in R$ with $aba = a$. Then $e = ba$ is idempotent and $Re \subset Ra$. Since $a = aba \in Re$ we have $Re = Ra$.

$(b) \Rightarrow (a)$ If $a \in R$ and $e \in R$ is idempotent with $Re = Ra$, then there exists $d \in R$ with $e = da$. Hence $a = ae = ada$.

$(b) \Leftrightarrow (c)$ is shown in Lemma 2.3, $(d) \Rightarrow (c)$ is trivial.

$(b) \Rightarrow (d)$ It is enough to show that, for any two idempotents $e, f \in R$, also $Re + Rf$ is generated by an idempotent: First observe

$Re + Rf = Re + R(f - fe)$.

Choose $x \in R$ with $f - fe = (f - fe)x(f - fe)$. Then $g = x(f - fe)$ is an idempotent with $ge = 0$ and

$$Re + Rf = Re + Rg = R(e + g - eg),$$

i.e. the left ideal $Re + Rf$ is generated by the idempotent $e + g - eg$.

A special type of regular rings are the *strongly regular* rings (Def. 3.1) which we characterize in our next theorem. For commutative rings, the notions 'regular' and 'strongly regular' are of course identical.

3.11 Strongly regular rings. Characterizations.

For a ring R with unit, the following properties are equivalent:

(a) R is strongly regular (for any $a \in R$ there exists $b \in R$ with $a = a^2b$);

(b) R is regular and contains no non-zero nilpotent elements;

(c) every left (right) principal ideal is generated by a central idempotent;

(d) R is regular and every left (right) ideal is an ideal.

Proof: $(a) \Rightarrow (b)$ From $a = a^2b$, $a, b \in R$, we get, for every $k \in I\!N$, $a = a^k b^{k-1}$. Hence R cannot contain non-zero nilpotent elements. Also $(a - aba)^2 = a^2 - a^2ba - aba^2 + aba^2ba = 0$, i.e. $a = aba$ and R is regular.

$(b) \Rightarrow (c)$ Since R has no non-zero nilpotent elements, every idempotent in R is central (see 2.1).

$(c) \Rightarrow (d)$ We already know from 3.10 that R is regular by (c). If L is a left ideal in R and $a \in L$ then, for some idempotent $e \in Z(R)$, we have $RaR = ReR = Re = Ra \subset L$, and hence L is a (two-sided) ideal.

$(d) \Rightarrow (a)$ For every $a \in R$, there exists $b \in R$ with $a = aba$. By assumption, Rab is also a right ideal and hence there exists $c \in R$ with $aba = cab$. Then $a = aba = cab = c(aba)b = a^2b$.

Observe that the strongly regular rings generalize division rings but not simple rings: (k,k)-matrix rings always have nilpotent elements for $k > 1$.

A ring R is called *prime* if, for two ideals $A, B \subset R$, the property $AB = 0$ implies A or B is zero. A *commutative* ring is prime if and only if it has no zero divisors ((integral) domain).

An ideal $P \subset R$ is *prime* if and only if the factor ring R/P is prime.

To point out the relationship between prime and semiprime rings it is helpful to consider a special product of rings:

3.12 Subdirect product of rings. Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a family of rings. The cartesian product $\prod_\Lambda S_\lambda = S$ of the S_λ becomes a ring by defining the ring operations in each component. With this structure the canonical projections $\pi_\lambda : S \to S_\lambda$ are ring homomorphisms.

A ring R is called a *subdirect product* of the rings S_λ if there is an injective ring homomorphism $\kappa : R \to S = \prod_{\lambda\in\Lambda} S_\lambda$ such that $\kappa \cdot \pi_\lambda$ is surjective for all $\lambda \in \Lambda$.

For a family of ideals $\{K_\lambda\}_{\lambda\in\Lambda}$ in R and the canonical mappings $\pi_\lambda : R \to R/K_\lambda$, we obtain a ring homomorphism

$$\kappa : R \to \prod\nolimits_\Lambda R/K_\lambda,\ r \mapsto (r + K_\lambda)_{\lambda\in\Lambda},\ r \in R,$$

with kernel $\kappa = \bigcap_\Lambda Ke\,\pi_\lambda = \bigcap_\Lambda K_\lambda$. Hereby the $\kappa \cdot \pi_\lambda$ are surjective and κ is injective if and only if $\bigcap_\Lambda K_\lambda = 0$. In this case R is a subdirect product of the rings R/K_λ.

In a semiprime ring R the intersection of the prime ideals is zero by definition. Thus R is a subdirect product of prime rings. We apply this in

3.13 Characterization of semiprime rings.
For a ring R with unit, the following are equivalent:
(a) R is semiprime (i.e. $P(R) = 0$);
(b) 0 is the only nilpotent (left) ideal in R;
(c) for ideals A, B in R with $AB = 0$ also $A \cap B = 0$;
(d) R is a subdirect product of prime rings.

Proof: $(a) \Rightarrow (b)$ is obvious, since all nilpotent (left) ideals of R are contained in $P(R)$ (see 2.13).

$(b) \Rightarrow (c)$ If $AB = 0$ then $(A \cap B)^2 \subset AB = 0$ and $A \cap B = 0$.

$(c) \Rightarrow (b)$ If $AA = 0$ then also $A \cap A = A = 0$.

$(b) \Rightarrow (a)$ Let $0 \neq a \in R$. Then $(Ra)^2 \neq 0$ and with $a = a_0$ there exists $0 \neq a_1 \in a_0 R a_0$. Then also $(Ra_1)^2 \neq 0$ and we find $0 \neq a_2 \in a_1 R a_1$, and so on. Hence a is not strongly nilpotent and $a \notin P(R)$ (see 2.13). Therefore $P(R) = 0$.

$(a) \Rightarrow (d)$ was outlined in 3.12.

$(d) \Rightarrow (a)$ Let $\kappa : R \to \prod_\Lambda S_\lambda$ be a subdirect product of prime rings $\{S_\lambda\}_{\lambda \in \Lambda}$ (with $\kappa\pi_\lambda : R \to S_\lambda$ surjective).
Then $Ke(\kappa\pi_\lambda)$ is a prime ideal and $P(R) \subset \bigcap_\Lambda Ke(\kappa\pi_\lambda) = Ke\,\kappa = 0$.

Having in mind that the prime radical in a commutative ring consists of all the nilpotent elements we derive from the above result:

3.14 Commutative semiprime rings.
A commutative ring R with unit is a subdirect product of integral domains if and only if it has no nilpotent elements.

As already remarked in 2.9, in a regular ring R every left principal ideal – and hence every left ideal – is idempotent, i.e. R is (left) fully idempotent. The converse need not be true. The left fully idempotent rings are also called *left weakly regular* (see 37.12,(1)). Our elementary approach gives the following

3.15 Characterization of left fully idempotent rings.
For a ring R, the following properties are equivalent:
(a) $L^2 = L$ for every left ideal $L \subset R$;
(b) $a \in RaRa$ for every $a \in R$;
(c) $IL = I \cap L$ for every left ideal L and ideal I in R.

Proof: $(a) \Rightarrow (b)$ For $a \in R$, we have $a \in Ra + \mathbb{Z}a$ and – since $(Ra + \mathbb{Z}a) = (Ra + \mathbb{Z}a)^2 \subset Ra$ – also $a \in Ra = RaRa$.

$(b) \Rightarrow (c)$ Let L be a left ideal and I an ideal in R. Then $IL \subset I \cap L$. For every $a \in I \cap L$ we have $a \in RaR \cdot a \subset IL$, i.e. $I \cap L \subset IL$.

$(c) \Rightarrow (a)$ For every left ideal L of R we have $RL + L = L$ and by (c) $L^2 = L(RL + L) = (LR + L)L = (LR + L) \cap L = L$.

3.16 Centre of a fully idempotent ring.
If R is fully idempotent, then its centre $Z(R)$ is regular.

Proof: For $a \in Z(R)$, by assumption $(Ra)^2 = Ra^2 = Ra$. Since $a \in Ra$ there exists $b \in R$ with $a = a^2 b = aba$ and therefore $(ba)^2 = ba$, $Ra = R(ba)$.

For any $c \in R$, we have

$$c(ba) - (ba)c = (c(ba) - (ba)c)ba = c(ba) - ba^2cb = (cb - cb)a = 0.$$

This means that ba belongs to the centre and $Z(R)a = Z(R)ba$. Hence $Z(R)$ is regular.

Finally we generalize left weakly regular rings:

3.17 Characterization of fully idempotent rings.
For a ring R the following properties are equivalent:
(a) R is fully idempotent (i.e. $I^2 = I$ for every ideal $I \subset R$);
(b) for all ideals I, J of R we have $I \cap J = IJ$;
(c) every factor ring of R is semiprime;
(d) every ideal I in R is an intersection of prime ideals, i.e. I is semiprime.

Proof: $(a) \Rightarrow (b)$ If I, J are ideals in R, then we get from (a) $IJ \supset (I \cap J)^2 = I \cap J$. Since always $IJ \subset I \cap J$, we have $IJ = I \cap J$.

$(b) \Rightarrow (a)$ For an ideal $I \subset R$, we get from (b) $I = I \cap I = I^2$.

$(a) \Rightarrow (c)$ Since every factor ring of R is also fully idempotent, it cannot contain a non-zero nilpotent ideal, i.e. it is semiprime.

$(c) \Rightarrow (a)$ Assume for an ideal $I \subset R$ we have $I^2 \neq I$. Then I/I^2 is a non-zero nilpotent ideal in the ring R/I^2, i.e. R/I^2 is not semiprime.

$(c) \Leftrightarrow (d)$ results from the relationship between prime ideals in R and R/I.

3.18 Exercises. *Verify for a ring R with unit :*

(1) The following assertions are equivalent:
(a) R is a regular ring;
(b) R is a direct sum of ideals $I_1, \ldots, I_n$, and every I_i is a regular ring.

(2) The following assertions are equivalent:
(a) R is fully idempotent;
(b) for ideals I and left ideals A in R, we have $A \cap I \subset AI$;
(c) for ideals I and left ideals A in R with $A \subset I$, we have $A \subset AI$.

(3) Every ideal A in the matrix ring $R^{(n,n)}$ is of the form $I^{(n,n)}$ for some ideal $I \subset R$. (Hint: Consider the set of coefficients of elements in A)

(4) In $R = \mathbb{Q}^{(2,2)}$, for every $r \in \mathbb{Q}$, the set $\{\begin{pmatrix} a & ar \\ b & br \end{pmatrix} \mid a, b \in \mathbb{Q}\}$ *forms a minimal left ideal.*

(5) Set $R = End(V)$, for an infinite dimensional left vector space V over a division ring. For every $k \in I\!N$, there is a decomposition of R into a direct sum of k cyclic left ideals isomorphic to R.

(6) The following are equivalent for a ring R :
(a) Every principal ideal is a direct summand of R;
(b) every principal ideal in R is generated by a central idempotent;
(c) every finitely generated ideal is a direct summand of R;
(d) every finitely generated ideal in R is generated by a central idempotent.
Rings with these properties are called biregular.

(7) Let R be a biregular ring with centre C. Then:
(i) R is left (and right) fully idempotent;
(ii) every prime ideal is maximal in R;
(iii) for every ideal $I \subset C$, we have $I = IR \cap C$;
(iv) for every ideal $A \subset R$, we have $A = (A \cap C)R$.

(8) The following properties are equivalent:
(a) R is strongly regular (vgl. 3.11);
(b) for any two left ideals $L_1, L_2 \subset R$, we have $L_1 \cap L_2 = L_1L_2$;
(c) if L is a left ideal and D a right ideal in R, then $L \cap D = LD$.

(9) R is called a Boolean ring *if each of its elements is idempotent. Assume R to have this property. Show:*
(i) R is commutative and $a = -a$ for all $a \in R$;
(ii) R is regular;
(iii) every subring and every factor ring of R is a Boolean ring;
(iv) if R is a simple ring, then $R \simeq \mathbb{Z}_2 (= \mathbb{Z}/2\mathbb{Z})$;
(v) for any index set Λ, the product R^Λ is a Boolean ring.

4 Chain conditions for rings

1.Left semisimple rings. 2.Nil radical and chain conditions. 3.Artinian rings with zero nil radical. 4.Structure of semiprime left artinian rings. 5.Properties of left artinian rings. 6.Exercises.

One possible way of classifying rings is through finiteness conditions. Later on we will have to investigate different conditions of this type. Two of these can already be treated with the techniques so far developed.

The ring R is said to satisfy the *descending chain condition (dcc)* on left ideals if every descending chain of left ideals $L_1 \supset L_2 \supset L_3 \supset \cdots$ becomes *stationary* after a finite number of steps, i.e. for some $k \in \mathbb{N}$ we get

$$L_k = L_{k+1} = L_{k+2} = \cdots \ .$$

The importance of this finiteness condition was first realized by Emil Artin. If it is satisfied, R is called a *left artinian ring.* Similarly *right artinian rings* are defined. An *artinian ring* is a ring which is left and right artinian.

We call R *left (right) noetherian* if R satisfies the *ascending chain condition (acc)* on left (right) ideals. These rings first were investigated by Emmy Noether. We will come to a more detailed study of both properties in § 27 and § 31. It is easy to see that these finiteness conditions are transferred from a ring to its factor rings. As a first example we have:

4.1 Left semisimple rings.
If R is a left semisimple ring with unit, then R is left artinian and left noetherian.

Proof: Consider a properly descending chain $L_1 \supset L_2 \supset L_3 \supset \cdots$ of left ideals. By 3.4, R is a direct sum of minimal left ideals $U_1, \ldots, U_k$ and, for $L_1 \neq R$, we find – numerating appropriately – $U_1, \ldots, U_{k_1}$ with
$R = L_1 \oplus (U_1 \oplus \cdots \oplus U_{k_1})$.

Since $L_2 \neq L_1$, the sum $U_1 \oplus \cdots \oplus U_{k_1}$ is not maximal with respect to $L_2 \cap (U_1 \oplus \cdots \oplus U_{k_1}) = 0$, because this would imply

$$R = L_2 \oplus (U_1 \oplus \cdots \oplus U_{k_1}) = L_1 \oplus (U_1 \oplus \cdots \oplus U_{k_1}),$$

and hence $L_2 = L_1$. Therefore $R = L_2 \oplus (U_1 \oplus \cdots \oplus U_{k_2})$, with $k_2 > k_1$. After a finite number of steps we arrive at $R = L_n \oplus (U_1 \oplus \cdots \oplus U_k)$, i.e. $L_n = 0$.

A similar argument shows that R is also left noetherian.

4.2 Nil radical and chain conditions.
The nil radical $N(R)$ of a left artinian or left noetherian ring R (not necessarily with unit) is nilpotent.

Proof: Set $N = N(R)$. If R is left artinian, the descending chain of left ideals $N \supset N^2 \supset N^3 \supset \cdots$ becomes stationary, i.e. for some $k \in I\!N$ we get $N^k = N^{k+1} = N^{2k}$. For the ideal $M = N^k$, this means $M = M^2 \neq 0$ in case N is not nilpotent.

Because of the descending chain condition the non-empty set of left ideals $\mathcal{A} = \{A \subset R \mid MA \neq 0\}$ has a minimal element B. Since $MB \neq 0$, there exists $b \in B$ with $Mb \neq 0$ and hence $0 \neq Mb = M^2b = M(Mb)$, i.e. $Mb \in \mathcal{A}$. Now $Mb \subset B$ and the minimality of B imply $Mb = B$ and there exists $m \in M$ with $mb = b$. $m \in N$ being nilpotent, this means $b = m^l b = 0$ for some $l \in I\!N$, a contradiction. Therefore N has to be nilpotent.

Now let R be left noetherian. Then every ascending chain of nilpotent left ideals is finite, the sum of all nilpotent left ideals *Np(R)* is a nilpotent ideal (see 2.10), and $\overline{R} = R/Np(R)$ has no nilpotent ideals. We already know from 2.12 that $Np(R) \subset N(R)$. To prove $N(R) \subset Np(R)$ it is sufficient to show that nil ideals in $\overline{R}$ are nilpotent.

Assume $I \subset \overline{R}$ to be a non-zero nil ideal. For $x \in \overline{R}$, the annihilators $An^l_{\bar{R}}(x) = \{a \in \overline{R} \mid ax = 0\}$ are left ideals in $\overline{R}$.

Hence the set $\{An^l_{\bar{R}}(x) \mid 0 \neq x \in I\}$ has a maximal element $An^l_{\bar{R}}(x_o)$ for some $x_o \in I$. For every $r \in \bar{R} \cup Z\!\!Z$, either $x_o r = 0$ or there exists $k \in I\!N$ with $(x_o r)^k \neq 0$ and $(x_o r)^{k+1} = 0$. By the choice of x_o this means $An^l_{\bar{R}}(x_o) = An^l_{\bar{R}}((x_o r)^k)$ and $x_o r x_o = 0$.

Therefore $Z\!\!Z x_o + \bar{R}x_o + x_o\bar{R} + \bar{R}x_o\bar{R}$ is a nilpotent ideal (square zero) and has to be zero, giving $x_o = 0$.

We will see in 31.4 that every left artinian ring with unit is left noetherian. In this case the first part of the proof is superfluous.

One of the most important consequences of the descending chain condition for left ideals in R is the existence of minimal left ideals. This is the crucial point in the proof of

4.3 Artinian rings with zero nil radical.
Let R be a left artinian ring (not necessarily with unit) and assume $N(R) = 0$. Then R is left semisimple.

Proof: We show that every left ideal I in R is generated by an idempotent. I contains a minimal left ideal A and $A^2 \neq 0$ since $N(R) = 0$. This is generated by an idempotent (see 2.7), i.e. there are idempotents $\neq 0$ in I

and the set of left annihilators of idempotents in I, $\{An^l(e) \mid e^2 = e \in I\}$, is non-empty. Because of the descending chain condition for left ideals there must be a minimal element in this set, say $An^l(f)$ with $f^2 = f \in I$.

Assume $I \cap An^l(f) \neq 0$. Then there is a minimal left ideal in this intersection which again is generated by an idempotent g, i.e. $0 \neq g \in I \cap An^l(f)$. Putting $h = f + g - fg$ we get (since $gf = 0$) $h^2 = h \in I$ and $hf = f$. This means $An^l(h) \subset An^l(f)$. Since $g \in An^l(f)$ but $gh = g \notin An^l(h)$ this inclusion is proper. By the minimality of $An^l(f)$, we get $An^l(h) = 0$. Hence h is a right unit in R and $I = R$. For $I \neq R$, this contradicts the assumption $I \cap An^l(f) \neq 0$, i.e. $I \cap An^l(f) = 0$. For every $a \in I$, we have $(a - af)f = 0$ and $a - af \in I \cap An^l(f) = 0$ which means $I = Rf$.

Combining the above information we can formulate the following structure theorem for left artinian rings which presents one of the important theorems of Classical Algebra and was the starting point for many generalizations. It was first proved by *J.H.M. Wedderburn* for finite dimensional algebras (see § 5) and then extended to left artinian rings by *Emil Artin*.

4.4 Structure theorem for semiprime left artinian rings.

For a ring R with unit, the following properties are equivalent:

(a) R is left artinian and semiprime ($N(R) = 0$);

(b) R is left semisimple;

(c) R is isomorphic to a finite product of finite matrix rings over division rings.

Proof: $(a) \Rightarrow (b)$ is shown in 4.3.

$(b) \Rightarrow (c)$ It is shown in 3.6 that a left semisimple ring is a finite product of simple rings. By 3.8, all these simple rings are finite matrix rings over division rings.

$(c) \Rightarrow (a)$ R being a finite product of simple rings, every ideal is generated by a central idempotent. Hence R has no nilpotent ideals, i.e. R is semiprime. Finite matrix rings over a division ring D are in particular finite dimensional vector spaces over D. Since left ideals are also D-subspaces, the descending chain condition for left ideals is obviously satisfied. Now every left ideal in R can be written as a direct sum of left ideals in the matrix rings. Hence R is left artinian.

4.5 Properties of left artinian rings.

Let R be a left artinian ring with unit. Then:

(1) $R/N(R)$ is a finite product of matrix rings over division rings.

(2) If P is a prime ideal, then P is maximal, i.e. R/P is simple.

(3) If R is regular, then R is left semisimple.

(4) If R is strongly regular, then R is a finite product of division rings.

Proof: (1) $R/N(R)$ is left artinian and semiprime and the assertion follows from 4.4.

(2) If P is a prime ideal, then R/P is left artinian and semiprime, i.e. a finite product of simple rings. A prime ring cannot be written as a direct sum of non-trivial ideals, i.e. R/P is simple and P is maximal.

(3) Regular rings are of course semiprime.

(4) R is a finite product of matrix rings without nilpotent elements.

4.6 Exercises. *Prove that*

(1)(i) $R = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix}$ *is a subring of* $\mathbb{R}^{(2,2)}$;

(ii) R is left artinian (and noetherian) but not right artinian.

(2)(i) $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ *is a subring of* $\mathbb{Q}^{(2,2)}$;

(ii) R is right noetherian but not left noetherian.

(3) A left artinian ring with unit is a finite direct sum of indecomposable left ideals.

(4) For a Boolean ring R with unit (see 3.18,(9)), the following properties are equivalent:

(a) R is artinian;

(b) R is noetherian;

(c) R is semisimple;

(d) R is finite.

(5) Assume the ring R with unit is a direct sum of ideals $I_1, \ldots, I_n$. Then R is left artinian (noetherian) if and only if all these ideals are left artinian (noetherian) rings.

5 Algebras and group rings

1.Structure of simple algebras. 2.Structural constants. 3.Semigroup ring. 4.R as subring of RG. 5.G as subsemigroup of RG. 6.Remarks. 7.Semisimple group algebras. 8.Exercises.

Let A be a vector space over the field K. A is called an *algebra over* K (or a K-*algebra*) if an associative product is defined on A which, together with addition in the vector space, turns A into a ring, and moreover

$$k(ab) = (ka)b = a(kb) \text{ for all } k \in K,\ a, b \in A.$$

If the product on A is not necessarily associative A is called a *non-associative algebra*.

In a K-algebra A with unit e, for every $k \in K$ we get $k \cdot e \in A$ and $(ke)a = k(ea) = k(ae) = (ka)e = a(ke)$ for all $a \in A$, i.e. $ke \in Z(A)$. Hence in this case we have a mapping

$$\varphi : K \to Z(A), \quad k \mapsto ke.$$

It is easy to verify that this is an (injective) ring homomorphism. On the other hand, such a ring homomorphism turns a ring A into a K-algebra.

Remark: For a commutative ring K with unit (not necessarily a field), K-algebras are defined as above replacing the vector space by a unitary K-*module* A (see § 6). The further considerations remain unchanged, however, φ need not be injective. Although we will begin to study modules only in the next section we want to keep an eye on this situation now.

An *ideal* in a K-algebra A is a subset I which is an ideal with respect to the ring structure of A and a K-subspace with respect to the vector space structure. This definition of an ideal $I \subset A$ means that the cosets A/I allow a ring structure as well as a K-vector space structure and A/I is also a K-algebra (*factor algebra*).

If A, B are two K-algebras, a mapping $f : A \to B$ is called a K-*algebra homomorphism* if it is both a ring and a K-vector space homomorphism. The kernel of f is an (algebra) ideal I in A and the canonical map $A \to A/I$ is a K-algebra homomorphism.

It is nice to observe that in a K-algebra A with unit e the algebra ideals are exactly the ring ideals: If I is 'only' a ring ideal, then, for every $k \in K$, we have $kI = (k \cdot e)I \subset I$. Hence I is also a K-subspace.

Every K-algebra with finite dimension as a K-vector space – it is called a *finite dimensional algebra* – satisfies the descending and ascending chain conditions for left (and right) ideals.

Examples of K-algebras are
- the endomorphism rings of K-vector spaces (resp. K-modules), in particular the matrix rings over K.
- the polynomial rings in one or more (commuting) indeterminates over K and their factor rings.

Semiprime finite dimensional algebras with unit are left artinian and hence can be written as a direct sum of matrix rings over division rings. In addition we see that these division rings are again finite dimensional K-vector spaces. For the structure of division rings this is an important property:

5.1 Structure of simple algebras.
Let A be a finite dimensional simple K-algebra with unit. Then:

(1) A is isomorphic to a finite matrix ring over a division ring D which is a finite dimensional K-algebra.

(2) If K is algebraically closed, then A is isomorphic to a finite matrix ring over K (and $K \simeq Z(A)$).

Proof: (1) For a minimal left ideal $I \subset A$, there is an idempotent $e \in A$ with $I = Ae$ and $D = eAe$ is a division ring (see 3.8). eAe is – as can easily be verified – a subspace of the finite dimensional vector space A, and hence finite dimensional.
(2) By (1), D is finite dimensional over K. For every $d \in D$, the ring extension $K[d] \subset D$ is finite dimensional, the powers $\{d^i\}_{i \in \mathbb{N}}$ are not linearly independent and hence d is algebraic over K (i.e. the zero of a polynomial in $K[X]$). K being algebraically closed, we get $d \in K$, hence $D = K$. Since the centre of a matrix ring over K is isomorphic to K we conclude $K \simeq Z(A)$.

5.2 Structural constants. Let A be an algebra over the field K and $\{a_i\}_{i \in \Lambda}$ a vector space basis of A. For $i, j \in \Lambda$ the product $a_i a_j$ can be written as

$$a_i a_j = \sum_{k \in \Lambda} \alpha_{ij}^{(k)} a_k \quad \text{with uniquely determined } \alpha_{ij}^{(k)} \in K.$$

These *structural constants* uniquely determine the algebra structure of A. On the other hand, given structural constants $\alpha_{ij}^{(k)}$, $i, j, k \in \Lambda$, the above relations define a product $a_i a_j$ which can be extended distributively to a product in A if $\alpha_{ij}^{(k)} \neq 0$ for only finitely many $k \in \Lambda$. The product obtained this way in general need not be associative. It is associative if

$$\sum_r \alpha_{ij}^{(r)} \alpha_{rk}^{(s)} = \sum_r \alpha_{jk}^{(r)} \alpha_{ir}^{(s)} \text{ for all } i, j, k, s \in \Lambda,$$

and it is commutative if $\alpha_{ij}^{(k)} = \alpha_{ji}^{(k)}$ for all $i, j, k \in \Lambda$.

5.3 Semigroup ring. Definition. Let G be a (multiplicative) semigroup with unit element e_G and R a ring with unit 1. Put

$$RG = \{f : G \to R \mid f(a) \neq 0 \text{ for only finitely many } a \in G\},$$

and define, for $f, g \in RG,\ a \in G$,

$$\begin{array}{c} f+g \text{ by } (f+g)(a) = f(a)+g(a), \\ f \cdot g \text{ by } f \cdot g(a) = \sum_{a_1a_2=a} f(a_1)g(a_2). \end{array}$$

This turns $(RG, +, \cdot)$ into a ring with *zero* $n\ :\ n(a) = 0$ for all $a \in G$, and *unit* $e\ :\ e(e_G) = 1,\ e(b) = 0$ for $e_G \neq b \in G$.
This ring is called the *semigroup ring* of G over R.

The ring axioms for these operations are easily verified, e.g. the associativity of multiplication is derived from associativity in G:

$$\begin{array}{rcl} [(fg)h](a) & = & \sum_{a_1a_2=a} fg(a_1)h(a_2) \\ & = & \sum_{a_1a_2=a} \sum_{a_3a_4=a_1} f(a_3)g(a_4)h(a_2) \\ & = & \sum_{a_3a_4a_2=a} f(a_3)g(a_4)h(a_2) \\ & = & \sum_{a_3b=a} f(a_3) \sum_{a_4a_2=b} g(a_4)h(a_2) \\ & = & \sum_{a_3b=a} f(a_3)gh(b) = [f(gh)](a). \end{array}$$

5.4 *R* as a subring of *RG*. In the following way R can operate on RG from the left and the right $(r, s \in R,\ f \in RG)$:

$$\begin{array}{l} R \times RG \to RG,\ (r, f) \mapsto rf :\ [rf](a) = rf(a) \text{ for all } a \in G, \\ RG \times R \to RG,\ (f, s) \mapsto fs :\ [fs](a) = f(a)s \text{ for all } a \in G. \end{array}$$

We obviously have $(rs)f = r(sf),\ (rf)s = r(fs),\ f(rs) = (fr)s$ for all $r, s \in R,\ f \in RG$ (RG is an (R, R)-bimodule).

The mapping $R \to RG,\ r \mapsto re,\ r \in R,$
yields an injective ring homomophism and hence we may consider R as a subring of RG.

For a commutative ring R, in this way RG becomes an R-algebra ($Re \subset Z(RG)$), the *semigroup algebra* of G over R. If in addition G is commutative, then RG is also commutative as a ring.

As a well-known example we obtain, for $G = (I\!N, +)$, the *polynomial ring* over R in one indeterminate: $RI\!N = R[X]$.

5.5 *G* as a subsemigroup of *RG*. Also the semigroup G can be embedded into RG for an arbitrary ring R. For this we define:

$$G \to RG,\ a \mapsto f_a:\ f_a(c) = \delta_{a,c} = \begin{cases} 1 & \textit{for}\ \ c = a, \\ 0 & \textit{otherwise}, \quad a, c \in G. \end{cases}$$

This is an injective semigroup homomorphism (with respect to multiplication in RG) and the f_a commute with the elements in R, since, for any $a, b, c \in G$ and $r \in R$,

$$f_a f_b(c) = \textstyle\sum_{c_1 c_2 = c} f_a(c_1) f_b(c_2) = \delta_{ab,c} = f_{ab}(c),$$
$$r f_a(c) = r\delta_{a,c} = \delta_{a,c} \cdot r = f_a(c) r.$$

The family $\{f_a\}_{a\in G}$ gives, for every element $f \in RG$, a unique representation

$$f = \sum_{a\in G} r_a f_a, \ \text{with}\ r_a \in R\ \text{being zero almost everywhere.}$$

This is possible since, for every $c \in G$,

$$[\sum_{a\in G} f(a) f_a](c) = \sum_{a\in G} f(a) f_a(c) = f(c), \ \text{i.e.}\ \sum_{a\in G} f(a) f_a = f,$$

and in the above representation we get $r_a = f(a)$.

If R is division ring (arbitrary ring), then the $\{f_a\}_{a\in G}$ form a basis of the vector space (free R-module) RG.

5.6 Remarks: (1) The embedding $G \to RG$ just constructed allows us to consider the elements of G as elements of RG and to write $f \in RG$ as $f = \sum_{a\in G} r_a a$, $r_a \in R$ zero almost everywhere. Therefore RG is also called the ring of *formal linear combinations* of elements of G with coefficients in R.

(2) If R is a field (commutative ring), then RG is an R-algebra and the multiplication in RG is determined by the structural constants obtained by multiplying the base elements $\{f_a\}_{a\in G}$ (see 5.2). These are 1 or 0 according to the multiplication table of G.

(3) Recall that, for a commutative ring R and $G = (I\!N, +)$, using the above notation an *indeterminate* $X = f_1$ is defined by

$$f_1 : R \to RI\!N = R[X],\ f_1(r) = \delta_{1,r},\ r \in R.$$

Since $(I\!N, +)$ is cyclic, we have in this case $f_n = f_1^n = X^n,\ n \in I\!N$.

(4) If G is a *group* and R a (commutative) ring, then RG is called the *group ring* (*group algebra*) of G over R. Since now every element of G has an inverse, the multiplication in RG can be written as:

$$(f \cdot g)(a) = \sum_{a_1 a_2 = a} f(a_1) g(a_2) = \sum_{b\in G} f(b) g(b^{-1}a),$$

with $f, g \in RG,\ a, a_1, a_2, b \in G$.

A classical result in the theory of group algebras is *Maschke's Theorem*:

5.7 Semisimple group algebras.

Let G be a finite group of order n and K a field of characteristic p. Then the group algebra KG is left semisimple if and only if p does not divide n.

Proof: Assume p does not divide n (or $p = 0$) and I is a left ideal in KG. We have to show that I is a direct summand. Since I is a subspace, there is a subspace $L \subset KG$ with $KG = I \oplus L$ as K-vector space. For $g \in G$ and $a \in KG$, there is a unique representation

$$ga = a_g + a' \text{ with } a_g \in I,\ a' \in L,$$

and we get the linear mappings

$$\varphi_g : KG \to KG,\ a \mapsto a',\ a \in KG,$$

which can be used to construct the following K-linear map:

$$\varphi : KG \to KG,\ \varphi = \frac{1}{n} \sum_{g \in G} g^{-1} \varphi_g.$$

Then $Im\,\varphi$ (= image of φ) is a subspace and in fact a left ideal, since for $a \in KG,\ b \in G$, we get

$$b\varphi(a) = \frac{1}{n} \sum_{g \in G} bg^{-1} \varphi_g(a) = \frac{1}{n} \sum_{g \in G} bg^{-1} \varphi_{gb^{-1}}(ba) = \varphi(ba).$$

For every $a \in KG$, we have the relation

$$\varphi(a) = \frac{1}{n} \sum_{g \in G} g^{-1}(ga - a_g) = a - b, \text{ for some } b \in I,$$

and hence $a = b + \varphi(a) \in I + Im\,\varphi$.

For $c \in I \cap Im\,\varphi$, there exists $a \in KG$ with $c = \varphi(a) \in I$, and by the above line this means $a \in I$. But for all $a \in I$, we know $\varphi_g(a) = 0$, i.e. $\varphi(a) = 0$. Hence $I \cap Im\,\varphi = 0$ and KG is a direct sum of I and $Im\,\varphi$, i.e. KG is left semisimple.

Now let p divide n. Putting $a = \sum_{g \in G} g$, we get $a \neq 0$ and $ax = xa = a$ for all $x \in G$, i.e. $a \in Z(KG)$ and aKG is an ideal in KG. For this ideal we obtain

$$(aKG)^2 = a^2 KG = (na)KG = 0,$$

since p divides n and $p1a = 0$. Therefore KG is not left semisimple.

5.8 Exercises.

(1) Let $G = \{e, a, b, c\}$ be a group with unit element e and the compositions $a^2 = b^2 = c^2 = e$, $ab = ba = c$ (Kleinian group).

Find a decomposition of the group algebra $\mathbb{Q}G$ as a direct sum of simple left ideals.

(2) Let R be a commutative ring with unit and G a cyclic group of order $n \in \mathbb{N}$. Prove that $RG \simeq R[X]/(X^n - 1)$.

Literature for Chapter 1: ANDERSON-FULLER, KASCH, ROTMAN; Aribaud, Baccella [1], Franzsen-Schultz, Hauptfleisch-Roos, Hirano, Okninski, Ramamurthi [1], Szeto-Wong.

Chapter 2

Module categories

6 Elementary properties of modules

1.Modules. 2.Submodules. 3.Rings without unit. 4.Homomorphisms and endomorphism rings. 5.Homomorphisms and factor modules. 6.Generating sets. 7.Maximal submodules. 8.Exercises.

We begin with presenting some basic notions, most of which are known from Linear Algebra. In particular we encounter familiar constructions with vector spaces.

6.1 Modules. Let M (more precisely $(M,+)$) be an abelian group. With the usual addition and composition of maps, the set of (group) endomorphisms of M form a ring. Writing homomorphisms as operations from the left, the product of two homomorphisms $f, g : M \to M$ is
$f \circ g(m) = f(g(m))$ for all $m \in M$,
while writing on the right yields
$(m)g * f = ((m)g)f$ for all $m \in M$.
Hence the endomorphism ring of M may operate on M from the left and we denote it by $End^l(M)$, or from the right and we write $End^r(M)$.

Obviously there is an anti-isomorphism between the two rings:

$$End^l(M) \to End^r(M), \quad f \mapsto f, \quad f \circ g \mapsto g * f.$$

Usually it is clear from the context what we mean and we simply let $End(M)$ denote $End^l(M)$ and $End^r(M)$ and write $f \circ g = fg$ and $g * f = gf$.

Let R be an associative ring and M an abelian group. If there is a ring homomorphism $\varphi : R \to End^l(M)$, then M (more precisely (M, R, φ)) is called a *left module over* R and is denoted by ${}_RM$. In case R has a unit 1 and $\varphi(1) = id_M$, then M is said to be a *unital left module.*

${}_RM$ is called *faithful* if φ is injective.

If R is a division ring, then ${}_RM$ is a *vector space* over R.

Putting $\varphi(r)(m) = rm$, for $r \in R,\ m \in M$, we obtain a map

$$\mu : R \times M \to M,\ (r, m) \mapsto rm,$$

with the properties

$$r(m + n) = rm + rn, \quad (r + s)m = rm + sm, \quad (rs)m = r(sm),$$

for $r, s \in R,\ m, n \in M$, and $1m = m$ if M is unital.

It is easy to check that for every such μ and $r \in R$ the map

$$\mu(r, -) : M \to M,\ m \mapsto \mu(r, m),\ m \in M,$$

is an endomorphism of the group M and

$$R \to End^l(M),\ r \mapsto \mu(r, -),\ r \in R,$$

defines a ring homomorphism, i.e. M becomes a left module.

A ring homomorphism $\psi : R \to End^r(M)$ turns M into a *right module* M_R and the above assertions hold similarly.

Every right R-module M_R can be considered as a left module over the *opposite ring* R^o. The ring R^o is based on the additive group $(R, +)$. The 'new' multiplication is obtained by multiplying in R with reverse order: $r \circ s := sr,\ r, s \in R$.

Now the map $R^o \xrightarrow{\psi} End^r(M) \longrightarrow End^l(M)$ is a ring homomorphism.

If R and S are rings and M is a left module over R and a right module over S, then M is called an *(R,S)-bimodule* if for all $r \in R,\ s \in S$ and $m \in M$ we have $(rm)s = r(ms)$.

Examples:

(1) For every abelian group M, the map $\mathbb{Z} \to End^l(M),\ n \mapsto n\, id_M$, is a ring homomorphism. Hence M is a $\mathbb{Z}$-module and module theory generalizes abelian groups. M is also a left module over $End^l(M)$.

(2) Of course, every ring R is an additive group and the maps

$R \to End^l(R),\ r \mapsto L_r :\ L_r(a) = ra$ for all $a \in R$,

$R \to End^r(R),\ r \mapsto R_r :\ (a)R_r = ar$ for all $a \in R$,

are ring homomorphisms which turn R into a left module ${}_RR$, a right module R_R and a bimodule ${}_RR_R$. Every left ideal in R is a left module over R, and every ideal in R is an (R, R)-bimodule.

The cartesian product R^Λ, Λ finite or infinite, is also a left R-module, right R-module and (R,R)-bimodule.

(3) If $\psi : R \to S$ is a ring homomorphism (e.g. $R \subset S$), then the map

$$R \times S \to S,\ (r,s) \mapsto \psi(r)\cdot s,\ r \in R,\ s \in S,$$

obviously satisfies all conditions to make S a left module over R. Similarly, S can be considered as a right module over R.

(4) For every ring R and every semigroup with unit G, the semigroup ring RG is an (R,R)-bimodule (see 5.4).

6.2 Submodules. Let M be a left module over R. A subgroup N of $(M,+)$ is called a *submodule* of M if N is closed under multiplication with elements in R, i.e. $rn \in N$ for all $r \in R$, $n \in N$. Then N is also an R-module by the operations induced from M:

$$R \times N \to N,\ (r,n) \mapsto rn,\ r \in R,\ n \in N.$$

M is called *simple* if $M \neq 0$ and it has no submodules except 0 and M. The submodules of ${}_RR$ (resp. ${}_RR_R$) are just the left (resp. two-sided) ideals.

For non-empty subsets $N_1, N_2, N \subset M$, $A \subset R$ we define:

$$\begin{array}{rcll} N_1 + N_2 & = & \{n_1 + n_2 \mid n_1 \in N_1,\ n_2 \in N_2\} & \subset\ M, \\ AN & = & \{\sum_{i=1}^k a_i n_i \mid a_i \in A,\ n_i \in N,\ k \in I\!N\} & \subset\ M. \end{array}$$

If N_1, N_2 are submodules, then $N_1 + N_2$ is also a submodule of M. For a left ideal $A \subset R$, the product AN is always a submodule of M.

For any infinite family $\{N_i\}_{i\in\Lambda}$ of submodules of M, a sum is defined by

$$\sum_{\lambda\in\Lambda} N_\lambda = \{\sum_{k=1}^{r} n_{\lambda_k} \mid r \in I\!N,\ \lambda_k \in \Lambda,\ n_{\lambda_k} \in N_{\lambda_k}\} \subset M.$$

This is a submodule in M. Also the intersection $\bigcap_{\lambda\in\Lambda} N_\lambda$ is a submodule of M. $\sum_{\lambda\in\Lambda} N_\lambda$ is the smallest submodule of M which contains all N_λ, $\bigcap_{\lambda\in\Lambda} N_\lambda$ is the largest submodule of M which is contained in all N_λ.

An important property of these constructions is the

Modularity condition.
If K, H, L are submodules of M and $K \subset H$, then

$$H \cap (K + L) = K + (H \cap L).$$

Proof: First observe

$$K + (H \cap L) = (H \cap K) + (H \cap L) \subset H \cap (K + L).$$

If $h = k + l \in H \cap (K + L)$ with $h \in H$, $k \in K$, $l \in L$, then $k \in K \subset H$, i.e. $l \in H \cap L$. Therefore $H \cap (K + L) \subset K + (H \cap L)$.

Remark: A (complete) *lattice* is an ordered set such that any two elements (any non-empty subset) have (has) a smallest upper and largest lower bound. The above considerations show that the submodules of a module form a *complete modular lattice* with respect to inclusion.

6.3 Rings without unit. Let R be a ring without unit, M an R-module, and $R^* = \mathbb{Z} \times R$ the Dorroh overring of R (see 1.5). By setting

$$R^* \times M \to M,\ (k, r) \cdot m = km + rm, \text{ for } k \in \mathbb{Z},\ r \in R,\ m \in M,$$

M becomes a unital R^*-module and a subgroup N of M is an R-submodule of ${}_RM$ if and only if it is an R^*-submodule of ${}_{R^*}M$. By this observation the structure theory of modules over rings without unit is reduced to modules over (different) rings with unit. Hence we make the

Convention: *In what follows usually R will be a ring with unit and R-modules will be unital.*

Observe that passing from R to R^* may change properties of the ring, e.g. there may occur (new) zero divisors.

6.4 Homomorphisms and endomorphism rings.
Let M and N be left modules over the ring R. A map $f : M \to N$ is called an *(R-module) homomorphism* (also *R-linear map*) if

$$\begin{array}{rcll} (m_1 + m_2)f & = & (m_1)f + (m_2)f & \text{for all } m_1, m_2 \in M, \\ (rm)f & = & r[(m)f] & \text{for all } m \in M,\ r \in R. \end{array}$$

There are certain advantages to writing homomorphisms of left modules on the right side and we will do so. Homomorphisms of right modules usually will be written on the left.

The set of R-homomorphisms of M in N is denoted by $Hom_R(M, N)$ or $Hom({}_RM, {}_RN)$ or simply $Hom(M, N)$, if it is clear which ring R is meant.

Addition of two $f, g \in Hom_R(M,N)$ is defined in an obvious way and again yields an R-homomorphism. This turns $Hom(M,N)$ into an abelian group. In particular, with this addition and the composition of mappings, $Hom_R(M,M) = End_R(M)$ becomes a ring, the *endomorphism ring* of M.

In our notation M is a right module over $S = End_R(M)$ $(\subset End^r_{\mathbb{Z}}(M))$ and an (R,S)-bimodule.

(R,S)-submodules of M are called *fully invariant* or *characteristic* submodules.

Also $B = End(M_S)$ is a ring and M a left module over B $(\subset End^l_{\mathbb{Z}}(M))$. B is called the *biendomorphism ring* of $_RM$ $(=Biend(_RM))$. Left multiplication with elements $r \in R$, $\tilde{r} : M \to M$, $m \mapsto rm$, $m \in M$, gives special S-endomorphisms of M ((R,S)-bimodule) and the map

$$\alpha : R \to B, \ r \mapsto \tilde{r}, \ r \in R,$$

is a ring homomorphism. M is faithful if and only if α is injective.

For commutative rings R, the map $\tilde{r}, r \in R$, is in fact an R-endomorphism and $S = End_R(M)$ and $B = End(M_S)$ are R-algebras.

For two R-modules M, N, the additive group $Hom_R(M,N)$ can be considered as a left module over $End_R(M)$ and a right module over $End_R(N)$ in a canonical way:

$$End(M) \times Hom(M,N) \to Hom(M,N), \ (\alpha, f) \mapsto \alpha f,$$
$$Hom(M,N) \times End(N) \to Hom(M,N), \ (f, \beta) \mapsto f\beta,$$

for $f \in Hom(M,N)$, $\alpha \in End(M)$, $\beta \in End(N)$.
Since $(\alpha f)\beta = \alpha(f\beta)$, $Hom(M,N)$ is an $(End(M), End(N))$-bimodule.

6.5 Homomorphisms and factor modules.
For $f \in Hom_R(M,N)$ we define the *kernel* and the *image* by

$$Ke\, f = \{m \in M \mid (m)f = 0\} \subset M, \quad Im\, f = \{(m)f \in N \mid m \in M\} \subset N.$$

$Ke\ f$ is a submodule of M, $Im\, f = (M)f$ is a submodule of N.

For any submodule U of M, the factor group $M/U = \{m + U \mid m \in M\}$ becomes an R-module by defining the operation of R on M/U

$$r(m+U) = rm + U, \ r \in R, \ m \in M,$$

and M/U is called the *factor* (or *quotient) module* of M by U. The map

$$p_U : M \to M/U, \ m \mapsto m + U, \ m \in M,$$

is surjective and a module homomorphism, called the *canonical homomorphism (projection)* of M onto M/U. It provides a bijection between the submodules of M containing U and the submodules of M/U.

Factor modules of M are also called *M-cyclic* modules.

From the factorization of group homomorphisms we obtain:

Factorization (Homomorphism) Theorem.
Let $f : M \to N$ be a homomorphism of R-modules. If U is a submodule of M with $U \subset Ke\, f$, then there is a unique homomorphism $\bar{f} : M/U \to N$ with $f = p_U \bar{f}$, i.e. the following diagram is commutative

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "p_U"'] & & N \\ & M/U \arrow[ur, "\bar{f}"'] & \end{tikzcd} .$$

Moreover, $Im\, f = Im\, \bar{f}$ and $Ke\, \bar{f} = Ke\, f \,/\, U$.

6.6 Generating sets. A subset L of a left R-module M is called a *generating set* of M if $RL = M$ (see 6.2). We also say L *generates* M or M *is generated by* L.

If there is a finite generating set in M, then M is called *finitely generated.* M is said to be *countably generated* if it has a generating set L with $card(L) \leq card(\mathbb{N})$. If M is generated by *one* element it is called *cyclic.*

For example, every ring is generated by its unit and the left principal ideals are just the cyclic submodules of ${}_RR$. The following properties are easily verified (compare vector spaces):

Let $f : M \to N$ be a module homomorphism and L a generating set in M. Then

(i) $(L)f$ is a generating set of $Im\, f$.

(ii) If M is finitely generated (cyclic), then also $Im\, f$ is finitely generated (resp. cyclic).

(iii) If $g : M \to N$ is another homomorphism, then $g = f$ if and only if $(l)g = (l)f$ for all $l \in L$.

6.7 Maximal submodules. A submodule $N \subset M$ is called *maximal* if $N \neq M$ and it is not properly contained in any proper submodule of M. By 6.5, N is maximal in M if and only if M/N is simple. Similar to 2.6 we obtain (using Zorn's Lemma):

In a finitely generated R-module every proper submodule is contained in a maximal submodule.

6.8 Exercises.

(1) Let M be a left R-module by $\varphi : R \to End^l(M)$ *and*
$An_R(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$. *Prove:*
(i) $An_R(M)$ *is an ideal in R;*
(ii) ${}_RM$ *is faithful if and only if* $An_R(M) = 0$*;*
(iii) M is faithful as a module over $R/An_R(M)$.
$An_R(M)$ *is called the* annihilator *of M.*

(2) Find all finitely generated sub- and factor modules of $\mathbb{Z}$ *and* $\mathbb{Q}$ *as* $\mathbb{Z}$*-modules.*

(3) Let R be a ring with unit and M an (R,R)-bimodule. Define a multiplication on the additive group $R \times M$ *by*
$(r, m) \cdot (s, n) = (rs, rn + ms)$. *Show that:*
(i) $R \times M$ *is a ring with unit (the* trivial extension *of R by M);*
(ii) there are ring homomorphisms $f : R \to R \times M$ *and* $g : R \times M \to R$ *with* $fg = id_R$*;*
(iii) (0,M) is an ideal in $R \times M$ *and* $(0, M)^2 = 0$.

(4) Let R,S be rings with unit and ${}_RM_S$ *an (R,S)-bimodule. Prove:*
(i) M becomes an $(R \times S, R \times S)$*-bimodule by setting* $(r, s)m = rm$ *and* $m(r, s) = ms$.
(ii) The trivial extension of $R \times S$ *by M (see(3)) is isomorphic to the (generalized) matrix ring* $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *with the usual matrix addition and multiplication.*
(iii) The left ideals of this ring are of the form

$$\{\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \mid (r, m) \in K,\ s \in I\}$$

for a left ideal $I \subset S$ *and an R-submodule* $K \subset R \times M$ *with* $MI \subset K$.
(iv) What form have the right ideals?

(5) Let R be an integral domain and Q a quotient field of R
(i.e. $Q = \{\frac{a}{b} \mid a \in R,\ 0 \neq b \in R\}$*).*
Prove: If Q is finitely generated as an R-module, then $R = Q$.

7 The category of R-modules

1.Definition. 2.Subcategory. 3.Examples. 4.Category of R-modules. 5.Special morphisms. 6.Properties of morphisms. 7.Special morphisms in R-MOD. 8.Special objects. 9.Properties of these objects. 10.Kernels and cokernels. 11.Kernels and cokernels in R-MOD. 12.Completion of a square. 13.Exact sequences. 14.Special morphisms and exact sequences. 15.Kernel Cokernel Lemma. 16.Diagram Lemma. 17.Isomorphism Theorem. 18.Five Lemma. 19.Exactness of rows. 20.Exercises.

After the general definition of a category we will mainly be concerned with categories of modules. A glance at other categories should serve to understand and appreciate the special nature of module categories. The proofs are mostly written in a way they can be transferred to more general categories with adequate properties. Occasionally the situation in a module category permits simpler proofs as in the general theory.

7.1 Definition. A *category* $\mathcal{C}$ is given by:

(1) A class of *objects*, $Obj(\mathcal{C})$.

(2) For every ordered pair (A, B) of objects in $\mathcal{C}$ there exists a set $Mor_{\mathcal{C}}(A, B)$, the *morphisms* of A to B, such that

$$Mor_{\mathcal{C}}(A, B) \cap Mor_{\mathcal{C}}(A', B') = \emptyset \text{ for } (A, B) \neq (A', B').$$

(3) A *composition* of morphisms, i.e. a map

$$Mor_{\mathcal{C}}(A, B) \times Mor_{\mathcal{C}}(B, C) \to Mor_{\mathcal{C}}(A, C), (f, g) \mapsto fg,$$

for every triple (A, B, C) of objects in $\mathcal{C}$, with the properties:

(i) It is associative: For A, B, C, D in $Obj(\mathcal{C})$ and $f \in Mor_{\mathcal{C}}(A, B)$, $g \in Mor_{\mathcal{C}}(B, C)$, $h \in Mor_{\mathcal{C}}(C, D)$ we have $(fg)h = f(gh)$;

(ii) there are *identities*: For every $A \in Obj(\mathcal{C})$ there is a morphism $id_A \in Mor_{\mathcal{C}}(A, A)$, the *identity* of A, with $id_A f = f id_B = f$ for every $f \in Mor_{\mathcal{C}}(A, B)$, $B \in Obj(\mathcal{C})$.

We often write $Mor_{\mathcal{C}}(A, B) = Mor(A, B)$ and, for short, $A \in \mathcal{C}$ instead of $A \in Obj(\mathcal{C})$.

For $f \in Mor(A, B)$, we call A the *source* of f, B the *target* of f, and we write $f : A \to B$ or $A \xrightarrow{f} B$.

For $A, B, C, D \in Obj(\mathcal{C})$, the morphisms $f \in Mor_{\mathcal{C}}(A, B)$, $g \in Mor_{\mathcal{C}}(B, D)$, $h \in Mor_{\mathcal{C}}(A, C)$, $k \in Mor_{\mathcal{C}}(C, D)$ can be presented in the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & D \end{array}$$

The diagram is *commutative* if $fg = hk$.

Remark: The definition of a category is based on a *class* of objects. The notion 'class' is (as 'set') defined in set theory by certain axioms using the relation 'element of' (i.e. $\in$). Every set is a class. The totality of all sets forms a class but not a set. Two classes are equal if they contain the same elements. The intersection of two classes is again a class. As for sets the inclusion ($\subset$) of two classes is defined and the cartesian product of two classes exists. If X is a set and $\{X_\lambda\}_\Lambda$ an indexed class of subsets $X_\lambda \subset X$, Λ a class, then intersection and union of these sets are subsets of X.

7.2 Subcategory. A category $\mathcal{D}$ is called a *subcategory* of $\mathcal{C}$ if

(i) $Obj(\mathcal{D}) \subset Obj(\mathcal{C})$,

(ii) $Mor_{\mathcal{D}}(A, B) \subset Mor_{\mathcal{C}}(A, B)$ for all $A, B \in Obj(\mathcal{D})$ and

(iii) the composition of morphisms in $\mathcal{D}$ is the restriction of the composition in $\mathcal{C}$.

If $Mor_{\mathcal{D}}(A, B) = Mor_{\mathcal{C}}(A, B)$, then $\mathcal{D}$ is called a *full subcategory* of $\mathcal{C}$. Hence a full subcategory of $\mathcal{C}$ is already determined by its objects.

7.3 Examples:

(1) Category of *sets with maps, ENS*
Objects: class of all sets, morphisms: all mappings, i.e.
$Mor_{ENS}(A, B) = Map(A, B)$ for sets A, B,
composition: composition of maps.
The finite sets form a full subcategory.

(2) Category of *sets with relations, ENS_R*
Objects: class of all sets, morphisms: all relations, i.e.
$Mor_{ENS_R}(A, B) = Rel(A, B) =$ power set of $(A \times B)$ for sets A, B,
composition: composition of relations.
ENS is a subcategory of ENS_R (not full).

(3) Category of *ordered sets*,
i.e. sets with a reflexive, transitive and antisymmetric binary relation.
Objects: class of all ordered sets,
morphisms: all order preserving maps between these sets,
composition: composition of maps.

(4) A *quasi-ordered set* $(X, \leq)$,
i.e. a set X with a reflexive and transitive binary relation $\leq$.
Objects: the elements of X,
morphisms: $Mor(x,y) = \begin{cases} \emptyset & \text{for } x \not\leq y, \quad x,y \in X \\ (x,y) & \text{for } x \leq y. \end{cases}$
For $x \leq y$, the set $Mor(x,y)$ has one element denoted by (x,y);
composition: $(x,y)(y,z) := (x,z)$ for $x \leq y \leq z$ in X.

(5) $\mathcal{C}^o$, the *dual category* of a category $\mathcal{C}$
Objects: objects in $\mathcal{C}$,
morphisms: $Mor_{\mathcal{C}^o}(A,B) = Mor_{\mathcal{C}}(B,A)$ for $A, B \in Obj(\mathcal{C})$,
composition: $Mor_{\mathcal{C}^o}(A,B) \times Mor_{\mathcal{C}^o}(B,C) \to Mor_{\mathcal{C}^o}(A,C), \quad (f,g) \mapsto gf$,
where gf is formed in $\mathcal{C}$.

(6) Category of *rings, RING*
Objects: all (associative) rings, morphisms: ring homomorphisms,
composition: composition of maps,
there is a faithful functor *RING* $\to$ *ENS* (not full).

(7) Category of *groups, GRP*
Objects: class of all groups, morphisms: group homomorphisms,
composition: composition of maps,
there is a faithful functor *GRP* $\to$ *ENS* (not full).

(8) Category of *abelian groups, AB*
Objects: class of all abelian groups; morphisms, composition as in (7).
AB is a full subcategory of *GRP*.

We now come to the example which will be of most importance to us:

7.4 Category of R-modules, R-MOD.
Objects: class of all unital left R-modules,
morphisms: module homomorphisms $Hom_R(A,B)$ for R-modules A, B,
composition: composition of maps.

R-MOD is a subcategory of *ENS, GRP and AB*. For $R = \mathbb{Z}$, we get $\mathbb{Z}$-*MOD* = *AB*.

Instead of *(R-module) homomorphism* we shall usually say *morphism (in R-MOD)*.

We denote by *R-mod* the full subcategory of the finitely generated modules in *R-MOD* (see 6.6). Similarly we write *MOD-R* (resp. *mod-R*) for the category of the (finitely generated) right R-modules.

As a special property of the categories *R-MOD, R-mod* and *AB*, the morphism sets $Hom_R(A,B)$ form an additive (abelian) group and we have the disributivity laws

$$(g_1+g_2)f = g_1f + g_2f,\ g(f_1+f_2) = gf_1 + gf_2,$$

and $g0 = 0$, $0f = 0$ whenever the compositions are defined. Categories with these properties are called *additive*. In these cases $Mor(A,A) = Hom(A,A)$ is a ring for every object A, the *endomorphism ring* $End(A)$ of A.

By a *module category* we will understand a full subcategory of the category *R-MOD* (or *MOD-R*).

7.5 Special morphisms. Definitions.

Let $\mathcal{C}$ be a category. A morphism $f : A \to B$ in $\mathcal{C}$ is called
a monomorphism if, for $g, h \in Mor(C,A)$, $C \in \mathcal{C}$:
$gf = hf$ implies $g = h$ (f is right cancellable);
an epimorphism if, for $g, h \in Mor(B,D)$, $D \in \mathcal{C}$:
$fg = fh$ implies $g = h$ (f is left cancellable);
a bimorphism if f is both a mono- and an epimorphism;
a retraction if there exists $g \in Mor(B,A)$ with $gf = id_B$;
a coretraction if there exists $g \in Mor(B,A)$ with $fg = id_A$;
an isomorphism if f is both a retraction and a coretraction;
a (left and right) zero morphism if, for any $g, h \in Mor(D,A)$, $D \in \mathcal{C}$,
$gf = hf$, and, for any $g', h' \in Mor(B,D')$, $D' \in \mathcal{C}$, $fg' = fh'$.

7.6 Properties of morphisms.

Let $\mathcal{C}$ be a category and $f : A \to B$, $g : B \to C$ morphisms in $\mathcal{C}$. Then

(1) If f and g are monomorphisms (epimorphisms), then fg is also a monomorphism (epimorphism).

(2) If fg is an epimorphism, then g is an epimorphism.

(3) If fg is a monomorphism, then f is a monomorphism.

(4) If f is a retraction, then f is an epimorphism.

(5) If f is a coretraction, then f is a monomorphism.

(6) If f is an isomorphism, then f is a bimorphism.

Proof: (1) is obvious.

(2) If $h_1, h_2 \in Mor(C,D)$ and $gh_1 = gh_2$, then $fgh_1 = fgh_2$, i.e. $h_1 = h_2$ since fg is an epimorphism. Hence g is an epimorphism.

(3) is shown as (2). (4),(5) and (6) follow from (2),(3), observing that identities are monomorphisms and epimorphisms.

Monomorphisms (epimorphisms) in $\mathcal{C}$ are also called *monic* (resp. *epic*) morphisms.

Two objects $A, B \in \mathcal{C}$ are called *isomorphic* if there is an isomorphism $f : A \to B$ in $\mathcal{C}$.

We also use the notation *splitting epimorphisms* for retractions and *splitting monomorphisms* for coretractions.

In *ENS* injective maps are just monomorphisms and surjective maps are (splitting) epimorphisms. Monomorphisms with non-empty source are splitting.

In some categories bimorphisms need not be isomorphisms (see 7.20,(4)).

Zero morphisms in $Mor(A, B)$ are denoted by $0_{(A,B)}$, or simply 0. Arbitrary categories need not have zero morphisms. In *R-MOD* we have:

7.7 Special morphisms in *R-MOD*.

Let $f : M \to N$ be a morphism in R-MOD or R-mod. Then f is
(i) a monomorphism if and only if it is injective;
(ii) an epimorphism if and only if it is surjective;
(iii) an isomorphism if and only if it is bijective (i.e. bimorphism);
(iv) a zero morphism if and only if $Im\, f = 0$.

Proof: (i) It is easy to verify that injective maps are right cancellable. Assume that f is not injective, i.e. there are $m_1 \neq m_2$ in M with $(m_1)f = (m_2)f$. Consider the maps

$$h_1 : R \to M,\ r \mapsto rm_1, \ \text{ and } h_2 : R \to M,\ r \mapsto rm_2, \text{ for } r \in R.$$

We get $h_1 f = h_2 f$ although $h_1 \neq h_2$, i.e. f is not monic.

(ii) Obviously surjective maps are left cancellable (also in *ENS*).

If f is not surjective, then $N/Im\, f \neq 0$ und the canonical map $p : N \to N/Im\, f$ is not the zero map. For the zero map $n : N \to N/Im\, f$ we get $fn = fp = 0$, i.e. f is not epic.

(iii) follows from (i),(ii). (iv) is evident.

The same proofs yield the corresponding assertions in *R-mod*.

Remark: Notions in a category which are obtained from each other by reversing the morphisms ('arrows') are called *dual.* E.g., the definition of an epimorphism is obtained by *dualising* the definition of a monomorphism. The dual notion often is denoted with the prefix *co-*. For example, *coretraction* is dual to *retraction.* The bimorphisms and the isomorphisms are dual to itself, they are called *self dual*. A similar notation will also be used for other categorical constructions.

7.8 Special objects. Definitions.
Let $\mathcal{C}$ be a category. An object A in $\mathcal{C}$ is called
an initial object if $Mor_{\mathcal{C}}(A, B)$, $B \in \mathcal{C}$, always has just one element;
a terminal object if $Mor_{\mathcal{C}}(C, A)$, $C \in \mathcal{C}$, always has just one element;
a zero object if A is an initial and a terminal object.

7.9 Properties of these objects.
Let $\mathcal{C}$ be a category and $A, B, C, N \in Obj(\mathcal{C})$. Then:

(1) All initial objects (resp. terminal objects) are isomorphic.

(2) If A is an initial object, then every $f \in Mor(A, B)$ is a right zero morphism and every $g \in Mor(C, A)$ is a retraction.

(3) If A is a terminal object, then every $g \in Mor(C, A)$ is a left zero morphism and every $f \in Mor(A, B)$ is a coretraction.

(4) If N is a zero object, then the (unique) morphism $N \to B$ is a coretraction and $B \to N$ is a retraction.

(5) If N is a zero object, then the (unique) morphisms $f : N \to B$, $g : C \to N$ and $gf : C \to B$ are zero morphisms, in particular $Mor(C, B) \neq \emptyset$, and there is only one zero morphism $C \to B$ (denoted by 0).

Proof: (1) If A and B are initial objects, then there are morphisms $f : A \to B$ and $g : B \to A$ and we get $fg = id_A$ and $gf = id_B$.
(2) Let A be an initial object, $f \in Mor(A, B)$, $g, h \in Mor(B, D)$, $A, B, C, D \in \mathcal{C}$. Since $Mor(A, D)$ has only one element we conclude $fh = fg$. For $c \in Mor(C, A)$, $d \in Mor(A, C)$, we get $dc = id_A$.
(3) is shown in a similar way to (2).
(4),(5) follow from (2),(3) and exercise 7.20,(1).

The initial object in *ENS* is the empty set, while every set consisting of one element is a terminal object. In *R-MOD (R-mod)* the zero module is (the only) initial, terminal and zero object.

7.10 Kernels and cokernels. Definitions.
Let $\mathcal{C}$ be a category with zero object and $f : A \to B$ a morphism in $\mathcal{C}$.
(1) A morphism $i : K \to A$ is called a *kernel of f* if
$if = 0$ and, for every morphism $g : D \to A$ with $gf = 0$, there is a unique

morphism $h : D \to K$ with $hi = g$, i.e. we have a commutative diagram

$$\begin{array}{ccccc} & & D & & \\ & {}^{h}\swarrow & \downarrow g & & \\ K & \xrightarrow{i} & A & \xrightarrow{f} & B. \end{array}$$

(2) A morphism $p : B \to C$ is called a *cokernel of f* if $fp = 0$ and, for every $g : B \to D$ with $fg = 0$, there is a unique morphism $h : C \to D$ with $ph = g$, i.e. we have the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{p} & C \\ & & g\downarrow & \swarrow h & \\ & & D & & . \end{array}$$

Properties: *For every morphism $f : A \to B$ in $\mathcal{C}$ we have:*

(1) A kernel of f (i.e. $i : K \to A$) is a monomorphism, and a cokernel of f (i.e. $p : B \to C$) is an epimorphism.

(2) The source of a kernel of f and the target of a cokernel of f are uniquely determined up to isomorphisms, more precisely:

If $i_1 : K_1 \to A$, $i_2 : K_2 \to A$ are two kernels of f, then there is an isomorphism $\alpha : K_1 \to K_2$ with $i_1 = \alpha i_2$.

A similar result holds for the cokernel of f.

Proof: (1) For $w_1, w_2 \in Mor(D, K)$ with $w_1 i = w_2 i$, we have the commutative diagram

$$\begin{array}{ccccc} & & D & & \\ & {}^{w_2}\swarrow & \downarrow w_1 i & & \\ K & \xrightarrow{i} & A & \xrightarrow{f} & B \end{array}$$

with $w_1 i f = 0$. Then w_2 is uniquely determined and hence $w_1 = w_2$, i.e. i is monic. Similarly we obtain that p is epic.

(2) follows from the definitions.

For morphisms in an arbitrary category with zero object, kernels or cokernels need not exist.

For a homomorphism $f : M \to N$ of R-modules we already have used *kernel of f* to denote the submodule

$$Ke\, f = \{m \in M \mid (m)f = 0\}.$$

Connected with it is the embedding as homomorphism $i : Ke\, f \to M$. This yields the morphism 'kernel of f' in R-MOD:

7.11 Kernels and cokernels in *R-MOD*.
Let $f : M \to N$ be a homomorphism in R-MOD (resp. R-mod). Then

(1) The inclusion $i : Ke\,f \to M$ is a kernel of f in R-MOD.

(2) The projection $p : N \to N/Im\,f$ is a cokernel of f in R-MOD (resp. R-mod).

Proof: (1) If $g : L \to M$ is given with $gf = 0$, then $(L)g \subset Ke\,f$. In R-mod not every f has a kernel, since $Ke\,f$ need not be finitely generated.
(2) If $h : N \to L$ is given with $fh = 0$, then $Im\,f \subset Ke\,h$ and the factorization theorem yields the desired map $N/Im\,f \to L$.
The same argument holds in *R-mod*.

Notation: For $f : M \to N$ in *R-MOD*, the symbol '$Ke\,f$' will denote the submodule $Ke\,f \subset M$ as well as the inclusion map $i : Ke\,f \to M$. '$Coke\,f$' will denote the factor module N/Mf or the canonical epimorphism $p : N \to N/Mf$.
As a first application of these notions we show:

7.12 Completion of a square.
Consider the following commutative diagram in R-MOD:

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_2 \\ \varphi_1 \downarrow & & \downarrow \varphi_2 \\ N_1 & \xrightarrow{g_1} & N_2 \end{array}$$

Then to every morphism $f_o : M_o \to M_1$ with $f_o f_1 = 0$ there is a unique $\varphi_o : M_o \to Ke\,g_1$, and to every $g_2 : N_2 \to N_3$ with $g_1 g_2 = 0$ there is a unique $\varphi_3 : Coke\,f_1 \to N_3$ yielding the commutative diagram

$$\begin{array}{ccccccc} M_o & \xrightarrow{f_o} & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{p} & Coke\,f_1 \\ \downarrow \varphi_o & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 \\ Ke\,g_1 & \xrightarrow{i} & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \end{array} .$$

Proof: The commutativity of the given diagrams and $f_o f_1 = 0$ imply $(f_o \varphi_1) g_1 = f_o f_1 \varphi_2 = 0$. Existence and uniqueness of φ_o are derived from the defining properties of $Ke\,g_1$ (see 7.10).
Similarly we obtain the existence and uniqueness of φ_3.

7.13 Exact sequences. Definition.
In a category $\mathcal{C}$ with zero object and kernels, a sequence of two morphisms

$A \xrightarrow{f} B \xrightarrow{g} C$ is called *exact* if $fg = 0$ and in the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & & B \xrightarrow{g} C \\ & {}_h\searrow & & \nearrow & \\ & & Ke\, g & & \end{array},$$

the uniquely determined h is an epimorphism.

A *sequence of morphisms* $\{f_i : A_i \to A_{i+1} \,|\, i \in I\!N\}$ in $\mathcal{C}$ is called *exact at* A_i if f_{i-1} and f_i form an exact sequence. It is called *exact* if it is everywhere exact.

A diagram is said to be *exact* if all its rows and columns are exact.

In *R-MOD* a sequence of two morphisms $M \xrightarrow{f} N \xrightarrow{g} L$ is exact if and only if $Im\, f = Ke\, g$. Hence we easily see:

7.14 Special morphisms and exact sequences.

For a homomorphism $f : M \to N$ in R-MOD we have:

(1) $0 \to M \xrightarrow{f} N$ *is exact if and only if f is monic;*

(2) $M \xrightarrow{f} N \to 0$ *is exact if and only if f is epic;*

(3) $0 \to M \xrightarrow{f} N \to 0$ *is exact if and only if f is an isomorphism;*

(4) $0 \to K \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{p} L \to 0$ *is exact if and only if i is the kernel of f and p is the cokernel of f.*

An exact sequence of R-modules of the form

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called a *short exact sequence* or an *extension of N by K*. It is obvious that in this case f is a kernel of g and g a cokernel of f. Hence usually K is considered as a submodule and N as a factor module of M.

Remark: A category $\mathcal{C}$ is called *exact* if $\mathcal{C}$ has a zero object and in $\mathcal{C}$ every morphism can be written as a composition of an epimorphism and a monomorphism.

The properties given in 7.14 more generally hold in any exact category. The same is true for the following important lemma, although we will argue in the given proof with elements, for the sake of simplicity:

7.15 Kernel Cokernel Lemma.

Consider the commutative diagram with exact rows and columns in R-MOD

$$
\begin{array}{ccccc}
Ke\,\varphi_1 & & Ke\,\varphi_2 & & Ke\,\varphi_3 \\
\downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 \\
M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 \\
\downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 \\
N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 \\
\downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 \\
Coke\,\varphi_1 & & Coke\,\varphi_2 & & Coke\,\varphi_3
\end{array} \quad .
$$

There are uniquely determined morphisms

$$Ke\,\varphi_1 \xrightarrow{\alpha_1} Ke\,\varphi_2 \xrightarrow{\alpha_2} Ke\,\varphi_3, \quad Coke\,\varphi_1 \xrightarrow{\beta_1} Coke\,\varphi_2 \xrightarrow{\beta_2} Coke\,\varphi_3$$

which render the completed diagram commutative and:
(1) if g_1 is monic, then the first row is exact;
(2) if f_1 is monic, then α_1 is monic;
(3) if f_2 is epic, then the last row is exact;
(4) if g_2 is epic, then β_2 is epic;
(5) if f_2 is epic and g_1 monic, then there exists $\delta : Ke\,\varphi_3 \to Coke\,\varphi_1$ (connecting morphism) which yields the exact sequence

$$Ke\,\varphi_2 \xrightarrow{\alpha_2} Ke\,\varphi_3 \xrightarrow{\delta} Coke\,\varphi_1 \xrightarrow{\beta_1} Coke\,\varphi_2\,.$$

Since the connecting morphism δ can be drawn in the diagram as a snaky curve, this is also called *Snake Lemma*.

Proof: Existence and uniqueness of the desired morphisms are obtained by the defining properties of kernel resp. cokernel.

(1) Exactness of the first row if g_1 is monic: $\alpha_1\alpha_2 i_3 = i_1 f_1 f_2 = 0$ implies $\alpha_1\alpha_2 = 0$. If $m \in Ke\,\alpha_2 \subset Ke\,\varphi_2$, then $mi_2 f_2 = m\alpha_2 i_3 = 0$, i.e. $mi_2 \in Ke\,f_2 = Im\,f_1$, and there exists $k \in M_1$ with $mi_2 = kf_1$. From this we get $k\varphi_1 g_1 = kf_1\varphi_2 = mi_2\varphi_2 = 0$, i.e. $k\varphi_1 \in Ke\,g_1 = 0$. This means $k \in Ke\,\varphi_1$ and $ki_1 f_1 = kf_1 = mi_2 = k\alpha_1 i_2$, i.e. $m = k\alpha_1$ and hence $Ke\,\alpha_2 = Im\,\alpha_1$.

(2) If f_1 is monic, then $i_1 f_1 = \alpha_1 i_2$ is monic and hence α_1 is monic.

(3) is proved in a similar way to (1) by 'diagram chasing'.

(4) If g_2 is epic, then $g_2 p_3 = p_2\beta_2$ and hence β_2 is epic.

(5)(i) Construction of $\delta : Ke\,\varphi_3 \to Coke\,\varphi_1$:
Take $m_3 \in Ke\,\varphi_3$. Then there exists $m_2 \in M_2$ with $(m_2)f_2 = m_3$ and we get $(m_2)\varphi_2 g_2 = (m_2)f_2\varphi_3 = 0$. Hence we find an element $n_1 \in N_1$ with

$(n_1)g_1 = (m_2)\varphi_2$. We put $(m_3)\delta = (n_1)p_1$ and show that this assignment is independent of the choice of m_2 and n_1:

Take $m_2' \in M_2$ with $(m_2')f_2 = m_3$ and $n_1' \in N_1$ with $(n_1')g_1 = (m_2')\varphi_2$. Then $(m_2 - m_2') \in Ke\, f_2$, i.e. $(m_2 - m_2') = (x)f_1$ for an $x \in M_1$. Further we have $(n_1 - n_1')g_1 = (m_2 - m_2')\varphi_2 = (x)f_1\varphi_2 = (x)\varphi_1 g_1$. Now we see that $n_1 - n_1' = (x)\varphi_1$ lies in $Im\,\varphi_1$ and $(n_1)p_1 = (n_1')p_1$.

It is left as an exercise to show that δ is a homomorphism.

(ii) Exactness of $Ke\,\varphi_2 \xrightarrow{\alpha_2} Ke\,\varphi_3 \xrightarrow{\delta} Coke\,\varphi_1$:
If $m_3 \in Im\,\alpha_2$, then there exists $y \in Ke\,\varphi_2$ with $y\alpha_2 = m_3$, and, with the notation from (i), we choose an $m_2 = yi_2$, where $m_2\varphi_2 = yi_2\varphi_2 = 0$ and $n_1 = 0$. We see $(m_3)\delta = (n_1)p_1 = 0$, i.e. $\alpha_2\delta = 0$.

Now take $m_3 \in Ke\,\delta$. By construction of δ, there exists $n_1 \in N_1$ with $0 = (m_3)\delta = (n_1)p_1$, i.e. $n_1 \in Im\,\varphi_1$, $n_1 = (z)\varphi_1$ for some $z \in M_1$. With the above notation we get $(m_2)\varphi_2 = (n_1)g_1 = (z\varphi_1)g_1 = (z)f_1\varphi_2$, i.e. $m_2 - (z)f_1 \in Ke\,\varphi_2$ and $(m_2 - (z)f_1)f_2 = (m_2)f_2 = m_3$. This implies $m_3 \in Im\,\alpha_2$ and $Im\,\alpha_2 = Ke\,\delta$.

The exactness of $Ke\,\varphi_3 \xrightarrow{\delta} Coke\,\varphi_1 \xrightarrow{\beta_1} Coke\,\varphi_2$ can be shown in a similar way.

As an application of the Kernel Cokernel Lemma we obtain the

7.16 Diagram Lemma.
Consider the commutative diagram with exact rows in R-MOD

$$\begin{array}{ccccccc} & & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0 \\ & & \downarrow\varphi_1 & & \downarrow\varphi_2 & & \downarrow\varphi_3 & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & & \end{array} .$$

(1) If φ_1 and φ_3 are monic (epic), then φ_2 is also monic (epic).
(2) If φ_1 is epic and φ_2 monic, then φ_3 is monic.
(3) If φ_2 is epic and φ_3 monic, then φ_1 is epic.
(4) The following assertions are equivalent:
(a) there exists $\alpha : M_3 \to N_2$ with $\alpha g_2 = \varphi_3$;
(b) there exists $\beta : M_2 \to N_1$ with $f_1\beta = \varphi_1$.

Proof: (1),(2) and (3) are immediately derived from the Kernel Cokernel Lemma.

(4) $(b) \Rightarrow (a)$ If $\beta : M_2 \to N_1$ has the given property, then $f_1\beta g_1 = \varphi_1 g_1 = f_1\varphi_2$, i.e. $f_1(\varphi_2 - \beta g_1) = 0$. Since f_2 is the cokernel of f_1, there exists $\alpha : M_3 \to N_2$ with $f_2\alpha = \varphi_2 - \beta g_1$. This implies $f_2\alpha g_2 = \varphi_2 g_2 - \beta g_1 g_2 = \varphi_2 g_2 = f_2\varphi_3$. f_2 being epic we conclude $\alpha g_2 = \varphi_3$.

$(a) \Rightarrow (b)$ is obtained similarly.

The assertion in (4) is also called the *Homotopy Lemma*.
A further application of the Kernel Cokernel Lemma yields

7.17 Isomorphism Theorem.
Assume K and N to be submodules of the R-module M with $K \subset N$. Then
$$M/N \simeq (M/K)/(N/K).$$

Proof: We have the commutative diagram with exact columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & N/K & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & M/N & \longrightarrow & M/K/N/K & \longrightarrow & 0 \quad .
\end{array}$$

The first two rows are exact by construction. The exactness of the last row is derived from the Kernel Cokernel Lemma.

Next we want to list some relations in diagrams with exact rows which will be useful:

7.18 Five Lemma.
Consider the following commutative diagram with exact rows in R-MOD:

$$\begin{array}{ccccccccc}
M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\
\downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_4 & & \downarrow \varphi_5 \\
N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5
\end{array}$$

(1) If φ_1 is epic and φ_2, φ_4 are monic, then φ_3 is monic.

(2) If φ_5 is monic and φ_2, φ_4 are epic, then φ_3 is epic.

(3) If φ_1 epic, φ_5 are monic and φ_2, φ_4 are isomorphisms, then φ_3 is an isomorphism.

Proof: (1) The assertion can be reduced to the Kernel Cokernel Lemma by writing $M_2 \to M_3$ and $N_2 \to N_3$ as a composition of epi- and monomorphisms. It can also be proved directly by diagram chasing.
(2) is shown similarly. (3) follows from (1) and (2).

Let us finally state the following way of deriving the exactness of one row from the exactness of the other row:

7.19 Exactness of rows.
Consider the following commutative diagram in R-MOD:

$$\begin{array}{ccccc}
M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 \\
\downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 \\
N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3
\end{array}$$

(1) Assume that the second row is exact, φ_1 is epic, φ_2 and φ_3 are monic. Then the first row is also exact.

(2) Assume that the first row is exact, φ_1 and φ_2 are epic, and φ_3 is monic. Then the second row is also exact.

(3) If φ_1 is epic, φ_3 monic and φ_2 an isomorphism, then the first row is exact if and only if the second row is exact.

Proof: (1) The assertion can be reduced to the Kernel Cokernel Lemma by writing f_1 and g_1 as a composition of an epi- and a monomorphism or shown directly by diagram chasing.

(2) is proved in a similar way. (3) follows from (1) and (2).

7.20 Exercises.

(1) Let $\mathcal{C}$ be a category with zero object 0.

(i) Prove that for $A \in \mathcal{C}$ the following assertions are equivalent:
(a) A is a zero object;
(b) id_A is a zero morphism;
(c) there is a monomorphism $A \to 0$;
(d) there is an epimorphism $0 \to A$.

(ii) Let $h : C \to B$ be a zero-morphism in $\mathcal{C}$, $f : 0 \to B$ and $g : C \to 0$. Show that $h = gf$. (Hence there is exactly one zero morphism $C \to B$.)

(2) Consider the following commutative diagram in R-MOD with exact row and column:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & A_2' & & & & \\
 & & {\scriptstyle f}\nearrow & & \downarrow & & \searrow{\scriptstyle\alpha} & & \\
0 & \rightarrow & A_1' & \rightarrow & A & \rightarrow & A_1'' & \rightarrow 0 & \\
 & & {\scriptstyle\beta}\searrow & & \downarrow & & \nearrow{\scriptstyle g} & & \\
 & & & & A_2'' & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Show that α, β are zero morphisms and f, g are isomorphisms.

(3) Prove: A sequence $L \xrightarrow{f} M \xrightarrow{g} N$ in R-MOD is exact if and only if $fg = 0$ and the uniquely determined morphism $Coke\, f \to N$ is monic.

(4) Show that in the category of rings (with units) the inclusion $\mathbb{Z} \to \mathbb{Q}$ is a bimorphism but not an isomorphism.

(5) In the commutative diagram in R-MOD

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & M_o & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \\
0 & \longrightarrow & M_o & \longrightarrow & M_1 & \longrightarrow & M_3 & &
\end{array}$$

assume the upper row to be exact. Prove that the lower row is exact if and only if the right column is exact.

8 Internal direct sum

1.Definitions. 2.Decomposition by morphisms. 3.Splitting sequences. 4.Idempotents and direct summands. 5.Internal direct sum. 6.Direct decomposition and idempotents. 7.Idempotents and endomorphisms. 8.Direct decomposition and generating sets. 9.Direct sums of finitely generated modules. 10.Summands of sums of countably generated modules. 11.Exercises.

8.1 Definitions. Let $M_1,\ M_2$ be submodules of the R-module M. If $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$, then M is called the *(internal) direct sum* of M_1 and M_2. This is written as $M = M_1 \oplus M_2$ and is called a *direct decomposition* of M.

In this case every $m \in M$ can be *uniquely* written as $m = m_1 + m_2$ with $m_1 \in M_1$, $m_2 \in M_2$. M_1 and M_2 are called *direct summands* of M. If M_1 is a direct summand, then in general there are various submodules M_2 with $M = M_1 \oplus M_2$.

M is called *(direct) indecomposable* if $M \neq 0$ and it cannot be written as a direct sum of non-zero submodules. Observe that $M = M \oplus 0$ always is a (trivial) decomposition of M.

Direct decompositions can also be obtained by certain morphisms:

8.2 Decomposition by morphisms.
Let $f : M \to N$, $g : N \to M$ be morphisms in R-MOD with $gf = id_N$ (i.e. f retraction, g coretraction). Then $M = Ke\, f \oplus Im\, g$.

Proof: If $m = (n)g \in Ke\, f \cap Im\, g$, $m \in M$, $n \in N$, then $0 = (m)f = (n)gf = n$ and $m = (n)g = 0$, i.e. $Ke\, f \cap Im\, g = 0$. For any $k \in M$, we have $(k - (k)fg)f = (k)f - (k)f = 0$ and hence
$k = (k - (k)fg) + (k)fg \in Ke\, f + Im\, g$, i.e. $M = Ke\, f + Im\, g$.

A short exact sequence (in an exact category)

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} L \longrightarrow 0$$

is said to *split on the left (right)* if f is a coretraction (resp. g a retraction). It is said to *split* if it splits on the left and the right.

In *R-MOD* we have several characterizations of this important class of sequences:

8.3 Splitting sequences. Characterization.

For a short exact sequence $0 \to K \xrightarrow{f} M \xrightarrow{g} L \to 0$ *of* R*-modules the following assertions are equivalent:*
(a) the sequence splits;
(b) f *is a coretraction (= splitting monomorphism);*
(c) g *is a retraction (= splitting epimorphism);*
(d) $Im\, f$ *(=* $Ke\, g$*) is a direct summand in* M*.*

Proof: $(a) \Rightarrow (b), (c)$ by definition.

$(b) \Rightarrow (c)$ and $(c) \Rightarrow (b)$ are derived from Lemma 7.16,(4), applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f} & M & \xrightarrow{g} & L & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & K & \xrightarrow{f} & M & \xrightarrow{g} & L & \longrightarrow & 0 \end{array} .$$

This also implies $(b) \Rightarrow (a)$ and $(c) \Rightarrow (a)$.

$(b) \Rightarrow (d)$ is shown in 8.2.

$(d) \Rightarrow (b)$ Let $M = Im\, f \oplus N$. Since f is monic, for every $m \in M$, there are unique elements $k \in K$ and $n \in N$ with $m = (k)f + n$. The assignment $m \mapsto k$ yields a homomorphism $h : M \to K$ with $fh = id_K$.

If $g : M \to N$ is a retraction and $f : N \to M$ with $fg = id_N$, then f is a coretraction and, by 8.2, $Im\, f$ is a direct summand in M. On the other hand, $(gf)^2 = g(fg)f = gf$, i.e. gf is an idempotent in $End_R(M)$ and $Im\, f = (M)gf$. The direct summands of M are always closely related to idempotents in $End_R(M)$:

8.4 Idempotents and direct summands.

Let M *be an* R*-module. Then:*

(1) If $f \in End_R(M)$ *is idempotent, then* $Ke\, f = Im\,(1 - f)$*,* $Im\, f = Ke\,(1 - f)$ *and* $M = Mf \oplus M(1 - f)$*.*

(2) If $M = K \oplus L$ *is a direct decomposition of* M *and*

$$p_K : M \to K, \quad m = k + l \mapsto k,$$

the projection of M *onto* K *(along* L*), then (with inclusion* $i_K : K \to M$*)* $f_K = p_K i_K \in End_R(M)$ *is idempotent and* $K = Mf_K$*,* $L = Ke\, f_K$*.*

Proof: (1) $M(1 - f)f = 0$ implies $Im\,(1 - f) \subset Ke\, f$. For every $m \in M$, we have $m = (m)f + m(1 - f)$, i.e. $M = Mf + M(1 - f)$. For $m \in Ke\, f$, we get $m = m(1 - f) \in Im\,(1 - f)$. For $m, n \in M$, we derive from $mf = n(1-f)$ (applying f) the equation $mf = mf^2 = n(1-f)f = 0$, i.e. $Mf \cap M(1-f) = 0$ and $M = Mf \oplus M(1-f)$.

(2) These assertions are easily verified. Observe that f_K does not only depend on K but also on the choice of L.

Now we look at the decomposition of a module in several summands:

8.5 Internal direct sum. Let M be an R-module and $\{M_\lambda\}_\Lambda$ a non-empty family of submodules of M. If

$$(i)\ \ M = \sum_{\lambda\in\Lambda} M_\lambda \ \text{ and } \ (ii)\ \ M_\lambda \cap (\sum_{\mu\neq\lambda} M_\mu) = 0 \ \text{ for all } \lambda \in \Lambda,$$

then M is called the *(internal) direct sum* of the submodules $\{M_\lambda\}_\Lambda$. This is written as $M = \bigoplus_\Lambda M_\lambda$ and the M_λ are called *direct summands* of M.

If only (ii) is satisfied, then $\{M_\lambda\}_\Lambda$ is called an *independent family* of submodules. The condition (ii) means that non-zero elements in M_λ cannot be written as a sum of elements in the other M_μ's. This is equivalent to the uniqueness of every representation $m_{\lambda_1} + \cdots + m_{\lambda_r}$ with $m_{\lambda_k} \in M_{\lambda_k}$, $\lambda_i \neq \lambda_j$.

If M is an internal direct sum of $\{M_\lambda\}_\Lambda$, then every element $m \in M$ can be written uniquely as a finite sum $m = m_{\lambda_1} + \cdots + m_{\lambda_r}$ with distinct λ_i and $m_{\lambda_k} \in M_{\lambda_k}$.

Also, for every $\lambda \in \Lambda$, we have $M = M_\lambda \oplus (\sum_{\mu\neq\lambda} M_\mu)$ and, by 8.4, there is an idempotent $e_\lambda \in End_R(M)$ with

$$M_\lambda = Im\, e_\lambda \quad \text{and} \quad \sum_{\mu\neq\lambda} M_\mu = Ke\, e_\lambda .$$

From this we see that $\lambda \neq \mu$ always implies $e_\mu e_\lambda = e_\lambda e_\mu = 0$, i.e. e_λ and e_μ are *orthogonal* and we get:

8.6 Direct decomposition and idempotents.

(1) For a family $\{M_\lambda\}_\Lambda$ of submodules of the R-module M, the following assertions are equivalent:

(a) M is the internal direct sum of the $\{M_\lambda\}_\Lambda$;

(b) there is a family of (orthogonal) idempotents in $End_R(M)$, $\{e_\lambda\}_\Lambda$, with

$$M_\lambda = Me_\lambda \quad \textit{and} \quad \sum_{\mu\neq\lambda} M_\mu = Ke\, e_\lambda,\ \lambda \in \Lambda.$$

(2) If $e_1, \cdots, e_k$ are orthogonal idempotents in $End_R(M)$ with $e_1 + \cdots + e_k = id_M$, then $M = Me_1 \oplus \cdots \oplus Me_k$.

Proof: (1) $(a) \Rightarrow (b)$ is pointed out above.

$(b) \Rightarrow (a)$ Since, by 8.4, $Im\, e_\lambda \cap Ke\, e_\lambda = 0$ for each $\lambda \in \Lambda$, the family of submodules $\{M_\lambda\}_\Lambda$ is independent.

Now $Im\, e_\lambda + Ke\, e_\lambda = M$ implies $\sum_\Lambda M_\lambda = M_\lambda + \sum_{\mu\neq\lambda} M_\mu = M$.

(2) is a consequence of (1).

8.7 Idempotents and endomorphisms.
Let M be an R-module, $S = End_R(M)$ and e, f idempotents in S. Then

$Hom_R(Me, Mf) \simeq eSf$ *(group isomorphism) and*

$Hom_R(Me, Me) \simeq eSe$ *(ring isomorphism).*

Proof: Every morphism $h : Me \to Mf$ can be extended to an endomorphism $h' : M \to M$ with $h = eh'f|_{Me}$.

On the other hand, for every $s \in S$, the composition esf is a morphism from Me to Mf.

The remaining assertions are also easily verified.

8.8 Direct decomposition and generating sets.
Let M be an R-module.

(1) If M is finitely generated, then every direct decomposition of M is finite.

(2) Let $M = \bigoplus_\Lambda M_\lambda$, $0 \neq M_\lambda \subset M$, with Λ an infinite index set.

(i) If X is a generating set of M, then $card(\Lambda) \leq card(X)$.

(ii) If, in addition, $M = \bigoplus_\Gamma N_\gamma$, $0 \neq N_\gamma \subset M$, and all M_λ, N_γ are cyclic, then $card(\Lambda) = card(\Gamma)$.

Proof: (1) Assume $M = \bigoplus_\Lambda M_\lambda$ for a family of submodules $\{M_\lambda\}_\Lambda$ and let $m_1, \ldots, m_k$ be a generating set of M. Then the $m_1, \ldots, m_k$ are contained in a finite partial sum $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_r}$ and we have

$M = \sum_{i\leq k} Rm_i = M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_r}$.

(2) $card(A)$ denotes the cardinality of a set A.

(i) For $x \in X$, let Λ_x be the finite set of the $\lambda \in \Lambda$ for which x has a non-zero component in M_λ. X being a generating set, we get $\bigcup_{x\in X} \Lambda_x = \Lambda$. Now the Λ_x are finite but Λ is infinite. Hence X has to be infinite and from the theory of cardinals we learn that $card(X)$ cannot be smaller than $card(\Lambda)$, i.e. $card(\Lambda) \leq card(X)$.

(ii) For every $\gamma \in \Gamma$, we can choose an element $n_\gamma \in N_\gamma$ with $Rn_\gamma = N_\gamma$. Then $\{n_\gamma\}_\Gamma$ is a generating set of M and, by (i), we have $card(\Lambda) \leq card(\Gamma)$. Hence Γ is also infinite and we get – again by (i) – that $card(\Gamma) \leq card(\Lambda)$, i.e. $card(\Gamma) = card(\Lambda)$.

Observe that in 8.8 nothing is said about the number of summands in a *finite* (cyclic) decomposition of M (see Exercise 3.18,(5)).

We conclude this paragraph with two assertions about direct sums of finitely (resp. countably) generated modules:

8.9 Direct sums of finitely generated modules.

For a countably generated module N, the following are equivalent:

(a) *N is a direct sum of finitely generated modules;*

(b) *every finitely generated submodule of N is contained in a finitely generated direct summand.*

Proof: $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (a)$ By assumption, there is a countable generating set $\{a_i\}_{\mathbb{N}}$ in N. Let N_1 be a finitely generated direct summand containing a_1, i.e. $N = N_1 \oplus K_1$ for some $K_1 \subset N$. Choose a finitely generated summand L_2 containing N_1 and a_2, i.e. $N = L_2 \oplus K_2$. Then N_1 is also a summand of L_2 and $N = N_1 \oplus N_2 \oplus K_2$ for some $N_2 \subset L_2$.

Inductively we arrive at a submodule $\bigoplus_{\mathbb{N}} N_i$ of N which contains all the a_i's, i.e. $N = \bigoplus_{\mathbb{N}} N_i$.

The subsequent theorem of *I. Kaplansky* will be particularly helpful for investigating projective modules:

8.10 Summands of sums of countably generated modules.

Assume the R-module M is a direct sum of countably generated modules. Then every direct summand of M is also a direct sum of countably generated modules.

Proof: Let $M = \bigoplus_\Lambda M_\lambda$, with each M_λ countably generated and $M = K \oplus L$. Let $\{K_\alpha\}_A$ and $\{L_\beta\}_B$ denote the countably generated submodules of K and L, respectively.

Consider the set $\mathcal{T}$ of all triples (Λ', A', B') with

$$\Lambda' \subset \Lambda,\ A' \subset A,\ B' \subset B \ \text{ and } \bigoplus\nolimits_{\Lambda'} M_\lambda = (\bigoplus\nolimits_{A'} K_\alpha) \oplus (\bigoplus\nolimits_{B'} L_\beta).$$

Define an order $\leq$ on $\mathcal{T}$ by

$$(\Lambda', A', B') \leq (\Lambda'', A'', B'') \ \Leftrightarrow\ \Lambda' \subset \Lambda'',\ A' \subset A'',\ B' \subset B''.$$

It is easily checked that $(\mathcal{T}, \leq)$ is inductive. Hence, by *Zorn's Lemma*, it has a maximal element (Λ^o, A^o, B^o). If we can show that $\Lambda^o = \Lambda$, then we have $K = \bigoplus_{A^o} K_\alpha$ and the assertion is verified.

Assume that $\Lambda^o \neq \Lambda$ and take $\mu \in \Lambda \setminus \Lambda^o$. Let $e \in End(M)$ denote the idempotent with $Me = K$, $M(1-e) = L$. For every countable subset $D \subset \Lambda$, the R-module $\bigoplus_D M_\lambda$ is countably generated and hence

$$(\bigoplus_D M_\lambda)e \quad \text{and} \quad (\bigoplus_D M_\lambda)(1-e)$$

are countably generated submodules of $M = \bigoplus_\Lambda M_\lambda$. Therefore we find a countable subset $D' \subset \Lambda$ such that $\bigoplus_{D'} M_\lambda$ contains these two submodules.

By recursion we construct an ascending sequence $D_1 \subset D_2 \subset \cdots$ of countable subsets of Λ with

$$M_\mu \subset M_\mu e + M_\mu(1-e) \subset \bigoplus_{D_1} M_\lambda,$$

$$\bigoplus_{D_n} M_\lambda \subset (\bigoplus_{D_n} M_\lambda)e + (\bigoplus_{D_n} M_\lambda)(1-e) \subset \bigoplus_{D_{n+1}} M_\lambda.$$

For $D = \bigcup_{I\!N} D_n$, we have $\mu \in D \not\subset \Lambda^o$,

$$(\bigoplus_D M_\lambda)e \subset \bigoplus_D M_\lambda, \quad (\bigoplus_D M_\lambda)(1-e) \subset \bigoplus_D M_\lambda,$$

and $\bigoplus_D M_\lambda$ is a countably generated R-module.

With the notation $M^o = \bigoplus_{\Lambda^o} M_\lambda$, $K^o = \bigoplus_{A^o} K_\alpha$ and $L^o = \bigoplus_{B^o} K_\beta$, we have, by construction, $M^o = K^o \oplus L^o$. Putting

$$M' = \bigoplus_{\Lambda^o \cup D} M_\lambda, \; K' = M'e \text{ and } L' = M'(1-e),$$

we obtain $M' \subset K' \oplus L'$, $K^o \subset K'$ and $L^o \subset L'$. In addition we have

$$\begin{aligned} K' &= (K^o + L^o + \bigoplus_D M_\lambda)e \subset K^o + \bigoplus_D M_\lambda \subset M', \\ L' &= (K^o + L^o + \bigoplus_D M_\lambda)(1-e) \subset L^o + \bigoplus_D M_\lambda \subset M'. \end{aligned}$$

From this we derive $M' = K' \oplus L'$. As direct summands in M, the modules K^o and L^o are also direct summands in K' resp. L', i.e. $K' = K^o \oplus K_1$ and $L' = L^o \oplus L_1$ for some $K_1 \subset K', L_1 \subset L'$. Now

$$M' = K' \oplus L' = M^o \oplus K_1 \oplus L_1 \text{ and } K_1 \oplus L_1 \simeq M'/M \simeq \bigoplus_{D \setminus \Lambda^o} M_\lambda.$$

This implies that K_1 and L_1 are countably generated and yields a contradiction to the maximality of (Λ^o, A^o, B^o).

8.11 Exercises.

(1) Let $K_1 = I\!R(1,0)$, $K_2 = I\!R(0,1)$, $K_3 = I\!R(1,1)$ *and* $K_4 = I\!R(3,1)$ *be submodules of the* $I\!R$*-module* $I\!R \times I\!R$.

(i) Prove $I\!R \times I\!R = K_1 \oplus K_2 = K_1 \oplus K_3 = K_1 \oplus K_4$;

(ii) write the element $(r,s) \in I\!R \times I\!R$ *as a sum of elements in* K_1 *and* K_i, $(i = 2,3,4)$.

(2) Let $f : M \to N$ *be an epimorphism of R-modules and* $M = K + L$. *Show that:*

(i) $N = (K)f + (L)f$.

(ii) If $Ke\, f = K \cap L$, *then* $N = (K)f \oplus (L)f$.

(3) Let M be an R-module, $S = End({}_RM)$, *and K an (R,S)-submodule of M (fully invariant submodule). Prove: If* $M = M_1 \oplus M_2$, *then*

$K = (K \cap M_1) \oplus (K \cap M_2)$ *and*

$M/K \simeq M_1/(K \cap M_1) \;\oplus\; M_2/(K \cap M_2)$.

(4) Let M be an R-module and $S = End(M)$. *Show that the following assertions are equivalent:*

(a) every idempotent in S is central;

(b) every direct summand of ${}_RM$ *is a fully invariant submodule;*

(c) if K is a direct summand of ${}_RM$, *then there is exactly one* $L \subset M$ *with* $M = K \oplus L$;

(d) if $M = K_1 \oplus L_1 = K_2 \oplus L_2$ *are decompositions of M, then* $M = (K_1 \cap K_2) \oplus (K_1 \cap L_2) \oplus (L_1 \cap K_2) \oplus (L_1 \cap L_2)$.

9 Product, coproduct and subdirect product

1.Product. 2.Product of morphisms. 3.Product in R-MOD. 4.Characterization of the product. 5.Coproduct. 6.Coproduct of morphisms. 7.Coproduct in R-MOD. 8.Characterization of the coproduct. 9.Relation between internal and external sum. 10.Free modules. 11.Subdirect product of modules. 12.Modules as products of factor modules. 13.R as product of rings. 14.Exercises.

In this section product and coproduct in arbitrary categories are defined. Then existence and special properties of these constructions in *R-MOD* are studied.

9.1 Product. Definition. Let $\{A_\lambda\}_\Lambda$ be a family of objects in the category $\mathcal{C}$. An object P in $\mathcal{C}$ with morphisms $\{\pi_\lambda : P \to A_\lambda\}_\Lambda$ is called the *product* of the family $\{A_\lambda\}_\Lambda$ if:

For every family of morphisms $\{f_\lambda : X \to A_\lambda\}_\Lambda$ in $\mathcal{C}$, there is a unique morphism $f : X \to P$ with $f\pi_\lambda = f_\lambda$ for all $\lambda \in \Lambda$.

For the object P, we usually write $\prod_{\lambda\in\Lambda} A_\lambda$, $\prod_\Lambda A_\lambda$ or $\prod A_\lambda$. If all A_λ are equal to A, then we put $\prod_\Lambda A_\lambda = A^\Lambda$.

The morphisms π_λ are called the λ-*projections* of the product. The definition can be described by the following commutative diagram:

$$\begin{tikzcd} \prod_\Lambda A_\mu \arrow[rr, "\pi_\lambda"] & & A_\lambda \\ & X \arrow[ul, "f"] \arrow[ur, "f_\lambda"'] & \end{tikzcd}$$

The product of a family of objects is - if it exists - uniquely determined up to isomorphism:
If $\{\pi_\lambda : P \to A_\lambda\}_\Lambda$ and $\{\pi'_\lambda : P' \to A_\lambda\}_\Lambda$ are products of the family $\{A_\lambda\}_\Lambda$, then there is an isomorphism $\gamma : P \to P'$ with $\gamma\pi'_\lambda = \pi_\lambda$ for all $\lambda \in \Lambda$.

If there are products of objects in a category, then also the product of a family of morphisms can be constructed in the following way:

9.2 Product of morphisms.
Let $\{f_\lambda : A_\lambda \to B_\lambda\}_\Lambda$ be a family of morphisms in a category $\mathcal{C}$ and $\{\pi_\lambda : \prod A_\mu \to A_\lambda\}_\Lambda$, $\{\tilde\pi_\lambda : \prod B_\mu \to B_\lambda\}_\Lambda$ the corresponding products. Then there is a unique morphism

$$f : \prod_\Lambda A_\lambda \to \prod_\Lambda B_\lambda \text{ with } f\tilde\pi_\lambda = \pi_\lambda f_\lambda \text{ for all } \lambda \in \Lambda.$$

If all f_λ are monic, then f is also monic.

Notation: $f = \prod_\Lambda f_\lambda$ or $f = \prod f_\lambda$.

Proof: The existence and uniqueness of f with the desired properties follow from the defining property of $\prod_\Lambda B_\lambda$. We can see this in the following commutative diagram:

$$\begin{array}{cccc} & \prod B_\mu & \xrightarrow{\tilde\pi_\lambda} & B_\lambda & \\ f & \uparrow & & \uparrow & f_\lambda \\ & \prod A_\mu & \xrightarrow{\pi_\lambda} & A_\lambda & \end{array}$$

Now consider $g, h : X \to \prod A_\mu$ with $gf = hf$. Then

$gf\tilde\pi_\lambda = g\pi_\lambda f_\lambda = hf\tilde\pi_\lambda = h\pi_\lambda f_\lambda$.

If all f_λ are monic, then $g\pi_\lambda = h\pi_\lambda$. By the defining property of $\prod A_\lambda$, there is only one morphism $\bar g : X \to \prod A_\mu$ with $\bar g\pi_\lambda = g\pi_\lambda = h\pi_\lambda$, i.e. $\bar g = g = h$ and f is monic.

In the category of sets *ENS*, the cartesian product of a family $\{A_\lambda\}_\Lambda$ of sets with the canonical projections is a product as defined above. For example, it can be represented in the following way:

$$\begin{array}{rcl} \prod_\Lambda A_\lambda & = & \{\alpha \in Map(\Lambda, \bigcup_{\lambda\in\Lambda} A_\lambda) \mid (\lambda)\alpha \in A_\lambda \text{ for all } \lambda \in \Lambda\} \\ & = & \{(a_\lambda)_\Lambda \mid a_\lambda \in A_\lambda\}. \end{array}$$

By the *axiom of choice*, this set is not empty if all A_λ's are non-empty.

Projections: $\pi_\mu : \prod_\Lambda A_\lambda \to A_\mu$, $\alpha \mapsto (\mu)\alpha$.

These π_μ are obviously surjective, i.e. they are retractions in *ENS*.

For any family of morphisms $\{f_\lambda : B \to A_\lambda\}$ we obtain the desired map f with $f\pi_\lambda = f_\lambda$ by

$$f : B \to \prod\nolimits_\Lambda A_\lambda, \quad b \mapsto ((b)f_\lambda)_{\lambda\in\Lambda}, \quad b \in B.$$

The products in *ENS* also allow us to construct products in *R-MOD*:

9.3 Product in *R-MOD*.

Let $\{M_\lambda\}_\Lambda$ be a family of R-modules and $(\prod_\Lambda M_\lambda, \pi_\lambda)$ the product of the M_λ in ENS. For $m, n \in \prod_\Lambda M_\lambda$, $r \in R$, using

$$(m+n)\pi_\lambda = (m)\pi_\lambda + (n)\pi_\lambda, \quad (rm)\pi_\lambda = r(m)\pi_\lambda,$$

a left R-module structure is defined on $\prod_\Lambda M_\lambda$ such that the π_λ are homomorphisms.

With this structure $(\prod_\Lambda M_\lambda, \pi_\lambda)$ is the product of the $\{M_\lambda\}_\Lambda$ in R-MOD.

Properties:

(1) If $\{f_\lambda : N \to M_\lambda\}_\Lambda$ is a family of morphisms, then we get the map

$$f : N \to \prod_\Lambda M_\lambda \quad n \mapsto ((n)f_\lambda)_{\lambda\in\Lambda},$$

and $Ke\, f = \bigcap_\Lambda Ke\, f_\lambda$ since $(n)f = 0$ if and only if $(n)f_\lambda = 0$ for all $\lambda \in \Lambda$.

(2) For every $\mu \in \Lambda$, we have a canonical embedding

$$\varepsilon_\mu : M_\mu \to \prod_\Lambda M_\lambda,\ \ m_\mu \mapsto (m_\mu\delta_{\mu\lambda})_{\lambda\in\Lambda},\ \ m_\mu \in M_\mu,$$

with $\varepsilon_\mu\pi_\mu = id_{M_\mu}$, i.e. π_μ is a retraction and ε_μ a coretraction.

This construction can be extended to larger subsets of Λ: For a subset $A \subset \Lambda$ we form the product $\prod_A M_\lambda$ and a family of homomorphisms

$$f_\mu : \prod_A M_\lambda \to M_\mu,\ \ f_\mu = \begin{cases} \pi_\mu & \text{for } \mu \in A, \\ 0 & \text{for } \mu \in \Lambda \setminus A. \end{cases}$$

Then there is a unique homomorphism

$$\varepsilon_A : \prod_A M_\lambda \to \prod_\Lambda M_\lambda \ \text{ with } \ \varepsilon_A\pi_\mu = \begin{cases} \pi_\mu & \text{for } \mu \in A, \\ 0 & \text{for } \mu \in \Lambda \setminus A. \end{cases}$$

The universal property of $\prod_A M_\lambda$ yields a homomorphism

$$\pi_A : \prod_\Lambda M_\lambda \to \prod_A M_\lambda \ \text{ with } \pi_A\pi_\mu = \pi_\mu \text{ for } \mu \in A.$$

Together this implies $\varepsilon_A\pi_A\pi_\mu = \varepsilon_A\pi_\mu = \pi_\mu$ for all $\mu \in A$, and, by the properties of the product $\prod_A M_\lambda$, we get $\varepsilon_A\pi_A = id_{\prod_A M_\lambda}$. By 8.2, the image of ε_A is a direct summand in $\prod_\Lambda M_\lambda$ and we have shown:

(3) If $\{M_\lambda\}_\Lambda$ is a family of R-modules and $\Lambda = A \cup B$ with $A \cap B = \emptyset$, then

(*i*) $\prod_\Lambda M_\lambda = (\prod_A M_\lambda)\varepsilon_A \oplus (\prod_B M_\lambda)\varepsilon_B$;

(*ii*) the following sequence is exact and splits:

$$0 \longrightarrow \prod_A M_\lambda \xrightarrow{\varepsilon_A} \prod_\Lambda M_\lambda \xrightarrow{\pi_B} \prod_B M_\lambda \longrightarrow 0 \quad .$$

The following assertions are readily verified. Observe that they are stronger than the assertions about products in arbitrary categories in 9.2:

(4) If $\{f_\lambda : M_\lambda \to N_\lambda\}_\Lambda$ is a family of morphisms in *R-MOD* and $\prod_\Lambda f_\lambda : \prod_\Lambda M_\lambda \to \prod_\Lambda N_\lambda$ the product of the f_λ (see 9.2), then

$$Ke \prod_\Lambda f_\lambda = \prod_\Lambda Ke\, f_\lambda,\ \ Im \prod_\Lambda f_\lambda = \prod_\Lambda Im\, f_\lambda.$$

(5) If $0 \longrightarrow L_\lambda \xrightarrow{f_\lambda} M_\lambda \xrightarrow{g_\lambda} N_\lambda \longrightarrow 0$ is a family of exact sequences, then the following sequence is also exact

$$0 \longrightarrow \prod_\Lambda L_\lambda \xrightarrow{\prod f_\lambda} \prod_\Lambda M_\lambda \xrightarrow{\prod g_\lambda} \prod_\Lambda N_\lambda \longrightarrow 0\,.$$

9.4 Characterization of the product.

It is equivalent to the definition of the product of a family $\{M_\lambda\}_{\lambda\in\Lambda}$ *of* R*-modules, to claim that, for every* R*-module* N*, the following map is bijective:*

$$\Phi : Hom(N, \prod_\Lambda M_\lambda) \to \prod_\Lambda Hom(N, M_\lambda),\ \ f \mapsto (f\pi_\lambda)_{\lambda\in\Lambda}.$$

Here the product on the left is to be formed in *R-MOD* and the product on the right is in *ENS*.

$Hom(N, \prod_\Lambda M_\lambda)$ and $Hom(N, M_\lambda)$ are left modules over $End_R(N)$ and hence $\prod_\Lambda Hom(N, M_\lambda)$ can also be considered as a left $End_R(N)$-module (see above). Φ respects these structures, i.e. Φ is an $End_R(N)$-morphism.

Dual to the notion of a product we define:

9.5 Coproduct. Definition. Let $\{A_\lambda\}_\Lambda$ be a family of objects of a category $\mathcal{C}$. An object K in $\mathcal{C}$ with morphisms $\{\varepsilon_\lambda : A_\lambda \to K\}_\Lambda$ is called the *coproduct* of the family $\{A_\lambda\}_\Lambda$ if:

For every family of morphisms $\{g_\lambda : A_\lambda \to Y\}_\Lambda$ in $\mathcal{C}$, there is a unique morphism $g : K \to Y$ with $\varepsilon_\lambda g = g_\lambda$.

For this object K, we usually write $\coprod_{\lambda\in\Lambda} A_\lambda$, $\coprod_\Lambda A_\lambda$, or $\coprod A_\lambda$. If all the A_λ are equal to A, then we use the notation $\coprod_\Lambda A_\lambda = A^{(\Lambda)}$.

The morphisms ε_λ are called the λ-*injections* of the coproducts. The properties of the coproduct are described in the diagram

$$\begin{array}{ccc} A_\lambda & \xrightarrow{\varepsilon_\lambda} & \coprod_\Lambda A_\mu \\ & {}_{g_\lambda}\searrow \quad \swarrow_{g} & \\ & Y & \end{array} \quad .$$

Coproducts are uniquely determined up to isomorphisms. Although the definitions of product and coproduct in any category are dual to each other, the existence of one of them in general need not imply the existence of the other. In categories with coproducts, the coproduct of morphisms can also be constructed, and dual to 9.2 we obtain:

9.6 Coproduct of morphisms.

Let $\{g_\lambda : A_\lambda \to B_\lambda\}_\Lambda$ *be a family of morphisms in a category* $\mathcal{C}$ *and* $\{\varepsilon_\lambda : A_\lambda \to \coprod_\Lambda A_\mu\}$, $\{\tilde{\varepsilon}_\lambda : B_\lambda \to \coprod_\Lambda B_\mu\}$ *the corresponding coproducts. Then there is a unique morphism*

$$g : \coprod\nolimits_\Lambda A_\lambda \to \coprod\nolimits_\Lambda B_\lambda \quad \textit{with} \quad g_\lambda \tilde{\varepsilon}_\lambda = \varepsilon_\lambda g \textit{ for all } \lambda \in \Lambda.$$

If all the g_λ *are epic, then* g *is also epic.*

Notation: $g = \coprod_\Lambda g_\lambda$ or $g = \coprod g_\lambda$.

In some categories, e.g. in the category of non-commutative groups, the proof of the existence of coproducts might be quite tedious. However, in *R-MOD* coproducts are obtained without effort from products:

9.7 Coproduct in *R-MOD*.

Let $\{M_\lambda\}_\Lambda$ *be a family of R-modules. Then*

$$\coprod\nolimits_\Lambda M_\lambda = \{m \in \prod\nolimits_\Lambda M_\lambda \mid (m)\pi_\lambda \neq 0 \textit{ only for finitely many } \lambda \in \Lambda\}$$

forms an R-module and together with the injections

$$\varepsilon_\mu : M_\mu \to \coprod\nolimits_\Lambda M_\lambda, \;\; m_\mu \mapsto (m_\mu \delta_{\mu\lambda})_{\lambda\in\Lambda},$$

is the coproduct of $\{M_\lambda\}_\Lambda$ *in R-MOD.*

$\coprod_\Lambda M_\lambda$ *also is called the* (external) direct sum *of the* $\{M_\lambda\}_\Lambda$ *and is denoted by* $\bigoplus_{\lambda\in\Lambda} M_\lambda$, $\bigoplus_\Lambda M_\lambda$ *or* $\bigoplus M_\lambda$.

Proof: Assume the family of morphisms $\{g_\lambda : M_\lambda \to Y\}_\Lambda$ in *R-MOD* is given. The desired homomorphism is obtained by

$$g : \bigoplus\nolimits_\Lambda M_\lambda \to Y, \;\; m \mapsto \sum_{\mu\in\Lambda} (m)\pi_\mu g_\mu.$$

Since $m \in \bigoplus_\Lambda M_\lambda$, the sum to be formed is in fact finite. For $m_\lambda \in M_\lambda$, we have $(m_\lambda)\varepsilon_\lambda g = \sum_{\mu\in\Lambda}(m_\lambda \varepsilon_\lambda)\pi_\mu g_\mu = m_\lambda g_\lambda$ (since $\varepsilon_\lambda \pi_\mu = \delta_{\lambda\mu}$), i.e. $\varepsilon_\lambda g = g_\lambda$.

Properties:

(1) For finite index sets $\Lambda = \{1, \cdots, k\}$, the objects $\coprod_\Lambda M_\lambda$ and $\prod_\Lambda M_\lambda$ are isomorphic (they are even represented by the same object) and we write

$$M_1 \times \cdots \times M_k = M_1 \oplus \cdots \oplus M_k = \bigoplus\nolimits_{1\leq i\leq k} M_i.$$

(2) If $\{g_\lambda : M_\lambda \to N\}_\Lambda$ is a family of R-module homomorphisms and $g : \bigoplus_\Lambda M_\lambda \to N$ the corresponding map from the coproduct to N, then

$$Im\, g = \sum\nolimits_\Lambda Im\, g_\lambda = \sum\nolimits_\Lambda (M_\lambda) g_\lambda \subset N.$$

(3) It follows from the considerations in 9.3,(2) that $\varepsilon_\mu : M_\mu \to \bigoplus_\Lambda M_\lambda$ is a coretraction.

(4) For subsets A, B of the index set Λ with $A \cup B = \Lambda$ and $A \cap B = \emptyset$, the restriction of the maps (see 9.3,(3))

$$\varepsilon_A : \prod\nolimits_A M_\lambda \to \prod\nolimits_\Lambda M_\lambda \text{ and } \pi_A : \prod\nolimits_\Lambda M_\lambda \to \prod\nolimits_A M_\lambda$$

yields corresponding maps of the direct sums. Denoting these by the same symbols we get

(*i*) $\bigoplus_\Lambda M_\lambda = (\bigoplus_A M_\lambda)\varepsilon_A \oplus (\bigoplus_B M_\lambda)\varepsilon_B$;

(*ii*) the following sequence is exact and splits:
$0 \longrightarrow \bigoplus_A M_\lambda \xrightarrow{\varepsilon_A} \bigoplus_\Lambda M_\lambda \xrightarrow{\pi_B} \bigoplus_B M_\lambda \longrightarrow 0 \quad .$

(5) If $\{f_\lambda : M_\lambda \to N_\lambda\}_\Lambda$ is a family of morphisms in R-MOD, and $\bigoplus_\Lambda f_\lambda$ the coproduct of the f_λ (see 9.6), then

$$Ke \bigoplus\nolimits_\Lambda f_\lambda = \bigoplus\nolimits_\Lambda Ke\, f_\lambda, \quad Im \bigoplus\nolimits_\Lambda f_\lambda = \bigoplus\nolimits_\Lambda Im\, f_\lambda.$$

(6) If $0 \longrightarrow L_\lambda \xrightarrow{f_\lambda} M_\lambda \xrightarrow{g_\lambda} N_\lambda \longrightarrow 0$ is a family of exact sequences, then the following sequence is also exact

$$0 \longrightarrow \bigoplus\nolimits_\Lambda L_\lambda \xrightarrow{\oplus f_\lambda} \bigoplus\nolimits_\Lambda M_\lambda \xrightarrow{\oplus g_\lambda} \bigoplus\nolimits_\Lambda N_\lambda \longrightarrow 0 \ .$$

9.8 Characterization of the coproduct.

It is equivalent to the definition of the coproduct of a family $\{M_\lambda\}_\Lambda$ *of* R*-modules, to demand that, for every* N *in* R*-MOD, the following map is bijective:*

$$\Psi : Hom(\bigoplus\nolimits_\Lambda M_\lambda, N) \to \prod\nolimits_\Lambda Hom(M_\lambda, N), \quad g \mapsto (\varepsilon_\lambda g)_{\lambda \in \Lambda}.$$

With the canonical module structure on both sides (see 9.4), Ψ is in fact a homomorphism of right $End_R(N)$-modules.

9.9 Relationship between internal and external sum.

If $\{M_\lambda\}_\Lambda$ is a family of submodules of the R-module M, then the coproduct (= external direct sum) $\coprod_\Lambda M_\lambda$ can be formed and, for the inclusions $in_\mu : M_\mu \to M$, $\mu \in \Lambda$, we get a morphism

$$h : \coprod_\Lambda M_\lambda \to M, \text{ with } \varepsilon_\mu h = in_\mu,\ Im\, h = \sum_\Lambda M_\lambda.$$

h is surjective if and only if $\sum_\Lambda M_\lambda = M$. h is injective if $m_{\lambda_1} + \cdots + m_{\lambda_k} = 0$ with $m_{\lambda_r} \in M_{\lambda_r}$, $\lambda_r \in \Lambda$, only if all m_{λ_r} are zero, i.e. if the $\{M_\lambda\}_\Lambda$ are an independent family of submodules. Hence:

h is an isomorphism if and only if M is an internal direct sum of the $\{M_\lambda\}_\Lambda$.

In this case we get $\coprod_\Lambda M_\lambda \simeq \bigoplus_\Lambda M_\lambda$ (internal direct sum), i.e. usually it is not necessary to distinguish between internal and external direct sums.

Observe that $\coprod M_\lambda$ also can be considered as an internal direct sum of the $M_\lambda \varepsilon_\lambda$, i.e. $\coprod_\Lambda M_\lambda = \bigoplus_\Lambda M_\lambda \varepsilon_\lambda$.

9.10 Free modules. A generating set $\{m_\lambda\}_\Lambda$ of an R-module ${}_RM$ is called a *basis* of M if the representation of any element $m \in M$ as

$$m = a_1 m_{\lambda_1} + \cdots + a_k m_{\lambda_k} \text{ with } a_i \in R$$

is uniquely determined. If there exists a basis in M, then M is called a *free* R-module.

If $\{m_\lambda\}_\Lambda$ is a basis, then $Rm_\lambda \simeq R$, M is an internal direct sum of the $\{Rm_\lambda\}_\Lambda$, and there are isomorphisms of left R-modules

$$R^{(\Lambda)} \simeq \bigoplus_\Lambda Rm_\lambda \simeq M.$$

The map $\gamma : \Lambda \to M$, $\lambda \mapsto m_\lambda$, has the following properties, which could be used to define free modules over Λ:

For every (set) map from Λ into an R-module ${}_RN$, $\alpha : \Lambda \to N$, there is a unique R-homomorphism $f : M \to N$ with $\gamma f = \alpha$.

This is just the fact that vector space homomorphisms $V \to W$ can be obtained by assigning arbitrary images to a base of V.

If a module M has an infinite basis, then every other basis of M has the same cardinality. This follows immediately from 8.8.

$R^{(\Lambda)}$ can be represented as

$$R^{(\Lambda)} = \{f : \Lambda \to R \mid (\lambda)f = 0 \text{ for almost all } \lambda \in \Lambda\},$$

and the family $\{f_\lambda\}_\Lambda$ with $f_\lambda(\mu) = \delta_{\lambda\mu}$ forms a basis, the *canonical basis* of $R^{(\Lambda)}$.

Given a generating set $\{n_\lambda\}_\Lambda$ of an R-module N, then the map $R^{(\Lambda)} \to N,\ f_\lambda \mapsto n_\lambda$, is an epimorphism, i.e.:

Every (finitely generated) R-module is a factor module of a (finitely generated) free R-module.

Finally we want to consider another product of modules which we already know for rings (see 3.12):

9.11 Subdirect product of modules.
Let $\{M_\lambda\}_\Lambda$ be a family of R-modules. A submodule $M \subset \prod_\Lambda M_\lambda$ is called the *subdirect product* of the M_λ if, for every $\lambda \in \Lambda$, the restriction of the projection π_λ to M, $\pi_\lambda|_M : M \to M_\lambda$, is an epimorphism.

Referring to the properties of the product it is readily verified:

(1) A module N is isomorphic to a subdirect product of $\{M_\lambda\}_\Lambda$ if and only if there is a family of epimorphisms $f_\lambda : N \to M_\lambda$ with $\bigcap_\Lambda Ke\, f_\lambda = 0$.

(2) If $\{N_\lambda\}_\Lambda$ is a family of submodules of the R-module N, then $N/\bigcap_\Lambda N_\lambda$ is isomorphic to a subdirect product of the modules $\{N/N_\lambda\}_\Lambda$.

An R-module N is called *subdirectly irreducible*, if it is not a subdirect product of proper factor modules. This is the case if and only if the intersection of all non-zero submodules is again non-zero (see 14.8).

Examples of subdirect products are the product $\prod_\Lambda M_\lambda$ and the direct sum $\bigoplus_\Lambda M_\lambda$ of any family $\{M_\lambda\}_\Lambda$ of R-modules.

Dual to the representation of a module as the coproduct of submodules we obtain for finite families:

9.12 Modules as products of factor modules.
Let M be an R-module and $K_1, \ldots, K_n$ submodules of M. Then the following assertions are equivalent:

(a) The canonical map $p : M \to \prod_{i\le n} M/K_i,\ m \mapsto (m + K_i)_{i\le n}$, is epic (and monic);

(b) for every $j \le n$ we have $K_j + \bigcap_{i\ne j} K_i = M$ (and $\bigcap_{i\le n} K_i = 0$).

Proof: $(a) \Rightarrow (b)$ Let p be epic and $m \in M$. For $j \le n$ we form the element $(\cdots, 0, m+K_j, 0, \cdots) \in \prod_{i\le n} M/K_i$ and choose a preimage $m' \in M$ under p.

Then $m' - m \in K_j$ and $m' \in \bigcap_{i\ne j} K_i$, i.e. $m \in K_j + \bigcap_{i\ne j} K_i$.

$(b) \Rightarrow (a)$ Consider $(m_i + K_i)_{i\le n} \in \prod_{i\le n} M/K_i$. By (b), we can find

$$k_j \in K_j \ \text{ and } \ \tilde{k}_j \in \bigcap\nolimits_{i\ne j} K_i \ \text{ with } \ m_j = k_j + \tilde{k}_j\,.$$

For the element $m = \tilde{k}_1 + \cdots + \tilde{k}_n \in M$, we get

$$(m)p\pi_j = m + K_j = \tilde{k}_j + K_j = m_j + K_j \text{ for all } j \leq n.$$

Since $Ke\,p = \bigcap_{i\leq n} K_i$, the map p is monic if and only if $\bigcap_{i\leq n} K_i = 0$.

Considering the ring R as a bimodule, the submodules are just the ideals in R and, for $M = {}_RR_R$, we get from 9.12 a representation of R as a product of rings (see 3.12). The following version of this fact is (in number theory) known as *Chinese Remainder Theorem*:

9.13 *R* as a product of rings.

For ideals $I_1, \cdots, I_n$ in a ring R with unit, the following are equivalent:

(a) The canonical map $p : R \to \prod_{i\leq n} R/I_i$ is epic (and monic);

(b) for $i \neq j$ we have $I_i + I_j = R$ (and $\bigcap_{i\leq n} I_i = 0$).

Proof: We have to show that $I_1 + \bigcap_{i>1} I_i = R$ is equivalent to $I_1 + I_i = R$ for all $1 < i \leq n$. The first implication is clear.

Let $I_1 + I_i = R$ for $1 < i \leq n$, i.e. $a_i + b_i = 1$ for some $a_i \in I_1$ and $b_i \in I_i$. Then $1 = (a_2 + b_2)\cdots(a_n + b_n) = a_0 + b_2 \cdots b_n$ for an $a_0 \in I_1$, i.e. $1 \in I_1 + I_2 \cdots I_n \subset I_1 + \bigcap_{i>1} I_i$.

9.14 Exercises.

(1) Let $\mathcal{C}$ be a category. Prove:

(i) If $\{\pi_\mu : \prod_\Lambda A_\lambda \to A_\mu\}_\Lambda$ is a product in $\mathcal{C}$, then π_μ is a retraction if and only if, for every $\lambda \in \Lambda$, there is a morphism $A_\mu \to A_\lambda$.

(ii) If $\{\varepsilon_\mu : A_\mu \to \coprod_\Lambda A_\lambda\}_\Lambda$ is a coproduct in $\mathcal{C}$, then ε_μ is a coretraction if and only if, for every $\lambda \in \Lambda$, there is a morphism $A_\lambda \to A_\mu$.

(2) Show that $\mathbb{Z}/(30)$ with the canonical projections to $\mathbb{Z}/(2)$, $\mathbb{Z}/(3)$ and $\mathbb{Z}/(5)$ is a product of these modules in $\mathbb{Z}$-MOD.

(3) Let $f : M \to N$ be an isomorphism of R-modules. Prove:
If $M = \bigoplus_\Lambda M_\lambda$, then $N = \bigoplus_\Lambda (M_\lambda)f$.

(4) Show that $\mathbb{Z}$ is a subdirect product of the modules
$\{\mathbb{Z}/(p) \mid p$ a prime number $\}$.

10 Pullback and pushout

1.Pullback. 2.Existence. 3.Properties. 4.Pushout. 5.Existence. 6.Properties. 7.Characterizations. 8.Exercises.

In this section we give two constructions with universal properties for pairs of morphisms and consider some related diagram properties.

10.1 Pullback. Definition. Let $f_1 : M_1 \to M$, $f_2 : M_2 \to M$ be two morphisms in *R-MOD*. A commutative diagram in *R-MOD*

$$\begin{array}{ccc} P & \xrightarrow{p_2} & M_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow{f_1} & M \end{array}$$

is called the *pullback (or fibre product, cartesian square)* for the pair (f_1, f_2) if, for every pair of morphisms

$$g_1 : X \to M_1,\;\; g_2 : X \to M_2 \;\text{ with }\; g_1 f_1 = g_2 f_2\,,$$

there is a unique morphism $g : X \to P$ with $gp_1 = g_1$ and $gp_2 = g_2$.

For a pair of morphisms the pullback is uniquely determined up to isomorphism: If $p_1' : P' \to M_1$, $p_2' : P' \to M_2$ is also a pullback for f_1, f_2 given above, then there is an isomorphism $h : P' \to P$ with $hp_i = p_i'$, $i = 1, 2$.

10.2 Existence. *For every pair $f_1 : M_1 \to M$, $f_2 : M_2 \to M$ of morphisms in R-MOD there exists a pullback:*
With the projections $\pi_i : M_1 \oplus M_2 \to M_i$, $i = 1, 2$, we obtain a morphism

$$p^* = \pi_1 f_1 - \pi_2 f_2 :\; M_1 \oplus M_2 \to M,$$

and with the restriction π_i' of π_i to $Ke\,p^* \subset M_1 \oplus M_2$, the square

$$\begin{array}{ccc} Ke\,p^* & \xrightarrow{\pi_2'} & M_2 \\ {\scriptstyle \pi_1'}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow{f_1} & M \end{array}$$

becomes a pullback for (f_1, f_2). By construction

$$Ke\,p^* = \{(m_1, m_2) \in M_1 \oplus M_2 \mid (m_1)f_1 = (m_2)f_2\}.$$

Proof: Let $g_1 : X \to M_1$, $g_2 : X \to M_2$ be given with $g_1 f_1 - g_2 f_2 = 0$ and $\tilde{g} : X \to M_1 \oplus M_2$ the corresponding map into the product. Then

$$\tilde{g}p^* = \tilde{g}\pi_1 f_1 - \tilde{g}\pi_2 f_2 = g_1 f_1 - g_2 f_2 = 0,$$

and hence $\tilde{g}$ factorizes over $Ke\,p^*$.

If M_1, M_2 are submodules of M and $M_i \to M$ the natural embeddings, then we have as pullback

$$\begin{array}{ccc} M_1 \cap M_2 & \longrightarrow & M_1 \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & M \end{array} .$$

10.3 Properties of the pullback.
Consider the following commutative diagram in R-MOD:

$$QU: \quad \begin{array}{ccc} P & \xrightarrow{h_2} & M_2 \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow{f_1} & M \end{array}$$

(1) If QU is a pullback diagram, then:
(i) The following commutative diagram with exact rows exists:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{h_2} & M_2 \\ & & \| & & {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ 0 & \longrightarrow & K & \longrightarrow & M_1 & \xrightarrow{f_1} & M \end{array}$$

(ii) If f_1 is monic, then h_2 is monic.
(iii) If f_1 is epic, then h_2 is epic.

(2) If f_1 is monic, then for the commutative diagram with exact lower row

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \xrightarrow{h_2} & M_2 & \xrightarrow{f_2 p} & C & & \\ & & {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \| & & \\ 0 & \longrightarrow & M_1 & \xrightarrow{f_1} & M & \xrightarrow{p} & C & \longrightarrow & 0 \end{array}$$

we have: QU is a pullback if and only if the first row is exact.
If f_2 is epic, then $f_2 p$ is also epic.

Proof: (1) Assume QU to be a pullback. Using the presentation and notation given in 10.1 we may assume

$$P = Ke\,p^* = \{(m_1, m_2) \in M_1 \oplus M_2 \mid (m_1)f_1 = (m_2)f_2\}, \quad h_1 = \pi_1', \ h_2 = \pi_2'.$$

(i) Setting $K = Ke\,f_1$ and $K \to P$, $k \mapsto (k, 0)$, we get the desired diagram.

(ii) is a consequence of (i).

(*iii*) Let f_1 be epic. Then, for $m_2 \in M_2$, there is an $m_1 \in M_1$ with $(m_1)f_1 = (m_2)f_2$. Then $(m_1, m_2) \in P$ and $(m_1, m_2)h_2 = m_2$.

(2) Let f_1 be monic. If QU is a pullback, then, choosing a representation as in (1), we first obtain that h_2 is monic. For $m_2 \in Ke\, f_2 p$, there exists $m_1 \in M_1$ with $(m_1)f_1 = (m_2)f_2$. This means $(m_1, m_2) \in P$ and $(m_1, m_2)h_2 = m_2$. Hence the first row is exact.

Now assume the first row to be exact and $g_1 : X \to M_1$, $g_2 : X \to M_2$ with $g_1 f_1 = g_2 f_2$. Then $g_2 f_2 p = g_1 f_1 p = 0$, i.e. there is a unique $k : X \to P = Ke\, f_2 p$ with $kh_2 = g_2$.

We also have $kh_1 f_1 = kh_2 f_2 = g_1 f_1$. f_1 being monic, this implies $kh_1 = g_1$ and QU is a pullback.

Recalling that the pullback of two submodules is just their intersection we obtain from 10.3:

Noether Isomorphism Theorem.

For two submodules M_1, M_2 of an R-module M, we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 \cap M_2 & \longrightarrow & M_2 & \longrightarrow & M_2/M_1 \cap M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\simeq & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_1 + M_2 & \longrightarrow & M_1 + M_2 \,/\, M_1 & \longrightarrow & 0 \quad .
\end{array}$$

The pullback was formed for two morphisms with the same target. Dually, we define for two morphisms with the same source:

10.4 Pushout. Definition. Let $g_1 : N \to N_1$, $g_2 : N \to N_2$ be two morphisms in *R-MOD*. A commutative diagram in *R-MOD*

$$\begin{array}{ccc}
N & \xrightarrow{g_2} & N_2 \\
{\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle q_2} \\
N_1 & \xrightarrow{q_1} & Q
\end{array}$$

is called the *pushout* for the pair (g_1, g_2) if, for every pair of morphisms

$$h_1 : N_1 \to Y,\ h_2 : N_2 \to Y \text{ with } g_1 h_1 = g_2 h_2\,,$$

there is a unique morphism $h : Q \to Y$ with $q_1 h = h_1$, $q_2 h = h_2$.

Again Q is uniquely determined up to isomorphism.

The pushout is also called the *fibre sum, amalgamated sum* or *cocartesian square* for (g_1, g_2).

Dually to 10.2, the existence of the pushout for any two morphisms with same source in *R-MOD* is obtained. It is useful to repeat the construction explicitely for this case:

10.5 Existence. *For every pair $g_1 : N \to N_1$, $g_2 : N \to N_2$ of morphisms in R-MOD a pushout exists:*
With the injections $\varepsilon_i : N_i \to N_1 \oplus N_2$, $i = 1, 2$, we obtain a morphism

$$q^* = g_1\varepsilon_1 + g_2\varepsilon_2 : \; N \to N_1 \oplus N_2.$$

With the canonical maps $\bar{\varepsilon}_i : N_i \to N_1 \oplus N_2 \to Coke\, q^*$ the square

$$\begin{array}{ccc} N & \xrightarrow{g_2} & N_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle -\bar{\varepsilon}_2} \\ N_1 & \xrightarrow{\bar{\varepsilon}_1} & Coke\, q^* \end{array}$$

is a pushout for (g_1, g_2). By construction,
$Im\, q^* = N(g_1\varepsilon_1 + g_2\varepsilon_2) = \{((n)g_1, (n)g_2) \mid n \in N\} \subset N_1 \oplus N_2$ and
$Coke\, q^* = N_1 \oplus N_2 / Im\, q^*$.

Proof: Assume for $i = 1, 2$ that we have morphisms $h_i : N_i \to Y$ with $g_1 h_1 = g_2 h_2$. We get the diagram

$$\begin{array}{ccccc} N & \xrightarrow{q^*} & N_1 \oplus N_2 & \longrightarrow & Coke\, q^* \\ & & {\scriptstyle \pi_1 h_1 - \pi_2 h_2} \searrow & & \\ & & & Y & \end{array}$$

with $q^*(\pi_1 h_1 - \pi_2 h_2) = (g_1\varepsilon_1 + g_2\varepsilon_2)(\pi_1 h_1 - \pi_2 h_2) = g_1 h_1 - g_2 h_2 = 0$.
By the cokernel property, there is a unique morphism $h : Coke\, q^* \to Y$ which yields the desired commutative diagram.

As a special case it is easily verified:

Let N_1, N_2 be submodules of N and $N \to N/N_i$ for $i = 1, 2$ the canonical projections. Then the pushout is given by

$$\begin{array}{ccc} N & \longrightarrow & N/N_1 \\ \downarrow & & \downarrow \\ N/N_2 & \longrightarrow & N/(N_1 + N_2) \end{array} \quad .$$

10.6 Properties of the pushout.
Consider the following commutative diagram in R-MOD:

$$QU: \quad \begin{array}{ccc} N & \xrightarrow{f_2} & N_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ N_1 & \xrightarrow{g_1} & Q \end{array}$$

(1) If QU is a pushout, then:
(i) We have the commutative diagram with exact rows

$$\begin{array}{ccccccc} N & \xrightarrow{f_2} & N_2 & \longrightarrow & C & \longrightarrow & 0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} & & \| & & \\ N_1 & \xrightarrow{g_1} & Q & \longrightarrow & C & \longrightarrow & 0\ . \end{array}$$

(ii) If f_2 is epic, then g_1 is epic.
(iii) If f_2 is monic, then g_1 is monic.

(2) If f_2 is epic, then for the commutative diagram with exact upper row

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & N & \xrightarrow{f_2} & N_2 & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} & & \\ & & K & \xrightarrow{if_1} & N_1 & \xrightarrow{g_1} & Q & \longrightarrow & 0 \end{array}$$

we have: QU is a pushout if and only if the lower row is exact.

If f_1 is monic, then if_1 is also monic.

Proof: (1)(i) From the representation of the pushout in 10.5, we have $Im\, q^* = \{((n)f_1, (n)f_2) \mid n \in N\}$ and we get a morphism

$$g : Q = N_1 \oplus N_2/Im\, q^* \to N_2/(N)f_2,\ \ (n_1, n_2) + Im\, q^* \mapsto n_2 + (N)f_2,$$

which leads to the desired diagram

$$\begin{array}{ccccccc} N & \xrightarrow{f_2} & N_2 & \longrightarrow & N_2/Nf_2 & \longrightarrow & 0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} & & \| & & \\ N_1 & \xrightarrow{g_1} & Q & \xrightarrow{g} & N_2/Nf_2 & \longrightarrow & 0. \end{array}$$

(ii) is obvious.
(iii) If f_2 is monic and $(n_1)g_1 = (n_1, 0) + Im\, q^* = 0 \in N_1 \oplus N_2/Im\, q^*$, then there exists $n \in N$ with $(n_1, 0) = ((n)f_1, (n)f_2)$, i.e. $n = 0$ and hence $n_1 = (0)f_1 = 0$.

(2) Let f_2 be epic. If QU is a pushout, then, by (1), g_1 is epic. Choose $K = Ke\, f_2$. If $n_1 \in Ke\, g_1$, then $(n_1, 0) \in Im\, q^*$, i.e. there exists $n \in N$ with $(n_1, 0) = ((n)f_1, (n)f_2)$, i.e. $n \in K = Ke\, f_2$. Hence the lower row is exact.

Now assume the lower row to be exact and let $h_i : N_i \to Y$, $i = 1, 2$, be morphisms with $f_1 h_1 = f_2 h_2$. Then $if_1 h_1 = 0$, and the cokernel property of Q yields a unique $h : Q \to Y$ with $g_1 h = h_1$. Then $f_2 h_2 = f_1 h_1 = f_1 g_1 h = f_2 g_2 h$ and hence $h_2 = g_2 h$ since f_2 is epic. Consequently the square is a pushout.

The representations of the pullback and pushout in *R-MOD* given in 10.2 and 10.5 can be combined as follows:

10.7 Characterizations of pullback and pushout.
Consider the following diagram in R-MOD:

$$QU: \quad \begin{array}{ccc} P & \xrightarrow{g_2} & M_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow{f_1} & Q \end{array}$$

Putting $p^* = \pi_1 f_1 - \pi_2 f_2,\ q^* = g_1\varepsilon_1 + g_2\varepsilon_2$ *we have the sequence*

$$P \xrightarrow{q^*} M_1 \oplus M_2 \xrightarrow{p^*} Q$$

with the properties:

(1) $q^*p^* = 0$ *if and only if QU is commutative;*

(2) q^* *is the kernel of* p^* *if and only if QU is a pullback;*

(3) p^* *is the cokernel of* q^* *if and only if QU is a pushout;*

(4) $0 \longrightarrow P \xrightarrow{q^*} M_1 \oplus M_2 \xrightarrow{p^*} Q \longrightarrow 0$ *is exact if and only if QU is a pullback and a pushout.*

10.8 Exercises.

(1) Show that, for $f : K \to L$ in R-MOD, the diagrams

$$\begin{array}{ccc} Ke\,f & \to & 0 \\ \downarrow & & \downarrow \\ K & \to & L \end{array} \qquad\qquad \begin{array}{ccc} K & \to & L \\ \downarrow & & \downarrow \\ 0 & \to & L/Im f \end{array}$$

represent a pullback, resp. pushout, for $(f, 0)$.

(2) Show that in R-MOD the pullback for $M_1 \to 0$, $M_2 \to 0$ and the pushout for $0 \to M_1$, $0 \to M_2$ can be represented by $M_1 \oplus M_2$.

(3) Let

$$\begin{array}{ccccc} K' & \longrightarrow & K & \longrightarrow & K'' \\ \downarrow & & \downarrow & & \downarrow \\ L' & \longrightarrow & L & \longrightarrow & L'' \end{array}$$

be a commutative diagram in R-MOD. Prove:

(i) If every partial square is a pullback (pushout), then the whole rectangle is also a pullback (pushout).

(ii) If the whole rectangle is a pullback and $K \to K''$ is monic, then the left square is a pullback.

(iii) If the whole rectangle is a pushout and $L' \to L$ is epic, then the right square is a pushout.

11 Functors, Hom-functors

1.Definitions. 2.Special functors. 3.Properties of covariant functors. 4.Properties of contravariant functors. 5.Mor-functors. 6.Properties. 7.Special functors in module categories. 8.Characterization of exact functors. 9.Properties of additive functors. 10.Hom-functors. 11.The functor $Hom_R(R,-)$. 12.Exercises.

Between groups or modules, the structure preserving maps, the homomorphisms, are of great interest. Similarly we consider connections between categories which respect their structure:

11.1 Functors. Definitions. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.
A *covariant functor* $T : \mathcal{C} \to \mathcal{D}$ consists of assignments for

objects: $Obj(\mathcal{C}) \to Obj(\mathcal{D}),\ A \mapsto T(A)$,
morphisms: $Mor(\mathcal{C}) \to Mor(\mathcal{D}),\ [f : A \to B] \mapsto [T(f) : T(A) \to T(B)]$,

with the properties
(i) $T(id_A) = id_{T(A)}$,
(ii) $T(gf) = T(g)T(f)$ if gf is defined in $\mathcal{C}$.

A *contravariant functor* $S : \mathcal{C} \to \mathcal{D}$ consists of assignments for

objects: $Obj(\mathcal{C}) \to Obj(\mathcal{D}),\ A \mapsto S(A)$,
morphisms: $Mor(\mathcal{C}) \to Mor(\mathcal{D}),\ [f : A \to B] \mapsto [S(f) : S(B) \to S(A)]$,

with the properties
(i) $S(id_A) = id_{S(A)}$,
(ii) $S(gf) = S(f)S(g)$ if gf is defined in $\mathcal{C}$.

An example of a contravariant functor is the transition from $\mathcal{C}$ to the dual category $\mathcal{C}^o$, $D : \mathcal{C} \to \mathcal{C}^o$.

The composition of two functors again yields a functor. In particular, for every contravariant functor $S : \mathcal{C}^o \to \mathcal{D}$ the composition of D and S is a covariant functor from $\mathcal{C}$ to $\mathcal{D}$.

A functor $T : \mathcal{C} \to \mathcal{D}$ is said to *preserve* properties of an object $A \in Obj(\mathcal{C})$ or a morphism $f \in Mor(\mathcal{C})$, if $T(A)$, resp. $T(f)$, again have the same properties.

The functor T *reflects* a property of A, resp. of f, if: Whenever $T(A)$, resp. $T(f)$, has this property, then this is also true for A, resp. f.

By definition, *covariant* functors preserve identities and composition of morphisms. From this we see immediately that retractions, coretractions, isomorphisms and commutative diagrams are also preserved.

Contravariant functors also preserve identities and commutative diagrams and hence isomorphisms. However, they convert retractions into coretractions and coretractions into retractions.

Observe that in general a functor need not preserve or reflect either monomorphisms or epimorphisms.

An arbitrary functor provides only a loose connection between two categories. Of more interest are functors with special properties. Let us first concentrate on the fact that every covariant (contravariant) functor $T : \mathcal{C} \to \mathcal{D}$ assigns to a morphism from A to B in $\mathcal{C}$ a morphism from $T(A)$ to $T(B)$ ($T(B)$ to $T(A)$) in $\mathcal{D}$, i.e. for every pair A, B in $Obj(\mathcal{C})$ we have a (set) map

$$T_{A,B} : Mor_{\mathcal{C}}(A, B) \to Mor_{\mathcal{D}}(T(A), T(B)),$$

$$\text{resp. } T_{A,B} : Mor_{\mathcal{C}}(A, B) \to Mor_{\mathcal{D}}(T(B), T(A)).$$

Since these maps are significant for the properties of the functor, they yield the first three of the following specifications of functors. The others refer to the assignment of the objects:

11.2 Special functors. Definitions.

Let $T : \mathcal{C} \to \mathcal{D}$ be a covariant (or contravariant) functor between the categories $\mathcal{C}$ and $\mathcal{D}$. T is called

faithful if $T_{A,B}$ is injective for all $A, B \in Obj(\mathcal{C})$,
full if $T_{A,B}$ is surjective for all $A, B \in Obj(\mathcal{C})$,
fully faithful if T is full and faithful,
an *embedding* if the assignment $T : Mor(\mathcal{C}) \to Mor(\mathcal{D})$ is injective,
representative if for every $D \in Obj(\mathcal{D})$ there is an $A \in Obj(\mathcal{C})$ with $T(A)$ isomorphic to D.

Observe that a faithful functor need not be an embedding. It is an embedding if and only if $T : Obj(\mathcal{C}) \to Obj(\mathcal{D})$ is injective. Let us list some of the pecularities of the functors just defined:

11.3 Properties of covariant functors.

Let $T : \mathcal{C} \to \mathcal{D}$ be a covariant functor between the categories $\mathcal{C}$ and $\mathcal{D}$.

(1) If T is faithful it reflects monomorphisms, epimorphisms, bimorphisms and commutative diagrams.

(2) If T is fully faithful it also reflects retractions, coretractions and isomorphisms.

(3) If T is fully faithful and representative it preserves and reflects mono-, epi- and bimorphisms (retractions, coretractions, isomorphisms and commutative diagrams).

Proof: (1) Let T be faithful, $f : A \to B$ a morphism in $\mathcal{C}$ and $T(f)$ monic. For $g_1, g_2 \in Mor_{\mathcal{C}}(C, A)$ with $g_1 f = g_2 f$ we know

$T(g_1)T(f) = T(g_2)T(f)$, i.e. $T(g_1) = T(g_2)$ and hence $g_1 = g_2$ which means that f is monic.

Similarly we see that T reflects epimorphisms and hence bimorphisms. Consider the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & C \arrow[ur, "h"'] & \end{tikzcd}$$

and assume $T(f) = T(g)T(h) = T(gh)$. Then $f = gh$.

(2) Let T be fully faithful and $f \in Mor_{\mathcal{C}}(A, B)$ with $T(f)$ a retraction in $\mathcal{D}$. Then there exists $\gamma \in Mor_{\mathcal{D}}(T(B), T(A))$ with $\gamma T(f) = id_{T(B)}$. T being full, we can find $g \in Mor_{\mathcal{C}}(B, A)$ with

$T(g) = \gamma$ and $T(g)T(f) = T(gf) = id_{T(B)}$.
This means that $gf = id_B$ (since T is faithful) and f is a retraction.

Similarly we see that T reflects coretractions and isomorphisms.

(3) Let T be fully faithful and representative. By (1) and (2), it remains to show that T preserves mono- and epimorphisms. Let $f : A \to B$ be a monomorphism in $\mathcal{C}$ and $\gamma_1, \gamma_2 \in Mor_{\mathcal{D}}(X, T(A))$ with $\gamma_1 T(f) = \gamma_2 T(f)$. T being representative, there is a $C \in Obj(\mathcal{C})$ with an isomorphism

$\alpha : T(C) \to X$ in $Mor(\mathcal{D})$ and

$\alpha\gamma_1 T(f) = \alpha\gamma_2 T(f) : T(C) \to T(B)$.

Since T is full, there exist $g_1, g_2 \in Mor_{\mathcal{C}}(C, A)$ with $T(g_1) = \alpha\gamma_1$ and $T(g_2) = \alpha\gamma_2$ and hence

$$T(g_1 f) = T(g_1)T(f) = T(g_2)T(f) = T(g_2 f).$$

This implies $g_1 f = g_2 f$ (T is faithful) and we get $g_1 = g_2$, $\alpha\gamma_1 = \alpha\gamma_2$, and finally $\gamma_1 = \gamma_2$. Hence $T(f)$ is monic.

Dually we see that epimorphisms are also preserved.

From the preceding proofs we also obtain the following

11.4 Properties of contravariant functors.

Let $S : \mathcal{C} \to \mathcal{D}$ be a contravariant functor between the categories $\mathcal{C}$ and $\mathcal{D}$ and $f \in Mor(\mathcal{C})$. Then:

(1) If S is faithful it reflects bimorphisms and commutative diagrams; if $S(f)$ is monic (epic), then f is epic (monic).

(2) If S is fully faithful and $S(f)$ a retraction (coretraction, isomorphism), then f is a coretraction (retraction, isomorphism).

(3) If S is fully faithful and representative, then f is a mono-, epi-, resp. bimorphism if and only if $S(f)$ is an epi-, mono-, resp. bimorphism.

If $f : B \to C$ is a morphism in a category $\mathcal{C}$ and A an object in $\mathcal{C}$, then, by composition, f yields the following maps between morphism sets

$$\begin{aligned} Mor(A,f) : Mor_{\mathcal{C}}(A,B) \to Mor_{\mathcal{C}}(A,C),\ \ u \mapsto uf,\ u \in Mor_{\mathcal{C}}(A,B), \\ Mor(f,A) : Mor_{\mathcal{C}}(C,A) \to Mor_{\mathcal{C}}(B,A),\ \ v \mapsto fv,\ v \in Mor_{\mathcal{C}}(C,A), \end{aligned}$$

and we easily get:

11.5 *Mor*-functors.

Let $\mathcal{C}$ be a category, A an object in $\mathcal{C}$. The assignments

$$\begin{array}{llll} Mor(A,-): & Obj(\mathcal{C}) \to Obj(ENS), & B \mapsto Mor_{\mathcal{C}}(A,B), & B \in Obj(\mathcal{C}), \\ & Mor(\mathcal{C}) \to Map, & f \mapsto Mor_{\mathcal{C}}(A,f), & f \in Mor(\mathcal{C}), \end{array}$$

define a covariant functor $Mor(A,-) : \mathcal{C} \to ENS$.
The assignments

$$\begin{array}{llll} Mor(-,A): & Obj(\mathcal{C}) \to Obj(ENS), & B \mapsto Mor_{\mathcal{C}}(B,A), & B \in Obj(\mathcal{C}), \\ & Mor(\mathcal{C}) \to Map, & f \mapsto Mor_{\mathcal{C}}(f,A), & f \in Mor(\mathcal{C}), \end{array}$$

define a contravariant functor $Mor(-,A) : \mathcal{C} \to ENS$.

Special properties of an object A yield special properties of the functors $Mor(A,-)$, resp. $Mor(-,A)$. This is an interesting starting-point for characterizing special objects and we will pursue this for module categories in Chapter 3.

The following properties are generally valid:

11.6 Properties of the *Mor*-functors.
Let $\mathcal{C}$ be a category. For any $A \in Obj(\mathcal{C})$ we have:

(1) The covariant functor $Mor(A, -) : \mathcal{C} \to ENS$ preserves monomorphisms.

(2) The contravariant functor $Mor(-, A) : \mathcal{C} \to ENS$ converts epimorphisms into monomorphisms.

Proof: (1) Let $f : B \to C$ be a monomorphism in $\mathcal{C}$. If $u_1, u_2 \in Mor_{\mathcal{C}}(A, B)$ and $Mor(A, f)(u_1) = Mor(A, f)(u_2)$, then, by definition, $u_1 f = u_2 f$, i.e. $u_1 = u_2$. Hence $Mor(A, f)$ is an injective map, i.e. a monomorphism in ENS.

(2) is seen dually to (1).

The properties of functors defined in 11.2 and the consequences derived in 11.3, 11.4 are valid in arbitrary categories. In categories with additional properties, of course, those functors are of interest which respect these properties. Especially for module categories – or suitable subcategories – the following functors are of importance:

11.7 Special functors in module categories. Definitions.
Let R and S be rings and $T : R\text{-}MOD \to S\text{-}MOD$ a covariant functor. Then we call T

additive if for all R-modules M, N the map
$T_{M,N} : Hom_R(M, N) \to Hom_S(T(M), T(N))$ is additive,
i.e. $T(f_1 + f_2) = T(f_1) + T(f_2)$ for all $f_1, f_2 \in Hom_R(M, N)$;

exact with respect to an exact sequence $M_1 \to M_2 \to M_3 \to \cdots$ if
$T(M_1) \to T(M_2) \to T(M_3) \to \cdots$ is an exact sequence;

half exact if, for every exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, also
$T(M_1) \to T(M_2) \to T(M_3)$ is exact;

left exact if T is exact with respect to all exact sequences
$0 \to M_1 \to M_2 \to M_3$;

right exact if T is exact with respect to all exact sequences
$M_1 \to M_2 \to M_3 \to 0$;

exact if T is exact with respect to all exact sequences.

For contravariant T the 'arrows' are to be reversed. Then T is *left exact* if it is exact with respect to all exact sequences $M_1 \to M_2 \to M_3 \to 0$.

Having in mind the characterizations of kernels and cokernels in $R\text{-}MOD$ by exact sequences, we obtain the following characterizations of exact functors:

11.8 Characterization of exact functors.
Let R and S be rings and

$$(*)\ \ 0 \to K \to L \to M \to 0$$

a short exact sequence in R-MOD. For a covariant functor

$$T : R\text{-}MOD \to S\text{-}MOD\,,$$

the following properties are respectively equivalent:

(1) (*a*) *T is left exact;*
(*b*) *T preserves kernels;*
(*c*) *for all sequences $(*)$, $0 \to T(K) \to T(L) \to T(M)$ is exact.*

(2) (*a*) *T is right exact;*
(*b*) *T preserves cokernels;*
(*c*) *for all sequences $(*)$, $T(K) \to T(L) \to T(M) \to 0$ is exact.*

(3) (*a*) *T is exact;*
(*b*) *T is left and right exact;*
(*c*) *T is exact with respect to all sequences $(*)$;*
(*d*) *T is exact with respect to all exact sequences $M_1 \to M_2 \to M_3$.*

Proof: (1) $(a) \Leftrightarrow (b)$ is implied by the fact that $T(M_1) \to T(M_2)$ is the kernel of $T(M_2) \to T(M_3)$ if and only if $0 \to T(M_1) \to T(M_2) \to T(M_3)$ is exact (see 7.14).

$(a) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$ Let $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be an exact sequence in *R-MOD*. Then also $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} (M_2)g \to 0$ and $0 \to (M_2)g \xrightarrow{i} M_3$ are exact sequences and, by (c),

$0 \longrightarrow T(M_1) \xrightarrow{T(f)} T(M_2) \xrightarrow{T(g)} T((M_2)g)$ and

$0 \longrightarrow T((M_2)g) \xrightarrow{T(i)} T(M_3)$ are also exact (in *S-MOD*).

Since $Im\,T(f) = Ke\,T(g) = Ke\,(T(g)T(i)) = Ke\,T(g)$, we finally see that

$$0 \longrightarrow T(M_1) \xrightarrow{T(f)} T(M_2) \xrightarrow{T(g)} T(M_3)$$

is exact.

(2) is shown dually to (1).

(3) $(a) \Leftrightarrow (d)$ and $(a) \Rightarrow (c)$ are obvious.

$(b) \Leftrightarrow (c)$ follows from (1) and (2).

$(c) \Rightarrow (d)$ Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be exact. Then the factorization of f via $Ke\,g$, $M_1 \to Ke\,g$, is epic. Hence in the diagram

$$\begin{array}{ccccc} T(M_1) & & \xrightarrow{T(f)} & & T(M_2) & \xrightarrow{T(g)} & T(M_3) \\ & \searrow & & \nearrow & & & \\ & & T(Ke\,g) & & & & \end{array}$$

the morphism $T(M_1) \to T(Ke\,g)$ is epic. Since kernels are preserved the row is exact (see 7.13).

For additive functors we note:

11.9 Properties of additive functors.
Let $T : R\text{-}MOD \to S\text{-}MOD$ be a functor.

(1) T is additive if and only if it preserves finite (co) products.

(2) If T is half exact, then it is additive.

Proof: (1) If T is additive it preserves zero morphisms since $T_{M,N}$ maps zero to zero. For a zero object X in R-*MOD*, we have $id_X = 0$ and hence $id_{T(X)} = T(id_X) = T(0) = 0$, i.e. $T(X)$ is a zero object in S-*MOD*.

Let M_1, M_2 be in R-*MOD*, $M_1 \oplus M_2$ the product with the canonical projections π_i and injections ε_i, $i = 1, 2$. Then $T(M_1 \oplus M_2)$, with the projections $T(\pi_i)$, is the product of $T(M_1)$ and $T(M_2)$:

For morphisms $X \xrightarrow{f_i} T(M_i)$, $i = 1, 2$, in S-*MOD* and

$$f = f_1 T(\varepsilon_1) + f_2 T(\varepsilon_2) : \ X \longrightarrow T(M_1 \oplus M_2)\,,$$

we get $fT(\pi_i) = f_i$. For every morphism $g : X \to T(M_1 \oplus M_2)$ with $gT(\pi_i) = f_i$, we deduce from

$T(\pi_1\varepsilon_1 + \pi_2\varepsilon_2) = T(\pi_1)T(\varepsilon_1) + T(\pi_2)T(\varepsilon_2) = id_{T(M_1 \oplus M_2)}$, that

$$g = gT(\pi_1)T(\varepsilon_1) + gT(\pi_2)T(\varepsilon_2) = f_1T(\varepsilon_1) + f_2T(\varepsilon_2) = f.$$

Hence $T(M_1 \oplus M_2)$ is the direct sum of $T(M_1)$ and $T(M_2)$.

The reverse conclusion in (1) is contained in the proof of (2).

(2) For M in R-*MOD*, the sequence $0 \to 0 \to M \xrightarrow{id} M \to 0$ is exact. Since T is half exact, $T(0) \to T(M)$ is a zero morphism. Hence T preserves zero morphisms and (see (1)) zero objects. With the notation of (1), we obtain that $T(\varepsilon_1)$ is a coretraction, $T(\pi_2)$ a retraction and consequently the sequence

$$0 \longrightarrow T(M_1) \xrightarrow{T(\varepsilon_1)} T(M_1 \oplus M_2) \xrightarrow{T(\pi_2)} T(M_2) \longrightarrow 0$$

is exact and splits. Hence T preserves finite products.

For $f, g \in Hom_R(M, N)$, we obtain from the diagonal map

$$\Delta : M \to M \oplus M, \quad m \mapsto (m, m), \text{ and}$$

$$(f, g) : M \oplus M \to N, \quad (m_1, m_2) \mapsto (m_1)f + (m_2)g,$$

the commutative diagram

$$\begin{tikzcd} T(M) \arrow[r, "T(\Delta)"] \arrow[dr] & T(M \oplus M) \arrow[r, "T(f,g)"] \arrow[d, equal] & T(N) \\ & T(M) \oplus T(M) \arrow[ur, "(T(f),T(g))"'] & \end{tikzcd}$$

From this we see that T is additive.

For an R-module M with $S = End_R(M)$, the *Mor*-functors considered in 11.5 yield the covariant and contravariant *Hom*-functors

$Hom_R(M, -) : R\text{-}MOD \to S\text{-}MOD$,
$Hom_R(-, M) : R\text{-}MOD \to MOD\text{-}S$,

regarding, for $N \in R\text{-}MOD$, the group $Hom_R(M, N)$ as a left S-module and $Hom_R(N, M)$ as a right S-module (see 6.4). They have the following properties:

11.10 *Hom*-functors. Properties.
Let R be a ring, M an R-module and $S = End_R(M)$. Then:

(1) The functors $Hom_R(M, -)$ and $Hom_R(-, M)$ are additive and left exact.

(2) For a family $\{N_\lambda\}_\Lambda$ of R-modules, we have

$$(i) \quad Hom_R(M, \prod_\Lambda N_\lambda) \simeq \prod_\Lambda Hom_R(M, N_\lambda) \text{ in } S\text{-}MOD,$$

i.e. $Hom_R(M, -)$ preserves products;

$$(ii) \quad Hom_R(\bigoplus_\Lambda N_\lambda, M) \simeq \prod_\Lambda Hom_R(N_\lambda, M) \text{ in } MOD\text{-}S,$$

i.e. $Hom_R(-, M)$ converts coproducts into products.

Proof: (1) For $f, g \in Hom(N_1, N_2)$, we have by definition

$$Hom(M, f + g) : Hom(M, N_1) \to Hom(M, N_2), \ \varphi \mapsto \varphi(f + g).$$

$\varphi(f + g) = \varphi f + \varphi g$ implies $Hom(M, f + g) = Hom(M, f) + Hom(M, g)$.

Similarly we see that $Hom(-, M)$ is additive. From this we know, by 11.9, that the two functors preserve finite products. The assertions in (2) are even stronger.

To test the exactness of $Hom(M, -)$ we apply it to the exact sequence $0 \to K \xrightarrow{f} L \xrightarrow{g} N$ in R-MOD to obtain

$$0 \longrightarrow Hom(M,K) \xrightarrow{Hom(M,f)} Hom(M,L) \xrightarrow{Hom(M,g)} Hom(M,N).$$

By 11.6, $Hom(M, f)$ is monic.

From $fg = 0$ we obtain $Hom(M, fg) = Hom(M, f)Hom(M, g) = 0$, i.e.

$$Im\, Hom(M, f) \subset Ke\, Hom(M, g).$$

For $t \in Ke\, Hom(M, g)$, we have $tg = 0$, i.e. there exists $u : M \to K$ with $uf = t$, $Hom(M, f)(u) = uf = t$ and

$$Im\, Hom(M, f) \supset Ke\, Hom(M, g).$$

Hence the sequence is exact.

Dually we see that $Hom(-, M)$ is also left exact.

(2) These isomorphisms are known from 9.8.

Observe that the functors $Hom(M, -)$ and $Hom(-, M)$ in general have none of the properties considered in 11.2 (full, faithful, representative). These may occur for special modules M. We shall investigate this later on but want to point out an important case now:

11.11 The functor $Hom_R(R, -)$.

Considering the ring R as an (R,R)-bimodule, for every R-module M, the abelian group $Hom_R(R, M)$ becomes a left R-module. For $m \in M$, we define an R-homomorphism $\varphi_m : R \to M$, $r \mapsto rm$, and the assignment

$$\mu : M \to Hom_R(R, M), \quad m \mapsto \varphi_m,$$

yields an R-isomorphism.

If ${}_RM_S$ is an (R,S)-bimodule, then μ is an (R,S)-isomorphism.

This is easily verified and we obtain that the functor

$$Hom_R(R, -) : \ R\text{-}MOD \to R\text{-}MOD$$

is full, faithful and representative.

Properties of $Hom_R(-, R)$ will be investigated later on. They are not as nice as those of $Hom_R(R, -)$.

11.12 Exercises.

(1) Let $\varphi : R \to S$ be a ring homomorphism. Every S-module ${}_SN$ becomes an R-module by defining $rn = \varphi(r)n$. Show that this determines a (non-trivial) covariant functor from S-MOD to R-MOD.
Is this functor exact, faithful, full, an embedding?

(2) Let $e \neq 0$ be an idempotent in the ring R. For every R-module N, we consider eN as a left eRe-module. Show that this yields a (non-trivial) covariant functor from R-MOD to eRe-MOD.
Is this functor exact, faithful, full, an embedding?

(3) Let Z(R) be the centre of the ring R (see § 2). Show that, for an R-module M, the functors Hom(M,–) and Hom(–,M) represent functors from R-MOD to Z(R)-MOD.

(4) Let r be a central element in the ring R (see § 2). Show that the canonical map $N \to N/rN$ yields a covariant functor from R-MOD to R/Rr-MOD (see 12.11).

(5) A covariant functor $T : R\text{-}MOD \to R\text{-}MOD$ is called a subfunctor of the identity *if $T(N) \subset N$ for all N in R-MOD. Prove that these functors preserve coproducts.*

12 Tensor product, tensor functor

1.Definitions. 2.Existence. 3.TP of homomorphisms. 4.TP and direct sums. 5.Module structure of the TP. 6.TP with R. 7.Associativity of the TP. 8.Tensor functors. 9.TP and direct product. 10.Zero in TP. 11.TP with cyclic modules. 12.Hom-Tensor relation. 13.M-flat modules. 14.Direct sum of M-flat modules. 15.Properties of M-flat modules. 16.Flat modules. 17.Faithfully flat modules. 18.TP over commutative rings. 19.Exercises.

Similar to the *Hom-functors* the *tensor functors* are of immense importance in module theory. They are derived from the *tensor product* which is known for vector spaces from Linear resp. Multilinear Algebra. To ensure the generality desired for our purposes we give an account of the construction of the tensor product over rings. In this paragraph we do not generally presume the existence of a unit in R.

12.1 Definitions. Let M_R be a right module, ${}_RN$ a left module over the ring R and G an abelian group.

A $\mathbb{Z}$-bilinear map $\beta : M \times N \to G$ is called *R-balanced* if, for all $m \in M$, $n \in N$ and $r \in R$, we have: $\beta(mr, n) = \beta(m, rn)$.

An abelian group T with an R-balanced map $\tau : M \times N \to T$ is called the *tensor product of M and N* if every R-balanced map

$$\beta : M \times N \to G,\ G \text{ an abelian group },$$

can be uniquely factorized over τ, i.e. there is a unique $\mathbb{Z}$-Homomorphism $\gamma : T \to G$ which renders the following diagram commutative:

$$\begin{array}{ccc} & M \times N & \\ {}^{\tau}\swarrow & & \searrow^{\beta} \\ T & \xrightarrow{\gamma} & G \end{array}$$

With standard arguments applied for universal constructions it is easily seen that the tensor product (T, τ) for a pair of modules M_R, ${}_RN$ is uniquely determined up to isomorphism (of $\mathbb{Z}$-modules).

12.2 Existence of tensor products. For the R-modules M_R, ${}_RN$, we form the direct sum of the family of $\mathbb{Z}$-modules $\{\mathbb{Z}_{(m,n)}\}_{M\times N}$ with $\mathbb{Z}_{(m,n)} \simeq \mathbb{Z}$, the free $\mathbb{Z}$-module over $M \times N$,

$$F = \bigoplus\nolimits_{M\times N} \mathbb{Z}_{(m,n)} \simeq \mathbb{Z}^{(M\times N)}.$$

By construction, there is a (canonical) basis $\{f_{(m,n)}\}_{M\times N}$ in F (see 9.10). We simply write $f_{(m,n)} = [m,n]$. Let K denote the submodule of F generated by elements of the form

$$[m_1 + m_2, n] - [m_1, n] - [m_2, n], \quad [m, n_1 + n_2] - [m, n_1] - [m, n_2],$$

$$[mr, n] - [m, rn], \text{ with } m, m_i \in M, \; n, n_i \in N, \; r \in R.$$

Putting $M \otimes_R N := F/K$ we define the map

$$\tau : M \times N \to M \otimes_R N, \quad (m,n) \mapsto m \otimes n := [m,n] + K \; .$$

By definition of K, the map τ is R-balanced. Observe that τ is not surjective but the image of τ, $Im\,\tau = \{m \otimes n \mid m \in M, \; n \in N\}$, is a generating set of $M \otimes_R N$ as a $\mathbb{Z}$-module.

If $\beta : M \times N \to G$ is an R-balanced map we obtain a $\mathbb{Z}$-homomorphism $\tilde{\gamma} : F \to G$, $[m,n] \mapsto \beta(m,n)$, and obviously $K \subset Ke\,\tilde{\gamma}$. Hence $\tilde{\gamma}$ factorizes over τ and we have the commutative diagram

$$\begin{array}{ccccc} & & M \times N & & \\ & {}^{\tau}\swarrow & & \searrow^{\beta} & \\ M \otimes_R N & & \xrightarrow{\gamma} & & G \end{array} \; .$$

γ is unique since its values on the generating set $Im\,\tau$ of T are uniquely determined.

Observe that every element in $M \otimes_R N$ can be written as a finite sum

$$m_1 \otimes n_1 + \cdots + m_k \otimes n_k.$$

However this presentation is not unique. $m \otimes n$ only represents a coset and m, n are not uniquely determined. Also a presentation of zero in $M \otimes_R N$ is not unique. We may even have that $M \otimes_R N$ is zero for non-zero M and N, e.g. $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = 0$.

12.3 Tensor product of homomorphisms.

For two R-homomorphisms $f : M_R \to M'_R$ and $g : {}_RN \to {}_RN'$, there is a unique $\mathbb{Z}$-linear map $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ with

$$(m \otimes n) f \otimes g = f(m) \otimes (n)g, \; m \in M, \; n \in N.$$

$f \otimes g$ is called the *tensor product of the homomorphisms* f and g. Here the homomorphism of right modules f is written on the left and the homomorphism of left modules g is written on the right (see 6.4).

Proof: Define a map

$$f \times g : M \times N \to M' \otimes_R N' \quad \text{by } (m,n) \mapsto f(m) \otimes (n)g.$$

It is $\mathbb{Z}$-bilinear and R-balanced since $f(mr) \otimes (n)g = f(m) \otimes (rn)g$. Hence the map $f \times g$ can be factorized over $M \otimes_R N$ and we obtain the desired map $f \otimes g$.

From the definitions we readily obtain the following

Properties:

(1) $id_M \otimes id_N = id_{M\otimes N}$; $f \otimes 0 = 0 \otimes g = 0$.

(2) If $f' : M'_R \to M''_R$, $g' : {}_RN' \to {}_RN''$ are homomorphisms, we obtain for the composition: $(f \otimes g)(f' \otimes g') = f'f \otimes gg'$.

(3) If f and g are retractions, coretractions or isomorphisms, then the same is true for $f \otimes g$, e.g. we get in the last case $(f \otimes g)^{-1} = f^{-1} \otimes g^{-1}$.

(4) For $f_1, f_2 : M_R \to M'_R$ and $g_1, g_2 : {}_RN \to {}_RN'$ we have:

$$(f_1 + f_2) \otimes g = f_1 \otimes g + f_2 \otimes g \text{ and } f \otimes (g_1 + g_2) = f \otimes g_1 + f \otimes g_2.$$

12.4 Tensor product and direct sums.
Let M_R be an R-module and ${}_RN = \bigoplus_\Lambda N_\lambda$, with the canonical injections $\varepsilon_\lambda : {}_RN_\lambda \to {}_RN$ and projections $\pi_\lambda : {}_RN \to {}_RN_\lambda$.
Then $(M \otimes_R N, id_M \otimes \varepsilon_\lambda)$ is a direct sum of $\{M \otimes_R N_\lambda\}_\Lambda$, i.e.

$$M \otimes_R (\bigoplus\nolimits_\Lambda N_\lambda) \simeq \bigoplus\nolimits_\Lambda (M \otimes_R N_\lambda).$$

We say the tensor product *commutes with direct sums.*

Proof: For the maps $id_M \otimes \pi_\lambda : M \otimes_R N \to M \otimes_R N_\lambda$, we derive from properties of tensor products of homomorphisms

$$(id_M \otimes \varepsilon_\lambda)(id_M \otimes \pi_\mu) = \delta_{\lambda\mu} id_{M\otimes N_\lambda}.$$

For a family $\{f_\lambda : M \otimes N_\lambda \to X\}_\Lambda$ of $\mathbb{Z}$-linear maps, we define $f : M \otimes_R N \to X$ by

$$(m \otimes n)f = \sum\nolimits_{\lambda\in\Lambda} (m \otimes n)(id_M \otimes \pi_\lambda) f_\lambda\,,$$

where the sum is always finite.

Obviously, $(id_M \otimes \varepsilon_\lambda)f = f_\lambda$ and $(M \otimes_R N, id_M \otimes \varepsilon_\lambda)$ is a direct sum of the $\{M \otimes_R N_\lambda\}_\Lambda$.

By symmetry, we obtain, for $M_R = \bigoplus_{\Lambda'} M_\mu$,

$$(\bigoplus\nolimits_{\Lambda'} M_\mu) \otimes_R N \simeq \bigoplus\nolimits_{\Lambda'} (M_\mu \otimes_R N),$$

$$(\bigoplus\nolimits_{\Lambda'} M_\mu) \otimes_R (\bigoplus\nolimits_{\Lambda} N_\lambda) \simeq \bigoplus\nolimits_{\Lambda' \times \Lambda} (M_\mu \otimes N_\lambda).$$

12.5 Module structure of tensor products.

By construction, the tensor product $M \otimes_R N$ of M_R and ${}_RN$ is only an abelian group. However, if ${}_TM_R$ or ${}_RN_S$ are bimodules, then we may define module structures on $M \otimes_R N$:

If ${}_TM_R$ is a (T, R)-bimodule, then the elements of T may be regarded as R-endomorphisms of M and the tensor product with id_N yields a map

$$T \to End_{\mathbb{Z}}(M \otimes_R N), \quad t \mapsto t \otimes id_N.$$

From the properties of this construction noted in 12.3 we see that this is a ring homomorphism. Hence ${}_TM \otimes_R N$ becomes a *left T-module* and the action of $t \in T$ on $\sum m_i \otimes n_i \in M \otimes N$ is given by

$$t(\sum m_i \otimes n_i) = \sum (tm_i) \otimes n_i.$$

For an (R, S)-bimodule ${}_RN_S$, we obtain in the same way that $M \otimes_R N_S$ is a right S-module.

If ${}_TM_R$ and ${}_RN_S$ are bimodules, the structures defined above turn ${}_TM \otimes_R N_S$ into a (T, S)-bimodule since we have, for all $t \in T$, $s \in S$ and $m \otimes n \in M \otimes_R N$, that $(t(m \otimes n))s = (tm) \otimes (ns) = t((m \otimes n)s)$.

12.6 Tensor product with *R*.

Regarding R as an (R, R)-bimodule, for every R-module ${}_RN$, there is an R-epimorphism

$$\mu_R : R \otimes_R N \to RN, \; \sum r_i \otimes n_i \mapsto \sum r_i n_i.$$

The map exists since the map $R \times N \to RN$, $(r, n) \mapsto rn$ is balanced, and obviously has the given properties.

Assume that, for every finite subset $r_1, \ldots, r_k$ of R, there is an idempotent $e \in R$ with $er_1 = r_1, \ldots, er_k = r_k$ (we say ${}_RR$ *has many idempotents*). In this case μ_R is an isomorphism since, from $\sum r_i n_i = 0$, we deduce

$$\sum r_i \otimes n_i = \sum e \otimes r_i n_i = e \otimes \sum r_i n_i = 0.$$

For a ring R with unit we have $RN = N$ and $\mu_R : R \otimes_R N \to N$ is an R-isomorphism. Since the tensor product commutes with direct sums (see 12.4), we obtain, for every free right R-module $F_R \simeq R_R^{(\Lambda)}$, Λ an index set, a $\mathbb{Z}$-isomorphism $F \otimes_R N \simeq N^{(\Lambda)}$.

12.7 Associativity of the tensor product.
Assume three modules M_R, $_RN_S$ and $_SL$ are given. Then $(M \otimes_R N) \otimes_S L$ and $M \otimes_R (N \otimes_S L)$ can be formed and there is an isomorphism

$$\sigma : (M \otimes_R N) \otimes_S L \to M \otimes_R (N \otimes_S L), \quad (m \otimes n) \otimes l \mapsto m \otimes (n \otimes l)\,.$$

Proof: We only have to show the existence of such a map σ. Then, by symmetry, we obtain a corresponding map in the other direction which is inverse to σ:
We first define, for $l \in L$, a morphism $f_l : N \to N \otimes_S L$, $n \mapsto n \otimes l$, then form the tensor product $id_M \otimes f_l : M \otimes_R N \to M \otimes_R (N \otimes_S L)$ and obtain

$$\beta : (M \otimes_R N) \times L \to M \otimes_R (N \otimes_S L), \quad (m \otimes n, l) \mapsto id_M \otimes f_l\,(m \otimes n).$$

It only remains to verify that β is balanced to obtain the desired map.

12.8 Tensor functors. *For an (S,R)-bimodule $_SU_R$, the assignments*

$$\begin{array}{rlcll} {_SU} \otimes_R - : & Obj(R\text{-}MOD) & \longrightarrow & Obj(S\text{-}MOD), & {_RM} \mapsto {_SU} \otimes M, \\ & Mor(R\text{-}MOD) & \longrightarrow & Mor(S\text{-}MOD), & f \mapsto id_U \otimes f, \end{array}$$

yield a covariant functor $_SU \otimes_R - : R\text{-}MOD \to S\text{-}MOD$ with the properties
(1) $_SU \otimes_R -$ is additive and right exact;
(2) $_SU \otimes_R -$ preserves direct sums.
Similarly we obtain a functor $- \otimes_S U_R : MOD\text{-}S \to MOD\text{-}R$ with the same properties.

Proof: Applying 12.3 it is easily checked that the given assignments define an additive functor. In 12.4 we have seen that it preserves direct sums. It remains to show that it is right exact. From the exact sequence $K \xrightarrow{f} L \xrightarrow{g} N \to 0$ in R-MOD, $_SU \otimes_R -$ yields the sequence

$$U \otimes_R K \xrightarrow{id \otimes f} U \otimes_R L \xrightarrow{id \otimes g} U \otimes_R N \longrightarrow 0\,.$$

For $u \otimes n \in U \otimes N$, we choose an $l \in L$ with $n = (l)g$ and obtain $id \otimes g(u \otimes l) = u \otimes (l)g = u \otimes n$. Since the $\{u \otimes n\}$ form a generating set in $U \otimes N$, we see that $id \otimes g$ is surjective.

From $(id \otimes f)(id \otimes g) = id \otimes fg = 0$ we get $Im\,(id \otimes f) \subset Ke\,(id \otimes g)$ and, by the Factorization Theorem, we have the commutative diagram in S-MOD

$$\begin{tikzcd}
U \otimes L \arrow[rr, "id\otimes g"] \arrow[dr, "p"'] & & U \otimes N \\
& U \otimes L/Im\,(id \otimes f) \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

with the canonical projection p. If we can show that α is monic (i.e. an isomorphism), then we get $Im\,(id \otimes f) = Ke\,(id \otimes g)$. For this we first consider an assignment

$$\tilde{\beta} : U \times N \to U \otimes L/Im\,(id \otimes f), \quad (u, n) \mapsto u \otimes l + Im\,(id \otimes f),$$

with $(l)g = n$, $l \in L$. $\tilde{\beta}$ is a map, since, for any $l' \in L$ with $(l')g = n$, we deduce (from $l' - l \in Ke\,g = Im\,f$) that

$$u \otimes l + Im\,(id \otimes f) = u \otimes l + u \otimes (l' - l) + Im\,(id \otimes f) = u \otimes l' + Im\,(id \otimes f).$$

It is readily checked that $\tilde{\beta}$ is balanced and hence there exists $\beta : U \otimes N \to U \otimes L/Im\,(id \otimes f)$ with $\alpha\beta = id$ since

$$((u \otimes l) + Im\,(id \otimes f))\alpha\beta = (u \otimes (l)g)\beta = u \otimes l + Im\,(id \otimes f)\,.$$

The relation between tensor products and direct products is more complicated than that between tensor products and direct sums:

12.9 Tensor product and direct product.

Let R be a ring with unit, U_R a right R-module and $\{L_\lambda\}_\Lambda$ a family of left R-modules. With the canonical projections we have the maps

$$id_U \otimes \pi_\mu : U \otimes_R (\prod\nolimits_\Lambda L_\lambda) \to U \otimes_R L_\mu$$

and, by the universal property of the product,

$$\varphi_U : U \otimes_R (\prod\nolimits_\Lambda L_\lambda) \to \prod\nolimits_\Lambda U \otimes_R L_\lambda, \quad u \otimes (l_\lambda)_\Lambda \mapsto (u \otimes l_\lambda)_\Lambda.$$

It is easy to see that, for $U = R$, and hence also for $U = R^n$, φ_U is an isomorphism.

(1) The following assertions are equivalent:

(a) U is finitely generated;

(b) φ_U is surjective for every family $\{L_\lambda\}_\Lambda$;

(c) $\tilde{\varphi}_U : U \otimes R^\Lambda \to (U \otimes R)^\Lambda \simeq U^\Lambda$ is surjective for any set Λ (or $\Lambda = U$).

(2) The following assertions are also equivalent:

(a) There is an exact sequence $R^m \to R^n \to U \to 0$ with $m, n \in I\!N$ (U is finitely presented in R-MOD, see § 25);

(b) φ_U is bijective for every family $\{L_\lambda\}_\Lambda$ (i.e. $U \otimes_R -$ preserves direct products);

(c) $\tilde\varphi_U : U \otimes_R R^\Lambda \to U^\Lambda$ is bijective for every set Λ.

Proof: (1) $(a) \Rightarrow (b)$ If U is finitely generated and $R^{(A)} \xrightarrow{f} R^n \xrightarrow{g} U \to 0$ is exact, we can form the commutative diagram with exact rows (see 12.8 and 9.3,(5)):

$$\begin{array}{ccccccc}
R^{(A)} \otimes \prod_\Lambda L_\lambda & \xrightarrow{f\otimes id} & R^n \otimes \prod_\Lambda L_\lambda & \xrightarrow{g\otimes id} & U \otimes \prod_\Lambda L_\lambda & \to 0 \\
\downarrow \varphi_{R^{(A)}} & & \downarrow \varphi_{R^n} & & \downarrow \varphi_U & \\
\prod_\Lambda (R^{(A)} \otimes L_\lambda) & \xrightarrow{\prod(f\otimes id)} & \prod_\Lambda (R^n \otimes L_\lambda) & \xrightarrow{\prod(g\otimes id)} & \prod_\Lambda (U \otimes L_\lambda) & \to 0
\end{array}$$

As pointed out above, φ_{R^n} is bijective and hence φ_U is surjective.

$(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$ Assume (c). Then, for $\Lambda = U$, the map $\tilde\varphi : U \otimes R^U \to U^U$ is surjective. For the element $(u_u)_U$ $(= id_U$ in $Map(U,U) = U^U)$, we choose $\sum_{i\le k} m_i \otimes (r_u^i)$ as a preimage under $\tilde\varphi_U$, with $r_u^i \in R$, $m_i \in U$, i.e.

$$(u_u)_U = \sum\nolimits_{i\le k} (m_i r_u^i)_U = (\sum\nolimits_{i\le k} m_i r_u^i)_U .$$

Hence, for every $u \in U$, we get $u = \sum_{i\le k} m_i r_u^i$, i.e. $m_1, \dots, m_k$ is a generating set of U.

(2) $(a) \Rightarrow (b)$ In the proof (1) $(a) \Rightarrow (b)$ we can choose a finite index set A. Then $\varphi_{R^{(A)}}$ is an isomorphism and hence also φ_U.

$(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$ From (1) we already know that U is finitely generated. Hence there is an exact sequence $0 \to K \to R^n \to U \to 0$, $n \in I\!N$. From this we obtain – for any set Λ – the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & K \otimes R^\Lambda & \longrightarrow & R^n \otimes R^\Lambda & \longrightarrow & U \otimes R^\Lambda & \longrightarrow & 0 \\
 & & \downarrow \tilde\varphi_K & & \downarrow \tilde\varphi_{R^n} & & \downarrow \tilde\varphi_U & & \\
0 & \longrightarrow & K^\Lambda & \longrightarrow & (R^n)^\Lambda & \longrightarrow & U^\Lambda & \longrightarrow & 0 \quad .
\end{array}$$

Here $\tilde\varphi_{R^n}$ is an isomorphism (see above) and $\tilde\varphi_U$ is an isomorphism by (c). According to the Kernel Cokernel Lemma, $\tilde\varphi_K$ is surjective and, by(1), K is finitely generated. Therefore, for some $m \in I\!N$, we get an exact sequence $R^m \to K \to 0$, and $R^m \to R^n \to U \to 0$ is also exact.

As a consequence of the right exactness of the tensor functor the following two results can be shown:

12.10 Zero in the tensor product.
Let R be a ring with unit, $\{n_i\}_{i\in\Lambda}$ a generating set of the R-module ${}_RN$ and $\{m_i\}_{i\in\Lambda}$ a family of elements in the R-module M_R with only finitely many $m_i \neq 0$.

Then $\sum_\Lambda m_i \otimes n_i = 0$ in $M \otimes_R N$ if and only if there are finitely many elements $\{a_j\}_{j\in\Lambda'}$ in M and a family $\{r_{ji}\}_{\Lambda'\times\Lambda}$ of elements in R with the properties
(i) $r_{ji} \neq 0$ for only finitely many pairs (j,i),
(ii) $\sum_{i\in\Lambda} r_{ji}n_i = 0$ for every $j \in \Lambda'$,
(iii) $m_i = \sum_{j\in\Lambda'} a_j r_{ji}$.

Proof: For elements with these properties we see

$$\sum\nolimits_\Lambda m_i \otimes n_i = \sum\nolimits_\Lambda \sum\nolimits_{\Lambda'} a_j r_{ji} \otimes n_i = \sum\nolimits_{\Lambda'} (a_j \otimes \sum\nolimits_\Lambda r_{ji}n_i) = 0.$$

Now assume $\sum_\Lambda m_i \otimes n_i = 0$. With the canonical basis $\{f_i\}_{i\in\Lambda}$ and the map $g : R^{(\Lambda)} \to {}_RN,\ \ f_i \mapsto n_i$, we obtain the exact sequence

$$0 \longrightarrow {}_RK \xrightarrow{\varepsilon} R^{(\Lambda)} \xrightarrow{g} {}_RN \longrightarrow 0.$$

Tensoring with $M \otimes_R -$ yields the exact sequence

$$M \otimes_R K \xrightarrow{id\otimes\varepsilon} M \otimes R^{(\Lambda)} \xrightarrow{id\otimes g} M \otimes_R N \longrightarrow 0\ .$$

By assumption, $(\sum_\Lambda m_i \otimes f_i)id\otimes g = \sum_\Lambda m_i \otimes n_i = 0$ and there is an element $\sum_{j\in\Lambda'} a_j \otimes k_j \in M \otimes_R K$ with $(\sum_{j\in\Lambda'} a_j \otimes k_j)id \otimes \varepsilon = \sum_{i\in\Lambda} m_i \otimes f_i$.

Every $k_j \in K \subset R^{(\Lambda)}$ can be written as $k_j = \sum_{i\in\Lambda} r_{ji}f_i$ with only finitely many $r_{ji} \neq 0$. This implies $0 = (k_j)\varepsilon g = \sum_{i\in\Lambda} r_{ji}n_i$ for all $j \in \Lambda'$, and in $M \otimes_R R^{(\Lambda)}$ we get

$$\sum\nolimits_{i\in\Lambda} m_i \otimes f_i = \sum\nolimits_{j\in\Lambda'} a_j \otimes k_j = \sum\nolimits_{i\in\Lambda} (\sum\nolimits_{j\in\Lambda'} a_j r_{ji}) \otimes f_i.$$

From this the projections onto the components yield the desired condition $m_i = \sum_{j\in\Lambda'} a_j r_{ji}$.

12.11 Tensor product with cyclic modules. *Let I be a right ideal of a ring R with many idempotents and ${}_RM$ a left module. Then*

$$R/I \otimes_R M \simeq RM/IM \ \ (\simeq M/IM \ \textit{if}\, 1 \in R).$$

Proof: From the exact sequence $0 \to I \to R \to R/I \to 0$ we obtain the first row exact in the commutative diagram

$$\begin{array}{ccccccccc} & & I\otimes M & \longrightarrow & R\otimes M & \longrightarrow & R/I\otimes M & \longrightarrow & 0 \\ & & \downarrow \mu_I & & \downarrow \mu_R & & \downarrow \gamma & & \\ 0 & \longrightarrow & IM & \longrightarrow & RM & \longrightarrow & RM/IM & \longrightarrow & 0 \end{array}$$

with the map $\mu_I : I\otimes M \to IM,\ \ i\otimes m \mapsto im$. By 12.6, μ_R is an isomorphism and hence γ is an isomorphism by the Kernel Cokernel Lemma.

An interesting connection between Hom- and tensor functors is derived from the definition of the tensor product:

12.12 Hom-tensor relation. Let U_R and $_RM$ be R-modules, N a $\mathbb{Z}$-module and denote by $Ten(U \times M, N)$ the set of the R-balanced maps from $U \times M$ into N. By the definition of $U \otimes_R M$ (see 12.1), the canonical map $\tau : U \times M \to U \otimes_R M$ yields a $\mathbb{Z}$-isomorphism

$$\psi_1 : Hom_{\mathbb{Z}}(U \otimes_R M, N) \to Ten(U \times M, N),\ \ \alpha \mapsto \tau\alpha.$$

On the other hand, every $\beta \in Ten(U \times M, N)$ defines an R-homomorphism

$$h_\beta : M \to Hom_{\mathbb{Z}}(U, N),\ \ m \mapsto (-, m)\beta,$$

where $Hom_{\mathbb{Z}}(U, N)$ is regarded as a left R-module in the usual way. From this we obtain a $\mathbb{Z}$-isomorphism

$$\psi_2 : Ten(U \times M, N) \to Hom_R(M, Hom_{\mathbb{Z}}(U, N)),\ \ \beta \mapsto h_\beta.$$

Now every $\varphi \in Hom_R(M, Hom_{\mathbb{Z}}(U, N))$ determines an R-balanced map

$$\tilde{\varphi} : U \times M \to N,\ \ (u, m) \mapsto (u)(m)\varphi,$$

and the assignment $\varphi \mapsto \tilde{\varphi}$ is a map inverse to ψ_2. The composition of ψ_1 and ψ_2 leads to the *$\mathbb{Z}$-isomorphism*

$$\psi_M : Hom(U \otimes_R M, N) \to Hom_R(M, Hom(U, N)),\ \ \delta \mapsto [m \mapsto (- \otimes m)\delta],$$

with inverse map $\psi_M^{-1} : \varphi \mapsto [u \otimes m \mapsto (u)(m)\varphi]$.

If $_SU_R$ is an (S, R)-bimodule and $_SN$ an S-module, then $_SU \otimes_R M$ is also a left S-module and with respect to this structure ψ_M becomes a $\mathbb{Z}$-isomorphism

$$\psi_M : Hom_S(U \otimes_R M, N) \to Hom_R(M, Hom_S(U, N)).$$

It is readily verified that, for every R-homomorphism $g : {}_RM \to {}_RM'$, the following diagram is commutative:

$$\begin{array}{ccc}
Hom_S(U \otimes_R M', N) & \xrightarrow{Hom(id \otimes g, N)} & Hom_S(U \otimes_R M, N) \\
\downarrow \psi_{M'} & & \downarrow \psi_M \\
Hom_R(M', Hom_S(U, N)) & \xrightarrow{Hom(g, Hom_S(U,N))} & Hom_R(M, Hom_S(U, N))
\end{array}$$

Similarly we obtain, for modules ${}_RU_S$, M_R and N_S, a $\mathbb{Z}$-isomorphism

$$\psi'_M : Hom_S(M \otimes_R U, N) \to Hom_R(M, Hom_S(U, N))$$

and a corresponding commutative diagram.

Further relations between Hom- and tensor functors will be derived later for special modules (see 25.5). We now turn to the question: for which right modules U_R is the functor $U \otimes_R - : R\text{-}MOD \to \mathbb{Z}\text{-}MOD$ exact? For this we need:

12.13 M-flat modules. Definitions. Let M be a left R-module. A right R-module U_R is called *M-flat* if, for every exact sequence $0 \to K \to M$ in R-MOD, the sequence $0 \to U \otimes_R K \to U \otimes_R M$ is exact.

U_R is said to be *flat (with respect to R-MOD)* if U is M-flat for every $M \in R\text{-}MOD$.

Since $U \otimes_R -$ is always right exact, U_R is flat (with respect to R-MOD) if and only if the functor $U \otimes_R - : R\text{-}MOD \to \mathbb{Z}\text{-}MOD$ is exact.

12.14 Direct sum of M-flat modules.

Let $\{U_\lambda\}_\Lambda$ be a family of right R-modules and ${}_RM \in R\text{-}MOD$. The direct sum $\bigoplus_\Lambda U_\lambda$ is M-flat if and only if U_λ is M-flat for every $\lambda \in \Lambda$.

Proof: From the exact sequence $0 \to K \xrightarrow{f} M$ we form the commutative diagram

$$\begin{array}{ccc}
(\bigoplus_\Lambda U_\lambda) \otimes_R K & \xrightarrow{id \otimes f} & (\bigoplus_\Lambda U_\lambda) \otimes_R M \\
\downarrow & & \downarrow \\
\bigoplus_\Lambda (U_\lambda \otimes_R K) & \xrightarrow{\oplus(id_\lambda \otimes f)} & \bigoplus_\Lambda (U_\lambda \otimes_R M)
\end{array} \quad ,$$

in which the vertical maps are the canonical isomorphisms (see 12.4). Hence $id \otimes f$ is monic if and only if all $id_\lambda \otimes f$ are monic (see 9.7).

12.15 Properties of M-flat modules.
Let U_R be a right R-module. Then:

(1) U_R is M-flat if and only if $U \otimes_R -$ is exact with respect to every exact sequence $0 \to K' \to M$ with K' finitely generated.

(2) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in R-MOD. If U_R is M-flat, then U_R is also M'- and M''-flat.

(3) Let $\{M_\lambda\}_\Lambda$ be a family of left R-modules. If U_R is M_λ-flat for every $\lambda \in \Lambda$, then U_R is also $\bigoplus_\Lambda M_\lambda$-flat.

Proof: (1) Let $0 \to K \xrightarrow{\varepsilon} M$ be exact and $\sum_{i \leq n} u_i \otimes k_i \in U \otimes_R K$ with $(\sum_{i\leq n} u_i \otimes k_i) id \otimes \varepsilon = 0 \in U \otimes_R M$. Let K' denote the submodule of K generated by $k_1, \ldots, k_n$. Since the map

$$id \otimes \varepsilon' : U \otimes_R K' \to U \otimes_R K \to U \otimes_R M$$

is monic by assumption, we get $\sum_{i\leq n} u_i \otimes k_i = 0$ in $U \otimes_R K'$. Then it also has to be zero in $U \otimes_R K$, i.e. $id \otimes \varepsilon$ is monic.

(2) Let U_R be M-flat. If $0 \to K \xrightarrow{\varepsilon'} M'$ is exact, the canonical map $U \otimes_R K \xrightarrow{id \otimes \varepsilon'} U \otimes_R M' \longrightarrow U \otimes_R M$ is monic and U_R is M'-flat.

If $0 \to L \xrightarrow{f} M''$ is exact, we obtain, by a pullback, the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \quad .
\end{array}$$

Tensoring with U_R yields the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & U \otimes_R M' & \longrightarrow & U \otimes_R P & \longrightarrow & U \otimes_R L & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow id \otimes f & & \\
0 & \longrightarrow & U \otimes_R M' & \longrightarrow & U \otimes_R M & \longrightarrow & U \otimes_R M'' & \longrightarrow & 0 \quad .
\end{array}$$

By the Kernel Cokernel Lemma, $id \otimes f$ has to be monic, i.e. U is M''-flat.

(3) We show that U_R is $M_1 \oplus M_2$-flat if it is both M_1- and M_2-flat. Then we get assertion (3) for finite index sets Λ by induction. For arbitrary sets

Λ we use (1): A finitely generated submodule $K' \subset \oplus_\Lambda M_\lambda$ is contained in a finite partial sum. Since the tensor product preserves direct summands, the assertion follows from the finite case.

Let U_R be M_1- and M_2-flat and $0 \to K \xrightarrow{f} M_1 \oplus M_2$ exact. Forming a pullback we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & K & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & M_2 & \longrightarrow & 0 \quad .
\end{array}$$

Tensoring with U_R yields the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & U \otimes_R P & \longrightarrow & U \otimes_R K & \longrightarrow & U \otimes_R L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle id \otimes f} & & \downarrow & & \\
0 & \longrightarrow & U \otimes_R M_1 & \longrightarrow & U \otimes_R (M_1 \oplus M_2) & \longrightarrow & U \otimes_R M_2 & \longrightarrow & 0 \quad .
\end{array}$$

By the Kernel Cokernel Lemma, $id \otimes f$ has to be monic.

12.16 Flat modules. Characterizations.

Let R be a ring with unit. For a right R-module U_R, the following assertions are equivalent:

(a) U_R is flat (with respect to R-MOD);

(b) $U \otimes_R -$ is exact with respect to all exact sequences $0 \to {}_RI \to {}_RR$ (with ${}_RI$ finitely generated);

(c) for every (finitely generated) left ideal ${}_RI \subset R$, the canonical map $\mu_I : U \otimes_R I \to UI$ is monic (and hence an isomorphism).

Proof: The equivalence of (a) and (b) follows from 12.13.

$(b) \Leftrightarrow (c)$ For every (finitely generated) left ideal $I \subset R$, we have the commutative diagram with exact rows (see 12.11)

$$\begin{array}{ccccccccc}
 & & U \otimes_R I & \xrightarrow{id \otimes \varepsilon} & U \otimes_R R & \longrightarrow & U \otimes_R R/I & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \mu_I} & & \| & & \| & & \\
0 & \longrightarrow & UI & \longrightarrow & U & \longrightarrow & U/UI & \longrightarrow & 0 \quad .
\end{array}$$

Hence μ_I is monic (an isomorphism) if and only if $id \otimes \varepsilon$ is monic.

In a ring R with unit, for every left ideal $I \subset R$, we have $R \otimes_R I \simeq I$. Hence R_R is a flat module (with respect to R-MOD). Then, by 12.14, all free R-modules and their direct summands (= projective modules) are flat (with respect to R-MOD).

Further properties of flat modules will be obtained in 17.14 and later on, by studying 'pure exact' sequences (Chapter 7).

An R-module U_R is called *faithfully flat (with respect to R-MOD)* if U_R is flat (w.r.t. to R-MOD) and, for $N \in R$-MOD, the relation $U \otimes_R N = 0$ implies $N = 0$.

12.17 Faithfully flat modules. Characterizations.

Let R be a ring with unit. For a right R-module U_R the following assertions are equivalent:

(a) U_R is faithfully flat;

(b) U_R is flat and, for every (maximal) left ideal $I \subset R$, $I \neq R$, we have $U \otimes_R R/I \neq 0$ (i.e. $UI \neq U$);

(c) the functor $U \otimes_R - : R\text{-}MOD \to \mathbb{Z}\text{-}MOD$ is exact and reflects zero morphisms;

(d) the functor $U \otimes_R - : R\text{-}MOD \to \mathbb{Z}\text{-}MOD$ preserves and reflects exact sequences.

Proof: $(a) \Rightarrow (b)$ Because of the isomorphism $U \otimes_R R/I \simeq U/UI$ (see 12.11), $U \otimes_R R/I \neq 0$ is equivalent to $UI \neq U$. By (a), $U \otimes_R R/I = 0$ would imply $I = R$.

$(b) \Rightarrow (a)$ If $UI \neq U$ for every maximal left ideal $I \subset R$, then this is also true for every proper left ideal $I \subset R$. Hence $U \otimes_R K \neq 0$ for every cyclic R-module K. Since every R-module N contains a cyclic submodule and U_R is flat, we have $U \otimes_R N \neq 0$.

$(a) \Rightarrow (c)$ Let $f : L \to N$ be a morphism in R-MOD and $id \otimes f : U \otimes_R L \to U \otimes_R N$ a zero morphism, i.e. $0 = (U \otimes_R L)id \otimes f \simeq U \otimes_R (L)f$ (U_R flat). By (a), this implies $(L)f = 0$, i.e. $f = 0$.

$(c) \Rightarrow (d)$ Since U_R is flat, $U \otimes_R -$ preserves exact sequences.
Let $K \xrightarrow{f} L \xrightarrow{g} N$ be a sequence in R-MOD and assume

$$U \otimes_R K \xrightarrow{id\otimes f} U \otimes_R L \xrightarrow{id\otimes g} U \otimes_R N$$

to be exact. Then $id \otimes fg = (id \otimes f)(id \otimes g) = 0$ implies $fg = 0$. From the exact sequence $0 \to Im\ f \to Ke\,g \to Ke\ g/Im\,f \to 0$ we obtain the exact sequence

$$0 \to U \otimes_R Im\,f \to U \otimes_R Ke\,g \to U \otimes_R (Ke\,g/Im\,f) \to 0\,.$$

Since $U \otimes K \to U \otimes L \to U \otimes N$ is exact, we may identify $U \otimes_R Im\, f$ and $U \otimes_R Ke\, g$, thus obtaining $U \otimes_R (Ke\, g/Im\, f) = 0$. By (c), this means $Ke\, g = Im\, f$, i.e. the given sequence is exact.

$(d) \Rightarrow (a)$ Assume (d) and consider any $N \in R\text{-}MOD$ with $U \otimes_R N = 0$. From $0 \to N \to 0$ we obtain the exact sequence $0 \to U \otimes_R N \to 0$. By (d), $0 \to N \to 0$ also has to be exact, i.e. $N = 0$. Since $U \otimes_R -$ is exact, U_R is faithfully flat.

12.18 Tensor product over commutative rings.

If M_R, ${}_RN$ and ${}_RL$ are modules over a commutative ring R, then M_R can be regarded as a left R-module by defining $rm := mr$ for $r \in R$, $m \in M$.

An R-balanced map $\beta : M \times N \to L$ is called *R-bilinear* if

$$(rm, n)\beta = r(m, n)\beta \text{ for all } r \in R, m \in M, n \in N.$$

By 12.5, ${}_RM \otimes_R N$ is a left R-module with $r(m \otimes n) = (rm) \otimes n$ and we see that the balanced map

$$\tau : M \times N \to M \otimes N, \quad (m, n) \mapsto m \otimes n,$$

is bilinear: $\tau(rm, n) = (rm) \otimes n = r\tau(m, n)$. Hence we have, for commutative rings R:

A map $\beta : M \times N \to L$ is R-bilinear if and only if there is an R-linear map $\bar{\beta} : M \otimes_R N \to L$ with $\beta = \tau\bar{\beta}$.

With the notation $Bil_R(M \times N, L) = \{\beta : M \times N \to L \mid \beta\ R\text{-}bilinear\}$ we have an isomorphism of R-modules

$$Hom_R(M \otimes_R N, L) \simeq Bil_R(M \times N, L).$$

If M and N are vector spaces over a field K, then, by the above considerations, $M \otimes_K N$ is also a K-vector space and since the tensor product commutes with direct sums we find

$$dim_K(M \otimes_K N) = dim_K M \cdot dim_K N.$$

Every free R-module ${}_RF$ is isomorphic to $R^{(\Lambda)}$ for a suitable index set Λ. Over non-commutative rings the cardinality of Λ need not be uniquely determined. However, over a commutative ring with unit we have:

If $R^{(\Lambda)} \simeq R^{(\Lambda')}$, then Λ and Λ' have the same cardinality.

Proof: For a maximal ideal m of R, tensoring with $- \otimes R/m$ yields $(R/m)^{(\Lambda)} \simeq (R/m)^{(\Lambda')}$. For vector spaces over a field $(= R/m)$ it is known that the cardinality of a basis is uniquely determined.

12.19 Exercises.

(1) Consider two exact sequences in R-MOD and MOD-R respectively:

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0, \quad 0 \to N' \xrightarrow{h} N \xrightarrow{k} N'' \to 0\,.$$

Show that: (i) $Ke\,(g \otimes k) = Im\,(f \otimes id_N) + Im(id_M \otimes h)$;

(ii) $(M/M'f) \otimes_R (N/N'h) \simeq M \otimes_R N/Ke\,(g \otimes k)$.

(2) Let ${}_RM_S$ *be a bimodule and* ${}_SN$ *an S-module. Prove: If* ${}_RM$ *and* ${}_SN$ *are flat modules, then* ${}_RM \otimes_S N$ *is a flat R-module.*

(3) Show: If M,N are modules over a commutative ring R, then there is an R-isomorphism $M \otimes_R N \to N \otimes_R M$.

(4) Let A,B be algebras over a commutative ring R. Show that $A \otimes_R B$ *is an R-algebra with multiplication* $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

(5) Show that, for polynomial rings over a commutative ring R, $R[X] \otimes_R R[Y] \simeq R[X, Y]$.

(6) Let $\mu_{\mathbb{Q}} : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ *and* $\mu_{\mathbb{C}} : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ *be the maps defined by multiplication. Prove:*

(i) $\mu_{\mathbb{Q}}$ *and* $\mu_{\mathbb{C}}$ *are ring homomorphisms;*

(ii) $\mu_{\mathbb{Q}}$ *is an isomorphism,* $\mu_{\mathbb{C}}$ *is not monic.*

(7) Show: (i) $\mathbb{Q}$ *is flat as a* $\mathbb{Z}$*-module.*

(ii) For abelian torsion groups M (every element has finite order), we have $M \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

(iii) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

(iv) For finite $\mathbb{Z}$*-modules K, L, we have* $K \otimes_{\mathbb{Z}} L \simeq Hom_{\mathbb{Z}}(K, L)$.

Literature for Chapter 2: ANDERSON-FULLER, COHN, FAITH [1], HILTON-STAMMBACH, KASCH, NĂSTĂSESCU, ROTMAN; Gouguenheim [1,2], Hill [1], Raynaud-Gruson, Wilson.

Chapter 3

Modules characterized by the Hom-functor

13 Generators, trace

1.Definitions. 2.Reformulation. 3.Sets of generators for a module. 4.Definitions. 5.Properties of the trace. 6.Generators in module categories. 7.Generators in R-MOD. 8.Finitely generated modules. 9.Properties of finitely generated modules. 10.Exercises.

We will first study generators in arbitrary categories and then investigate their properties in module categories.

13.1 Definitions. Let $\mathcal{U}$ be a non-empty set (class) of objects in a category $\mathcal{C}$. An object A in $\mathcal{C}$ is said to be *generated by* $\mathcal{U}$ or $\mathcal{U}$*-generated* if, for every pair of distinct morphisms $f, g : A \to B$ in $\mathcal{C}$, there is a morphism $h : U \to A$ with $U \in \mathcal{U}$ and $hf \neq hg$. In this case $\mathcal{U}$ is called a *set (class) of generators* for A.

If $\mathcal{U}$ generates every object of a full subcategory $\mathcal{C}'$ of $\mathcal{C}$, then it is called a *set (class) of generators* or a *generating set (class)* for $\mathcal{C}'$.

In case $\mathcal{U}$ consists of just one $U \in Obj(\mathcal{C})$, we call U a *generator for* A, resp. *for* (or *in*) $\mathcal{C}'$, if $\{U\}$ has the corresponding property (and $U \in \mathcal{C}'$).

Observe that every category $\mathcal{C}$ has a *class* of generators (e.g. $Obj(\mathcal{C})$) but not necessarily a *set* of generators.

By definition of the functor $Mor_{\mathcal{C}}(U, -) : \mathcal{C} \to ENS$, these properties can be expressed differently:

13.2 Reformulation of the generator property.
Let U be an object in the category $\mathcal{C}$.

(1) For an object A in $\mathcal{C}$, the following assertions are equivalent:
(a) U generates A;
(b) the mappings $Mor(U,-) : Mor(A,B) \to Map(Mor(U,A), Mor(U,B))$ are injective for every B in $\mathcal{C}$.

(2) The following properties are equivalent:
(a) U is a generator for $\mathcal{C}$;
(b) the functor $Mor(U,-) : \mathcal{C} \to ENS$ is faithful.

In (subcategories of) *R-MOD* there are various ways for characterizing generators. Two morphisms $f, g : M \to N$ in *R-MOD* are distinct if and only if $f - g \neq 0$, and for a morphism $h : U \to M$ the inequality $hf \neq hg$ just means $h(f-g) \neq 0$.

Hence we have:

13.3 Sets of generators for a module in *R-MOD*.

Let $\mathcal{U}$ be a non-empty set of R-modules. For an R-module N the following assertions are equivalent:
(a) N is generated by $\mathcal{U}$;
(b) for every non-zero morphism $f : N \to L$ in R-MOD, there is an $h : U \to N$ with $U \in \mathcal{U}$ and $hf \neq 0$;
(c) there is an epimorphism $\bigoplus_\Lambda U_\lambda \to N$ with modules $U_\lambda \in \mathcal{U}$;
(d) N is a sum of submodules which are homomorphic images of modules in $\mathcal{U}$;
(e) $\bigoplus_{U\in\mathcal{U}} U$ is a generator for N.

Proof: $(a) \Leftrightarrow (b)$ follows from the preceding remark, $(c) \Leftrightarrow (d)$ is obvious and $(c) \Leftrightarrow (e)$ is easily seen.

$(c) \Rightarrow (b)$ Let $\varphi : \bigoplus_\Lambda U_\lambda \to N$ be an epimorphism, $U_\lambda \in \mathcal{U}$. For $f : N \to L$ with $f \neq 0$ also $\varphi f \neq 0$. Then, for at least one injection $\varepsilon_\mu : U_\mu \to \bigoplus_\Lambda U_\lambda$, $\mu \in \Lambda$, we have $\varepsilon_\mu \varphi f \neq 0$ and $\varepsilon_\mu \varphi \in Hom(U_\mu, N)$.

$(b) \Rightarrow (c)$ For $U \in \mathcal{U}$ and $\alpha \in Hom(U,N)$ set $U_\alpha = U$. From the homomorphisms $U_\alpha \to N$, $u \mapsto (u)\alpha$, we obtain a homomorphism $\varphi_U : U^{(Hom(U,N))} \to N$ with $\varepsilon_\alpha \varphi_U = \alpha$ for $\alpha \in Hom(U,N)$. We use it to form

$$\varphi = \sum_{U\in\mathcal{U}} \varphi_U : \bigoplus_{U\in\mathcal{U}} U^{(Hom(U,N))} \to N$$

and show that this is an epimorphism: For $g, h \in Hom(N,L)$ with $\varphi g = \varphi h$, i.e. $\varphi(g-h) = 0$, we get $0 = \varepsilon_\alpha \varphi_U (g-h) = \alpha(g-h)$ for all $\alpha \in Hom(U,N)$ and all $U \in \mathcal{U}$. By (b), this implies $g = h$ and hence φ is epimorphic.

Since every module N is a sum of its finitely generated (or cyclic) submodules, the module N is generated by its finitely generated (cyclic) submodules (in the above sense).

13.4 Definitions. Let $\mathcal{U}$ be a non-empty set (class) of R-modules. An R-module ${}_RN$ is said to be *finitely generated by* $\mathcal{U}$, or *finitely* $\mathcal{U}$*-generated*, if there exists an epimorphism $\bigoplus_{i\leq k} U_i \to N$ with finitely many $U_1, \ldots, U_k \in \mathcal{U}$.

$Gen(\mathcal{U})$ denotes the class of R-modules generated by $\mathcal{U}$, $gen(\mathcal{U})$ the class of R-modules finitely generated by $\mathcal{U}$. For an R-module L, the submodule

$$Tr\,(\mathcal{U}, L) = \sum\{Im\, h \mid h \in Hom(U, L),\ U \in \mathcal{U}\} \subset L$$

is called the *trace of* $\mathcal{U}$ *in* L. If $\mathcal{U}$ consists of a single module U we simply write $Tr(U, L)$ and $Gen(U)$ instead of $Tr(\{U\}, L)$ or $Gen(\{U\})$.

From the characterizations of $\mathcal{U}$-generated modules in 13.3 we immediately obtain: The full subcategory $Gen(\mathcal{U})$ ($gen(\mathcal{U})$) of *R-MOD* is closed under (finite) direct sums and homomorphic images. In both categories every morphism has a cokernel but not necessarily a kernel, since submodules of $\mathcal{U}$-generated modules need not be $\mathcal{U}$-generated.

13.5 Properties of the trace.
Let $\mathcal{U}$ be a set of R-modules and L an R-module.

(1) $Tr(\mathcal{U}, L)$ is the largest submodule of L generated by $\mathcal{U}$.

(2) $L = Tr(\mathcal{U}, L)$ if and only if L is $\mathcal{U}$-generated.

(3) $Tr(\mathcal{U}, L)$ is an $End_R(L)$-submodule of L (since for $U \in \mathcal{U}$, Hom(U,L) is a right $End_R(L)$-module).

(4) If $\mathcal{U}$ contains just one module U, then

$$Tr(U, L) = U\,Hom(U, L) = \{\sum_{i=1}^{k} u_i\varphi_i \mid u_i \in U,\ \varphi_i \in Hom(U, L), k \in I\!N\}.$$

For two R-modules U, L, we get from 13.3 that U generates L if and only if the functor $Hom(U, -) : Gen(L) \to AB$ is faithful. In special subcategories of *R-MOD* we have:

13.6 Generators in module categories. Characterizations.
Let $\mathcal{C}$ be a full subcategory of R-MOD closed under factor modules and submodules. Then the following are equivalent for an R-module U:
(a) U is a generator for $\mathcal{C}$;

(b) the functor $Hom_R(U,-) : \mathcal{C} \to AB$ *is faithful;*
(c) $Hom_R(U,-)$ *reflects zero morphisms in* $\mathcal{C}$*;*
(d) $Hom_R(U,-)$ *reflects epimorphisms in* $\mathcal{C}$*;*
(e) $Hom_R(U,-)$ *reflects exact sequences in* $\mathcal{C}$*.*

Proof: The equivalence of (a), (b) and (c) is derived from 13.3.

$(a) \Rightarrow (e)$ Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence in $\mathcal{C}$ for which

$$Hom(U,A) \xrightarrow{Hom(U,f)} Hom(U,B) \xrightarrow{Hom(U,g)} Hom(U,C)$$

is exact. Then $Hom(U,fg) = Hom(U,f)Hom(U,g) = 0$, i.e. $fg = 0$ (since $(a) \Leftrightarrow (c)$), and $Im\, f \subset Ke\, g$. By assumption, $K := Ke\, g \subset B$ is U-generated, hence $K = UHom(U,K)$, $UHom(U,K)g = Kg = 0$ and

$$Hom(U,K) \subset Ke\,(Hom(U,g)) = Im\,(Hom(U,f)) = Hom(U,A)f.$$

This implies $K = UHom(U,K) \subset UHom(U,A)f = Af$ and $Im\, f = Ke\, g$.

$(e) \Rightarrow (d)$ is obtained from the sequence $A \to B \to 0$.

$(d) \Rightarrow (a)$ We show that, for every $A \in \mathcal{C}$, we have $Tr(U,A) = A$: Putting $S = Tr(U,A)$ we form the exact sequence $0 \to S \xrightarrow{i} A \xrightarrow{p} A/S \to 0$ and obtain the exact sequence

$$0 \longrightarrow Hom(U,S) \xrightarrow{Hom(U,i)} Hom(U,A) \xrightarrow{Hom(U,p)} Hom(U,A/S).$$

For $\alpha \in Hom(U,A)$, we have $U\alpha p = 0$ which means

$$\alpha \in Ke\,(Hom(U,p)) = Im\,(Hom(U,i))$$

and hence $Hom(U,i)$ is surjective. By assumption (d), the inclusion i has to be surjective, i.e. $A = S = Tr(U,A)$.

We already know that every R-module is a factor module of a free R-module $R^{(\Lambda)}$ and hence (by 13.3) ${}_RR$ is a finitely generated generator in R-MOD (here $1 \in R$ is important). Of course, this is not the only generator in R-MOD:

13.7 Generators in *R-MOD*. Characterizations.

For a left R-module G the following statements are equivalent:
(a) G is a generator in R-MOD;
(b) $Hom(G,-) : R\text{-}MOD \to AB$ *is faithful;*

(c) G generates all finitely generated modules in R-MOD;

(d) G generates R;

(e) there exist finitely many $\alpha_1, \dots, \alpha_k \in Hom(G, R)$ *and* $g_1, \dots, g_k \in G$ *with*

$$(g_1)\alpha_1 + \cdots + (g_k)\alpha_k = 1\,(\in R);$$

(f) R is a direct summand of G^k *for some* $k \in \mathbb{N}$.

In this case, G_S *is a direct summand of* S^k *for* $S = End(_RG)$.

Proof: The equivalences of (a) to (d) are obvious, having in mind that every module generating a generator in *R-MOD* is also a generator.

$(d) \Rightarrow (e)$ G generates R means $GHom(G, R) = R$, and from this we obtain the elements desired.

$(e) \Rightarrow (f)$ For $\alpha_i \in Hom(G, R)$, we obtain a homomorphism

$$\alpha : G^k \to R, \;\; (g'_1, \dots, g'_k) \mapsto \sum_{i=1}^{k} (g'_i)\alpha_i \; .$$

α is surjective since, by (e), $1 = \sum (g_i)\alpha_i \in Im\,\alpha$ and hence $R = R1 \subset Im\,\alpha$. For $\beta : R \to G^k$, $r \mapsto r(g_1, \dots, g_k)$, we get $\beta\alpha = id_R$ and therefore $R = Im\,\alpha$ is a direct summand of G^k.

$(f) \Rightarrow (a)$ is trivial.

By (f), we have a splitting sequence $0 \to R \to G^k$ which turns into a splitting sequence in *MOD-S* under $Hom(-, G)$:

$$Hom(G^k, G) \to Hom(R, G) \to 0.$$

Hence $Hom(R, G)_S \simeq G_S$ is a direct summand in $Hom(G^k, G)_S \simeq S^k$ (and finitely generated).

A further description of generators in *R-MOD* will be obtained in 18.8.

An R-module M is called *finitely generated* if it contains a finite generating set (see 6.6). This internal property of a module has the following categorical meaning and characterizations which can be immediately derived from the definitions. The last two properties were shown in 12.9:

13.8 Finitely generated modules. Characterizations.

For an R-module M the following assertions are equivalent:

(a) M is finitely generated (see 6.6);

(b) every generating set of M contains a finite subset generating M;

(c) M is finitely generated by R;

(d) if M is generated by some set $\mathcal{U}$ of R-modules, then M is finitely generated by $\mathcal{U}$;

(e) if $M = \sum_\Lambda M_\lambda$ for submodules $M_\lambda \subset M$, then there exists a finite subset $E \subset \Lambda$ with $M = \sum_E M_\lambda$;

(f) if $\varphi : \bigoplus_\Lambda U_\lambda \to M$ is epic with R-modules $\{U_\lambda\}_\Lambda$, then there exists a finite subset $E \subset \Lambda$ such that the following composition of maps is epic

$$\bigoplus\nolimits_E U_\lambda \xrightarrow{\varepsilon_E} \bigoplus\nolimits_\Lambda U_\lambda \xrightarrow{\varphi} M;$$

(g) for every family of right R-modules $\{L_\lambda\}_\Lambda$, the canonical map

$$\varphi_M : (\prod\nolimits_\Lambda L_\lambda) \otimes_R M \to \prod\nolimits_\Lambda (L_\lambda \otimes_R M)$$

is surjective;

(h) for any set Λ, the canonical map $\varphi_M : R^\Lambda \otimes_R M \to M^\Lambda$ is surjective.

The properties (d) and (f) can be used to define 'finitely generated' objects in arbitrary categories with coproducts.

Obviously, finite (direct) sums and homomorphic images of finitely generated R-modules are again finitely generated. Their submodules, however, need not be finitely generated.

13.9 Properties of finitely generated modules.

(1) Let $0 \to K \xrightarrow{f} L \xrightarrow{g} N \to 0$ be an exact sequence of R-modules. If K and N are finitely generated, then L is also finitely generated.

(2) Let K be a finitely generated R-module and $\{U_\lambda\}_\Lambda$ a family of R-modules. Then:

(i) For every morphism $f : K \to \bigoplus_\Lambda U_\lambda$, there is a finite subset $E \subset \Lambda$ with $(K)f \subset \bigoplus_E U_\lambda$.

(ii) $Hom_R(K, \bigoplus_\Lambda U_\lambda) \simeq \bigoplus_\Lambda Hom(K, U_\lambda)$, i.e. $Hom_R(K, -)$ preserves direct sums.

Proof: (1) Let $K_o \subset K$, $N_o \subset N$ be finite generating sets of K resp. N. Then there is a finite subset $L_o \subset L$ with $(L_o)g = N_o$, and $(K_o)f \bigcup L_o$ is a finite generating set of L since:

For every $m \in L$, there exist $l_i \in L_o$ and $r_i \in R$ with $(m)g = \sum r_i(l_i)g$. The element $m - \sum r_i l_i \in Ke\, g = Im\, f$ is a linear combination of elements in K_o and $m = \sum r_i l_i + (m - \sum r_i l_i)$.

(2)(i) Every element $(k)f,\ k \in K$, is contained in a finite partial sum of the $\{U_\lambda\}_\Lambda$. Hence a finite generating set of $(K)f$ is contained in a finite sum $\bigoplus_E U_\lambda$ and $(K)f \subset \bigoplus_E U_\lambda$.

(ii) The isomorphism (see 9.4)

$$\phi : Hom(K, \prod_\Lambda U_\lambda) \to \prod_\Lambda Hom(K, U_\lambda)$$

yields, by restriction, a monomorphism

$$\phi' : Hom(K, \bigoplus_\Lambda U_\lambda) \to \prod_\Lambda Hom(K, U_\lambda).$$

By (i), we find, for every $f \in Hom(K, \bigoplus_\Lambda U_\lambda)$, a finite subset $E \subset \Lambda$ with $f \in Hom(K, \bigoplus_E U_\lambda)$ and hence $(f)\phi' \in \bigoplus_E Hom(K, U_\lambda)$.

It is easy to see that $Im\,\phi' = \bigoplus_\Lambda Hom(K, U_\lambda)$.

13.10 Exercises.

(1) For $M \in R\text{-}MOD$ and a left ideal $I \subset R$ define $An_M(I) = \{m \in M \mid Im = 0\}$. Prove: $Tr(R/I, M) = R\,An_M(I)$.

(2) Let $M \in R\text{-}MOD$ and e,f be idempotents in $S = End(_RM)$. Show: $Tr(Me, Mf) = MeSf, \quad Tr(Me, M) = MeS$.

(3) Let $f : M \to N$ be a monomorphism in R-MOD and $U \in R\text{-}MOD$. Show: If $Tr(U, N) \subset Im\,f$, then $Tr(U, M)f = Tr(U, N)$.

(4) If $\{N_\lambda\}_\Lambda$ is a family of R-modules, then, for every R-module U: $Tr(U, \bigoplus_\Lambda N_\lambda) = \bigoplus_\Lambda Tr(U, N_\lambda)$.

(5) Let $\mathcal{U}$ be a class of R-modules. Prove: If $\mathcal{U}' \subset Gen(\mathcal{U})$, then $Tr(\mathcal{U}', N) \subset Tr(\mathcal{U}, N)$ for every $N \in R\text{-}MOD$.

(6) Let $\mathcal{U}$ be a class of R-modules. Show that $Tr(\mathcal{U}, -)$ defines a functor from R-MOD into R-MOD (see exercise (4) and 11.12,(5)).

(7) Show that, for any family $\{U_\lambda\}_\Lambda$ of R-modules and $N \in R\text{-}MOD$, $Tr(\bigoplus_\Lambda U_\lambda, N) = \sum_\Lambda Tr(U_\lambda, N)$.

Literature: ANDERSON-FULLER; Yao.

14 Cogenerators, reject

1.Definitions. 2.Reformulation. 3.Set of cogenerators for a module. 4.Definitions. 5.Properties of the reject. 6.Cogenerators for module categories. 7.Finitely cogenerated modules. 8.Cocyclic modules. 9.Subdirect product of cocyclic modules. 10.Exercises.

Dual to the notion of generators first we define cogenerators in arbitrary categories and then derive special properties of cogenerators in *R-MOD*.

14.1 Definitions. Let $\mathcal{U}$ be a non-empty set (class) of objects of a category $\mathcal{C}$. An object B in $\mathcal{C}$ is said to be *cogenerated by* $\mathcal{U}$ or $\mathcal{U}$*-cogenerated* if, for every pair of distinct morphisms $f, g : A \to B$ in $\mathcal{C}$, there is a morphism $h : B \to U$ with $U \in \mathcal{U}$ and $fh \neq gh$. Then $\mathcal{U}$ is called a *set (class) of cogenerators for* B.

$\mathcal{U}$ is said to be *a set (class) of cogenerators for a subcategory* $\mathcal{C}'$ of $\mathcal{C}$ if every object in $\mathcal{C}'$ is cogenerated by $\mathcal{U}$.

In case $\mathcal{U}$ has only one element $U \in Obj(\mathcal{C})$, then U is called a *cogenerator for A, resp. for* $\mathcal{C}'$, if $\{U\}$ has the corresponding property.

Dual to 13.2 the definitions give

14.2 Reformulation of the cogenerator property.
Let U be an object in the category $\mathcal{C}$.

(1) For $B \in Obj(\mathcal{C})$, the following are equivalent:

(a) U cogenerates B;

(b) the map $Mor(-,U) : Mor(A,B) \to Map(Mor(B,U), Mor(A,U))$ is injective for every $A \in Obj(\mathcal{C})$.

(2) The following assertions are equivalent:

(a) U is a cogenerator for $\mathcal{C}$;

(b) the functor $Mor(-,U) : \mathcal{C} \to ENS$ is faithful.

Again we obtain special characterizations for cogenerators in *R-MOD*: For two R-modules N, U and the family of mappings

$$\{\beta : N \to U_\beta = U,\ n \mapsto (n)\beta \mid \beta \in Hom(N,U)\},$$

we obtain a homomorphism

$$\psi : N \to \prod \{U_\beta \mid \beta \in Hom(N,U)\} = U^{Hom(N,U)}.$$

By 9.3, $Ke\,\psi = \bigcap \{Ke\,\beta \mid \beta \in Hom(N,U)\} \subset N$.

Dually to 13.3 we get:

14.3 Set of cogenerators for a module in *R-MOD*.
Let $\mathcal{U}$ be a set of R-modules. For an R-module N the following assertions are equivalent:
(a) N is $\mathcal{U}$-cogenerated;
(b) for every non-zero morphism $f : L \to N$ in R-MOD, there exists $h \in Hom(N,U)$ with $U \in \mathcal{U}$ and $fh \neq 0$;
(c) there is a monomorphism $N \to \prod_\Lambda U_\lambda$ with modules $U_\lambda \in \mathcal{U}$;
(d) $\bigcap\{Ke\,f \mid f \in Hom(N,U),\ U \in \mathcal{U}\} = 0$;
(e) $\prod_{U\in\mathcal{U}} U$ is a cogenerator for N.

In case $\mathcal{U}$ has only one element the characterizations become simpler.

14.4 Definitions. Let $\mathcal{U}$ be a non-empty set (class) of R-modules. An R-module N is said to be *finitely cogenerated by* $\mathcal{U}$, or *finitely* $\mathcal{U}$*-cogenerated*, if there is a monomorphism

$$N \to \prod_{i\leq k} U_i = \bigoplus_{i\leq k} U_i \text{ with finitely many } U_i \in \mathcal{U}.$$

Let $Cog(\mathcal{U})$ denote the class of $\mathcal{U}$-cogenerated R-modules, $cog(\mathcal{U})$ the class of R-modules finitely cogenerated by $\mathcal{U}$.

For any R-module L, the submodule

$$Re(L,\mathcal{U}) = \bigcap \{Ke\,f \mid f \in Hom(L,U),\ U \in \mathcal{U}\} \subset L$$

is called the *reject of* $\mathcal{U}$ *in* L.

From the properties stated in 14.3 we have immediately: $Cog(\mathcal{U})$ (resp. $cog(\mathcal{U})$) is closed under arbitrary (resp. finite) products and submodules. Hence both categories have kernels but not necessarily cokernels. In $Cog(\mathcal{U})$ there also exist arbitrary coproducts.

14.5 Properties of the reject.
Let $\mathcal{U}$ be a non-empty set of R-modules and L an R-module.

(1) $Re(L,\mathcal{U})$ is the smallest submodule K of L for which L/K is $\mathcal{U}$-cogenerated.

(2) $Re(L,\mathcal{U}) = 0$ if and only if L is $\mathcal{U}$-cogenerated.

(3) $Re(L,\mathcal{U})$ is an $End_R(L)$-submodule of L (since, for any $U \in \mathcal{U}$, the group Hom(L,U) is an $End_R(L)$-module).

(4) In case $\mathcal{U}$ consists of one module U, then $Re(L,U) = \bigcap\{Ke\,f \mid f \in Hom(L,U)\}$.

With these notions it is obvious how to dualize the characterizations of generators in module categories given in 13.6:

14.6 Cogenerators for module categories.

Let $\mathcal{C}$ be a full subcategory of R-MOD closed under factor modules and submodules. Then, for any R-module U, the following are equivalent:

(a) U is a cogenerator for $\mathcal{C}$;

(b) the functor $Hom_R(-,U) : \mathcal{C} \to AB$ is faithful;

(c) $Hom_R(-,U)$ reflects zero morphisms in $\mathcal{C}$;

(d) if, for $f : N \to L$ in $\mathcal{C}$, $Hom(f,U)$ is epic, then f is monic;

(e) $Hom_R(-,U)$ reflects exact sequences in $\mathcal{C}$.

We have seen in § 13 that there is a natural generator in *R-MOD*, i.e. $_RR$. To prove the existence of cogenerators in *R-MOD* is not so simple. This will be shown later (in connection with the study of injective modules). In particular it cannot be obtained by formally dualizing the results in § 13.

In this context it is also of interest how to dualize the notion of 'finitely generated modules'. From their characterizations in 13.8 we get, for example, from (e):

Definition. We call an R-module N *finitely cogenerated* if for every monomorphism $\psi : N \to \prod_\Lambda U_\lambda$ (in *R-MOD*) there is a finite subset $E \subset \Lambda$ such that

$$N \xrightarrow{\psi} \prod_\Lambda U_\lambda \xrightarrow{\pi_E} \prod_E U_\lambda$$

is monic (with π_E as in 9.3,(3)).

In the proof of 13.8 we use in fact that there is a finitely generated generator in *R-MOD*. Hence we cannot simply dualize those assertions. However we can show:

14.7 Finitely cogenerated modules. Characterizations.

For an R-module N the following assertions are equivalent:

(a) N is finitely cogenerated;

(b) for every family $\{V_\lambda\}_\Lambda$ of submodules of N with $\bigcap_\Lambda V_\lambda = 0$, there is a finite subset $E \subset \Lambda$ with $\bigcap_E V_\lambda = 0$;

(c) for every family of morphisms $\{f_\lambda : N \to U_\lambda\}$ in R-MOD with $\bigcap_\Lambda Ke\, f_\lambda = 0$, there is a finite subset $E \subset \Lambda$ with $\bigcap_E Ke\, f_\lambda = 0$;

(d) every submodule of N is finitely cogenerated.

Proof: $(a) \Rightarrow (b)$ Let $\{V_\lambda\}_\Lambda$ be a family of submodules of N with $\bigcap_\Lambda V_\lambda = 0$. Then there is a monomorphism $\psi : N \to \prod_\Lambda N/V_\lambda$ (see 9.11).

By (a), there is a finite subset $E \subset \Lambda$ such that

$$N \xrightarrow{\psi} \prod_{\Lambda} N/V_\lambda \xrightarrow{\pi_E} \prod_{E} N/V_\lambda$$

is monic, i.e. $\bigcap_E V_\lambda = Ke\,\psi\pi_E = 0$.

The other assertions are proved in a similar way.

A cyclic module $N = Rn_o$ can be characterized by the property that every morphism $f : M \to N$ with $n_o \in Im\, f$ is epic. Dually we define:

Definition. An R-module N is called *cocyclic* if there is an $n_o \in N$ with the property: every morphism $g : N \to M$ with $n_o \notin Ke\, g$ is monic.

Recall that a non-zero module N is simple if it has no non-zero proper submodules. Obviously simple modules are cocyclic.

14.8 Cocyclic modules. Characterizations.

For a non-zero R-module N the following assertions are equivalent:

(a) N is cocyclic;

(b) N has a simple submodule K which is contained in every non-zero submodule of N;

(c) the intersection of all non-zero submodules of N is non-zero;

(d) N is subdirectly irreducible;

(e) for any monomorphism $\varphi : N \to \prod_\Lambda U_\lambda$ in R-MOD, there is a $\lambda_o \in \Lambda$ for which $\varphi\pi_{\lambda_o} : N \to U_{\lambda_o}$ is monic;

(f) N is an essential extension of a simple module (see § 17).

Proof: $(a) \Rightarrow (b)$ If U is a submodule of N, $n_o \in N$ as in the definition above and $n_o \notin U$, then $N \to N/U$ is monic, i.e. $U = 0$. Hence Rn_o is contained in every non-zero submodule of N and is therefore simple.

$(b) \Rightarrow (a)$ If $K = Rn_1$ and $g : N \to M$ is a morphism with $n_1 \notin Ke\, g$, then we conclude $Ke\, g = 0$.

$(b) \Leftrightarrow (c) \Leftrightarrow (d)$ are trivial (see 9.11).

$(a) \Rightarrow (e)$ Choose $n_o \in N$ as in the definition above.
Since $0 = Ke\,\varphi = \bigcap_\Lambda Ke\,\varphi\pi_\lambda$ we must have $n_o \notin Ke\,\varphi\pi_{\lambda_o}$ for some $\lambda_o \in \Lambda$. Then $\varphi\pi_{\lambda_o}$ is monic.

$(e) \Rightarrow (a)$ is obtained from the next proposition and the trivial observation that submodules of cocyclic modules are again cocyclic.

$(b) \Leftrightarrow (f)$ is immediately derived from the definitions (see 17.1).

14.9 Subdirect product of cocyclic modules.

(1) Every non-zero module is isomorphic to a subdirect product of its cocyclic factor modules.

(2) An R-module is finitely cogenerated if and only if it is isomorphic to a subdirect product of finitely many cocyclic modules.

Proof: (1) Let N be a non-zero R-module and $0 \neq n \in N$. The set of submodules U of N with $n \notin U$ ordered by inclusion is inductive and hence has a maximal element $U_n \subset N$. The submodules L/U_n of N/U_n correspond to the submodules L of N containing U_n, hence $n \in L$ if $L \neq U_n$. Therefore $(Rn+U_n)/U_n$ is contained in every non-zero submodule of N/U_n, i.e. N/U_n is cocyclic.

The canonical morphisms $\varphi_n : N \to N/U_n, 0 \neq n \in N$, yield a morphism $\varphi : N \to \prod_{N\setminus 0} N/U_n$ with $Ke\,\varphi = \bigcap_{N\setminus 0} Ke\,\varphi_n = 0$.

(2) We will show in 21.4 that a finite direct sum of finitely cogenerated (in particular cocyclic) modules is finitely cogenerated. This implies the assertion by (1).

As a consequence we observe that the class of cocyclic modules is a *class* of cogenerators in *R-MOD*. We will see soon that there is also a *set* of cogenerators in *R-MOD* (see 17.12).

Further properties and characterizations of finitely cogenerated and cocyclic modules will occur in § 17 in the course of investigating *essential extensions* and *injective hulls*.

14.10 Exercises.

(1) For an R-module N, put $An_R(N) = \{r \in R \mid rN = 0\}$. *Show:*

(i) If U is an R-module which generates or cogenerates N, then $An_R(U) \subset An_R(N)$.

(ii) The following are equivalent:

(a) $An_R(N) = 0$ *(N is faithful);*

(b) N cogenerates R;

(c) N cogenerates a generator in R-MOD.

(2) Assume $M \in$ *R-MOD and let* e, f *be idempotents in* $S = End(_RM)$. *Show:* $Re(Me, Mf) = \{m \in Me \mid meSf = 0\}$.

(3) Let $f : M \to N$ *be an epimorphism in R-MOD and* $U \in$ *R-MOD. Show: If* $Ke\,f \subset Re(M,U)$, *then* $Re(M,U)f = Re(N,U)$.

(4) Prove that, for any family of R-modules $\{N_\lambda\}_\Lambda$ *and any R-module U,* $Re(\bigoplus_\Lambda N_\lambda, U) = \bigoplus_\Lambda Re(N_\lambda, U)$.

(5) Let $\mathcal{U}$ be a class of R-modules. Show: If $\mathcal{U}' \subset Cog(\mathcal{U})$, then $Re(N,\mathcal{U}) \subset Re(N,\mathcal{U}')$ for every $N \in$ R-MOD.

(6) Let $\mathcal{U}$ be a class of R-modules. Show that $Re(-,\mathcal{U})$ defines a functor of R-MOD to R-MOD (see exercise (4) and 11.12,(5)).

(7) Prove for a family $\{U_\lambda\}_\Lambda$ of R-modules and any $N \in$ R-MOD:

$$Re(N, \prod_\Lambda U_\lambda) = \bigcap_\Lambda Re(N, U_\lambda) = Re(N, \bigoplus_\Lambda U_\lambda).$$

(8) Let $N = \bigoplus \{\mathbb{Z}_p \mid p$ a prime number in $\mathbb{N}\}$. Show:

(i) If N is cogenerated by a $\mathbb{Z}$-module U, then N is isomorphic to a submodule of U;

(ii) N is not finitely cogenerated.

Literature: ANDERSON-FULLER; Kasch-Pareigis, Onodera [3], Vámos [2].

15 Subgenerators, the category $\sigma[M]$

1.Properties of $\sigma[M]$. 2.Subgenerators in $\sigma[M]$. 3.Subgenerators in R-MOD. 4.Special subgenerators in R-MOD. 5.M as a generator in $\sigma[M]$. 6.M-generated R-modules as B-modules. 7.Density Theorem. 8.Characterization of density. 9.Modules flat over End($_RM$). 10.Torsion modules over $\mathbb{Z}$. 11.Exercises.

Having provided a number of tools we now want to define, for an R-module M, a category closely connected with M and hence reflecting properties of M. Many investigations about the module M will in fact be a study of this category.

Definitions. Let M be an R-module. We say that an R-module N is *subgenerated by* M, or that M *is a subgenerator for* N, if N is isomorphic to a submodule of an M-generated module.

A subcategory $\mathcal{C}$ of *R-MOD* is *subgenerated by M*, or M is a *subgenerator for* $\mathcal{C}$, if every object in $\mathcal{C}$ is subgenerated by M.

We denote by $\sigma[M]$ the full subcategory of *R-MOD* whose objects are all R-modules subgenerated by M.

By definition, M is a subgenerator in $\sigma[M]$, and a module is subgenerated by M if and only if it is a kernel of a morphism between M-generated modules. Hence we obtain the first of the

15.1 Properties of $\sigma[M]$. *For an R-module M we have:*

(1) For N in $\sigma[M]$, all factor modules and submodules of N belong to $\sigma[M]$, i.e. $\sigma[M]$ has kernels and cokernels.

(2) The direct sum of a family of modules in $\sigma[M]$ belongs to $\sigma[M]$ and is equal to the coproduct of these modules in $\sigma[M]$.

(3) The sets

$$\mathcal{M}_f = \{U \subset M^{(\mathbb{N})} \mid U \text{ finitely generated}\} \text{ and } \mathcal{M}_c = \{Rm \mid m \in M^{(\mathbb{N})}\}$$

are sets of generators in $\sigma[M]$.

Therefore $\sigma[M]$ is called a locally finitely generated *category.*

(4) $U_f = \bigoplus\{U \mid U \in \mathcal{M}_e\}$ and $U_c = \bigoplus\{Z \mid Z \in \mathcal{M}_z\}$ are generators in $\sigma[M]$.

(5) Pullback and pushout of morphisms in $\sigma[M]$ belong to $\sigma[M]$.

(6) For a family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M]$, the product in $\sigma[M]$ exists and is given by $\prod^M_\Lambda N_\lambda := Tr(U_f, \prod_\Lambda N_\lambda)$.

Proof: (1) follows from the preceding remark.

(2) If $\{N_\lambda\}$ is a family of R-modules in $\sigma[M]$ and $N_\lambda \subset M_\lambda$ for M-generated M_λ, then $\bigoplus_\Lambda N_\lambda \subset \bigoplus_\Lambda M_\lambda$ with $\bigoplus_\Lambda M_\lambda$ obviously M-generated, i.e. $\bigoplus_\Lambda N_\lambda$ belongs to $\sigma[M]$. This is also the coproduct of $\{N_\lambda\}_\Lambda$ in $\sigma[M]$.

(3) Let N be in $\sigma[M]$. It is enough to show that every cyclic submodule $Rn \subset N$, $n \in N$, is generated by $\mathcal{M}_z$ (and hence by $\mathcal{M}_e$): By definition of $\sigma[M]$, there is an M-generated module $\widetilde{N}$ with $N \subset \widetilde{N}$. Let $\varphi : M^{(\Lambda)} \to \widetilde{N}$ be epic and $m \in M^{(\Lambda)}$ with $(m)\varphi = n \in N$. Then $m \in M^{(\mathbb{N})}$, i.e. $Rm \in \mathcal{M}_z$ and the restriction $\varphi|_{Rm} : Rm \to Rn$ is epic.

(4) follows immediately from (3) (see 13.3).

(5) is a consequence of (1) and (2) .

(6) Let $\{f_\lambda : X \to N_\lambda\}$ be a family of morphismen in $\sigma[M]$. By the property of products in R-MOD, we have the commutative diagram

$$\begin{array}{ccc} \prod_\Lambda N_\lambda & \xrightarrow{\pi_\lambda} & N_\lambda \\ & {}^{f}\nwarrow \quad \nearrow^{f_\lambda} & \\ & X & \end{array} .$$

Since X is in $\sigma[M]$, also $(X)f \in \sigma[M]$, i.e.

$$(X)f \subset Tr(\mathcal{M}_f, \prod\nolimits_\Lambda N_\lambda) = Tr(U_f, \prod\nolimits_\Lambda N_\lambda).$$

Hence $Tr(U_f, \prod_\Lambda N_\lambda)$, together with the restrictions of the canonical projections π_λ, is the product of $\{N_\lambda\}_\Lambda$ in $\sigma[M]$.

From the definitions and properties just stated we easily get:

15.2 Subgenerators in $\sigma[M]$.

For two R-modules M, N the following are equivalent:

(a) N is a subgenerator in $\sigma[M]$;

(b) $\sigma[M] = \sigma[N]$;

(c) $N \in \sigma[M]$ and $M \in \sigma[N]$;

(d) $N \in \sigma[M]$ and the (cyclic) submodules of $N^{(\mathbb{N})}$ provide a set of generators for $\sigma[M]$.

Observe that M need not be a generator in $\sigma[M]$ and that, in general, $\sigma[M]$ does not have a finitely generated generator. However this is the case if R belongs to $\sigma[M]$:

15.3 Subgenerators in *R-MOD*.

For any R-module M the following assertions are equivalent:

(a) *R is subgenerated by M (i.e.* $R \in \sigma[M]$*);*

(b) $\sigma[M] = R$*-MOD;*

(c) $R \subset M^k$ *for some* $k \in I\!N$*;*

(d) $\{U \subset M^{(I\!N)} \mid U \text{ cyclic}\}$ *is a set of generators in R-MOD.*

Proof: The equivalence of (a), (b) and (d) and $(c) \Rightarrow (d)$ are obvious.

$(d) \Rightarrow (c)$ By (d), there exist $u \in M^k$, $k \in I\!N$, and an epimorphism $\alpha : Ru \to R$ with $(u)\alpha = 1$. For $\beta : R \to Ru$, $(r)\beta := ru$, we get $\beta\alpha = id_R$, i.e. α is a retraction and $R \simeq (R)\beta$ is a direct summand in Ru and hence a submodule of M^k.

Let us point out two special cases in which $\sigma[M]$ coincides with a full module category:

15.4 Special subgenerators in *R-MOD*.

(1) If the R-module M is finitely generated as a module over $S = End({}_RM)$*, then* $\sigma[M] = R/An_R(M)$*-MOD.*

(2) If R is commutative, then, for every finitely generated R-module M, we have $\sigma[M] = R/An_R(M)$*-MOD.*

Proof: (1) For a generating set $m_1, \ldots, m_k$ of M_S consider the map

$$\rho : R \to R(m_1, \ldots, m_k) \subset M^k, \quad r \mapsto r(m_1, \ldots, m_k).$$

We have $Ke\,\rho = \bigcap_{i \le k} An_R(m_i) = An_R(M)$, i.e. $R/An_R(M) \subset M^k$.

(2) is a consequence of (1) since we have $R/An_R(M) \subset S$ canonically.

Whether M is a generator in $\sigma[M]$ will often be of interest. We give some descriptions of this case derived from § 13 and the definition:

An R-module is called a *self-generator* (*self-cogenerator*) if it generates all its submodules (cogenerates all its factor modules).

15.5 M as a generator in $\sigma[M]$.

For any R-module M with $S = End({}_RM)$*, the following are equivalent:*

(a) *M is a generator in* $\sigma[M]$*;*

(b) *the functor* $Hom(M, -) : \sigma[M] \to S$*-MOD is faithful;*

(c) *M generates every (cyclic) submodule of* $M^{(I\!N)}$*;*

(d) $M^{(I\!N)}$ *is a self-generator;*

(e) *for every submodule* $U \subset M^k$*,* $k \in I\!N$*, we have* $U = M\,Hom(M, U)$*.*

We shall encounter further characterizations of generators in $\sigma[M]$ resp. *R-MOD* in the course of studying *projective* modules. An interesting property, the *Density Theorem*, can be shown now. For this let us recall that the image of the defining ring homomorphism of an R-module M,

$$\varphi : R \to End_{\mathbb{Z}}(M), \;\; \varphi(r)[m] = rm,$$

is a subring of the biendomorphism ring (with $S = End(_R M)$)

$$B = Biend(_R M) = End(M_S) \subset End(_{\mathbb{Z}} M)\,.$$

Since M is a left B-module, $M^{(\Lambda)}$ is a left B-module for any set Λ.

Every $f \in End(_R M^{(\Lambda)})$ can formally be written, with the canonical injections and projections ε_i, π_i (since $\sum_\Lambda \pi_i \varepsilon_i = id_{M^{(\Lambda)}}$), as

$$f = \sum_{i,j} \pi_i \varepsilon_i f \pi_j \varepsilon_j ,$$

where the $\varepsilon_i f \pi_j$ are elements in $S = End(_R M)$ and the sum is in fact finite for every element in $M^{(\Lambda)}$.

For $m \in M^{(\Lambda)}$ and $b \in B = Biend(_R M)$, this yields

$$\begin{array}{rcccl} b(mf) & = & \sum_{i,j} b((m\pi_i)\varepsilon_i f \pi_j \varepsilon_j) & = & \sum_{i,j} (b(m\pi_i))\varepsilon_i f \pi_j \varepsilon_j \\ & = & \sum_{i,j} (bm)\pi_i \varepsilon_i f \pi_j \varepsilon_j & = & (bm)f\,. \end{array}$$

Hence the elements in $Biend(_R M)$ can also be considered as elements of $Biend(_R M^{(\Lambda)})$ and we obtain:

15.6 M-generated R-modules as B-modules.

Let M be an R-module, $B = Biend(_R M)$, and Λ an index set.
Then every M-generated R-submodule of $M^{(\Lambda)}$ is a B-submodule of $M^{(\Lambda)}$.

Proof: For an M-generated submodule U of $M^{(\Lambda)}$, we have

$$U = Tr(M^{(\Lambda)}, U) = M^{(\Lambda)} Hom(M^{(\Lambda)}, U),$$

and assuming $Hom(M^{(\Lambda)}, U) \subset End(_R M^{(\Lambda)})$ we get

$$BU = B\,[M^{(\Lambda)} Hom(M^{(\Lambda)}, U)] = [BM^{(\Lambda)}]\, Hom(M^{(\Lambda)}, U) = U.$$

This is used in the proof of the first part of the

15.7 Density Theorem.
Let M be an R-module with one of the following properties:

(i) M is a generator in $\sigma[M]$, or

(ii) for every cyclic submodule $U \subset M^n$, $n \in I\!N$, the factor module M^n/U is cogenerated by M. Then:

(1) For any finitely many $m_1, \ldots, m_n$ in M and $\beta \in B = Biend(_RM)$, there exists $r \in R$ with $\beta(m_i) = rm_i$ for all $i = 1, \ldots, n$.

(2) If M is finitely generated over $S = End(_RM)$, then the defining morphism (see above) $\varphi : R \to B$ is surjective.

Property (1) is also expressed by saying $\varphi(R)$ *is dense in* B. This can be regarded as 'dense' in a certain topology on B.

Proof: (1)(i) For elements $m_1, \ldots, m_n$ in M, the R-submodule $U = R(m_1, \ldots, m_n)$ of M^n is M-generated and hence a B-submodule by 15.6, in particular $B(m_1, \ldots, m_n) \subset R(m_1, \ldots, m_n)$. Consequently, for every $\beta \in B$, there exists $r \in R$ with $(\beta(m_1), \ldots, \beta(m_n)) = (rm_1, \ldots, rm_n)$.

(ii) Assume that, for some $m_1, \ldots, m_n \in M$ and $\beta \in B$, we have

$$b := \beta(m_1, \ldots, m_n) \notin R(m_1, \ldots, m_n) =: U \subset M^n.$$

Since M^n/U is M-cogenerated, there is a morphism $h' : M^n/U \to M$ with $(b + U)h' \neq 0$. With the projection $p : M^n \to M^n/U$, the morphism $h = ph' : M^n \to M$ has the properties $(U)h = 0$ and $(b)h \neq 0$. Regarding h as an element in $Hom(M^n, M) \subset End(M^n)$ yields

$$(b)h = \beta((m_1, \ldots, m_n)h) = 0,$$

a contradiction to our assumption above.

(2) If $m_1, \ldots, m_n$ is a generating set of M_S, then β is uniquely determined by the $\beta(m_i)$ and, according to (1), every $\beta = \varphi(r)$ for some $r \in R$.

Again let M be an R-module defined by $\varphi : R \to End_{\mathbb{Z}}(M)$ and $B = Biend(_RM)$. The map $\varphi : R \to B$ turns every B-module in a canonical way into an R-module and obviously $\sigma[_BM] \subset \sigma[_RM]$. The coincidence of these categories yields a

15.8 Characterization of density.

For an R-module M the following are equivalent (notation as above):

(a) $\varphi(R)$ is dense in B;

(b) the categories $\sigma[_RM]$ and $\sigma[_BM]$ coincide, i.e.

(i) every R-module in $\sigma[_RM]$ is a B-module canonically;

(ii) for any K, L in $\sigma[_RM]$, we have $Hom_R(K, L) = Hom_B(K, L)$.

Proof: $(a) \Rightarrow (b)$ Of course, all direct sums ${}_RM^{(\Lambda)}$ are B-modules.

If $U \subset {}_RM^{(\Lambda)}$ is an R-submodule, $u = (m_1, \dots, m_k) \in U$, $m_i \in M$, and $\beta \in B$, then, by (a), there exists $r \in R$ with

$$\beta u = \beta(m_1, \dots, m_k) = r(m_1, \dots, m_k) = ru \in U.$$

Hence U is a B-module. Arbitrary R-modules in $\sigma[{}_RM]$ are of the form U/V with R-submodules U, $V \subset {}_RM^{(\Lambda)}$. Since U and V are B-modules, U/V is also a B-module.

For K, L in $\sigma[{}_RM]$, take $f \in Hom_R(K, L)$ and $a \in K$. Then we have $(a, (a)f) \in K \oplus L$ and, for some $n_1, \dots, n_k \in M$, there is a B-morphism

$$B(n_1, \dots, n_k) \to B(a, (a)f), \;\; (n_1, \dots, n_k) \mapsto (a, (a)f).$$

For every $\beta \in B$, we now find an $r \in R$ with $\beta(a, (a)f) = r(a, (a)f)$ and hence $(\beta a)f = (ra)f = r(a)f = \beta(a)f$. This means $f \in Hom_B(K, L)$.

$(b) \Rightarrow (a)$ For $m_1, \dots, m_k \in M$, the R-module $R(m_1, \dots, m_k) \subset M^k$ is a B-submodule by (b), i.e. $B(m_1, \dots, m_k) \subset R(m_1, \dots, m_k)$ (see 15.7).

If the module M is a generator in $\sigma[M]$, then besides the density theorem we see that M is a flat module over $S = End({}_RM)$ (with respect to *S-MOD*), i.e. the functor $M_S \otimes - : S\text{-}MOD \to \sigma[M]$ is exact. This follows from the more general observation:

15.9 Modules M flat over $End({}_RM)$.

For an R-module M with $S = End({}_RM)$, the following are equivalent:

(a) M_S is flat (with respect to S-MOD);

(b) for every R-morphism $f : M^n \to M^k$, $n, k \in I\!N$, the module $Ke\,f$ is M-generated;

(c) for every R-morphism $f : M^n \to M$, $n \in I\!N$, the module $Ke\,f$ is M-generated.

Proof: $(a) \Rightarrow (b)$ Let M_S be flat and $f : M^n \to M^k$ a morphism. We have the exact sequences $0 \to Ke\,f \to M^n \to M^k$ and

$$0 \to Hom_R(M, Ke\,f) \to Hom_R(M, M^n) \to Hom_R(M, M^k),$$

which yield in a canonical way the commutative exact diagram

$$\begin{array}{ccccccc} 0 \to & M \otimes Hom(M, Ke\,f) & \to & M \otimes Hom(M, M^n) & \to & M \otimes Hom(M, M^k) \\ & \downarrow \mu & & \downarrow \simeq & & \downarrow \simeq \\ 0 \to & Ke\,f & \to & M^n & \to & M^k \end{array} .$$

From this we see that μ is an isomorphism and $Ke\, f$ is M-generated.

$(b) \Rightarrow (c)$ is trivial.

$(c) \Rightarrow (a)$ By 12.16, we have to show that, for every finitely generated left ideal $J = Ss_1 + \cdots + Ss_n \subset S$, the canonical map $\mu_J : M \otimes_S J \to MJ$ is monic. We form the exact sequence

$$0 \longrightarrow K \longrightarrow M^n \stackrel{\sum s_i}{\longrightarrow} MJ \longrightarrow 0$$

and obtain, with the functor $Hom_R(M, -)$, the exact sequence

$$0 \longrightarrow Hom(M, K) \longrightarrow Hom(M, M^n) \stackrel{\sum s_i}{\longrightarrow} J \longrightarrow 0.$$

Applying the functor $M \otimes_S -$ and canonical mappings we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} & & M \otimes_S Hom(M,K) & \longrightarrow & M \otimes_S S^n & \longrightarrow & M \otimes_S J & \longrightarrow & 0 \\ & & \downarrow \mu_K & & \downarrow \simeq & & \downarrow \mu_J & & \\ 0 & \longrightarrow & K & \longrightarrow & M^n & \longrightarrow & MJ & \longrightarrow & 0 \end{array} .$$

Since, by assumption (c), μ_K is epic (observe $MJ \subset M$), the Kernel Cokernel Lemma implies that μ_J is monic.

Dually to 15.9, it can be shown that M_S is *weakly S-injective* (see 16.9, 35.8) if and only if, for every morphism $f : M^n \to M^k$, the module $Coke\, f$ is cogenerated by ${}_RM$ (see 47.7).

Now let us consider the notions just introduced in $\mathbb{Z}$-*MOD*, i.e. for abelian groups. Let M be a $\mathbb{Z}$-module.

M is called a *torsion module (torsion group)* if, for every $a \in M$, there exists a non-zero $n \in \mathbb{N}$ with $na = 0$.

M is called a *p-torsion module (p-group)*, for a prime number p, if, for every $a \in M$, there exists $k \in \mathbb{N}$ with $p^k a = 0$.

The *torsion submodule of* M is defined as

$$t(M) = \{a \in M \mid na = 0 \text{ for some non-zero } n \in \mathbb{N}\},$$

the *p-component of* M is

$$p(M) = \{a \in M \mid p^k a = 0 \text{ for some } k \in \mathbb{N}\}.$$

If $t(M) = 0$, then M is called *torsion free*.
Recall that we use the notation $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$.

15.10 Torsion modules over $\mathbb{Z}$.

(1) Every torsion module M over $\mathbb{Z}$ is a direct sum of its p-components: $M = \bigoplus\{p(M) \mid p$ a prime number$\}$.

(2) The p-component of $\mathbb{Q}/\mathbb{Z}$ is denoted by $\mathbb{Z}_{p^\infty}$ (Prüfer group) and

$$\mathbb{Q}/\mathbb{Z} = \bigoplus\{\mathbb{Z}_{p^\infty} \mid \; p \text{ a prime number}\}.$$

(3) $\sigma[\mathbb{Z}_{p^\infty}] = \sigma[\bigoplus_{\mathbb{N}}\mathbb{Z}_{p^n}]$ is the subcategory of the p-torsion modules in $\mathbb{Z}$-MOD.

$\mathbb{Z}_{p^\infty}$ is a cogenerator and $\bigoplus_{\mathbb{N}}\mathbb{Z}_{p^n}$ is a generator in this category.

(4) $\sigma[\mathbb{Q}/\mathbb{Z}] = \sigma[\bigoplus_{\mathbb{N}}\mathbb{Z}_n]$ is the subcategory of the torsion modules in $\mathbb{Z}$-MOD.

$\mathbb{Q}/\mathbb{Z}$ is a cogenerator and $\bigoplus_{\mathbb{N}}\mathbb{Z}_n$ is a generator in this category.

Proof: (1) First let us show $M = \sum\{p(M) \mid p \textit{ a prime number}\}$:

For $a \in M$, set $An_{\mathbb{Z}}(a) = n\mathbb{Z}$ with $n = p_1^{k_1} \cdots p_r^{k_r}$ for different prime numbers $p_1, \ldots, p_r$.

The numbers $n_i = n/p_i^{k_i}$, $i = 1, \ldots, r$, have greatest common divisor 1, and hence there are $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}$ with $\sum \alpha_i n_i = 1$.

Hence $a = \alpha_1 n_1 a + \cdots + \alpha_r n_r a$ and, by construction, $\alpha_i n_i a \in p_i(M)$.

It remains to show that $\{p(M) \mid p \textit{ prime number}\}$ is an independent family of submodules: Let $a \in p_1(M) \cap (p_2(M) + \cdots + p_l(M))$ for different prime numbers $p_1, \ldots, p_l$. Then $p_1^{k_1} a = 0$ and $p_2^{k_2} \cdots p_l^{k_l} a = 0$ for some $k_i \in \mathbb{N}$.

Since $p_1^{k_1}$ and $p_2^{k_2} \cdots p_l^{k_l}$ are relatively prime, this implies $a = 1a = 0$.

(2) This follows from (1). For further properties of $\mathbb{Z}_{p^\infty}$ see 17.13.

(3) Every cyclic p-torsion module is of the form $\mathbb{Z}_{p^k}$ for some $k \in \mathbb{N}$. Hence $\bigoplus_{\mathbb{N}}\mathbb{Z}_{p^n}$ is a generator for the p-torsion modules. Since the map

$$\mathbb{Z}_{p^k} \to p(\mathbb{Q}/\mathbb{Z}), \;\; z + p^k\mathbb{Z} \mapsto \frac{z}{p^k} + \mathbb{Z}$$

is monic, we may regard the $\mathbb{Z}_{p^k}$ as submodules of $\mathbb{Z}_{p^\infty}$. This yields $\sigma[\mathbb{Z}_{p^\infty}] = \sigma[\bigoplus_{\mathbb{N}}\mathbb{Z}_{p^n}]$.

It will follow from 16.5 and 17.13 that $\mathbb{Z}_{p^\infty}$ is a cogenerator in this category.

(4) This can be shown similarly to the proof of (3).
We will see in 16.7 that $\mathbb{Q}/\mathbb{Z}$ is in fact a cogenerator in $\mathbb{Z}$-MOD.

15.11 Exercises.

(1) Set $R = \begin{pmatrix} I\!R & I\!R \\ 0 & \mathbb{Q} \end{pmatrix}$.
For which left ideals $N \subset R$ *is* $\sigma[N] = R$*-MOD, for which is this not true?*

(2) Let $\{K_\lambda\}_\Lambda$ *be a family of submodules of the* R*-module* M *with* $\bigcap_\Lambda K_\lambda = 0$ *and* $N = \bigoplus_\Lambda M/K_\lambda$. *Show:*
(i) If Λ *is finite, then* $\sigma[M] = \sigma[N]$.
(ii) For infinite Λ *this need not be true (example).*

(3) Let M *be an* R*-module,* $S = End({}_RM)$ *and* $N \in S$*-MOD. Show:*
(i) $M \otimes_S N \in Gen({}_RM)$ *(hence in particular* $\in \sigma[{}_RM]$*).*
(ii) If N is a generator in S-MOD, then $\sigma[{}_RM \otimes_S N] = \sigma[{}_RM]$.

(4) Prove

$$\mathbb{Z}_{p^\infty} = \{q + \mathbb{Z} \mid q \in \mathbb{Q},\ p^k q \in \mathbb{Z} \text{ for some } k \in I\!N\} = \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z},$$

where $\mathbb{Z}[\frac{1}{p}]$ *denotes the subring of* $\mathbb{Q}$ *generated by* $\mathbb{Z}$ *and* $\frac{1}{p}$.

(5) Show that, for a finitely generated $\mathbb{Z}$*-module M, either* $\sigma[M] = \mathbb{Z}$*-MOD or* $\sigma[M] = \mathbb{Z}_n$*-MOD for some* $n \in I\!N$.
What can be said about $\mathbb{Z}$*-modules which are not finitely generated?*

(6) Show that in $\sigma[\mathbb{Q}/\mathbb{Z}]$ *and in* $\sigma[\mathbb{Z}_{p^\infty}]$ *there are no finitely generated subgenerators.*

(7) Show that, for every $\mathbb{Z}$*-module M,* $t(M) = Tr(\bigoplus_{I\!N} \mathbb{Z}_n, M)$.

(8) For a prime number p and $k \in I\!N$ *let*

$$\mathcal{E}(p^k) = \{c \in \mathbb{C} \mid c^{p^k} = 1\} = \{e^{2\pi i \nu p^{-k}} \mid \nu \in I\!N\}$$

be the set of p^k*-th roots of units in* $\mathbb{C}$. *Show:*

(i) $\bigcup_{k \in I\!N} \mathcal{E}(p^k)$ *is a group with respect to multiplication in* $\mathbb{C}$.

(ii) The map $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}$, $q + \mathbb{Z} \mapsto e^{2\pi i q}$, *yields a group isomorphism* $\mathbb{Z}_{p^\infty} \to \bigcup_{k \in I\!N} \mathcal{E}(p^k)$.

(9) Show that in general for an R-module M the category $\sigma[M]$ *is not closed under extensions and products in R-MOD, i.e. for an exact sequence of R-modules* $0 \to K \to L \to N \to 0$, $K, N \in \sigma[M]$ *does not imply* $L \in \sigma[M]$, *and for a family of modules in* $\sigma[M]$ *the product formed in R-MOD need not belong to* $\sigma[M]$.

Literature: Camillo-Fuller, Damiano [2], Fuller [2], Harada [1], Lambek [1], Onodera [3,5], Roux [3], Rowen [1] , Zelmanowitz [3,4], Zimmermann-Huisgen.

16 Injective modules

1.Product of M-injective modules. 2.Properties. 3.Characterizations. 4.Injective modules in R-MOD. 5.Injective cogenerators. 6.Divisible modules. 7.Injective cogenerators in $\mathbb{Z}$-MOD. 8.Injective cogenerators in R-MOD and $\sigma[M]$. 9.Weakly M-injective modules. 10.Direct sums of weakly M-injective modules. 11.Weakly M-injective implies M-generated. 12.Exercises.

Let M and U be two R-modules. U is called *M-injective* if every diagram in *R-MOD* with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & M \\ & & \downarrow & & \\ & & U & & \end{array}$$

can be extended commutatively by a morphism $M \to U$. This property is obviously equivalent to the condition that the map

$$Hom_R(f,U) : Hom_R(M,U) \to Hom_R(K,U)$$

is surjective for every monomorphism $f : K \to M$. Since we already know that the functor $Hom_R(-,U)$ is left exact we have:

U is M-injective if and only if $Hom_R(-,U)$ is exact with respect to all exact sequences $0 \to K \to M \to N \to 0$.

The module U is called *self-injective* (or *quasi-injective*) if it is U-injective.

If $\mathcal{C}$ is a (full) subcategory of *R-MOD*, we call an R-module U *injective for* $\mathcal{C}$ (or *injective in* $\mathcal{C}$ in case $U \in Obj(\mathcal{C})$) if U is M-injective for all $M \in Obj(\mathcal{C})$.

Before proceding to characterizations of M-injective modules let us list some basic facts:

16.1 Product of M-injective modules.

Let M be an R-module and $\{U_\lambda\}_\Lambda$ a family of R-modules.

(1) The product $\prod_\Lambda U_\lambda$ (in R-MOD) is M-injective if and only if every U_λ is M-injective.

(2) If all U_λ are in $\sigma[M]$, then (1) is also true for the product $\prod^M_\Lambda U_\lambda$ in $\sigma[M]$.

Proof: (1) Let $0 \to K \xrightarrow{f} M$ be an exact sequence in *R-MOD*. If all U_λ

are M-injective, then, for every $\mu \in \Lambda$, a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & M \\ & & g\downarrow & & \\ & & \prod_\Lambda U_\lambda & \xrightarrow{\pi_\mu} & U_\mu \end{array}$$

can be extended commutatively by $h_\mu : M \to U_\mu$. Hence we obtain (by the property of products) an $h : M \to \prod_\Lambda U_\lambda$ with $h\pi_\mu = h_\mu$ and $fh\pi_\mu = fh_\mu = g\pi_\mu$, which implies $fh = g$.

On the other hand, if $\prod_\Lambda U_\lambda$ is M-injective, then a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & M \\ & & \gamma\downarrow & & \\ & & U_\mu & \xrightarrow{\varepsilon_\mu} & \prod_\Lambda U_\lambda \end{array}$$

can be extended commutatively by $\delta : M \to \prod_\Lambda U_\lambda$ and $\gamma\varepsilon_\mu = f\delta$ immediately yields $\gamma = \gamma\varepsilon_\mu\pi_\mu = f\delta\pi_\mu$.

While the preceding assertion allows us to construct from M-injective modules further M-injective modules, the next result shows that M-injective modules are also injective with respect to sub- and factor modules of M:

16.2 Properties of injective modules.

(1) If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence in R-MOD and the R-module U is M-injective, then U is also M' and M''-injective.

(2) If the R-module U is M_λ-injective for a family $\{M_\lambda\}_\Lambda$ of R-modules, then U is also $\bigoplus_\Lambda M_\lambda$-injective.

Proof: (1) Let U be M-injective. Show: U is M'-injective.
Every diagram with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & M' \\ & & \downarrow & & \downarrow f \\ & & U & & M \end{array}$$

can be commutatively extended by some $M \to U$.

Show: U is M''-injective. If $0 \to L \xrightarrow{h} M''$ is exact, we obtain, by forming a pullback, the following commutative exact diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow h & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array} .$$

Since U is M-injective, $Hom(-,U)$ yields the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Hom(M'',U) & \longrightarrow & Hom(M,U) & \longrightarrow & Hom(M',U) & \longrightarrow & 0 \\
 & & {\scriptstyle Hom(h,U)}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Hom(L,U) & \longrightarrow & Hom(P,U) & \longrightarrow & Hom(M',U) & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & & .
\end{array}$$

Now the Kernel Cokernel Lemma implies that $Hom(h,U)$ is epic, i.e. U is M''-injective.

(2) Let U be M_λ-injective for all $\lambda \in \Lambda$, $M = \bigoplus_\Lambda M_\lambda$ and $K \subset M$. For a morphism $g : K \to U$, we consider the set

$$\mathcal{F} = \{h : L \to U \mid K \subset L \subset M \text{ and } h|_K = g\}.$$

This set is ordered by

$$[h_1 : L_1 \to U] < [h_2 : L_2 \to U] \Leftrightarrow L_1 \subset L_2 \text{ and } h_2|_{L_1} = h_1.$$

It is easily seen that $\mathcal{F}$ is inductive and, hence by Zorn's Lemma, has a maximal element $h_o : L_o \to U$. To prove $M = L_o$ it is enough to show $M_\lambda \subset L_o$ for all $\lambda \in \Lambda$: Every diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & L_o \cap M_\lambda & \longrightarrow & M_\lambda \\
 & & \downarrow & & \\
 & & L_o & \xrightarrow{h_o} & U
\end{array}$$

can, by assumption, be commutatively extended by some $h_\lambda : M_\lambda \to U$. The assignment

$$h^* : L_o + M_\lambda \to U, \;\; l + m_\lambda \mapsto (l)h_o + (m_\lambda)h_\lambda,$$

is independent of the presentation $l + m_\lambda$ since, for $l + m_\lambda = 0$, we get $l = -m_\lambda \in L_o \cap M_\lambda$ and hence $(l + m_\lambda)h^* = (l)h_o - (l)h_\lambda = 0$.

Therefore $h^* : L_o + M_\lambda \to U$ is a morphism belonging to $\mathcal{F}$ and obviously is larger than $h_o : L_o \to U$.

Because of the maximality of $h_o : L_o \to U$, the morphisms h^* and h_o must be equal and, in particular, $L_o + M_\lambda = L_o$ and $M_\lambda \subset L_o$.

This enables us to prove the following characterizations of M-injective modules:

16.3 M-injective modules. Characterizations.

For R-modules U and M the following are equivalent:

(a) U is M-injective;

(b) U is N-injective for every (finitely generated, cyclic) submodule N of M;

(c) U is N-injective for any $N \in \sigma[M]$ *(i.e. U is injective for* $\sigma[M]$*);*

(d) the functor $Hom(-,U) : \sigma[M] \to AB$ *is exact.*

If U belongs to $\sigma[M]$*, then (a)-(d) are also equivalent to:*

(e) every exact sequence $0 \to U \to L \to N \to 0$ *in* $\sigma[M]$ *splits;*

(f) every exact sequence $0 \to U \to L \to N \to 0$ *in* $\sigma[M]$*, in which N is a factor module of M (or R), splits.*

In this case U is generated by M.

Proof: The equivalence of (a), (b) and (c) readily follows from 16.2 (M is generated by its cyclic submodules).

$(c) \Leftrightarrow (d)$ follows from the definition of M-injective.

$(c) \Rightarrow (e)$ If U is injective for $\sigma[M]$, then every diagram in $\sigma[M]$ with exact row

$$\begin{array}{ccccccccc} 0 & \to & U & \to & L & \to & N & \to & 0 \\ & & {\scriptstyle id}\downarrow & & & & & & \\ & & U & & & & & & \end{array}$$

can be extended commutatively by an $L \to U$, i.e. the sequence splits.

$(e) \Rightarrow (f)$ is trivial.

$(f) \Rightarrow (a)$ From a diagram with exact row $\begin{array}{ccccc} 0 & \to & K & \to & M \\ & & \downarrow & & \\ & & U & & \end{array}$

we get – forming the pushout – the commutative exact diagram in $\sigma[M]$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & U & \longrightarrow & Q & \longrightarrow & M/K & \longrightarrow & 0 \end{array} .$$

Since the second row splits because of (f), we get the morphism desired by $M \to Q \to U$.

If all the sequences $0 \to U \to L \to N \to 0$ with cyclic modules N are splitting, the same proof tells us that U is L-injective for all cyclic submodules L of M.

Since every module in $\sigma[M]$ is a submodule of an M-generated module, the injective modules U in $\sigma[M]$ are direct summands of M-generated modules and hence M-generated.

In particular, for *R-MOD* we get from 16.3:

16.4 Injective modules in *R-MOD* (Baer's Criterion).

For an R-module U the following properties are equivalent:

(a) U is injective in R-MOD;

(b) U is R-injective;

(c) for every left ideal $I \subset R$ and every morphism $h : I \to U$, there exists $u \in U$ with $(a)h = au$ for all $a \in I$.

Proof: It remains to show $(b) \Leftrightarrow (c)$.

$(b) \Rightarrow (c)$ If U is R-injective, then there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "h"'] & R \arrow[dl, "h'"] \\ & U & \end{tikzcd}$$

Putting $u = (1)h'$ we get $(a)h = (a)h' = a(1)h' = au$ for all $a \in I$.

$(c) \Rightarrow (b)$ Assume, for a morphism $h : I \to U$, that there exists $u \in U$ with $(a)h = au$ for all $a \in I$. Then $h' : R \to U$, $r \mapsto ru$, is obviously an extension of $h : I \to U$.

Before proving the existence of injective cogenerators in $\sigma[M]$ we want to find out which properties they have:

16.5 Injective cogenerators.

An injective module Q in $\sigma[M]$ is a cogenerator in $\sigma[M]$ if and only if it cogenerates every simple module in $\sigma[M]$, or equivalently, Q contains every simple module in $\sigma[M]$ as a submodule (up to isomorphism).

Proof: Assume that every simple module in $\sigma[M]$ is Q-cogenerated. For every non-zero morphism $f : L \to N$ in $\sigma[M]$ we have to find an $h : N \to Q$ with $fh \neq 0$. For an element $l \in L$ with $(l)f \neq 0$ and the inclusion $i : Rl \to L$, the composed morphism $if : Rl \to N$ is not zero and there is a maximal submodule $K \subset Rl$ with $Ke\,if \subset K$ (see 6.7). As a simple module in $\sigma[M]$, Rl/K is (isomorphic to) a submodule of Q, and with the projection $p : Rl/Ke\,if \to Rl/K$ we obtain the exact diagram

$$\begin{tikzcd} 0 \arrow[r] & Rl/Ke\,if \arrow[r] \arrow[d, "p"'] & N \\ 0 \arrow[r] & Rl/K \arrow[r] & Q \end{tikzcd} .$$

This can be extended to a commutative diagram by some $h : N \to Q$. Since $ifh \neq 0$, also $fh \neq 0$.

Injective cogenerators in R-*MOD* can be constructed by way of injective cogenerators in $\mathbb{Z}$-*MOD*. In $\mathbb{Z}$-*MOD* we note the fact that injectivity can be characterized by a further property. We first define for any ring R:

An R-module N is called *divisible* if, for every $s \in R$ which is not a zero divisor and every $n \in N$, there exists $m \in N$ with $sm = n$.

16.6 Divisible modules. Properties.

(1) Every R-injective module is divisible.

(2) Every factor module of a divisible module is divisible.

(3) If every left ideal is cyclic in R and R has no non-trivial zero divisors, then an R-module is injective if and only if it is divisible.

Proof: (1) Assume that ${}_RN$ is R-injective, $n \in N$ and $s \in R$ is not a zero divisor. Then for $n \in N$, the map $h : Rs \to N$, $rs \mapsto rn$, is a morphism and, by 16.4, there exists $m \in N$ with $n = (s)h = sm$.

(2) is easily verified.

(3) Let ${}_RN$ be divisible and $h : Rs \to N$, $0 \neq s \in R$, a morphism. There is an $m \in N$ with $sm = (s)h \in N$, and hence $(a)h = am$ for all $a \in Rs$, i.e. N is injective by 16.4.

By 16.6, a $\mathbb{Z}$-module is $\mathbb{Z}$-injective if and only if it is divisible. Hence ${}_{\mathbb{Z}}\mathbb{Q}$ (rational numbers) and ${}_{\mathbb{Z}}\mathbb{R}$ (real numbers) are examples of $\mathbb{Z}$-injective modules, however they are not cogenerators.

16.7 Injective cogenerators in $\mathbb{Z}$-*MOD*.

$\mathbb{Q}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ are injective cogenerators in $\mathbb{Z}$-MOD.

Proof: As factor modules of divisible modules, $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ are divisible and hence $\mathbb{Z}$-injective. The simple $\mathbb{Z}$-modules are of the form $\mathbb{Z}/p\mathbb{Z}$, p a prime number, and the mappings

$$\mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}, \quad z + p\mathbb{Z} \mapsto \frac{z}{p} + \mathbb{Z},$$

are monomorphisms. Hence, by 16.5, $\mathbb{Q}/\mathbb{Z}$ is a cogenerator in $\mathbb{Z}$-*MOD*. Since $\mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$ the same is true for $\mathbb{R}/\mathbb{Z}$.

To obtain injective cogenerators in R-*MOD* we consider, for an abelian group B, the morphism set $Hom_{\mathbb{Z}}(R_R, B)$ as a left R-module:

For $s \in R$ and $f \in Hom_{\mathbb{Z}}(R_R, B)$ the multiplication sf is defined by

$(r)sf = (rs)f$, for all $r \in R$.

Regarding the ring R as a $(\mathbb{Z}, R)$-bimodule, we obtain from the Hom-tensor relations 12.12, for any $N \in R\text{-}MOD$ (since $N \simeq R \otimes_R N$) the isomorphisms

$$\bar{\psi}_N : Hom_{\mathbb{Z}}(N, B) \simeq Hom_{\mathbb{Z}}(R \otimes_R N, B) \xrightarrow{\psi_N} Hom_R({}_RN, Hom_{\mathbb{Z}}(R_R, B))\,,$$

and, for every morphism $f : N \to N'$, the commutative diagram

$$\begin{array}{ccc}
Hom_{\mathbb{Z}}(N', B) & \xrightarrow{Hom(f,B)} & Hom_{\mathbb{Z}}(N, B) \\
\downarrow \bar{\psi}_{N'} & & \downarrow \bar{\psi}_N \\
Hom_R(N', Hom_{\mathbb{Z}}(R, B)) & \xrightarrow{Hom(f,Hom(R,B))} & Hom_R(N, Hom_{\mathbb{Z}}(R, B))
\end{array}$$

with isomorphisms $\bar{\psi}_{N'}$ and $\bar{\psi}_N$. If B is an injective cogenerator in $\mathbb{Z}$-*MOD*, the functor $Hom_{\mathbb{Z}}(-, B)$ is exact and reflects zero morphisms. From the diagram above we see that this is also true for the functor

$$Hom_R(-, Hom_{\mathbb{Z}}(R, B)) : R\text{-}MOD \to AB.$$

Hence ${}_RHom_{\mathbb{Z}}(R, B)$ is R-injective and a cogenerator (see 16.4, 14.6).

16.8 Injective cogenerators in R-MOD and $\sigma[M]$.
Let M be an R-module.

(1) ${}_RHom_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z})$ is an injective cogenerator in R-MOD.

(2) If Q is an injective module in R-MOD, then Tr(M,Q) is an injective module in $\sigma[M]$. If Q is an injective cogenerator in R-MOD, then Tr(M,Q) is an injective cogenerator in $\sigma[M]$.

(3) Every R-module (in $\sigma[M]$) is a submodule of an injective module (in $\sigma[M]$). We say: There are enough injectives in R-MOD and $\sigma[M]$.

Proof: (1) has been outlined in the preceding remarks.

(2) $Tr(M, Q)$ is M-injective: Every diagram with exact row

$$\begin{array}{ccccc}
0 & \longrightarrow & U & \longrightarrow & M \\
 & & \downarrow & & \\
 & & Tr(M, Q) & \subset & Q
\end{array}$$

can be extended commutatively by an $h : M \to Q$ and we obtain $Im\, h \subset Tr(M, Q)$. Hence $Tr(M, Q)$ is M-injective and injective in $\sigma[M]$. If Q is a cogenerator in R-*MOD*, all simple modules in $\sigma[M]$ are contained in $Tr(M, Q)$.

(3) Every module cogenerated by the injective module Q (resp. $Tr(M,Q)$) is a submodule of an injective module Q^Λ (resp. $Tr(M,Q^\Lambda)$ in $\sigma[M]$).

In general a direct sum of M-injective modules need not be M-injective. We are going to introduce a weaker notion of injectivity which is closed under forming direct sums. Its importance will become clear in the investigation of finiteness conditions and also in the study of pure exact sequences.

16.9 Weakly M-injective modules. Definition.

Let M and U be R-modules. U is called *weakly M-injective* if every diagram in R-MOD

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & M^{(I\!N)} \\ & & \downarrow & & \\ & & U & & \end{array}$$

with exact row and K *finitely generated*, can be extended commutatively by a morphism $M^{(I\!N)} \to U$, i.e. $Hom(-,U)$ is exact with respect to the given row.

If $M = R$, then weakly R-injective modules are also called *FP-injective*. Here 'FP' abbreviates 'finitely presented'. The meaning of this notation will become clear in § 25. It follows from the proof of 16.6,(1) that FP-injective modules are divisible.

As for 'M-injective' (see 16.1) it is easily seen that a product of modules is weakly M-injective if and only if this is true for every component. Moreover we now get:

16.10 Direct sums of weakly M-injective modules.
For every R-module M we have:

(1) The direct sum of any family of weakly M-injective R-modules $\{U_\lambda\}_\Lambda$ is weakly M-injective.

(2) If $U_1 \subset U_2 \subset \ldots$ is an ascending chain of weakly M-injective submodules of a module N, then $\bigcup_{I\!N} U_i$ is also weakly M-injective.

Proof: (1) Let K be a finitely generated submodule of $M^{(I\!N)}$ and $f : K \to \bigoplus_\Lambda U_\lambda$ a morphism. Then $(K)f$ is a finitely generated submodule of $\bigoplus_\Lambda U_\lambda$, i.e. there is a finite subset $E \subset \Lambda$ with $(K)f \subset \bigoplus_E U_\lambda$.

Since $\bigoplus_E U_\lambda$ is a direct product of weakly M-injective modules and hence is weakly M-injective, there exists a morphism $M^{(I\!N)} \to \bigoplus_E U_\lambda$ with the desired properties.

(2) can be seen by the same proof as (1).

Weakly M-injective modules have the following property which we already know for injective modules in $\sigma[M]$:

16.11 Weakly M-injective implies M-generated.
Let M be an R-module. Every weakly M-injective module in $\sigma[M]$ is M-generated.

Proof: Let $N \in \sigma[M]$ be weakly M-injective and $K \subset M^{(\mathbb{N})}$ finitely generated. Then $Tr(K,N) \subset Tr(M,N)$. Since the finitely generated submodules of $M^{(\mathbb{N})}$ form a set of generators in $\sigma[M]$ we get

$$N = \sum\{Tr(K,N) \mid K \subset M^{(\mathbb{N})},\ K \text{ finitely generated}\} = Tr(M,N).$$

Hence N is generated by M.

16.12 Exercises.

(1) Let U, M be R-modules. Show that the following assertions are equivalent:
(a) U is M-injective;
(b) every morphism $f : I \to U$, I a left ideal in R, with $R/Ke\, f \in \sigma[M]$, can be extended to R.

(2) Show that for an R-module M the following are equivalent (see 20.3):
(a) every module in $\sigma[M]$ is self-injective;
(b) every module in $\sigma[M]$ is M-injective.

(3) Let M be a self-injective R-module. Show:
(i) For a simple R-module N, the group $Hom_R(N,M)$ is zero or a simple right module over End(M).
(ii) M is injective in the category $\sigma[{}_BM]$ with B = Biend(M).
(iii) For every right ideal I in R, the set $An_M(I) = \{m \in M \mid Im = 0\}$ is a self-injective submodule of M.

(4) Show that, for a self-injective R-module M with S = End(M), the following assertions are equivalent:
(a) $\sigma[M] = R/An(M)$-MOD;
(b) M_S is finitely generated.

(5) Let M be an R-module and $\mathcal{F}$ the set (filter) of left ideals $J \subset R$ with $R/J \in \sigma[M]$. Show that, for any R-module N, the following assertions are equivalent:
(a) N is injective with respect to exact sequences $0 \to J \to R$ with $J \in \mathcal{F}$;
(b) N is injective with respect to exact sequences $0 \to K \to L$ with

$L/K \in \sigma[M]$;
(c) every exact sequence $0 \to N \to L$ in R-MOD with $L/N \in \sigma[M]$ splits.
(Hint: see proof of 16.2,(2).)

Literature: ALBU-NĂSTĂSESCU, FAITH [1,2];
Alamelu, Azumaya-Mbuntum, Beachy-Weakley [2], Bican-Jambor, Birkenmeier [1,3], Cailleau-Renault, Döman-Hauptfleisch, Goodearl [1], Hill [1], Hiremath [1], Jain-Singh S., Kraemer, Lambek [1], Li-Zelmanowitz, Martin, Ramamurthi-Rangaswamy [2], de Robert, de la Rosa-Viljoen, Roux [3], Singh [1], Smith [3,4], Stenström, Tsukerman, Tuganbaev [1,4-9], Vamos [3], Yue [2].

17 Essential extensions, injective hulls

1.Definitions. 2.Essential monomorphisms. 3.Essential extensions. 4.Direct sums of essential extensions. 5.Complements. 6.Complements and essential submodules. 7.Complement of a complement. 8.Injective hulls. 9.Existence. 10.Properties. 11.Self-injective modules. 12.Characterization of cogenerators. 13.Injective hulls of simple $\mathbb{Z}$-modules. 14.Characterization of M-flat modules. 15.Exercises.

Having seen in the preceding paragraph that every module in $\sigma[M]$ (resp. *R-MOD*) is a submodule of an injective module in $\sigma[M]$ (resp. *R-MOD*), we will show now that every module N is contained in a 'smallest' injective module in $\sigma[M]$, resp. *R-MOD*. Of course, this module can only be unique up to isomorphism. Fundamental for our investigations are the

17.1 Definitions. A submodule K of an R-module M is called *essential* or *large in* M if, for every non-zero submodule $L \subset M$, we have $K \cap L \neq 0$.

Then M is called an *essential extension of* K and we write $K \trianglelefteq M$. A monomorphism $f : L \to M$ is said to be *essential* if $Im\, f$ is an essential submodule of M.

Hence a submodule $K \subset M$ is essential if and only if the inclusion $K \to M$ is an essential monomorphism. For example, in $\mathbb{Z}$ every non-zero submodule (=ideal) is essential. We will come across further examples later. An interesting categorical characterization of essential monomorphisms is given in

17.2 Essential monomorphisms.

A monomorphism $f : L \to M$ in R-MOD is essential if and only if, for every (epi-)morphism $h : M \to N$ in R-MOD (or $\sigma[M]$):

fh monic implies that h is monic.

Proof: $\Rightarrow$ Let f be essential and fh monic.
Then $0 = Ke\, fh = (Ke\, h \cap Im\, f)f^{-1}$, implying $Ke\, h \cap Im\, f = 0$ and hence $Ke\, h = 0$.

$\Leftarrow$ Assume that f has the given property and $K \subset M$ is a submodule with $Im\, f \cap K = 0$. With the canonical projection $p : M \to M/K$ the composition $L \xrightarrow{f} M \xrightarrow{p} M/K$ is monic. Then, by assumption, p is monic and $K = 0$.

17.3 Properties of essential extensions.
Let K, L and M be R-modules.

(1) If $K \subset L \subset M$, then $K \trianglelefteq M$ if and only if $K \trianglelefteq L \trianglelefteq M$.

(2) Two monomorphisms $f : K \to L$, $g : L \to M$ are essential if and only if fg is essential.

(3) If $h : K \to M$ is a morphism and $L \trianglelefteq M$, then $(L)h^{-1} \trianglelefteq K$, i.e. the preimage of an essential submodule is an essential submodule .

(4) If $K_1 \trianglelefteq L_1 \subset M$ and $K_2 \trianglelefteq L_2 \subset M$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.

(5) The intersection of two (finitely many) essential submodules is an essential submodule in M.

Proof: (2) If f, g are essential monomorphisms and $h : M \to M'$ is a morphism with fgh monic, then gh and also h is monic, i.e. fg is essential.

Let fg be an essential monomorphism. If $h : M \to M'$ is a morphism with gh monic, then fgh is monic and hence h is monic. Therefore g is essential.

For any $k : L \to L'$ with fk monic, we form the pushout diagram

$$\begin{array}{ccccc} K & \xrightarrow{f} & L & \xrightarrow{g} & M \\ & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle p_2} \\ & & L' & \xrightarrow{p_1} & P \end{array} \quad .$$

With g also p_1 and $fkp_1 = fgp_2$ are monic. Therefore p_2 is monic (fg essential) and $gp_2 = kp_1$ implies that k is monic.

(1) follows from (2) applied to the inclusions $K \to L$, $L \to M$. It also can be shown directly.

(3) Assume $U \subset K$. If $(U)h = 0$ then $U \subset Ke\,h \subset (L)h^{-1}$. In case $(U)h \neq 0$ we get $(U)h \cap L \neq 0$. Then there is a non-zero $u \in U$ with $(u)h \in L$ and $0 \neq u \in U \cap (L)h^{-1}$, i.e. $(L)h^{-1}$ is essential in K.

(4) For $0 \neq X \subset L_1 \cap L_2$ we get $0 \neq X \cap K_1 \subset L_2$ and

$$0 \neq (X \cap K_1) \cap K_2 = X \cap (K_1 \cap K_2), \text{ i.e. } K_1 \cap K_2 \trianglelefteq L_1 \cap L_2.$$

(5) is an immediate consequence of (4).

For later use we want to prove the following assertion about essential submodules:

17.4 Direct sums of essential submodules.

Let $\{K_\lambda\}_\Lambda$ and $\{L_\lambda\}_\Lambda$ be families of submodules of the R-module M. If $\{K_\lambda\}_\Lambda$ is an independent family of submodules in M and $K_\lambda \trianglelefteq L_\lambda$ for all $\lambda \in \Lambda$, then $\{L_\lambda\}_\Lambda$ also is an independent family and $\bigoplus_\Lambda K_\lambda \trianglelefteq \bigoplus_\Lambda L_\lambda$.

Proof: If $K_1 \trianglelefteq L_1$, $K_2 \trianglelefteq L_2$ are submodules of M with $K_1 \cap K_2 = 0$, then, by 17.3, $0 \trianglelefteq L_1 \cap L_2$, i.e. $L_1 \cap L_2 = 0$. Applying 17.3,(3) to the projections $L_1 \oplus L_2 \to L_1$ and $L_1 \oplus L_2 \to L_2$, we obtain the relations $K_1 \oplus L_2 \trianglelefteq L_1 \oplus L_2$ and $L_1 \oplus K_2 \trianglelefteq L_1 \oplus L_2$ and then, by 17.3,(4),

$$K_1 \oplus K_2 = (K_1 \oplus L_2) \cap (L_1 \oplus K_2) \trianglelefteq L_1 \oplus L_2.$$

Hereby we have shown the assertion of 17.4 for families with two elements and, by induction, we get it for families with finitely many elements. For an arbitrary index set Λ, a family $\{L_\lambda\}_\Lambda$ is independent if every finite subfamily is independent and this is what we have just proved.

For any non-zero $m \in \bigoplus_\Lambda L_\lambda$, we have $m \in \bigoplus_E L_\lambda$ for some finite subset $E \subset \Lambda$. Since $\bigoplus_E K_\lambda \trianglelefteq \bigoplus_E L_\lambda$ we get $0 \neq Rm \cap \bigoplus_E K_\lambda \subset Rm \cap \bigoplus_\Lambda K_\lambda$. Hence the intersection of a non-zero submodule of $\bigoplus_\Lambda L_\lambda$ with $\bigoplus_\Lambda K_\lambda$ is again non-zero, i.e. $\bigoplus_\Lambda K_\Lambda \trianglelefteq \bigoplus_\Lambda L_\lambda$.

Remarks: (1) From 17.4 we have: If $\{K_\lambda\}_\Lambda$, $\{L_\lambda\}_\Lambda$ are families of R-modules with $K_\lambda \trianglelefteq L_\lambda$ for all $\lambda \in \Lambda$, then we have, for external direct sums, $\bigoplus_\Lambda K_\lambda \trianglelefteq \bigoplus_\Lambda L_\lambda$.

(2) The intersection of a family of essential submodules of M need not be essential in M. For example, in $\mathbb{Z}$ we have $\mathbb{Z}n \trianglelefteq \mathbb{Z}$, for all $n \in \mathbb{N}$, but $\bigcap_{n \in \mathbb{N}} \mathbb{Z}n = 0$ and hence is not essential in $\mathbb{Z}$.

Important non-trivial cases of essential extensions can be constructed applying Zorn's Lemma. For this the following notion is useful:

17.5 Complements. Definition.

Let K be a submodule of the R-module N. A submodule $K' \subset N$ is called an *(intersection) complement* of K in N if it is maximal in the set of submodules $L \subset N$ with $K \cap L = 0$.

Since the set of submodules $L \subset N$ with $K \cap L = 0$ is not empty and inductive (with respect to inclusion), by Zorn's Lemma every submodule $K \subset N$ has complements. In general these are not uniquely determined.

If $L \subset N$ is a submodule with $K \cap L = 0$, then we may find a complement K' of K in N with $L \subset K'$. The complements of $K \subset N$ are zero if and only if $K \trianglelefteq N$.

If K is a direct summand of N, i.e. $N = K \oplus L$, then L is a complement of K in N: Assume $L \subset L' \subset N$ with $L \neq L'$ and $l' \in L' \setminus L$. Then $l' = l + k$ with $l \in L$, $k \in K$, and $0 \neq l' - l \in L' \cap K$.

We know that also in this case L is unique only up to isomorphism.

The connection between essential extensions and complements is illuminated in

17.6 Complements and essential submodules.

Let K be a submodule of an R-module N and K' a complement of K in N. Then

$$(K+K')/K' \trianglelefteq N/K' \text{ and } K+K' \trianglelefteq N.$$

Proof: For a submodule $L \subset N$ with $K' \subset L \subset N$ and $L/K' \cap ((K+K')/K') = 0$, we get, by modularity,

$$(K+K') \cap L = (K \cap L) + K' \subset K',$$

hence $K \cap L \subset K'$, $K \cap L \subset K \cap K' = 0$ and finally $L = K'$ because of the maximality of K'. This means $(K+K')/K' \trianglelefteq N/K'$. As preimage of the essential submodule $(K+K')/K'$ of N/K' (under the canonical projection $p : N \to N/K'$), $K+K'$ is essential in N (see 17.3).

17.7 Complement of a complement.

Let K be a submodule of the R-module N, K' a complement of K in N, and K'' a complement of K' in N with $K \subset K''$. Then

(1) K' is a complement of K'' in N.

(2) K'' is a maximal essential extension of K in N, i.e. K'' is maximal in the set $\{L \subset N \mid K \trianglelefteq L\}$.

(3) If N is self-injective, then $N = K' \oplus K''$.

Proof: (1) If L is a submodule of N with $K' \subset L$ and $L \cap K'' = 0$, then also $L \cap K = 0$, i.e. $L = K'$ by definition of K'. Hence K' is maximal with respect to $K'' \cap K' = 0$ and therefore a complement of K'' in N.

(2) First we show $K \trianglelefteq K''$: Let $U \subset K''$ be a submodule with $U \cap K = 0$. For $k = u + k' \in K \cap (U + K')$, with $u \in U$, $k' \in K'$, we have $k - u = k' \in K'' \cap K' = 0$, hence $k \in K \cap U = 0$ and $K \cap (U + K') = 0$. By maximality of K', we derive $K' + U = K'$ and $U \subset K' \cap K'' = 0$. Hence $K \trianglelefteq K''$.

K'' is maximal in $\{L \subset N \mid K \trianglelefteq L\}$: Assume $K'' \subset L \subset N$ with $K \trianglelefteq L$. Then $(L \cap K') \cap K = 0$. Since $K \trianglelefteq L$, this means $L \cap K' = 0$ and (by definition of K'') we conclude $L = K''$.

(3) Since $K' \cap K'' = 0$, the composition of the inclusion map $i : K'' \to N$ and the projection $p : N \to N/K'$ is monic. N being self-injective, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K'' & \xrightarrow{ip} & N/K' \\ & & i\downarrow & & \\ & & N & & \end{array}$$

can be extended commutatively by an $h : N/K' \to N$, i.e. $iph = i$. Since $Im\, ip = (K'' + K')/K'$ is essential in N/K' by 17.6, the morphism h must be monic and $(N/K')h \simeq N/K'$ is an essential extension of K'' and K. By (2), this implies $(N/K')h = K''$ and therefore ip is a coretraction and hence an isomorphism yielding $N = K' + K''$.

17.8 Injective hulls. Definition.

Let N be a module in $\sigma[M]$, $M \in R$-*MOD*. An injective module E in $\sigma[M]$ (or R-*MOD*) together with an essential monomorphism $\varepsilon : N \to E$ is called an *injective hull (envelope) of* N *in* $\sigma[M]$ *(resp. R-MOD).*

The injective hull of N in $\sigma[M]$ is also called an *M-injective hull of N* and is usually denoted by $\widehat{N}$. With this terminology the injective hull of N in R-*MOD* is the *R-injective hull* and often denoted by $E(N)$. In general $\widehat{N} \neq E(N)$.

Recalling that every module is a submodule of an injective module (in $\sigma[M]$ resp. R-*MOD*), the existence of injective hulls is derived from 17.7:

17.9 Existence of injective hulls. *Let M be an R-module.*

(1) Every module N in $\sigma[M]$ *has an injective hull* $\widehat{N}$ *in* $\sigma[M]$.

(2) Every module N in R-MOD has an injective hull E(N) in R-MOD. If $N \in \sigma[M]$, *then* $\widehat{N} \simeq Tr(M, E(N))$.

(3) The injective hulls of a module (in $\sigma[M]$ *or R-MOD) are unique up to isomorphism.*

Proof: (1) Assume $N \in \sigma[M]$, and let Q be an injective module in $\sigma[M]$ with $N \subset Q$ (see 16.8). If N' is a complement of N in Q and N'' a complement of N' with $N'' \supset N$, then, by 17.7, $N \trianglelefteq N''$ and N'' is a direct summand of Q and hence injective in $\sigma[M]$. Therefore the inclusion $i : N \to N''$ is an M-injective hull of N.

(2) For $M = R$ we obtain the injective hulls in R-*MOD* from (1).

For $N \in \sigma[M]$ let $E(N)$ be an injective hull of N in R-*MOD* with $N \subset E(N)$. Then $N \subset Tr(M, E(N)) \subset E(N)$ and hence $N \trianglelefteq Tr(M, E(N))$. By 16.8, $Tr(M, E(N))$ is M-injective and $N \to Tr(M, E(N))$ is an injective hull of N in $\sigma[M]$. It will follow from (3) that $\widehat{N} \simeq Tr(M, E(N))$.

(3) Assume that $\varepsilon_1 : N \to \widehat{N}_1$ and $\varepsilon_2 : N \to \widehat{N}_2$ are injective hulls of N in $\sigma[M]$. Then there exists $f : \widehat{N}_1 \to \widehat{N}_2$ with $\varepsilon_1 f = \varepsilon_2$. Since ε_1 is essential, f is monic. Hence $(\widehat{N}_1)f (\simeq \widehat{N}_1)$ is an injective and essential submodule of $\widehat{N}_2$, i.e. $(\widehat{N}_1)f = \widehat{N}_2$ and f is epic.

The following properties will be needed frequently:

17.10 Properties of injective hulls. *Let M be an R-module and L, N in $\sigma[M]$ with M-injective hulls $\widehat{L}$, $\widehat{N}$, respectively.*

(1) If $L \trianglelefteq N$, then $\widehat{L} \simeq \widehat{N}$. In particular, N is isomorphic to a submodule of $\widehat{L}$.

(2) If $L \subset N$ and N is M-injective, then $\widehat{L}$ is isomorphic to a direct summand of N.

(3) If, for a family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M]$, the direct sum $\bigoplus_\Lambda \widehat{N}_\lambda$ is M-injective, then $\bigoplus_\Lambda \widehat{N}_\lambda$ is an M-injective hull of $\bigoplus_\Lambda N_\lambda$.

Proof: (1) $L \trianglelefteq N$ implies $L \trianglelefteq \widehat{N}$.

(2) The diagram
$$\begin{array}{ccccc} 0 & \rightarrow & L & \xrightarrow{\varepsilon} & \widehat{L} \\ & & i\downarrow & & \\ & & N & & \end{array}$$
can be extended to a commutative diagram by some $f : \widehat{L} \rightarrow N$. Since f is monic, $(\widehat{L})f$ $(\simeq \widehat{L})$ is an injective submodule of N and hence a direct summand.

(3) By 17.4, $\bigoplus_\Lambda N_\lambda \trianglelefteq \bigoplus_\Lambda \widehat{N}_\lambda$.

Assertion (1) in 17.10 characterizes the injective hull $\widehat{L}$ as a 'maximal' essential extension of L. Observe that we could not construct it directly as a maximal element in the totality of all essential extensions of L, since this need not be a set. Because of (2), $\widehat{L}$ may be regarded as a 'minimal' injective extension of L.

17.11 Self-injective modules.

Let M be an R-module, $\widehat{M}$ the M-injective hull of M in $\sigma[M]$, $E(M)$ the R-injective hull of M in R-MOD and $M \subset \widehat{M} \subset E(M)$.

(1) The following assertions are equivalent:
(a) M is M-injective;
(b) $M = \widehat{M}$;
(c) $M = MEnd_R(E(M))$, i.e. M is a fully invariant submodule of $E(M)$.

(2) Fully invariant submodules of self-injective modules are again self-injective.

Proof: (1) $(a) \Leftrightarrow (b)$ is clear since, for modules in $\sigma[M]$, 'M-injective' and 'injective in $\sigma[M]$' are equivalent properties.

$(a) \Rightarrow (c)$ If M is self-injective we get, by 17.9,

$$M = Tr(M, E(M)) = MHom_R(M, E(M)) = MEnd_R(E(M)).$$

$(c) \Rightarrow (a)$ If M is fully invariant in $E(M)$, i.e. $M = MEnd_R(E(M))$, then the above equation yields $M = Tr(M, E(M))$ and, by 16.8, M is M-injective.

(2) Let L be a fully invariant submodule of the self-injective module N, $K \subset L$ and $f : K \to L$ a morphism, i.e.

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & L \subset N \\ & & {\scriptstyle f}\downarrow & & \\ & & L & \subset & N \quad . \end{array}$$

Then there exists $g : N \to N$ with $g|_K = f$. By assumption we get $(L)g \subset L$. Hence L is self-injective.

By 16.5, an M-injective module Q is a cogenerator in $\sigma[M]$ if and only if it contains all simple modules in $\sigma[M]$. We see from 17.10 that in this case Q also contains their M-injective hulls as submodules. We will show instantly that this is a characterizing property for every cogenerator. Since every simple R-module E is isomorphic to a factor module R/N for a maximal left ideal $N \subset R$, the *set*

$$\{R/N \mid N \text{ is a maximal left ideal in } R\}$$

is a representing set for the simple modules in *R-MOD*. For the simple modules in $\sigma[M]$ we obtain a representing set as a subset of this set. A representing set is called *minimal* if any two distinct elements are not isomorphic.

17.12 Characterization of cogenerators.
Let M be a non-zero R-module, $\{E_\lambda\}_\Lambda$ a minimal representing set of the simple modules in $\sigma[M]$ and $\widehat{E}_\lambda$ the M-injective hull of E_λ, $\lambda \in \Lambda$.

(1) A module $Q \in \sigma[M]$ is a cogenerator in $\sigma[M]$ if and only if it contains, for every $\lambda \in \Lambda$, a submodule isomorphic to $\widehat{E}_\lambda$.

(2) $\{\widehat{E}_\lambda\}_\Lambda$ is a set of cogenerators in $\sigma[M]$.

(3) Every cogenerator in $\sigma[M]$ contains a submodule isomorphic to $\bigoplus_\Lambda \widehat{E}_\lambda$ (which is a 'minimal' cogenerator).

Proof: (1) Let Q be a cogenerator in $\sigma[M]$. Since the $\widehat{E}_\lambda$ are cocyclic modules (see 14.8) we get, for every $\lambda \in \Lambda$, a monomorphism $\widehat{E}_\lambda \to Q$.

Now assume that the module $Q \in \sigma[M]$ contains all $\widehat{E}_\lambda$ as submodules. Since every cocyclic module is an essential extension of some E_λ (see 14.8), it is a submodule of $\widehat{E}_\lambda$ (see 17.10). But every module is cogenerated by its cocyclic factor modules and hence is cogenerated by Q.

(2) follows from (1) (see 14.3).

(3) Let Q be a cogenerator in $\sigma[M]$. Because of (1), we can assume $\widehat{E}_\lambda \subset Q$, for every $\lambda \in \Lambda$, and also $\sum_\Lambda \widehat{E}_\lambda \subset Q$. For every element $\lambda \in \Lambda$, we get $\widehat{E}_\lambda \cap \sum_{\lambda \neq \lambda'} \widehat{E}_{\lambda'} = 0$ (otherwise $E_\lambda \subset \sum_{\lambda\neq\lambda'} \widehat{E}_{\lambda'}$, hence $E_\lambda \subset \widehat{E}_{\lambda'}$ for some $\lambda' \neq \lambda$, contradicting the minimality of $\{E_\lambda\}_\Lambda$). Therefore $\{\widehat{E}_\lambda\}_\Lambda$ is an independent family and $\bigoplus_\Lambda \widehat{E}_\lambda = \sum_\Lambda \widehat{E}_\lambda \subset Q$.

Observe that $\bigoplus_\Lambda \widehat{E}_\lambda$ need not be injective.

We already have seen that $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in $\mathbb{Z}$-*MOD*. Now let us determine injective hulls of simple $\mathbb{Z}$-modules, $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, p prime number, which are embedded into $\mathbb{Q}/\mathbb{Z}$ by $z + p\mathbb{Z} \mapsto \frac{z}{p} + \mathbb{Z}$ (see 16.7). By definition, the p-components of $\mathbb{Q}/\mathbb{Z}$ are just the Prüfer groups $\mathbb{Z}_{p^\infty}$ (see 15.10).

17.13 Injective hulls of simple $\mathbb{Z}$-modules.

(1) For any prime number $p \in \mathbb{N}$, $\mathbb{Z}_{p^\infty}$ is an injective hull of $\mathbb{Z}_p$.

(2) Every proper submodule K of $\mathbb{Z}_{p^\infty}$ is finite: there exists $n \in \mathbb{N}$ such that K is generated by $\frac{1}{p^n} + \mathbb{Z}$.

(3) $\mathbb{Z}_{p^\infty} = \sum_{n\in\mathbb{N}} \mathbb{Z}(\frac{1}{p^n} + \mathbb{Z}) = \bigcup_{n\in\mathbb{Z}} \mathbb{Z}(\frac{1}{p^n} + \mathbb{Z}) \subset \mathbb{Q}/\mathbb{Z}$.

(4) For any submodules K_1, K_2 of $\mathbb{Z}_{p^\infty}$, $K_1 \subset K_2$ or $K_2 \subset K_1$.

(5) In $\mathbb{Z}_{p^\infty}$ there are infinite ascending chains of submodules.
Every descending chain of submodules is finite.

(6) Every non-zero factor module of $\mathbb{Z}_{p^\infty}$ is isomorphic to $\mathbb{Z}_{p^\infty}$.

(7) Every non-zero proper submodule of $\mathbb{Z}_{p^\infty}$ is self-injective but not $\mathbb{Z}$-injective.

Proof: (1) As a direct summand of $\mathbb{Q}/\mathbb{Z}$, the module $\mathbb{Z}_{p^\infty}$ is divisible.

$\mathbb{Z}_p \simeq \{\frac{z}{p} + \mathbb{Z} \mid z \in \mathbb{Z}\}$ is a submodule of $\mathbb{Z}_{p^\infty}$ and is contained in every proper submodule of $\mathbb{Z}_{p^\infty}$ (see (2)).

(2) Let K be a proper submodule of $\mathbb{Z}_{p^\infty}$, and choose $n \in \mathbb{N}$ such that

$$\frac{1}{p^n} + \mathbb{Z} \in K \text{ but } \frac{1}{p^{n+1}} + \mathbb{Z} \notin K.$$

For any element $\frac{k}{p^m} + \mathbb{Z} \in K$ with $k \in \mathbb{Z}$, p not dividing k, and $m \in \mathbb{N}$, we can find $r, s \in \mathbb{Z}$ with $kr + p^m s = 1$. This yields

$$\frac{1}{p^m} + \mathbb{Z} = \frac{kr + p^m s}{p^m} + \mathbb{Z} = r(\frac{k}{p^m} + \mathbb{Z}) \in K.$$

By the choice of n, this means $m \leq n$ and $K = \mathbb{Z}(\frac{1}{p^n} + \mathbb{Z})$.

(3)-(6) are easily derived from (2).

(7) follows from 17.11 since, by (2), every submodule of $\mathbb{Z}_{p^\infty}$ is fully invariant.

We can use our knowledge about injective cogenerators to obtain new characterizations of (M-) flat modules:

Let ${}_SU_R$ be an (S,R)-bimodule, ${}_RM$ in R-*MOD* and ${}_SN$ in S-*MOD*. The group $Hom_S({}_SU_R, {}_SN)$ becomes a left R-module by defining (see 12.12)

$$(u)rf := (ur)f \text{ for } f \in Hom_S(U,N),\ r \in R,\ u \in U.$$

Referring to the Hom-tensor relations 12.12 we can show:

17.14 Characterization of M-flat modules.

Let R and S be rings, ${}_RM$ in R-MOD, ${}_SU_R$ an (S,R)-bimodule and ${}_SD$ an injective cogenerator for $\sigma[{}_SU]$. Then the following are equivalent:

(a) U_R is M-flat;

(b) U_R is N-flat for any $N \in \sigma[M]$;

(c) ${}_RHom_S(U,D)$ is (weakly) ${}_RM$-injective.

Proof: $(a) \Leftrightarrow (b)$ follows from the properties 12.15 of M-flat modules (and is only stated for completeness' sake).

$(a) \Leftrightarrow (c)$ For an exact sequence $0 \to K \to M$ (with K finitely generated), we obtain, by 12.12, the commutative diagram

$$\begin{array}{ccccc} Hom_S(U\otimes_R M, D) & \longrightarrow & Hom_S(U\otimes_R K, D) & \longrightarrow & 0 \\ \downarrow\simeq & & \downarrow\simeq & & \\ Hom_R(M, Hom_S(U,D)) & \longrightarrow & Hom_R(K, Hom_S(U,D)) & \longrightarrow & 0 \end{array}.$$

Observe that for any R-module ${}_RN$ we have $U\otimes_R N \in \sigma[{}_SU]$.

If U_R is M-flat, then the first row is exact (${}_SD$ is injective). Hence the second row also has to be exact, i.e. ${}_RHom_S(U,D)$ is M-injective.

On the other hand, if ${}_RHom_S(U,D)$ is weakly M-injective, then, for finitely generated K, the second – and also the first – row is exact. Since ${}_SD$ is a cogenerator for $\sigma[{}_SU]$ we see from 14.6 that $0 \to U\otimes K \to U\otimes M$ is exact, i.e. U is M-flat.

17.15 Exercises.

(1) Show for R-modules U, M:
U is M-injective if and only if it is injective with respect to exact sequences $0 \to K \xrightarrow{f} M$ with essential monomorphism f.

(2) Let N be a finitely cogenerated R-module and L an essential extension of N. Show that L is also finitely cogenerated.

(3) Prove: (i) A direct sum of self-injective R-modules need not be self-injective.

(ii) For an R-module M the following assertions are equivalent:
(a) the direct sum of two self-injective modules in $\sigma[M]$ is self-injective.
(b) every self-injective module in $\sigma[M]$ is M-injective.

(4) Let $\{p_i\}_{\mathbb{N}}$ be a family of different prime numbers and $\{l_i\}_{\mathbb{N}}$ a family of non-zero natural numbers.

Show that $\bigoplus_{\mathbb{N}} \mathbb{Z}_{p_i^{l_i}}$ is self-injective.

(5) Let N be a finitely generated torsion module over $\mathbb{Z}$. Show that N has an N-injective direct summand.

(6) Show that, for a torsion module U over $\mathbb{Z}$, the following assertions are equivalent:
(a) U is $\mathbb{Z}$-injective;
(b) U is $\mathbb{Q}/\mathbb{Z}$-injective;
(c) U is injective in the category of torsion modules.

(7) Show that, for a torsion $\mathbb{Z}$-module N, the following are equivalent:
(a) N is $\mathbb{Z}$-injective;
(b) N is $\mathbb{Z}_{p^\infty}$-injective for every prime number p;
(c) every exact sequence $0 \to N \to L \to \mathbb{Q}/\mathbb{Z} \to 0$ in $\mathbb{Z}$-MOD splits.

(8) Let M be a torsion free $\mathbb{Z}$-module ($t(M) = 0$). Show that the M-injective (= $\mathbb{Z}$-injective) hull $\widehat{M} \simeq M \otimes_{\mathbb{Z}} \mathbb{Q}$.

(9) Consider $M = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ as a $\mathbb{Z}_4$-module. Prove:
(i) $U = \mathbb{Z}_4(2,2)$ is a self-injective submodule which is not M-injective;
(ii) U is the intersection of injective submodules of M;
(iii) M contains more than one copy of the $\mathbb{Z}_4$-injective hull of U.

(10) Show that, for a self-injective R-module M, the following assertions are equivalent:
(a) every submodule of M has exactly one M-injective hull in M;
(b) the intersection of any two M-injective submodules of M is again M-injective.

(11) Show that, for a self-injective R-module M, the following assertions are equivalent:
(a) M is a cogenerator in $\sigma[M]$;
(b) M is a self-cogenerator (= cogenerates all its factor modules).
(Hint: Comp. 18.5.)

(12) Show for a ring R:

(i) The following assertions are equivalent:

(a) $_RR$ *is finitely cogenerated;*

(b) every cogenerator is a subgenerator in R-MOD;

(c) every faithful R-module is a subgenerator in R-MOD.

(ii) $_RR$ *is injective if and only if every subgenerator is a generator in R-MOD.*

(13) Show for an R-module M:

(i) M is a cogenerator in $\sigma[M]$ *if and only if, for every finitely cogenerated module* $N \in \sigma[M]$*, the M-injective hull* $\widehat{N}$ *is a direct summand of* $M^{(\mathbb{N})}$*.*

(ii) If M is a cogenerator in $\sigma[M]$ *and is finitely generated, then the M-injective hulls of simple modules in* $\sigma[M]$ *are finitely generated.*

(14) Let R be an integral domain with quotient field Q. Show that $_RQ$ *is an injective hull of R.*

Literature: ALBU-NĂSTĂSESCU, ANDERSON-FULLER; Enochs [2], Müller-Rizvi, Roux [3].

18 Projective modules

1.Direct sums of M-projective modules. 2.Properties. 3.Projective modules in $\sigma[M]$. 4.Further properties. 5.M-projective generators in $\sigma[M]$. 6.Projectives in R-MOD. 7.Trace ideals of projective modules. 8.Generators in R-MOD. 9.Faithful modules over commutative R. 10.Trace ideal for commutative R. 11.Projective generators over commutative R. 12.$\sigma[M]$ without projectives. 13.Exercises.

Definitions and basic properties of projective modules are dual to those of injective modules (see § 16). However, we shall also encounter problems of a different type and not all assertions about injective modules can be dualized.

Let M and P be R-modules. P is called *M-projective* if every diagram in *R-MOD* with exact row

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

can be extended commutatively by a morphism $P \to M$. This condition is equivalent to the surjectivity of the map

$$Hom_R(P, g) : Hom_R(P, M) \to Hom_R(P, N)$$

for every epimorphism $g : M \to N$. Since the functor $Hom_R(P, -)$ is always left exact we have:

P is M-projective if and only if $Hom_R(P, -)$ is exact with respect to all exact sequences $0 \to K \to M \to N \to 0$.

If P is P-projective, then P is also called *self-* (or *quasi-) projective*. Dually to 16.1 we can show:

18.1 Direct sums of projective modules.
Assume $M \in$ R-MOD and that $\{U_\lambda\}_\Lambda$ is a family of R-modules. The direct sum $\bigoplus_\Lambda U_\lambda$ is M-projective if and only if every U_λ is M-projective.

Again dual to the corresponding proof for injective modules we obtain the first assertion in the following proposition. The other assertions demand their own proofs:

18.2 Properties of projective modules. *Let P be an R-module.*

(1) If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence in R-MOD and P is M-projective, then P is M'- and M''-projective.

(2) If P is M_i-projective for finitely many modules $M_1, \ldots, M_k$, then P is also $\bigoplus_{i=1}^k M_i$-projective.

(3) If P is finitely generated and M_λ-projective for any family $\{M_\lambda\}_\Lambda$ of R-modules, then P is also $\bigoplus_\Lambda M_\lambda$-projective.

(4) If P is self-projective and $K \subset P$ is a fully invariant submodule, then P/K is also self-projective.

Proof: (2) Let P be M_i-projective, for $i = 1, 2$, and $g : M_1 \oplus M_2 \to N$ an epimorphism. With a pushout and Lemma 10.6 we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{\varepsilon_1} & M_1 \oplus M_2 & \xrightarrow{\pi_2} & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow g & & \downarrow & & \\
0 & \longrightarrow & N_1 & \longrightarrow & N & \longrightarrow & N_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

Observing that the first row splits and that P is M_1- and M_2-projective, the functor $Hom_R(P, -)$ yields the following commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \to & Hom(P, M_1) & \to & Hom(P, M_1 \oplus M_2) & \to & Hom(P, M_2) & \to & 0 \\
 & & \downarrow & & \downarrow Hom(P,g) & & \downarrow & & \\
0 & \to & Hom(P, N_1) & \to & Hom(P, N) & \to & Hom(P, N_2) & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & & .
\end{array}$$

By the Kernel Cokernel Lemma, $Hom(P, g)$ is surjective, i.e. P is $M_1 \oplus M_2$-projective.

By induction we obtain the assertion for all finite index sets.

(3) Let P be finitely generated and M_λ-projective, $\lambda \in \Lambda$. In every diagram with exact row

$$\begin{array}{ccccc}
 & & P & & \\
 & & \downarrow h & & \\
\bigoplus_\Lambda M_\lambda & \xrightarrow{g} & N & \longrightarrow & 0
\end{array}$$

the image $(P)h$ is finitely generated. Hence there is a finite subset $E \subset \Lambda$

which leads to the following diagram with exact row

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow h & & \\ \bigoplus_E M_\lambda & \xrightarrow{g} & N' & \longrightarrow & 0 \quad \text{with} \quad (P)h \subset N' \subset N\,. \end{array}$$

By (2), this can be extended commutatively. This also yields the desired extension of the first diagram.

(4) is shown dually to 17.11,(2).

A module in $\sigma[M]$ is called *projective in* $\sigma[M]$ if it is N-projective for every $N \in \sigma[M]$. From the preceding results we get:

18.3 Projective modules in $\sigma[M]$. Characterizations.

For R-modules P and M the following assertions are equivalent:

(a) P is $M^{(\Lambda)}$-projective for every index set Λ;

(b) P is N-projective for every $N \in \sigma[M]$;

(c) the functor Hom$(P,-) : \sigma[M] \to AB$ is exact.

If P is finitely generated, then (a)-(c) is also equivalent to:

(d) P is M-projective.

If P is in $\sigma[M]$, then (a)-(c) are equivalent to:

(e) P is projective in $\sigma[M]$;

(f) every exact sequence $0 \to K \to N \to P \to 0$ in $\sigma[M]$ splits.

If P is finitely generated and in $\sigma[M]$, then (a)-(f) are equivalent to:

(g) every exact sequence $0 \to K' \to N \to P \to 0$ in $\sigma[M]$ with $K' \subset M$ splits;

(h) every exact sequence $0 \to K \to N' \to P \to 0$ in $\sigma[M]$ with N' finitely generated splits.

Proof: The equivalence of (a), (b) (and (d) if P is finitely generated) follows from 18.2. The equivalence of (b), (c) and (e) is immediately derived from the definitions. For $P \in \sigma[M]$ the implication $(b) \Leftrightarrow (e)$ is obvious.

$(e) \Rightarrow (f)$ is seen from the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \| & & \\ N & \to & P & \to & 0. \end{array}$$

$(f) \Rightarrow (e)$ is shown dually to the proof $(f) \Rightarrow (a)$ of 16.3 (pullback). Also $(g) \Rightarrow (d)$ is obtained by forming a suitable pullback.

$(h) \Rightarrow (f)$ P is an epimorphic image of a finitely generated submodule N' of N.

18.4 Projective modules. Further properties.
Let M be an R-module and $S = End({}_RM)$.

(1) Every projective module in $\sigma[M]$ is a direct summand of a direct sum of finitely generated submodules of $M^{(\mathbb{N})}$.

(2) Every projective module in $\sigma[M]$ is isomorphic to a submodule of a direct sum $M^{(\Lambda)}$.

(3) Let M be self-projective and N in R-MOD:

(i) For every finitely generated S-submodule $I \subset Hom_R(M,N)$, we have $I = Hom_R(M, MI)$;

(ii) if M is finitely generated, then (i) holds for every S-submodule $I \subset Hom_R(M,N)$.

(4) Let M be projective in $\sigma[M]$ and $N \in \sigma[M]$. Then, for any submodules L_1, $L_2 \subset N$,

$$Hom(M, L_1 + L_2) = Hom(M, L_1) + Hom(M, L_2).$$

Proof: (1) and (2) follow immediately from 18.3 since the finitely generated submodules of $M^{(\mathbb{N})}$ are a set of generators in $\sigma[M]$.

(3)(i) Assume $I = \sum_{i=1}^{k} Sf_i$, $f_i \in I$. We may I regard as a subset of $Hom_R(M, MI)$. For every $g \in Hom_R(M, MI)$, the diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow g & & \\ M^k & \xrightarrow{\sum f_i} & MI & \longrightarrow & 0 \end{array}$$

can be commutatively extended by an $h = (h_1, \dots, h_k) : M \to M^k$, $h_i \in S$, i.e. $g = \sum_{i=1}^{k} h_i f_i \in I$. Hence $I = Hom_R(M, MI)$.

(ii) Assume $I = \sum_{\lambda\in\Lambda} Sf_\lambda$ and $g \in Hom_R(M, MI)$. Then $(M)g$ is a finitely generated submodule of $MI = \sum_{\lambda\in\Lambda} Mf_\lambda$ and hence is contained in a finite partial sum $\sum_{i=1}^{k} Mf_{\lambda_i}$. The rest follows from (i).

(4) This is derived from the diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow & & \\ L_1 \oplus L_2 & \longrightarrow & L_1 + L_2 & \longrightarrow & 0. \end{array}$$

Similarly to injective cogenerators (see 16.5), projective generators can also be characterized in various ways:

18.5 M-projective generators in $\sigma[M]$.
Let M be an R-module, $P \in \sigma[M]$ and $S = End({}_RP)$.

(1) If P is M-projective, then the following are equivalent:
(a) P is a generator in $\sigma[M]$;
(b) $Hom_R(P,E) \neq 0$ for every simple module $E \in \sigma[M]$;
(c) P generates every simple module in $\sigma[M]$;
(d) P generates every submodule of M.

(2) If P is finitely generated and a generator in $\sigma[M]$, then the following properties are equivalent:
(a) P is M-projective;
(b) P_S is faithfully flat.

Proof: (1) $(a) \Rightarrow (b) \Rightarrow (c)$ and $(a) \Rightarrow (d)$ are obvious.

$(d) \Rightarrow (c)$ We show that every simple module E in $\sigma[M]$ is a homomorphic image of a submodule of M: Since the M-injective hull $\widehat{E}$ of E is M-generated, there is (at least) one non-zero $f \in Hom(M, \widehat{E})$. For this f we get $E \subset (M)f$ and E is a homomorphic image of $(E)f^{-1} \subset M$.

$(c) \Rightarrow (a)$ It is enough to show that P generates every finitely generated submodule of $N \subset M^{(\mathbb{N})}$:

By 18.2, P is N-projective. Assume $Tr(P,N) \neq N$. Then there is a maximal submodule $K \subset N$ with $Tr(P,N) \subset K$ (see 6.7). By (c), there is an epimorphism $f : P \to N/K$ and, since P is N-projective, we obtain, with p the canonical projection, the commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & {}^{h}\swarrow & \downarrow f & & \\ N & \xrightarrow{p} & N/K & \to & 0 \end{array} .$$

From this we derive $(P)h \subset Tr(P,N) \subset K$, i.e. $f = hp = 0$, contradicting the choice of f. Hence we get $Tr(P,N) = N$.

$(2)(a) \Rightarrow (b)$ P_S is flat by 15.9. By 18.4, we have, for every proper left ideal $I \subset S$, the equality $Hom(P,PI) = I$, i.e. $PI \neq P$ and hence P_S is faithfully flat by 12.17.

$(b) \Rightarrow (a)$ Since P is finitely generated and a generator in $\sigma[M]$ it suffices to show that $Hom_R(P,-)$ is exact with respect to exact sequences of the form

$$P^n \xrightarrow{\sum f_i} P \longrightarrow 0,\ n \in \mathbb{N},\ f_i \in S,$$

i.e. the following sequence has to be exact:

$$Hom_R(P,P^n) \xrightarrow{Hom(P,\sum f_i)} Hom_R(P,P) \longrightarrow 0 .$$

Set ${}_SK = Coke\,(Hom(P, \sum f_i))$. Tensoring with $P_S \otimes -$ we obtain the commutative exact diagram with canonical isomorphisms μ_1, μ_2,

$$\begin{array}{ccccccc} P \otimes_S Hom(P, P^n) & \longrightarrow & P \otimes_S Hom(P, P) & \longrightarrow & P \otimes_S K & \longrightarrow & 0 \\ \downarrow \mu_1 & & \downarrow \mu_2 & & & & \\ P^n & \xrightarrow{\sum f_i} & P & \longrightarrow & 0 & & . \end{array}$$

From this we get $P \otimes_S K = 0$ and hence $K = 0$ because of (b). Therefore the above sequence is exact and P is M-projective.

We have seen in § 16 that there are enough injectives in $\sigma[M]$, i.e. every module is a submodule of an injective module in $\sigma[M]$. It is easy to see that – dually – there are enough projectives in R-*MOD* (if $1 \in R$!). However, this need not be true in $\sigma[M]$ (see 18.12).

18.6 Projectives in *R-MOD*. *Let R be a ring (with unit !).*

(1) ${}_RR$ is a projective generator in R-MOD and hence every R-module is an epimorphic image of a projective (free) R-module.

(2) For an R-module P, the following assertions are equivalent:
(a) P is projective in R-MOD;
(b) P is isomorphic to a direct summand of $R^{(\Lambda)}$, Λ an index set;
(c) there are elements $\{p_\lambda \in P \mid \lambda \in \Lambda\}$ and $\{f_\lambda \in Hom(P, R) \mid \lambda \in \Lambda\}$, such that for every $p \in P$:
(i) $(p)f_\lambda \neq 0$ for only finitely many $\lambda \in \Lambda$, and
(ii) $p = \sum (p)f_\lambda p_\lambda$ (dual basis).

Proof: (1) We know from § 13 that R is a (finitely generated) generator in R-*MOD*. To see that R is projective in R-*MOD*, by 18.3, it is enough to show that every epimorphism $f : N \to R$ in R-*MOD* splits: For an $n_1 \in N$ with $(n_1)f = 1$, we get a morphism $h : R \to N$, $r \mapsto rn_1$, $r \in R$, with $hf = id_R$.

(2) $(a) \Leftrightarrow (b)$ follows immediately from (1).

$(b) \Rightarrow (c)$ We have the mappings

$$R \xrightarrow{\varepsilon_\lambda} R^{(\Lambda)} \xrightarrow{g} P, \qquad R \xleftarrow{\pi_\lambda} R^{(\Lambda)} \xleftarrow{f} P$$

with $\sum_\Lambda \pi_\lambda \varepsilon_\lambda = id_{R^{(\Lambda)}}$ and $fg = id_P$. Putting $f_\lambda = f\pi_\lambda$ and $p_\lambda = (1)\varepsilon_\lambda g$ we get, for all $p \in P$,

$$p = (p)fg = \sum_\Lambda (p)f\pi_\lambda\varepsilon_\lambda g = \sum_\Lambda (p)f_\lambda((1)\varepsilon_\lambda g) = \sum_\Lambda (p)f_\lambda p_\lambda,$$

where $(p)f\pi_\lambda \neq 0$ for only finitely many $\lambda \in \Lambda$.

$(c) \Rightarrow (b)$ The $f_\lambda \in Hom(P,R)$ define a map $f: P \to R^\Lambda$ (property of products). Because of (i) we get $(P)f \subset R^{(\Lambda)}$. On the other hand, the mappings $R \to Rp_\lambda \subset P$ yield a morphism $g: R^{(\Lambda)} \to P$. From (ii) we derive $fg = id_P$ and hence P is isomorphic to a direct summand of $R^{(\Lambda)}$.

For every R-module M, the trace $Tr(M,R)$ of M in R is a two-sided ideal in R (see 13.5). It is called the *trace ideal of M in R*.

M is a generator in *R-MOD* if and only if $Tr(M,R) = R$.

Referring to the dual basis of projective modules, we obtain the following properties of trace ideals:

18.7 Trace ideals of projective modules.

If the R-module P is projective in R-MOD, then:
(1) $Tr(P,R)P = P$;
(2) $Tr(P,R)^2 = Tr(P,R)$ (idempotent ideal).

Proof: (1) In the representation $p = \sum_\Lambda (p)f_\lambda p_\lambda$ (see 18.6,(2)) all the $(p)f_\lambda$ are in $Tr(P,R)$.

(2) From the same representation we derive that, for every $g \in \mathrm{Hom}(P,R)$, we have $(p)g = \sum_\Lambda (p)f_\lambda (p_\lambda)g \in Tr(P,R)^2$.

Observe that for *self-projective* modules M, the trace ideal $Tr(M,R)$ need no longer have the properties given in 18.7. For example, for a simple (hence self-projective) module M we may have $Tr(M,R) = 0$.

The description of generators in *R-MOD* given in 13.7 can be extended by another useful

18.8 Characterization of a generator in *R-MOD*.

Let G be an R-module and $S = End_R(G)$.
G is a generator in R-MOD if and only if
(i) G_S is finitely generated and S-projective, and
(ii) $R \simeq Biend_R(G)$ $(:= End_S(G_S))$.

Proof: $\Rightarrow$ If G is a generator in *R-MOD*, then $G^k \simeq R \oplus K$ for some $k \in \mathbb{N}$ (see 13.7). The functor $Hom_R(-,G)$ yields the S-isomorphisms

$$S^k \simeq Hom_R(G^k, G) \simeq Hom_R(R,G) \oplus Hom_R(K,G) \simeq G_S \oplus Hom_R(K,G).$$

Hence G_S is finitely generated and projective as an S-module. Now the density theorem implies that $R \simeq Biend_R(G)$ (see 15.7).

$\Leftarrow$ Assume that for G, (i) and (ii) are true. Then by 18.6, for some $n \in \mathbb{N}$ we have $S^n \simeq G_S \oplus Q$ with $Q \in MOD\text{-}S$, and the functor $Hom_S(-, G_S)$ yields the R-isomorphisms

$$G^n \simeq Hom_S(S^n, G_S) \simeq Hom_S(G_S, G_S) \oplus Hom_S(Q, G_S).$$

By (ii), we have $R \simeq Hom_S(G_S, G_S)$, i.e. R is isomorphic to a direct summand of G^n and G is a generator in R-MOD.

Over commutative rings the characterization of projective generators becomes especially straightforward due to the following two propositions:

18.9 Faithful modules over commutative *R*.

Let R be a commutative ring and M a finitely generated R-module.

(1) If I is an ideal of R with $IM = M$, then there exists $r \in I$ such that $(1-r)M = 0$.

(2) If M is a faithful R-module, then M generates all simple modules in R-MOD.

Proof: (1) Let $m_1 \ldots, m_k$ be a generating set of M. Since $IM = M$, for every $i = 1, \ldots, k$ we can find elements $r_{ij} \in I$ with

$$m_i = \sum_{j=1}^{k} r_{ij} m_j.$$

This means $\sum_{j=1}^{k} (\delta_{ij} - r_{ij}) m_j = 0$ for $i = 1, \ldots, k$, and this can be written as multiplication of matrices

$$(\delta_{ij} - r_{ij}) \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix} = 0\,.$$

Multiplying from the left with the adjoint matrix of $(\delta_{ij} - r_{ij})$, we obtain $det(\delta_{ij} - r_{ij}) m_l = 0$ for $l = 1, \ldots, k$ und hence $det(\delta_{ij} - r_{ij})M = 0$.

The expression $det(\delta_{ij} - r_{ij})$ is of the form $1 - r$ for some $r \in I$.

(2) Any simple R-module is isomorphic to R/m for some maximal ideal m of R. Then, by (1), $M/mM \neq 0$ is a non-trivial vector space over the field R/m. As a direct summand of M/mM, the module R/m is a factor module of M/mM and hence M-generated.

18.10 Trace ideal for commutative R.

Let R be a commutative ring and P a projective R-module. If

(i) P is finitely generated, or

(ii) R is noetherian,

then the trace ideal Tr(P,R) is generated by an idempotent $e \in R$ and $An_R(P) = R(1-e)$, i.e.

$$R = Tr(P,R) \oplus An_R(P).$$

Proof: By assumption, there is a splitting sequence $R^k \to P \to 0$ with $k \in \mathbb{N}$. Then the sequence $0 \to Hom_R(P,R) \to R^k$ also splits and hence $Hom_R(P,R)$ is finitely generated (and projective) as an R-module.

This implies that $Tr(P,R) = P Hom_R(P,R)$ is finitely P-generated as an R-module. Therefore, in case (i) or (ii), $T = Tr(P,R)$ is finitely generated.

By 18.7, we have $TP = P$ and $T^2 = T$. The last equation, together with 18.9, implies the existence of an $e \in T$ with $T(1-e) = 0$, i.e. $e(1-e) = 0$ and $e^2 = e$, $T = Te = Re$. This yields $R(1-e) \subset An_R(P)$.

Moreover we observe $T An_R(P) = 0$:
Every $t \in T$ is of the form $t = \sum (p_i) f_i$ with $p_i \in P$, $f_i \in Hom(P,R)$. For $s \in An_R(P)$, we get $st = \sum (sp_i) f_i = 0$ which implies $R(1-e) \supset An_R(P)$.

18.11 Projective generators over commutative R.

Let R be a commutative ring, P a non-zero projective R-module. Assume

(i) P is finitely generated and faithful, or

(ii) P is finitely generated, R contains no non-trivial idempotents, or

(iii) P is faithful and R is noetherian.

Then P is a generator in R-MOD.

Proof: In all cases we get from 18.10 the equality $R = Tr(P,R)$.

For arbitrary $M \in$ *R-MOD* we cannot say anything about the existence of projective modules in $\sigma[M]$. In fact there need not exist any:

18.12 $\sigma[M]$ without projective objects.

In $\sigma[\mathbb{Q}/\mathbb{Z}]$, the category of torsion modules over $\mathbb{Z}$, and in $\sigma[\mathbb{Z}_{p^\infty}]$, the category of p-torsion modules over $\mathbb{Z}$ (see 15.10), there are no non-zero projective objects.

Proof: Assume $N = \mathbb{Z}^{(\Lambda)}/K$, Λ a index set and $K \subset \mathbb{Z}^{(\Lambda)}$, is a projective module in $\sigma[\mathbb{Q}/\mathbb{Z}]$. Choose a non-zero $a \in \mathbb{Z}^{(\Lambda)} \setminus K$, a prime number p and $k \in \mathbb{N}$ such that $p^k a \notin K$ but $p^{k+1} a \in K$. Then $p^{k+1} a \notin$

pK. Since $\mathbb{Z}^{(\Lambda)}/pK$ is a torsion module, by assumption the sequence with canonical mappings

$$0 \longrightarrow K/pK \longrightarrow \mathbb{Z}^{(\Lambda)}/pK \xrightarrow{\alpha} \mathbb{Z}^{(\Lambda)}/K \longrightarrow 0$$

splits. Hence there exists $\beta : \mathbb{Z}^{(\Lambda)}/K \to \mathbb{Z}^{(\Lambda)}/pK$ with $\beta\alpha = id_{\mathbb{Z}^{(\Lambda)}/K}$.

Now $a \in (a)\alpha\beta + Ke\,\alpha$ and $a + pK \subset (a + pK)\alpha\beta + Ke\,\alpha$, which implies $p^{k+1}a \in pK$, a contradiction.

The proof for $\sigma[\mathbb{Z}_{p^\infty}]$ is similar.

In case we need projective modules in $\sigma[M]$ in the future, we have to ensure their existence by appropriate assumptions (e.g. M finitely generated and R commutative, compare 15.3).

18.13 Exercises.

(1) Let M be an R-module and P a finitely generated module in $\sigma[M]$. Show: P is M-projective if and only if every exact sequence $0 \to K \to N \to P \to 0$ in $\sigma[M]$ with $K \subset M$ splits.

(2) Let p, q be different prime numbers. Show:

(i) For natural numbers k,n with $k \leq n-1$ we have: $\mathbb{Z}_{p^k}$ is self-projective but not $\mathbb{Z}_{p^n}$-projective (as a $\mathbb{Z}$-module).

(ii) For arbitrary k, $n \in \mathbb{N}$, the module $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{q^n}$ is self-projective.

(iii) For $n \in \mathbb{N}$, the module $\mathbb{Z}_n$ is self-projective but not $\mathbb{Z}$-projective.

(3) Show that, for a finitely generated torsion module M over $\mathbb{Z}$, the following assertions are equivalent:

(a) M is self-injective;

(b) M is self-projective;

(c) for any prime number p, the p-component of M is zero or isomorphic to $(\mathbb{Z}_{p^k})^r$ for k, $r \in \mathbb{N}$.

(4) Prove that $\mathbb{Q}/\mathbb{Z}$ and – for every prime number p – $\mathbb{Z}_{p^\infty}$ are not self-projective.

(5) Show:

(i) In the category of finitely generated $\mathbb{Z}$-modules there are enough projective but no non-trivial injective objects.

(ii) In the category of finitely generated torsion modules over $\mathbb{Z}$ there are no non-zero projective and no non-zero injective objects.

(6) Show for a ring R: ${}_RR$ is a cogenerator in R-MOD if and only if, for every finitely cogenerated ${}_RX$ in R-MOD, the R-injective hull ${}_R\widehat{X}$ is projective (see 17.15,(13)).

(7) Show: For an R-module M the following are equivalent (see 20.3):
(a) Every module is projective in $\sigma[M]$*;*
(b) every module is self-projective in $\sigma[M]$*;*
(c) every module is injective in $\sigma[M]$*.*

(8) Let M be a finitely generated, projective R-module, $S = End(M)$ *and* $B = Biend(M)$*. Show:*
(i) ${}_BM$ *is a finitely generated, projective B-module.*
(ii) M_S *is a generator in MOD-S.*

(9) Let M and N be R-modules. N is called strongly M-projective *if N is* M^Λ*-projective for every index set* Λ *(product in R-MOD).*

Show that N is strongly M-projective if and only if $N/An(M)N$ *is projective in* $R/An(M)$*-MOD.*

Literature: ALBU-NĂSTĂSESCU, ANDERSON-FULLER, KASCH; Azumaya-Mbuntum, Beck [2], Bican-Jambor, Bland, Colby-Rutter, Feigelstock-Raphael, Fuller-Hill, Garcia-Gomez [4], Harada [6], Hauptfleisch-Döman, Hill [1,5], Hiremath [2,6,7], Jirásková-Jirásko, McDonald, Miller, Nită, Rangaswamy [4], Rangaswamy-Vanaja [2], de Robert, Singh [1], Tiwary-Ghaubey, Tuganbaev [2], Ware, Whitehead, Zimmermann [2].

19 Superfluous epimorphisms, projective covers

1.Definitions. 2.Superfluous epimorphisms. 3.Properties of superfluous submodules. 4.Projective cover. 5.Properties. 6.Superfluous submodules and finitely generated modules. 7.Projective covers of simple modules. 8.Local rings. 9.Indecomposable injective modules. 10.Exercises.

Dual to essential extensions (monomorphisms) introduced in § 17 we define:

19.1 Definitions. A submodule K of an R-module M is called *superfluous* or *small in M*, written $K \ll M$, if, for every submodule $L \subset M$, the equality $K + L = M$ implies $L = M$.

An epimorphism $f : M \to N$ is called *superfluous* if $Ke\, f \ll M$.

Obviously $K \ll M$ if and only if the canonical projection $M \to M/K$ is a superfluous epimorphism.

It is easy to see that e.g. in $\mathbb{Z}$ there are no non-zero superfluous submodules. On the other hand, any nil (left) ideal I in a ring R is superfluous as a left module: Assume $R = I + L$ for some left ideal $L \subset R$. Then $1 = i + l$ for suitable $i \in I$, $l \in L$, and hence, for some $k \in \mathbb{N}$, we get

$0 = i^k = (1 - l)^k = 1 + l'$ for some $l' \in L$, so that $1 \in L = R$.

As a categorical characterization we obtain dually to 17.2:

19.2 Superfluous epimorphisms.

An epimorphism $f : M \to N$ in R-MOD is superfluous if and only if every (mono) morphism $h : L \to M$ in R-MOD (or $\sigma[M]$) with hf epic is epic.

Proof: $\Rightarrow$ If hf is epic and $m \in M$, then there exists $l \in L$ with $(m)f = (l)hf$, which means $m = (l)h + (m - (l)h) \in Im\, h + Ke\, f$ and hence $M = Im\, h + Ke\, f$. Now $Ke\, f \ll M$ implies $M = Im\, h$.

$\Leftarrow$ Assume $L \subset M$ with $L + Ke\, f = M$. With the inclusion $i : L \to M$ the map if is epic. By the given property, i has to be epic, i.e. $L = M$.

19.3 Properties of superfluous submodules.

Let K, L, N and M be R-modules.

(1) If $f : M \to N$ and $g : N \to L$ are two epimorphisms, then fg is superfluous if and only if f and g are superfluous.

(2) If $K \subset L \subset M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.

(3) If $K_1, \ldots, K_n$ are superfluous submodules of M, then $K_1 + \cdots + K_n$ is also superfluous in M.

(4) For $K \ll M$ and $f : M \to N$ we get $Kf \ll N$.

(5) If $K \subset L \subset M$ and L is a direct summand in M, then $K \ll M$ if and only if $K \ll L$.

Proof: (1) is seen dually to the proof of 17.3,(2).

(2) follows from (1) with the canonical mappings $M \to M/K \to M/L$.

(3) is obtained by induction.

(4) Assume $X \subset N$ with $Kf + X = N$. Then $M = K + Xf^{-1} = Xf^{-1}$ (since $K \ll M$), hence $Kf \subset X$ and $X = N$.

(5) follows from (4) with canonical mappings $L \to M$ and $M \to L$.

Dual to the notion of an injective hull of a module we define:

19.4 Projective cover. Definition.

Let M be an R-module and $N \in \sigma[M]$. A projective module P in $\sigma[M]$ together with a superfluous epimorphism $\pi : P \to N$ is called a *projective cover (hull) of N in* $\sigma[M]$ or a $\sigma[M]$*-projective cover of N.*

If $\sigma[M] = R\text{-}MOD$ we call it the *projective cover of N.*

Even if there are enough projectives in $\sigma[M]$ (e.g. in *R-MOD*), a module need not have a projective cover. The existence of injective hulls was shown using *(intersection) complements* of submodules whose existence was assured by Zorn's Lemma (§ 17). To get projective covers we need *supplements* which do not always exist. We will return to this problem in § 41.

The following assertions describe projective covers without saying anything about their existence:

19.5 Properties of projective covers.

Let M be an R-module and $\pi : P \to N$ a projective cover of N in $\sigma[M]$.

(1) If $f : Q \to N$ is epic with Q projective in $\sigma[M]$, then there is a decomposition $Q = Q_1 \oplus Q_2$, with $Q_1 \simeq P$, $Q_2 \subset Ke\,f$, and $f|_{Q_1} : Q_1 \to N$ is a $\sigma[M]$-projective cover of N.

(2) If (Q, f) is another projective cover of N in $\sigma[M]$, then there is an isomorphism $h : Q \to P$ with $h\pi = f$.

(3) If N is finitely generated, then P is also finitely generated.

(4) If M is projective in $\sigma[M]$ and N (finitely) M-generated, then P is also (finitely) M-generated.

(5) If $\pi_1 : P_1 \to N_1$, $\pi_2 : P_2 \to N_2$ are projective covers of N_1, N_2 in $\sigma[M]$, then $\pi_1 \oplus \pi_2 : P_1 \oplus P_2 \to N_1 \oplus N_2$ is a projective cover of $N_1 \oplus N_2$ in $\sigma[M]$.

Proof: (1) Because of the projectivity of Q, there exists $h : Q \to P$ with $h\pi = f$. Since π is superfluous, h is epic and hence h splits (P is projective). Therefore there exists some $g : P \to Q$ with $gh = id_P$ and hence $Q = Im\, g \oplus Ke\, h$. Putting $Q_1 = Im\, g$ and $Q_2 = Ke\, h$ we get the desired decomposition. Q_1 is projective in $\sigma[M]$ and, since $\pi = gf\,|_{Q_1}$, the epimorphism $f\,|_{Q_1}$ is superfluous.

(2) If f is superfluous, $Ke\, f$ cannot contain a non-zero direct summand and hence $Q_2 = 0$ in (1).

(3), (4) follow from (1) for the epimorphisms $R^n \to N$ resp. $M^{(\Lambda)} \to N$.

(5) is easy to see from 19.3.

Having in mind that every finitely generated R-module is a homomorphic image of a finite sum R^k, $k \in I\!N$, the proof of 19.5,(1) also yields:

19.6 Superfluous submodules and finitely generated modules.

Let K be a superfluous submodule of an R-module N. Then N is finitely generated if and only if N/K is finitely generated.

Non-trivial examples of projective covers are obtained by nil ideals $L \subset R$: Since $L \ll {}_RR$, the canonical projection $p : R \to R/L$ is a projective cover of R/L. Even simple R-modules M need not have a projective cover in *R-MOD*, however they are projective in $\sigma[M]$.

From 19.5,(1) we see that a non-zero factor module of $Z\!\!\!Z$ cannot have a projective cover in $Z\!\!\!Z$*-MOD*.

We know that a finitely generated module $N \in \sigma[M]$ is projective in $\sigma[M]$ if and only if it is M-projective (see 18.3). Since, by 19.5, the projective cover of N is also finitely generated we call it the *M-projective cover of N*.

In case simple modules do have projective covers they may be characterized in the following way:

19.7 Projective covers of simple modules.

Let M be an R-module. For a non-zero projective module P in $\sigma[M]$, the following assertions are equivalent:

(a) P is an M-projective cover of a simple module;

(b) P has a maximal submodule which is superfluous in P;

(c) every proper submodule is superfluous in P;

(d) every proper factor module of P is indecomposable;

(e) for any $f \in End_R(P)$, either f or $id - f$ is invertible (see 19.8).

In this case P is a finitely generated (cyclic) R-module (see 19.5).

Proof: $(a) \Leftrightarrow (b)$ $\pi : P \to E$ is a projective cover of a simple module E in $\sigma[M]$ if and only if $Ke\,\pi$ is maximal and superfluous in P.

$(b) \Rightarrow (c)$ Let $U \subset P$ be maximal and superfluous in P. For a submodule $V \subset P$ we have $V \subset U$ or $U + V = P$ and hence $V = P$. Every proper submodule of P is contained in U and hence superfluous in P.

$(c) \Leftrightarrow (d)$ Assume that U, V are submodules of P with $U+V=P$. Then $P/(U\cap V) = V/(U\cap V)\oplus U/(U\cap V)$. On the other hand, a decomposition of a factor module $P/X = X_1/X \oplus X_2/X$ with $X \subset X_i \subset P$, $i=1,2$, yields $X_1 \cap X_2 = X$ and $X_1 + X_2 = P$.

$(c) \Rightarrow (e)$ If f is epic, then it splits and $Ke\,f$ is a direct summand and (by (c)) superfluous in P, i.e. it is zero. If f is not epic, then $Pf \ll P$ and $P = P(id-f) + Pf$ implies $P = P(id-f)$. Hence $id - f$ is epic and – as seen above – an isomorphism.

$(e) \Rightarrow (c)$ Let U, V be submodules with $U+V=P$ and $\varepsilon : U \to P$, $\pi : P \to P/V$ the canonical mappings. The diagram

$$\begin{array}{ccccc} & & & & P \\ & & & & \downarrow \pi \\ U & \xrightarrow{\varepsilon} & P & \xrightarrow{\pi} & P/V \end{array}$$

can be extended commutatively by an $f : P \to U$, since $\varepsilon\pi$ is epic and P is projective. Then $f\varepsilon = \bar{f} \in End_R(P)$ and $Im\,\bar{f} \subset U$. The equality $\bar{f}\pi = f\varepsilon\pi = \pi$ implies $(id - \bar{f})\pi = 0$ and $Im\,(id-\bar{f}) \subset Ke\,\pi = V$. By assumption $\bar{f}$ or $id - \bar{f}$ is an isomorphism. The first case gives $U = P$, the second $V = P$.

$(c) \Rightarrow (b)$ We will show later on (in 22.3) that every projective module in $\sigma[M]$ has a maximal submodule.

The property of $End_R(P)$ given in 19.7(e) defines a class of rings which plays an important part in the study of decompositions of modules:

A ring R is called *local* if, for any $r \in R$, either r or $1-r$ is invertible. These rings can be described by various properties:

19.8 Local rings. Characterizations.

For a ring R the following properties are equivalent:
(a) R is local;
(b) R has a unique maximal left ideal;
(c) there is a maximal left ideal which is superfluous in R;
(d) the sum of two non-invertible elements in R is non-invertible;
(e) R has a unique maximal right ideal;
(f) there is a maximal right ideal which is superfluous in R.

Proof: Since $End({}_RR) \simeq R$ the equivalence of (a), (b) and (c) follows from 19.7.

$(b) \Rightarrow (d)$ Let M be the unique maximal left ideal of R. If $x, y \in R$ are not invertible, then $R(x+y) \subset Rx + Ry \subset M$ and $x+y$ is not invertible.

$(d) \Rightarrow (a)$ is trivial since $r + (1 - r) = 1$.

Since (a) is independent of sides, (e) and (f) are also equivalent to (a).

For any idempotent $e \neq 1$ in a ring we get $e(1 - e) = 0$. This means that neither e nor $1 - e$ is invertible. Hence a local ring has no non-trivial idempotents.

In the endomorphism ring $End_R(M)$ of an R-module M the idempotents determine the direct decompositions of M. If $End_R(M)$ is a local ring, then M is indecomposable.

However, even the endomorphism ring of a projective indecomposable M need not be local as the example $M = \mathbb{Z}$ with $End_{\mathbb{Z}}(\mathbb{Z}) = \mathbb{Z}$ shows.

The problem of which indecomposable modules have local endomorphism rings is of considerable interest for many investigations. An example:

19.9 Indecomposable injective modules. Characterizations.

For a self-injective R-module M the following are equivalent:
(a) M is indecomposable;
(b) every non-zero submodule is essential in M (we say: M is uniform);
(c) M is an M-injective hull for every non-zero cyclic submodule of M;
(d) $End_R(M)$ is a local ring.

Proof: $(a) \Rightarrow (b)$ For every submodule $U \subset M$, the M-injective hull $\widehat{U}$ is a direct summand of M (see 17.10). Since M is indecomposable, this implies $\widehat{U} = M$ and $U \trianglelefteq M$.

$(b) \Rightarrow (a)$, $(b) \Leftrightarrow (c)$ and $(d) \Rightarrow (a)$ are easily seen.

$(b) \Rightarrow (d)$ For any $f \in End_R(M)$, $Ke\,f \cap Ke\,(1-f) = 0$. If $Ke\,f = 0$, then $(M)f$ is M-injective and hence a direct summand in M, i.e. $(M)f = M$ and f is an isomorphism. For $Ke\,f \neq 0$, we see from (b) that $Ke\,(1-f) = 0$, and $1 - f$ is an isomorphism.

Observe that (self-) injective indecomposable modules need not be finitely cogenerated, i.e. they need not contain a simple submodule.

19.10 Exercises.

(1) Let M be an R-module and $N, L \in \sigma[M]$. Show: If N and $N \oplus L$ have projective covers in $\sigma[M]$, then L need not have a projective cover in $\sigma[M]$.

(2) Let I be a nilpotent left ideal in the ring R and M an R-module. Show: $IM \ll M$.

(3) Show that in the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ every finitely generated submodule is superfluous.

(4) Let p be a prime number. Show that the following rings are local:

(i) $\mathbb{Z}_{p^k}$;

(ii) $\{\frac{a}{b} \mid a, b \in \mathbb{Z},\ (a,b) = 1$ *and* $(b,p) = 1\} \subset \mathbb{Q}$.

(5) Let R be a commutative local ring and
$R[[X]] = \{\sum_{i=0}^{\infty} r_i X^i \mid r_i \in R\}$ *the ring of formal power series over R. Show:*

(i) $R[[X]]$ *is a local ring;*

(ii) the maximal ideal in $R[[X]]$ *is not a nil ideal.*

(6) Let R be a local ring and $M \in R$*-MOD. Show that M is a generator in R-MOD if and only if R is a direct summand of M.*

(7) Let M be an R-module and assume $\alpha : P \to M$ *is a projective cover in R-MOD. Prove:*

(i) The following assertions are equivalent:

(a) M is self-projective;

(b) $Ke\,\alpha$ *is a fully invariant submodule in P;*

(c) $Ke\,\alpha = Re(P, M)$;

(d) $M \simeq P/Re(P, M)$ *(compare 17.11, for reject see 14.4).*

(ii) If M is faithful and self-projective, then $Ke\,\alpha = 0$.

(8) Let us call an R-module M small projective *if* $Hom(M, -)$ *is exact with respect to the exact sequences* $0 \to K \to L \to N \to 0$ *in R-MOD with* $K \ll L$. *Show:*

(i) Direct sums and direct summands of small projective modules are small projective.

(ii) A small projective module which has a projective cover in R-MOD is projective.

Literature: ANDERSON-FULLER, KASCH; Beck [2], Enochs [2,3], Faticoni, Rangaswamy [4], Rayar [2], Tiwary-Chaubey.

Chapter 4

Notions derived from simple modules

Simple modules are those with no non-trivial submodules. They are to be regarded as basic building blocks in module theory. We already have encountered them in different places. In this chapter we are going to investigate direct sums of simple modules (*semisimple modules*) and to study the question of how arbitrary modules are connected with semisimple modules (*socle, radical*). In a certain sense dual to semisimple modules are the *co-semisimple* modules introduced at the end of this chapter.

20 Semisimple modules and rings

1.Sum of simple modules. 2.Characterization of semisimple modules in R-MOD. 3.Characterization of a semisimple module by $\sigma[M]$. 4.Properties of $\sigma[M]$. 5.Decomposition of semisimple modules. 6.Endomorphism rings. 7.Characterization of left semisimple rings. 8.Finiteness conditions. 9.Characterization of simple modules. 10.Characterization of division rings. 11.Simple generators in R-MOD. 12.Left primitive rings. 13.Left primitive non-simple rings. 14.Exercises.

For the study of simple modules it is also useful to deal with properties of direct sums of simple modules, the *semisimple* modules. Examples of semisimple modules are the left semisimple rings encountered in (§ 3). One of the fundamental properties of these modules is presented in

20.1 Sum of simple modules.

Let $\{N_\lambda\}_\Lambda$ be a family of simple submodules of the R-module M with $\sum_\Lambda N_\lambda = M$. Then:

For every submodule $K \subset M$, there is an index set $\Lambda_K \subset \Lambda$ such that

$$M = K \oplus (\bigoplus_{\Lambda_K} N_\lambda).$$

Proof: Let $K \subset M$ be a submodule. Choose a subset $\Lambda_K \subset \Lambda$ maximal with respect to the property that $\{N_\lambda\}_{\Lambda_K}$ is an independent family of submodules (see 8.5) with $K \cap \sum_{\Lambda_K} N_\lambda = 0$. Then $L = K + \sum_{\Lambda_K} N_\lambda$ is a direct sum, i.e. $L = K \oplus (\bigoplus_{\Lambda_K} N_\lambda)$. We show that $L = M$: For $\lambda \in \Lambda$ either $N_\lambda \cap L = N_\lambda$ or $N_\lambda \cap L = 0$. The latter yields a contradiction to the maximality of Λ_K. Hence we get $N_\lambda \subset L$ for all $\lambda \in \Lambda$ and $L = M$.

20.2 Characterization of semisimple modules in *R-MOD*.

For an R-module M the following properties are equivalent:

(a) M is a sum of simple (sub-) modules;

(b) M is a direct sum of simple modules (= semisimple);

(c) every submodule of M is a direct summand;

(d) M contains no proper essential submodules;

(e) every exact sequence $0 \to K \to M \to L \to 0$ in R-MOD splits;

(f) every (finitely generated, cyclic) R-module is M-projective;

(g) every R-module is M-injective.

Proof: $(a) \Rightarrow (b) \Rightarrow (c)$ follow from 20.1.

$(c) \Rightarrow (a)$ For every non-zero $m \in M$, the module $Rm \subset M$ contains a maximal submodule U. By (c), this is a direct summand in M and hence in Rm, i.e. $Rm = U \oplus V$ with a simple submodule $V \simeq Rm/U$. Therefore every non-zero submodule of M contains a simple submodule.

Let L be the sum of all simple submodules of M. By (c), L is a direct summand, i.e. there is a $P \subset M$ with $M = L \oplus P$. Since P cannot have any simple submodule it must be zero.

$(c) \Rightarrow (d)$ Direct summands are not essential.

$(d) \Rightarrow (c)$ For $K \subset M$, let $K' \subset M$ be a complement with $K + K' \trianglelefteq M$ (see 17.6). By (d), this yields $K + K' = M$, i.e. $M = K \oplus K'$.

$(c) \Leftrightarrow (e) \Leftrightarrow (g)$ are obvious and so is $(e) \Leftrightarrow (f)$ for arbitrary R-modules. That M is semisimple whenever every finitely generated R-module is M-projective will be seen in the next theorem.

From the above characterizations it is easily seen that direct sums, homomorphic images and submodules of semisimple modules are again semisimple and this implies:

20.3 Characterization of a semisimple module M by $\sigma[M]$.

For an R-module M the following assertions are equivalent:

(a) M is semisimple;

(b) every module in $\sigma[M]$ is semisimple;

(c) in $\sigma[M]$ every module is projective;

(d) in $\sigma[M]$ every module is injective;

(e) every short exact sequence in $\sigma[M]$ splits;

(f) $\sigma[M]$ has a semisimple generator;

(g) every finitely generated submodule of M is semisimple;

(h) in $\sigma[M]$ every finitely generated module is projective;

(i) in $\sigma[M]$ every simple module is projective.

Proof: The equivalences from (a) to (e) follow immediately from the preceding considerations.

$(a) \Leftrightarrow (f)$ Every (finitely generated) submodule of $M^{(\mathbb{N})}$ is a direct summand, hence M-generated and M is a generator in $\sigma[M]$ (see 15.1).

$(a) \Leftrightarrow (g)$ M is a sum of its finitely generated submodules.

$(c) \Rightarrow (h) \Rightarrow (i)$ is obvious.

$(i) \Rightarrow (f)$ The direct sum of all mutually non-isomorphic simple modules in $\sigma[M]$ is projective in $\sigma[M]$ and generates all its simple summands. By 18.5, it is a generator in $\sigma[M]$.

Remark: By a recent result of Osofsky-Smith, a module M is semisimple if and only if every *cyclic* module in $\sigma[M]$ is M-injective.

The following properties of semisimple modules are easily verified. None of them is sufficient to ensure that a module is semisimple. Some of them determine interesting classes of modules to be investigated later on.

20.4 Properties of $\sigma[M]$ for a semisimple module M.

For a semisimple R-module M we have:

(1) $\sigma[M]$ has a semisimple cogenerator (e.g. M);

(2) every (finitely generated) submodule of M is projective in $\sigma[M]$;

(3) every finitely generated submodule of M is a direct summand;

(4) every simple module in $\sigma[M]$ is M-injective;

(5) the modules in $\sigma[M]$ contain no non-zero superfluous submodules;

(6) every (finitely) M-generated module has a projective cover in $\sigma[M]$;

(7) M is a projective generator in $\sigma[M]$;

(8) M is an injective cogenerator in $\sigma[M]$.

We see from (7) that for semisimple modules the *Density Theorem* 15.7

applies. *Nathan Jacobson* had proved this theorem first for semisimple modules. Hence it is sometimes called the *Jacobson Density Theorem*.

Let M be a semisimple module and $\{E_\gamma\}_\Gamma$ a minimal representing set of the simple submodules of M. Then, for every $\gamma \in \Gamma$, the *trace* of E_γ in M $Tr(E_\gamma, M)$ is a fully invariant submodule (see 13.5) and obviously

$$Tr(E_\gamma, M) \cap Tr(E_\mu, M) = 0 \text{ if } E_\gamma \neq E_\mu.$$

$Tr(E_\gamma, M)$ are called the *homogeneous components of* M since they are (direct) sums of isomorphic simple modules. This yields the first part of

20.5 Decomposition of semisimple modules.
Let M be a semisimple R-module.

(1) If $\{E_\gamma\}_\Gamma$ is a minimal representing set of the simple submodules of M, then $M = \bigoplus_\Gamma Tr(E_\gamma, M)$.

(2) If $M = \bigoplus_\Lambda M_\lambda$ and $M = \bigoplus_{\Lambda'} N_\mu$ with simple modules M_λ, N_μ, then $card(\Lambda) = card(\Lambda')$.

Proof: (1) follows from the above remarks.

(2) If Λ is an infinite index set, the assertion follows from 8.8. If Λ is finite, by (1), it suffices to consider the decompositions of a finitely generated homogeneous module. We have to show that the number of the simple summands is always the same. This can be accomplished by induction on the number of simple summands. Later on we shall also obtain this by more general theorems (see modules *of finite length* 32.3, 32.4).

20.6 Endomorphism rings of semisimple modules.
Let M be a semisimple R-module and $S = End_R(M)$. Then:

(1) S is a regular ring.

(2) If M is simple, then S is a division ring (Schur's Lemma).

(3) If M is finitely generated, then S is a finite product of finite matrix rings over division rings (i.e. S is left semisimple, see 3.4, 20.7*).*

Proof: (1) If M is semisimple then, for every $f \in S$, the modules $Im\, f$ and $Ke\, f$ are direct summands and hence S is regular (see proof of 3.9).

(2) This follows from the fact that a non-trivial endomorphism of a simple module is an isomorphism.

(3) If $\{E_1, \ldots, E_k\}$ is a minimal representing set of the simple submodules of M, then, by 20.5,

$$M = \bigoplus_{i \leq k} Tr(E_i, M), \quad Tr(E_i, M) \simeq E_i^{n_i} \;\; for \;\; n_i \in I\!N.$$

Observing that $Hom(Tr(E_i, M), Tr(E_j, M)) = 0$ for $i \neq j$ and that Hom-functors commute with finite direct sums we get

$$End(M) = \bigoplus_{i \le k} End(Tr(E_i, M)) \simeq \bigoplus_{i \le k} End(E_i^{n_i}) \simeq \bigoplus_{i \le k} End(E_i)^{(n_i, n_i)},$$

where the last summands are (n_i, n_i)-matrix rings over the division rings $End(E_i)$ (see (2)).

For $M = R$ we obtain from 20.6 (and $R \simeq End({}_RR)$) the *Wedderburn-Artin Theorem* for left semisimple rings (see § 4). Together with 20.3 we now have the following descriptions for these rings:

20.7 Characterization of left semisimple rings.
For a ring R the following properties are equivalent:

(a) R is left semisimple (i.e. ${}_RR$ is semisimple);
(b) R is isomorphic to a finite product of finite matrix rings over division rings;
(c) ${}_RR$ is artinian and the nil radical N(R) is zero;
(d) every left ideal is a direct summand in R;
(e) every (finitely generated) R-module is projective;
(f) every R-module is injective;
(g) every short exact sequence in R-MOD splits;
(h) every simple R-module is projective;
(i) R is right semisimple (i.e. R_R is semisimple).

Proof: $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ has already been shown in 4.4. The implication $(a) \Rightarrow (b)$ can also be deduced from 20.6.

The equivalence of (a) with $(d) - (h)$ can be taken from 20.3.

$(a) \Leftrightarrow (i)$ is obtained in a similar fashion to $(a) \Leftrightarrow (b)$ (see 4.4).

The rings described in 20.7 are also called *artinian semisimple* or *classical semisimple*.

It is worth mentioning that for semisimple modules many of the finiteness conditions are equivalent:

20.8 Finiteness conditions for semisimple modules.
For a semisimple R-module M the following are equivalent:

(a) M is finitely generated;
(b) M is finitely cogenerated;
(c) M is a sum of finitely many simple submodules;
(d) $End_R(M)$ is a left semisimple ring.

Further characterizations will be obtained in 31.3.

Proof: $(a) \Leftrightarrow (c) \Rightarrow (d)$ are obvious.

$(d) \Rightarrow (c)$ If $End_R(M)$ is left semisimple, there are finitely many indecomposable idempotents $e_1, \ldots, e_k \in End_R(M)$ with $e_1 + \cdots + e_k = 1$. The Me_i are indecomposable submodules of M and hence simple, since: Every non-zero submodule of Me_i is a direct summand in M and therefore equal to Me_i. We get $M = Me_1 + \cdots + Me_k$.

$(a) \Rightarrow (b)$ By 14.7, we have to show that, for every family $\{V_\lambda\}_\Lambda$ of submodules of M with $\bigcap_\Lambda V_\lambda = 0$, already the intersection of finitely many of the V_λ's is zero. We do this by induction on the number of simple summands in a direct decomposition of M. If M is simple the assertion is obvious.

Assume the statement to be true for any modules which are (direct) sums of less than n simple modules. Consider $M = M_1 \oplus \cdots \oplus M_n$ with simple modules M_i and take the V_λ's as above. We get $V_\mu \cap M_n = 0$ for some $\mu \in \Lambda$. Now we derive from 20.1 that $V_\mu = M_1' \oplus \cdots \oplus M_k'$ with simple submodules M_i' and $i \leq n-1$. By hypothesis, finitely many elements of the family $\{V_\mu \cap V_\lambda\}_\Lambda$ has zero intersection. Hence M is finitely cogenerated.

$(b) \Rightarrow (a)$ Since the simple modules in $\sigma[M]$ are injective by assumption, they form a set of cogenerators (see 17.12). If M is finitely cogenerated, it is a submodule – hence a direct summand (see 20.2) - of a finite direct sum of simple modules.

Applying properties of semisimple modules we obtain:

20.9 Characterization of simple modules.

For an R-module M the following assertions are equivalent:

(a) M is simple;

(b) every module in $\sigma[M]$ is isomorphic to $M^{(\Lambda)}$ for some Λ;

(c) M is semisimple and $End_R(M)$ is a division ring.

Proof: $(a) \Rightarrow (b)$ M being a generator in $\sigma[M]$ by 20.4, every module in $\sigma[M]$ is of the form $N = \sum_\Lambda M_\lambda$ with $M_\lambda \simeq M$ for all $\lambda \in \Lambda$. By 20.1, for a suitable Λ the sum is direct.

$(b) \Rightarrow (a)$ If E is any simple module in $\sigma[M]$, the condition $E \simeq M^{(\Lambda)}$ implies $E \simeq M$.

$(a) \Leftrightarrow (c)$ is obtained from 20.6 and 20.8.

Since a ring R is a division ring if and only if ${}_RR$ is a simple module, 20.9 yields for $M = R$:

20.10 Characterization of division rings.

For a ring R the following properties are equivalent:

(a) R is a division ring;

(b) every R-module is free (has a basis).

The simple modules are the fundamental building blocks in module theory. They can be divided into two classes: those which are finitely generated as modules over their endomorphism rings and those which are not. This depends on properties of the base ring. In the first case we derive from our preceding considerations:

20.11 Simple generators in *R-MOD*.

For a simple R-module M with $S = End_R(M)$, the following assertions are equivalent:

(a) ${}_RM$ is a faithful module and M_S is finitely generated;

(b) M is a subgenerator in R-MOD ($\sigma[M] = R$-MOD);

(c) M is a generator in R-MOD;

(d) ${}_RR \simeq {}_RM^k$ for some $k \in I\!N$;

(e) R is a simple ring and has a minimal left ideal ($\simeq M$);

(f) $R \simeq S^{(k,k)}$ as a ring, $k \in I\!N$.

Rings which have a faithful simple left module are called *left primitive*.

20.12 Left primitive rings.

For a ring R the following are equivalent:

(a) R is left primitive;

(b) R has a maximal left ideal which contains no non-zero two-sided ideal;

(c) R is a dense subring of the endomorphism ring of a right vector space over a division ring.

Proof: $(a) \Rightarrow (b)$ Let M be a faithful simple left R-module. For a non-zero $m \in M$, the annihilator $L = An_R(m)$ is a maximal left ideal since $R/L \simeq M$. For any two-sided ideal $I \subset L$, $I \subset An_R(Rm) = An_R(M) = 0$.

$(a) \Rightarrow (c)$ With the notation of the above proof, $End_R(M)$ is a division ring (see 20.6) and the assertion follows from the Density Theorem 15.7.

$(b) \Rightarrow (a)$ Let $L \subset R$ be a maximal left ideal as in (b). Then R/L is a faithful simple left R-module.

$(c) \Rightarrow (a)$ Let D be a division ring, T_D a right D-vector space and R a dense subring of $End(T_D)$. Then T is a faithful left R-module. (If T_D has finite dimension, $R \simeq End(T_D)$.)

For any $u, v \in T$, there is a vector space homomorphism $\psi : T_D \to T_D$ with $\psi(u) = v$ and there exists $r \in R$ with $ru = \psi(u) = v$. This means that T is a simple R-module.

Left primitive rings may be artinian and simple or:

20.13 Left primitive non-simple rings.
Let R be a ring with a faithful simple left R-module M and $S = End_R(M)$. *Then the following statements are equivalent:*
(a) M_S *is not finitely generated;*
(b) $\sigma[M] \neq R\text{-}MOD$;
(c) R is (left primitive but) not left semisimple.

Proof: These equivalences are obtained from 20.11.

A *right primitive ring* is defined by the existence of a faithful simple right module. There are examples of (non-simple) right primitive rings which are not left primitive (see *Irving*).

Let M be a faithful simple left R-module and I, J ideals in R with $IJ = 0$. Then $I(JM) = 0$. If $JM = M$ then $I = 0$. Otherwise $JM = 0$ and $J = 0$. This shows that every left (or right) primitive ring is, in particular, a *prime ring*. Commutative primitive rings are obviously fields.

20.14 Exercises.

(1) Let $_RM$ *be a semisimple R-module and* $S = End(_RM)$. *Show that* M_S *also is semisimple.*

(2) Let $_RM$ *be a simple R-module and I a minimal left ideal in R. Show: If* $IM \neq 0$ *then* $I \simeq M$.

(3) Let $_RM$ *be a homogeneous semisimple R-module (all simple submodules isomorphic), L a simple submodule of M and* $D = End(L)$. *Show:*

If M is not finitely generated, then, for every $n \in \mathbb{N}$, *there is an idempotent* $e \in End(M)$ *with* $eEnd(M)e \simeq D^{(n,n)}$.

(4) Let V be a finite dimensional vector space over the field K and $f \in End(V)$. *The ring homomorphism* $K[X] \to End(V)$, $X \mapsto f$, *turns V into a* $K[X]$*-module. Show:*

V is semisimple as a $K[X]$*-module if and only if the minimal polynomial of f is a product of distinct irreducible factors in* $K[X]$.

(5) Let R be a prime ring. Show:
If R contains a simple left ideal, then R is left primitive.

(6) Let R be a semiprime ring. Show:

(i) If I is a finitely generated semisimple left ideal, then it is generated by an idempotent.

(ii) For any idempotent $e \in R$, the following assertions are equivalent:
(a) Re is a semisimple left module ;
(b) eR is a semisimple right module;
(c) eRe is a left (right) semisimple ring.

(7) Let M be a $\mathbb{Z}$-module. Show that M is semisimple if and only if: M is a torsion module and if $p^2a = 0$ for $a \in M$ and a prime number p, then also $pa = 0$.

Literature: DROZD-KIRICHENKO; Irving, Osofsky-Smith, Rososhek, Zelmanowitz [3,4].

21 Socle and radical of modules and rings

1.Characterization of the socle. 2.Properties. 3.Finitely cogenerated modules. 4.Extensions of finitely cogenerated modules. 5.Characterization of the radical. 6.Properties. 7.Remarks and examples. 8.Jacobson radical. 9.Quasi-regular elements and sets. 10.Quasi-regular left ideals. 11.Characterization of the Jacobson radical. 12.Properties of Jac(R). 13.Nakayama's Lemma. 14.Rings with Jac(R) = 0. 15.Rings with R/Jac(R) semisimple. 16.Characterizations of Jac(T), T without unit. 17.Exercises.

Let M be an R-module. As *socle of M* $(= Soc(M),\ Soc\,M)$ we denote the sum of all simple (minimal) submodules of M. If there are no minimal submodules in M we put $Soc(M) = 0$.

Let $\mathcal{E}$ be the class of simple R-modules. Then $Soc(M)$ is just the trace of $\mathcal{E}$ in M. By 20.1, $Soc(M)$ is a semisimple submodule of M.

21.1 Characterization of the socle. *For an R-module M we have*

$$\begin{aligned} Soc(M) &= Tr(\mathcal{E}, M) = \sum\{K \subset M \mid K \text{ is a simple submodule in } M\} \\ &= \bigcap\{L \subset M \mid L \text{ is an essential submodule in } M\}. \end{aligned}$$

Proof: The first row is just the definition.
If $L \trianglelefteq M$, then, for every simple submodule $K \subset M$, we have $0 \neq L \cap K = K$, i.e. $K \subset L$. This implies that $Soc(M)$ is contained in every essential submodule.

Put $L_o = \bigcap\{L \subset M \mid L \trianglelefteq M\}$. We show that L_o is semisimple: Let $K \subset L_o$ be a submodule and choose $K' \subset M$ maximal with respect to $K \cap K' = 0$. Then $K \oplus K' \trianglelefteq M$ and consequently

$$K \subset L_o \subset K \oplus K'.$$

By modularity, this yields

$$L_o = L_o \cap (K \oplus K') = K \oplus (L_o \cap K'),$$

i.e. K is direct summand of L_o and L_o is semisimple.

Observe that $Soc(M)$ need not be essential in M. By definition $Soc(M)$ is the largest semisimple submodule of M and $Soc(M) = M$ if and only if M is semisimple.

The following assertions are readily verified:

21.2 Properties of the socle.
Let M be an R-module.

(1) For any morphism $f : M \to N$, we have $Soc(M)f \subset Soc(N)$.

(2) For any submodule $K \subset M$, we have $Soc(K) = K \cap Soc(M)$.

(3) $Soc(M) \trianglelefteq M$ if and only if $Soc(K) \neq 0$ for every non-zero submodule $K \subset M$.

(4) $Soc(M)$ is an $End_R(M)$-submodule , i.e. $Soc(M)$ is fully invariant in M.

(5) $Soc(\bigoplus_\Lambda M_\lambda) = \bigoplus_\Lambda Soc(M_\lambda)$.

We have seen in 20.8 that for a semisimple R-module 'finitely generated' and 'finitely cogenerated' are equivalent conditions. From this we deduce:

21.3 Properties of finitely cogenerated modules.

(1) An R-module M is finitely cogenerated if and only if $Soc(M)$ is finitely generated and essential in M.

(2) Every finitely cogenerated module is a (finite) direct sum of indecomposable modules.

Proof: (1) If M is finitely cogenerated, then this is also true for every submodule and in particular $Soc(M)$ is finitely generated.

Assume $Soc(K) = \bigcap\{L \subset K \mid L \trianglelefteq K\} = 0$ for a submodule $K \subset M$. Then, for finitely many essential submodules $L_1, \dots L_r$ of K, the intersection $L_o := \bigcap_{i \leq r} L_i = 0$. Since $L_o \trianglelefteq K$ by 17.3, this means $K = 0$. We conclude that $Soc(M) \trianglelefteq M$.

On the other hand, every essential extension of a finitely cogenerated module is again finitely cogenerated.

(2) This is shown by induction on the number of simple summands (in a decomposition) of the socle: Obviously a finitely cogenerated module with simple socle is indecomposable. Assume the assertion is true for $n \in \mathbb{N}$, and let M be finitely cogenerated with $n+1$ simple summands in $Soc(M)$. If M is indecomposable nothing need be shown.

If $M = M_1 \oplus M_2$ with non-zero M_1 and M_2, then $Soc(M_1)$ and $Soc(M_2)$ are non-zero and have at most n simple summands. Hence by assumption M_1 and M_2 are direct sums of indecomposable modules.

Applying 21.3 we now can easily prove the following assertion which was used before in the proof of 14.9:

21.4 Extensions of finitely cogenerated modules.

(1) A finite direct sum of finitely cogenerated modules is again finitely cogenerated.

(2) If in an exact sequence $0 \to K \to L \to N \to 0$ *the modules* K *and* N *are finitely cogenerated, then* L *is also finitely cogenerated.*

Proof: (1) It suffices to show that the direct sum of two finitely cogenerated modules K and N is finitely cogenerated: First we see that $Soc(K \oplus N) = Soc(K) \oplus Soc(N)$ is finitely generated.

We know from 17.4 that $Soc(K \oplus N) \trianglelefteq K \oplus N$, and hence $K \oplus N$ is finitely cogenerated by 21.3.

(2) With K, the injective hull $\widehat{K}$ (in R-MOD) is also finitely cogenerated. Forming a pushout we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \widehat{K} & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \quad .
\end{array}$$

From this we see that L is a submodule of the module $P = \widehat{K} \oplus N$ which is finitely cogenerated by (1), and hence L finitely cogenerated.

Dual to the socle we define as *radical of an R-module M* ($= Rad(M)$, $Rad\,M$) the intersection of all maximal submodules of M. If M has no maximal submodules we set $Rad(M) = M$.

Let $\mathcal{E}$ be again the class of simple R-modules. Then $Rad(M)$ is just the reject of $\mathcal{E}$ in M (see 14.4).

21.5 Characterization of the radical. *For an R-module M we have*

$$\begin{array}{rcl}
Rad(M) & = & Re(M, \mathcal{E}) = \bigcap \{K \subset M \mid K \text{ is maximal in } M\} \\
 & = & \sum \{L \subset M \mid L \text{ is superfluous in } M\}.
\end{array}$$

Proof: The first row is just the definition.

If $L \ll M$ and K is a maximal submodule of M not containing L, then $K + L = M$ and $K = M$. Hence every superfluous submodule is contained in the radical.

Now assume $m \in Rad(M)$ and $U \subset M$ with $Rm + U = M$. If $U \neq M$ then, by Zorn's Lemma, there is a submodule $L \subset M$ maximal with respect to $U \subset L$ and $m \notin L$. Since $L + Rm = M$, the submodule L is maximal in M and $m \in Rad(M) \subset L$, a contradiction. Hence we have $U = M$ and $Rm \ll M$. Consequently $Rad(M)$ is the sum of superfluous submodules in M.

$Rad(M)$ need not be superfluous in M but every finitely generated submodule of $Rad\,M$ is superfluous in M. By definition, $Rad\,M$ is the smallest submodule $U \subset M$ for which the factor module M/U is cogenerated by simple modules (see 14.5). Hence we get $Rad\,M = 0$ if and only if M is cogenerated by simple modules, i.e. is a subdirect product of simple modules (see 9.11).

Observing that homomorphic images of superfluous submodules are again superfluous submodules, we obtain without difficulty:

21.6 Properties of the radical. *Let M be an R-module.*

(1) For a morphism $f : M \to N$ we have
(i) $(Rad\,M)f \subset Rad\,N$,
(ii) $Rad(M/Rad\,M) = 0$, and
(iii) $(Rad\,M)f = Rad(Mf)$, if $Ke\,f \subset Rad\,M$.

(2) $Rad\,M$ is an $End_R(M)$-submodule of M (fully invariant).

(3) If every proper submodule of M is contained in a maximal submodule, then $Rad\,M \ll M$ (e.g. if M is finitely generated, see 6.7).

(4) M is finitely generated if and only if $Rad\,M \ll M$ and $M/Rad\,M$ is finitely generated (see 19.6).

(5) If $M = \bigoplus_\Lambda M_\lambda$, then $Rad\,M = \bigoplus_\Lambda Rad\,M_\lambda$ and $M/Rad\,M \simeq \bigoplus_\Lambda M_\lambda/Rad\,M_\lambda$.

(6) If M is finitely cogenerated and $Rad\,M = 0$, then M is semisimple and finitely generated.

(7) If $\overline{M} = M/Rad\,M$ is semisimple and $Rad\,M \ll M$, then every proper submodule of M is contained in a maximal submodule.

Proof: (7) Let $U \subset M$ be a proper submodule. Denote by $p : M \to M/Rad\,M$ the canonical projection.

Since $Rad\,M \ll M$, we have $(U)p \neq \overline{M}$. Hence $(U)p$ is contained in a maximal submodule $X \subset \overline{M}$. Then U is a submodule of the maximal submodule $(X)p^{-1} \subset M$.

21.7 Remarks and examples for the radical and socle.

(1) The relation between radical and socle of a module M is not determined, we may have $Soc\,M \cap Rad\,M \neq 0$ (e.g. if M is finitely cogenerated and $Rad\,M \neq 0$).

(2) For a submodule $K \subset M$, in general $Rad\,K \neq K \cap Rad\,M$.

(3) Possibly $Rad(Rad\,M) \neq Rad\,M$, e.g. if $Rad\,M$ is finitely generated.

(4) For $\mathbb{Z}$ we have $Rad\,_{\mathbb{Z}}\mathbb{Z} = Soc\,_{\mathbb{Z}}\mathbb{Z} = 0$, since $\mathbb{Z}$ has no superfluous and no minimal submodules.

(5) For $_{\mathbb{Z}}\mathbb{Q}$ we have $Rad\,_{\mathbb{Z}}\mathbb{Q} = \mathbb{Q}$ and $Soc\,_{\mathbb{Z}}\mathbb{Q} = 0$, since $_{\mathbb{Z}}\mathbb{Q}$ has no maximal and no minimal $\mathbb{Z}$-submodules.

On the other hand, $Rad_{\mathbb{Q}}\mathbb{Q} = 0$ and $Soc_{\mathbb{Q}}\mathbb{Q} = \mathbb{Q}$.

21.8 The Jacobson radical. Definition.

The radical of $_RR$ is called the *Jacobson radical of* R, i.e.

$$Jac(R) = Rad(_RR).$$

As a fully invariant submodule of the ring, $Jac(R)$ is a two-sided ideal in R (see 21.6). For an internal characterization of $Jac(R)$ the following notions turn out to be useful:

21.9 Quasi-regular elements and sets. Definitions.

An element r in a ring R is called *left* (*right*) *quasi-regular* if there exists $t \in R$ with $r + t - tr = 0$ (resp. $r + t - rt = 0$).

r is called *quasi-regular* if it is left and right quasi-regular.

A *subset* of R is said to be *(left, right) quasi-regular* if every element in it has the corresponding property.

This terms are also used for rings without unit.

In rings with units the relation $r+t-tr = 0$ is equivalent to the equation $(1-t)(1-r) = 1$. Hence in such rings an element r is left quasi-regular if and only if $(1-r)$ is *left invertible.*

Examples of quasi-regular elements are nilpotent elements:
For any $r \in R$ with $r^n = 0$, $n \in \mathbb{N}$, we have

$$(1 + r + \cdots + r^{n-1})(1-r) = 1 = (1-r)(1 + r + \cdots + r^{n-1}).$$

In particular, nil ideals are quasi-regular ideals.

21.10 Quasi-regular left ideals. Properties.

(1) In a ring R (possibly without unit) every left quasi-regular left ideal is also right quasi-regular.

(2) In a ring R with unit, for a left ideal L the following are equivalent:
(a) L is left quasi-regular;
(b) L is quasi-regular;
(c) L is superfluous in $_RR$.

Proof: (1) Let L be a left quasi-regular left ideal in R and $a \in L$. Then there exists $b \in R$ with $a + b - ba = 0$. This implies $b \in L$ and we find some $c \in R$ with $b + c - cb = 0$. From these equations we obtain $ca + cb - cba = 0$ and $ba + ca - cba = 0$, hence $cb = ba$.

Now $a = ba - b = cb - b = c$ and therefore a is right quasi-regular.

(2) $(a) \Leftrightarrow (b)$ follows from (1).

$(a) \Rightarrow (c)$ Let K be a left ideal with $L + K = R$. Then $1 = l + k$ for some $k \in K$, $l \in L$. Since $k = 1 - l$ is left invertible we conclude $K = R$.

$(c) \Rightarrow (a)$ If $L \ll {}_RR$ then, for every $a \in L$, also $Ra \ll {}_RR$. From $R = Ra + R(1 - a)$ we get $R = R(1 - a)$ and hence $1 - a$ is left invertible.

These observations lead to the following

21.11 Characterization of the Jacobson radical.

In a ring R with unit, $Jac(R)$ can be described as the

(a) intersection of the maximal left ideals in R (= definition);
(b) sum of all superfluous left ideals in R;
(c) sum of all left quasi-regular left ideals;
(d) largest (left) quasi-regular ideal;
(e) $\{r \in R \mid 1 - ar$ is invertible for any $a \in R\}$;
(f) intersection of the annihilators of the simple left R-modules;
(a) intersection of the maximal right ideals.*

Replacing 'left' by 'right' further characterizations (b) -(f*) are possible.*

Proof: The equivalences of (a) to (e) are immediate consequences of 21.5 and 21.10.

$(a) \Leftrightarrow (f)$ Every simple left module E is isomorphic to R/K for a maximal left ideal $K \subset R$ and we have $An_R(E) = An_R(R/K) \subset K$, i.e.

$$\bigcap\{An_R(E) \mid E \text{ simple left module }\} \subset Jac(R).$$

On the other hand, $An_R(E) = \bigcap\{An_R(n) \mid n \in E\}$, where for non-zero $n \in E$, the $An_R(n)$ are maximal left ideals in R, i.e.

$$Jac(R) \subset \bigcap\{An_R(E) \mid E \text{ simple left module}\}.$$

$(d) \Leftrightarrow (a*)$ The property considered in (d) is left-right-symmetric.

The Jacobson radical of rings without unit will be described in 21.16.

21.12 Properties of $Jac(R)$.
For a ring R with unit we have:

(1) $N(R) \subset Jac(R)$, since nilpotent elements are quasi-regular;

(2) $Jac(R)$ contains no non-zero direct summands of ${}_RR$ and hence no non-zero idempotents;

(3) $Jac(R)K = 0$ for every R-module K with $Rad(K) = 0$;

(4) for every R-module M, we have $Jac(R)M \subset Rad(M)$ and $Jac(R)Soc(M) = 0$.

Particularly useful is the following characterization of ideals in $Jac(R)$ known as

21.13 Nakayama's Lemma.
For a left ideal I in a ring R, the following properties are equivalent:
(a) $I \subset Jac(R)$ (I is quasi-regular);
(b) For every finitely generated non-zero R-module M we have $IM \neq M$;
(c) for every finitely generated non-zero R-module M we have $IM \ll M$.

Proof: $(a) \Rightarrow (b)$ According to 21.12, $IM \subset Rad(M) \neq M$.

$(b) \Rightarrow (c)$ Let M be finitely generated and $N \subset M$ with $IM + N = M$. Then

$$I(M/N) = (IM + N)/N = M/N.$$

Now (b) implies $M/N = 0$ and hence $M = N$.
$(c) \Rightarrow (a)$ For $M = R$ condition (c) yields $I \subset IR \ll R$, which means $I \subset Jac(R)$.

A version of Nakayama's Lemma for modules which are not finitely generated will be given in 43.5.

A ring R with $Jac(R) = 0$ is – by definition – a subdirect product of simple *modules*. By 21.11, such a ring R is also a subdirect product of factor *rings* $R/An(E)$ with simple modules E. These are rings for which E is a simple faithful module, i.e. primitive rings (see 20.11, 20.12).
Hence rings R with $Jac(R) = 0$ are also called *semiprimitive* or *Jacobson semisimple* and we can state:

21.14 Rings with $Jac(R)$ zero. *Let R be a ring.*
(1) $Jac(R) = 0$ if and only if R is a subdirect product of primitive rings.
(2) If $Jac(R) = 0$ and ${}_RR$ is finitely cogenerated, then R is left semisimple.

We saw that the ring $R/Jac(R)$ need not be left semisimple. Of course it is noteworthy when this is the case. Such rings are called *semilocal* :

21.15 Rings with $R/Jac(R)$ left semisimple.
For a ring R the following assertions are equivalent:

(a) $R/Jac(R)$ is a left semisimple ring (R is semilocal);
(b) $R/Jac(R)$ is finitely cogenerated as a left R-module;
(c) every product of (semi-) simple (left) R-modules is semisimple;
(d) for every R-module M,
$Soc(M) = \{m \in M \mid Jac(R)m = 0\}$ $(= An_M(Jac(R)))$.

Proof: $(a) \Leftrightarrow (b)$ is obvious.

$(a) \Rightarrow (d)$ By 21.12, $Soc(M) \subset An_M(Jac(R))$.
Since $Jac(R)An_M(Jac(R)) = 0$ the annihilator $An_M(Jac(R))$ is an $R/Jac(R)$-module and hence semisimple (as an R-module) by (a) and contained in $Soc(M)$.

$(d) \Rightarrow (c)$ If M is a product of (semi-) simple modules, then $Jac(R)M = 0$ and $Soc(M) = M$ by (d), i.e. M is semisimple.

$(c) \Rightarrow (a)$ $R/Jac(R)$ is a submodule of a product of simple modules. This is semisimple by (c), and hence $R/Jac(R)$ is left semisimple.

Similarly to the radical, the *left socle* $Soc(_RR)$ of a ring R is also a two-sided ideal (fully invariant submodule) but in general $Soc(_RR) \neq Soc(R_R)$. The importance of the socle will become apparent when studying cogenerator rings.

For the characterization of the Jacobson radical of R in 21.11 we occasionly made use of the existence of a unit in R. For some applications it is of interest that the Jacobson radical can also be defined for rings without unit and has remarkable properties and characterizations. We elaborate this now.

Let T be an associative ring, possibly without unit. A T-module E is called *simple* if E has no non-trivial submodules and $TE \neq 0$ (hence $TE = E$).

Since the subset $\{a \in E \mid Ta = 0\}$ is a T-submodule of E we have, for every non-zero $a \in E$, the relation $Ta = E$. Hence there exists $c \in T$ with $a = ca$ and, for all $t \in T$, we have $(t - tc)a = 0$, i.e.
$t - tc \in An(a) = \{t \in T \mid ta = 0\}$.

A left ideal K in T is called *modular* if there exists $c \in T$ with $t - tc \in K$ for all $t \in T$.

In rings with unit, of course, we can always choose $c = 1$, i.e. every left ideal is modular.

It follows from the definition that every left ideal which contains a modular left ideal is itself modular.

From the above considerations we see that, for a simple T-module E and a non-zero $a \in E$, the ideal $An(a)$ is maximal and modular and $T/An(a) \simeq E$. Since $An(N) = \bigcap\{An(a) \mid a \in N\}$, for every T-module N we have:

The annihilators of simple left T-modules are intersections of maximal modular left ideals in T.

Defining the *Jacobson radical of* T ($= Jac(T)$) as the intersection of the annihilators of all simple T-modules we obtain:

21.16 Characterization of $Jac(T)$, T without unit.

In a ring T (without unit) $Jac(T)$ can be described as:

(a) $\bigcap\{An(E) \mid E$ *a simple left T-module*$\}$ $(=: Jac(T))$;

(b) $\bigcap\{K \subset T \mid K$ *a maximal modular left ideal in* $T\}$;

(c) *the largest left quasi-regular left ideal in T*;

(d) *the largest quasi-regular ideal in T*;

(a∗) $\bigcap\{An(E) \mid E$ *a simple right T-modul*$\}$;

(b∗) $\bigcap\{K \subset T \mid K$ *a maximal modular right ideal*$\}$;

(c∗) *the largest right quasi-regular right ideal.*

Proof: $(a) \Leftrightarrow (b)$ is derived from the above representation of the annihilator of a simple module as an intersection of maximal modular left ideals.

$(b) \Leftrightarrow (c)$ For this we first show: If $c \in T$ is not left quasi-regular, then there is a maximal modular left ideal $L \subset T$ with $c \notin L$. The subset $I_c = \{t - tc \mid t \in T\}$ is a modular left ideal and c is left quasi-regular if and only if $c \in I_c$ (hence $I_c = T$). If $c \notin I_c$, then the set of (modular) left ideals $\{K \subset {}_TT \mid I_c \subset K \text{ and } c \notin K\}$ is non-empty and obviously inductive (by inclusion). Therefore, by Zorn's Lemma, it contains a maximal element which is in fact a maximal modular left ideal. This shows that $Jac(T)$ is left quasi-regular.

Now let U be a left quasi-regular left ideal in T. Assume $U \not\subset Jac(T)$. Then there is a simple T-module E with $UE \neq 0$, i.e. $UE = E$. For every non-zero $a \in E$, we find $u \in U$ with $ua = a$. Since u is left quasi-regular, there exits $v \in T$ with $v + u - vu = 0$ and hence

$0 = (v + u - vu)a = ua = a$, a contradiction.

$(c) \Leftrightarrow (d)$ From the equivalence $(c) \Leftrightarrow (a)$ already shown we know that $Jac(T)$ is a (two-sided) ideal. In 21.10 we saw that left quasi-regular left ideals are right quasi-regular. By $(b) \Leftrightarrow (c)$, every quasi-regular ideal of T is contained in $Jac(T)$.

The equivalence of (d) with $(a*)$, $(b*)$ and $(c*)$ is derived similarly to the preceding remarks. The importance of (d) lies in the fact that this characterization of $Jac(T)$ is independent of sides.

Observe that some of the properties of the Jacobson radical of rings with unit are no longer true for rings without unit. For example, for nil rings T the radical $Jac(T) = T$.

21.17 Exercises.

(1) Let K be a submodule of $M \in R$-MOD. Show:

(i) $K = Rad(M)$ if and only if $K \subset Rad(M)$ and $Rad(M/K) = 0$.

(ii) $K = Soc(M)$ if and only if $Soc(M) \subset K$ and $Soc(K) = K$.

(iii) If $K \ll M$ and $Rad(M/K) = 0$, then $K = Rad(M)$.

(iv) If $K \trianglelefteq M$ and $Soc(K) = K$, then $K = Soc(M)$.

(2) Let P be a projective module in R-MOD. Show:
$Rad(P) = Jac(R)P$, $Soc(P) = Soc(R)P$.

(3) Let e, f be non-zero idempotents in a ring R with unit, $J = Jac(R)$. Show:

(i) $Rad(Re) = Je$, $\quad Jac(eRe) = eJe$.

(ii) The following assertions are equivalent:

(a) $Re \simeq Rf$; $\quad (c)$ $Re/Je \simeq Rf/Jf$;

(b) $eR \simeq fR$; $\quad (d)$ $eR/eJ \simeq fR/fJ$.

(iii) The following assertions are equivalent:

(a) Re/Je *is a simple left R-module*;

(b) Je *is the only maximal submodule of Re*;

(c) eRe *is a local ring*;

(d) eR/eJ *is a simple right R-module*;

(e) eJ *is the only maximal submodule of eR.*

(4) Show that for a ring R the following statements are equivalent:

(a) R is local (see 19.8);

(b) $Jac(R)$ is maximal as a left ideal;

(c) $R/Jac(R)$ is a division ring;

(d) $Jac(R) = \{a \in R \mid Ra \neq R\}$.

(5) Consider the rings $\begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$ and $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$.

Determine the Jacobson radical, the left socle and the right socle. Do the two socles coincide?

(6) For rings R, S and a bimodule ${}_RM_S$, the matrices $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *form a ring with the usual matrix operations.*

Determine the Jacobson radical of this ring.

(7) Let R be the subring of $\mathbb{Q}$ consisting of rational numbers with odd denominators. Show:

$Jac(R)$ consists of all rational numbers with odd denominator and even numerator.

(8) Show for a $\mathbb{Z}$-module M:
(i) If M is torsion free, then $Soc(M) = 0$;
(ii) if M is a torsion module, then $Soc(M) \trianglelefteq M$;
(iii) if M is divisible, then $Rad(M) = M$.

(9) Show for the $\mathbb{Z}$-module $\mathbb{Z}_4$ that $Rad(\mathbb{Z}_4) = Soc(\mathbb{Z}_4)$.

Literature: ANDERSON-FULLER, KASCH;
Ahsan [1], Baccella [3], Beidar, Hacque [1,2], Zöschinger [8].

22 The radical of endomorphism rings

1.End(M) of a self-injective M. 2.End(M) of a self-projective M. 3.The radical of projective modules. 4.Exercises.

In this section we want to describe the radical and properties of endomorphism rings of self-injective and self-projective modules. We also obtain a proof for the existence of maximal submodules in projective modules.

22.1 $End_R(M)$ of a self-injective module.
Let M be an M-injective R-module and $S = End_R(M)$. Then:

(1) $Jac(S) = \{f \in S \mid Ke\, f \trianglelefteq M\}$.

(2) $\overline{S} = S/Jac(S)$ is a regular ring and every idempotent in $\overline{S}$ is an image (under $S \to S/Jac(S)$) of an idempotent in S.

(3) If e, f are idempotents in S with $Se \cap Sf = 0$, then $Me \cap Mf = 0$.

(4) $\overline{S}$ is a left self-injective ring.

(5) If $Soc(M) \trianglelefteq M$, then

$$Jac(S) = Hom_R(M/Soc(M), M) = An_S(Soc(M))$$

and $\overline{S} \simeq End_R(Soc(M))$.

Proof: (1) Assume $f \in S$ and $Ke\, f \trianglelefteq M$. Since $Ke\, f \cap Ke(1-f) = 0$ we get $Ke(1-f) = 0$, and $Im(1-f) \simeq M$ is a direct summand of M. $Im(1-f)$ is also essential in M since, for every $m \in Ke\, f$, we have $m(1-f) = m$ and hence $Ke\, f \subset Im(1-f)$. This implies $Im(1-f) = M$.

The same considerations show that, for every $s \in S$, the element $1 - sf$ is invertible and hence $f \in Jac(S)$.

Now assume $f \in S$ and $K \subset M$ with $K \cap Ke\, f = 0$. Then the restriction $f' = f|_K : K \to M$ is monic and there exists $g \in S$ which leads to the commutative triangle

$$\begin{array}{ccc} K & \xrightarrow{f'} & M \\ \downarrow & \swarrow_g & \\ M & & \end{array} .$$

For every $k \in K$, we see $(k)fg = k$ and $K \subset Ke(1 - fg)$. If $f \in Jac(S)$, then $1 - fg$ is an isomorphism and $K = 0$ which means $Ke\, f \trianglelefteq M$.

(2) For any $f \in S$, we choose a submodule $K \subset M$ maximal with respect to $K \cap Ke\, f = 0$. Then $K + Ke\, f \trianglelefteq M$.

Since $K + Ke\, f \subset Ke(f - fgf)$, with g as in (1), $f - fgf \in Jac(S)$. This implies that $S/Jac(S)$ is a regular ring.

An idempotent in $S/Jac(S)$ comes from an $f \in S$ with $f - f^2 \in Jac(S)$, i.e. $L = Ke(f - f^2) \trianglelefteq M$. The M-injective hull of $(L)f$ is a direct summand in M and hence of the form Me for an idempotent $e \in S$. Since $fe = f$ on L, we get $Ke(fe - f) \trianglelefteq M$, i.e. $fe - f \in Jac(S)$.

For $g = e + (1-e)fe$, we see $g^2 = g$ and $g - fe = e - efe$ which obviously is zero on $Im(1 - e)$ and – by definition of L and e – also on $(L)f$.

Now $(L)f \trianglelefteq Me$ implies $Im(1 - e) + (L)f \trianglelefteq M$ and $g - fe \in Jac(S)$. Hence $(fe - f) + (g - fe) = g - f \in Jac(S)$ and $f + Jac(S)$ is the image of the idempotent $g \in S$.

(3) Assume $Me \cap Mf \neq 0$ and that K is a complement of $Me \cap Mf$ in M. Then we can find a maximal essential extension N of $Me \cap Mf$ in Me with $N \cap K = 0$. Since Me is self-injective, N is a direct summand in Me and in M (see 17.7). If $M = N \oplus K'$ is a decomposition with $K \subset K'$, then there exists an idempotent $g \in S$ with $N = Mg \subset Me$ and $Kg = 0$. This implies $g = ge$ and $Sg \subset Se$.

Similarly we find an idempotent $h \in S$ with $Me \cap Mf \subset Mh \subset Mf$ and $Kh = 0$, i.e. $h = hf$ and $Sh \subset Sf$.

$Ke(g - h)$ contains the essential submodule $(Me \cap Mf) + K \subset M$ and, by (1), $g - h \in Jac(S)$. Now we obtain $0 \neq \overline{S}g = \overline{S}h \subset \overline{S}e \cap \overline{S}f$.

(4) Let I be a left ideal in $\overline{S}$ and $\varphi : I \to \overline{S}$ an S-morphism. Choose a family $\{\bar{e}_\lambda\}_\Lambda$ of idempotents $\bar{e}_\lambda \in I$, maximal with respect to the property that $\sum_\Lambda \overline{S}\bar{e}_\lambda$ is direct. Since, by (2), $\overline{S}$ is regular we see $\bigoplus_\Lambda \overline{S}\bar{e}_\lambda \trianglelefteq I$.

The map φ is uniquely determined by its values on the $\bar{e}_\lambda$'s:
For $\varphi' : I \to \overline{S}$ with $(\bar{e}_\lambda)\varphi = (\bar{e}_\lambda)\varphi'$ for all $\lambda \in \Lambda$, we get $Ke(\varphi - \varphi') \trianglelefteq I$. If $Im(\varphi - \varphi') \neq 0$ there is a direct summand $P \subset \overline{S}$ with $0 \neq P \subset Im(\varphi - \varphi')$.

Then $P \simeq U/Ke(\varphi - \varphi')$ for a module U with $Ke(\varphi - \varphi') \subset U \subset I$, i.e. $Ke(\varphi - \varphi')$ is a direct summand in U (P being projective). This contradicts $Ke(\varphi - \varphi') \trianglelefteq U$, i.e. $\varphi = \varphi'$.

By (2), the idempotents $\bar{e}_\lambda \in I$ are images of idempotents $e_\lambda \in S$. Let us choose $a_\lambda \in S$ which are mapped to $\bar{a}_\lambda = (\bar{e}_\lambda)\varphi$ under the projection. By (3), the sum $\sum_\Lambda Me_\lambda$ is direct. The mappings $e_\lambda a_\lambda : Me_\lambda \to M$ define a morphism $a : \bigoplus_\Lambda Me_\lambda \to M$ which can be extended to $b : M \to M$.

By construction, we get for $\bar{b} = b + Jac(S)$:

$$\bar{e}_\lambda \bar{b} = \bar{e}_\lambda \bar{a} = \bar{e}_\lambda \bar{a}_\lambda = \bar{e}_\lambda(\bar{e}_\lambda)\varphi = (\bar{e}_\lambda)\varphi.$$

By Baer's Criterion 16.4, we conclude that $\overline{S}$ is left injective.

(5) From (1) we immediately see $Jac(S) \subset Hom(M/Soc(M), M)$. If $Soc(M) \trianglelefteq M$, then, also by (1), $Hom(M/Soc(M), M) \subset Jac(S)$.

From the exact sequence $0 \to Soc(M) \to M \to M/Soc(M) \to 0$ the functor $Hom_R(-, M)$ yields the exact sequence

$$0 \to Hom(M/Soc(M), M) \to Hom(M, M) \to Hom(Soc(M), M) \to 0.$$

Since $Hom(Soc(M), M) \simeq End(Soc(M))$, this implies the last assertion.

22.2 $End_R(M)$ of a self-projective module.
Let M be a self-projective R-module and $S = End_R(M)$. Then:

(1) $Jac(S) = \{f \in S \mid Im\, f \ll M\}$.

(2) The radical of $End_R(M/Rad\, M)$ is zero.

(3) $Jac(S) = Hom_R(M, Rad\, M)$ if and only if $Rad\, M \ll M$.

In this case $S/Jac(S) \simeq End_R(M/Rad\, M)$.

Proof: (1) Let $f \in S$ and $Im\, f \ll M$. We show $Sf \ll S$: If, for a left ideal $A \subset S$, the sum $A + Sf = S$, then $1 = sf + g$ for some $s \in S$, $g \in A$ and

$$M = Msf + Mg \subset Im\, f + Mg, \text{ i.e. } Mg = M.$$

Since M is self-projective, there exists $h \in S$ with $1 = hg \in A$ which means $A = S$.

Now assume $f \in Jac(S)$ and $K \subset M$ with $K + Im\, f = M$. Then the composition $M \xrightarrow{f} M \xrightarrow{p} M/K$ is an epimorphism and there exists $g \in S$ which extends the following diagram commutatively:

$$\begin{array}{ccccc} & & & & M \\ & & & & \downarrow p \\ M & \xrightarrow{f} & M & \xrightarrow{p} & M/K \end{array}$$

This means $0 = p - gfp = (1 - gf)p$. Since $(1 - gf)$ is invertible, this yields $p = 0$, i.e. $K = M$.

(2) $Rad\, M$ being a fully invariant submodule of M, the factor module $M/Rad\, M$ is also self-projective. It contains no superfluous submodules.

(3) If $Rad\, M \ll M$ the assertion follows from (1).
Now assume $Jac(S) = Hom(M, Rad\, M)$ and let $N \subset M$ be any submodule with $N + Rad\, M = M$. The diagram with canonical mappings

$$\begin{array}{ccccc} & & & & M \\ & & & & \downarrow \\ Rad\, M & \longrightarrow & M & \longrightarrow & M/N \end{array}$$

can be extended commutatively by an $h : M \to Rad\, M$ and $M = Im\, h + N$. By assumption and (1), we have $Im\, h \ll M$, i.e. $M = N$. This implies $Rad\, M \ll M$.

The isomorphism stated is a consequence of the M-projectivity of M (dual to 22.1,(5)).

We are now able to show the following assertion which completes the proof of the characterization of projective covers of simple modules (see 19.7, $(c) \Rightarrow (b)$):

22.3 The radical of projective modules.
Let M be a R-module and P a non-zero projective module in $\sigma[M]$. Then:

(1) There are maximal submodules in P, i.e. $Rad\, P \neq P$.

(2) If $P = P_1 \oplus P_2$ with $P_2 \subset Rad\, P$, then $P_2 = 0$.

Proof: (1) Assume $Rad\, P = P$. We show that every finitely generated submodule $N \subset P$ is zero:

Let $\{K_\lambda\}_\Lambda$ be a family of finitely generated (cyclic) modules in $\sigma[M]$ and $h : \bigoplus_\Lambda K_\lambda \to P$ an epimorphism. P being projective, there exists $g : P \to \bigoplus_\Lambda K_\lambda$ with $gh = id_P$. With N, and also $(N)g$, finitely generated, there is a finite subset $E \subset \Lambda$ with $(N)g \subset \bigoplus_E K_\lambda$.

With the canonical projection $\pi : \bigoplus_\Lambda K_\lambda \to \bigoplus_E K_\lambda$ we obtain an endomorphism $f := g\pi h$ of P with $nf = ng\pi h = ngh = n$ for all $n \in N$. $Im\, f$ is contained in the finitely generated submodule $(\bigoplus_E K_\lambda)h \subset P$ which is superfluous in P (since $Rad\, P = P$).

By 22.2, this implies $f \in Jac(End_R(P))$, i.e. $1 - f$ is an isomorphism and $N \subset Ke(1 - f) = 0$.

(2) Let $\pi_2 : P \to P_2$ denote the projection onto P_2. By 21.6,
$P_2 = P_2\pi_2 \subset Rad(P)\pi_2 \subset RadP_2$, i.e. $P_2 = Rad\, P_2$ and $P_2 = 0$ by (1).

22.4 Exercises.

(1) Let M be an R-module and $S = End(M)$. Show:
(i) If ${}_RM$ is self-injective and M_S is flat (see 15.9), then ${}_SS$ is FP-injective.
(ii) If ${}_RM$ is self-projective and M_S FP-injective, then S_S is FP-injective.
(Hint: Hom-tensor relation 12.12.)

(2) Let M be a self-projective R-module, $S = End(M)$ and $Soc(M) \trianglelefteq M$. Show: If $Hom_R(M, N) \neq 0$ for every non-zero submodule $N \subset M$, then $Soc({}_SS) \trianglelefteq S$.

(3) Let M be an R-module with M-injective hull $\widehat{M}$ and $S = End(\widehat{M})$. Show that the following assertions are equivalent:

(a) S is a regular ring;

(b) for every essential submodule $U \subset M$, we have $Hom(M/U, \widehat{M}) = 0$.

Literature: DROZD-KIRICHENKO;
Cailleau-Renault, Elliger, McDonald, Miller, Nită, Wisbauer [6,8], Zelmanowitz [6], Zöschinger [4].

23 Co-semisimple and good modules and rings

1.Co-semisimple modules. 2.Characterization of semisimple modules. 3.Good modules. 4.Direct sums of good and co-semisimple modules. 5.Left V-rings. 6.Example. 7.Left good rings. 8.Self-projective co-semisimple and good modules. 9.Exercises.

When listing the characterizations of semisimple modules in 20.3 it was pointed out that some of them are selfdual. The duals of several other properties are equivalent to each other but define a class of modules properly larger than the class of semisimple modules. Let us take one of these duals as definition:

We call an R-module M *co-semisimple* if every simple module (in $\sigma[M]$ or R-MOD) is M-injective.

Observe that any simple R-module E not belonging to $\sigma[M]$ is M-injective: For such an E the functor $Hom(-,E)$ turns every exact sequence in $\sigma[M]$ to zero.

Every semisimple module is of course co-semisimple.

23.1 Characterization of co-semisimple modules.

For an R-module M the following statements are equivalent:

(a) M is co-semisimple;

(b) every finitely cogenerated module in $\sigma[M]$ is M-injective;

(c) every module in $\sigma[M]$ is co-semisimple;

(d) every finitely cogenerated module in $\sigma[M]$ is semisimple;

(e) every finitely cogenerated factor module of M is semisimple;

(f) $\sigma[M]$ has a semisimple cogenerator;

(g) $\sigma[M]$ has a cogenerator Q with $Rad(Q) = 0$;

(h) for every module $N \in \sigma[M]$, $Rad(N) = 0$;

(i) for every factor module N of M, $Rad(N) = 0$;

(j) any proper submodule of M is an intersection of maximal submodules.

Proof: $(a) \Rightarrow (f)$ If every simple module in $\sigma[M]$ is M-injective, then the direct sum of the simple modules in $\sigma[M]$ is a semisimple cogenerator (see 17.12).

$(f) \Rightarrow (g) \Rightarrow (h) \Rightarrow (i)$ are obvious, $(i) \Leftrightarrow (j)$ follows immediately from the definition of the radical.

$(i) \Rightarrow (e)$ Every finitely cogenerated module N with $Rad(N) = 0$ is semisimple.

$(e) \Rightarrow (a)$ Let E be a simple module in $\sigma[M]$ and $\widehat{E}$ the M-injective hull of E. Any diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & M \\ & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & \widehat{E} \end{array}$$

can be extended to a commutative diagram by some $f : M \to \widehat{E}$. As a submodule of $\widehat{E}$, $(M)f$ is finitely cogenerated and hence semisimple by (e), i.e. $(M)f \subset Soc\,\widehat{E} = E$. Therefore E is M-injective.

$(b) \Rightarrow (a) \Rightarrow (d) \Rightarrow (e)$ and $(a) \Leftrightarrow (c)$ are obvious.

$(d) \Rightarrow (b)$ Since we have already seen $(d) \Leftrightarrow (a)$, condition (d) implies that every finitely cogenerated module in $\sigma[M]$ is a finite direct sum of simple M-injective modules and hence M-injective.

As a corollary we obtain

23.2 Further characterization of semisimple modules.
An R-module M is semisimple if and only if every module $N \in \sigma[M]$, with $Rad(N) = 0$, is M-injective.

Proof: It follows from 20.3 that semisimple modules have this property. Assume that every $N \in \sigma[M]$ with $Rad(N) = 0$ is M-injective. Then M is co-semisimple and hence every module in $\sigma[M]$ has zero radical. Therefore every module in $\sigma[M]$ is M-injective and M is semisimple.

One of the most important properties of the radical is that, for every morphism $f : M \to N$, we get $(RadM)f \subset Rad\,Mf$. In general we do not have equality.

M is said to be a *good module* if $(Rad\,M)f = Rad(Mf)$ for any f with source M.

23.3 Characterization of good modules.
For an R-module M the following statements are equivalent:
(a) $(Rad\,M)f = Rad(Mf)$ for every $f : M \to N$ in R-MOD (M is good);
(b) $Rad\,L = 0$ for every factor module L of $M/Rad\,M$;
(c) every M-generated R-module is good;
(d) $M/Rad\,M$ is co-semisimple.

Proof: $(a) \Rightarrow (b)$ Let $p : M \to M/Rad\,M$ be the canonical projection and $g : M/Rad\,M \to L$ an epimorphism. For $f = pg$, we get $Rad\,M \subset Ke\,f$ and $Rad\,L = Rad(Mf) = (RadM)f = 0$.

$(b) \Rightarrow (a)$ Let $f : M \to N$ be given.
The map $M \to Mf \to Mf/(Rad\, M)f$ factorizes over $M/RadM$. By (b), this implies $Rad(Mf/(Rad\, M)f) = 0$ and $(Rad\, M)f = Rad(Mf)$.

$(b) \Leftrightarrow (d)$ follows from 23.1.

$(a) \Leftrightarrow (c)$ If N is M-generated, then this is also true for $N/Rad\, N$.
An epimorphism $M^{(\Lambda)} \to N/Rad\, N$ can be factorized over
$M^{(\Lambda)}/Rad\, M^{(\Lambda)} \simeq (M/Rad\, M)^{(\Lambda)}$.
Hence $N/Rad\, N$ belongs to $\sigma[M/Rad\, M]$ and is co-semisimple by 23.1. Since $(b) \Leftrightarrow (d)$, this implies that N is a good module.

23.4 Direct sums of good and co-semisimple modules.

A direct sum of R-modules is good (co-semisimple) if and only if every summand is good (co-semisimple).

Proof: Assume $L = \bigoplus_\Lambda L_\lambda$. If L is good then, by 23.3, every factor module – and hence every direct summand – is a good module.

Now assume L_λ to be good for every $\lambda \in \Lambda$. Then the $\bar{L}_\lambda = L_\lambda/Rad\, L_\lambda$ are co-semisimple modules and hence every simple R-module is $\bar{L}_\lambda$-injective and therefore $\bigoplus_\Lambda \bar{L}_\lambda$-injective (see 16.2). By 23.1, $L/Rad\, L = \bigoplus_\Lambda \bar{L}_\lambda$ is co-semisimple and, by 23.3, L is good.

If $_RR$ is a co-semisimple module then the ring R is called *left co-semisimple* or a *left V-Ring*. The letter *'V'* refers to *O.E. Villamayor* who first drew attention to non-commutative rings of this type.

From 23.1 we get further descriptions of these rings by properties of *R-MOD*. Moreover they have the following interesting properties:

23.5 Properties of left V-rings.

(1) If R is a left V-ring, then $J^2 = J$ for every left ideal $J \subset R$ and the center $Z(R)$ is a (von Neumann) regular ring.

(2) A commutative ring is a (left) V-ring if and only if it is regular.

Proof: (1) By assumption, J^2 is an intersection of maximal left ideals $\{M_\lambda\}_\Lambda$. Assume there exists $r \in J$ with $r \notin J^2$. Then, for (at least) one $\lambda_o \in \Lambda$, the element $r \notin M_{\lambda_o}$ and $R = M_{\lambda_o} + Rr$, i.e. $1 = m + xr$ for some $m \in M_{\lambda_o}$, $x \in R$. This implies $r = rm + rxr$ and $rxr \in J^2 \subset M_{\lambda_o}$, a contradiction.

By 3.16, any left fully idempotent ring has a regular center.

(2) By (1), every commutative (left) V-ring is regular. On the other hand, for a commutative regular ring R every factor ring R/I is regular and $Jac(R/I) = 0$. Hence $_RR$ is co-semisimple (by 23.1).

In general regular rings need not be co-semisimple:

23.6 Example:

The endomorphism ring of an infinite dimensional (left) vector space is regular but not (right) co-semisimple.

Proof: Let V be an infinite dimensional vector space over a field K with basis $\{v_n\}$ and $S = End({}_KV)$. Then S is a regular ring (see 3.9) and V_S is a simple S-module. The set

$$I = \{f \in S \mid (v_k)f \neq 0 \ \text{ for only finitely many } k \in I\!N\}$$

is a right ideal in S and the map

$$\varphi : I \to V, \ f \mapsto \sum\nolimits_{I\!N}(v_k)f \quad \text{for } f \in I,$$

is an S-homomorphism.

Assume V_S to be S-injective. Then the diagram

$$\begin{array}{ccccc} 0 & \to & I & \to & S \\ & & \downarrow\varphi & & \\ & & V_S & & \end{array}$$

can be extended commutatively by some $\psi : S \to V_S$ and

$$\psi(id_V) = \sum\nolimits_{I\!N} r_k v_k$$

for some $r_k \in K$. For every $f \in I$, we have

$$\sum\nolimits_{I\!N}(v_k)f = \varphi(f) = \psi(f) = (\psi(id_V))f = \sum\nolimits_{I\!N} r_k(v_k)f.$$

Applying this formula to special morphisms $f \in I$ we find that $r_k = 1$ for *all* $k \in I\!N$, a contradiction.

By the way, co-semisimple rings need not be regular. The relationship between these two classes of rings (and modules) will be considered again later on (§ 37).

If ${}_RR$ is a good R-module, then R is called a *left good ring*. From the properties of the corresponding modules we obtain the following

23.7 Characterization of left good rings.

For a ring R the following assertions are equivalent:

(a) *R is a left good ring;*

(b) *every module in R-MOD is good;*

(c) *$Rad\, M = Jac(R)M$ for every module M in R-MOD;*

(d) *$Rad(L) = 0$ for every module L in $R/Jac(R)$-MOD;*

(e) *$R/Jac(R)$ is left co-semisimple.*

Proof: The equivalence of (a), (b), (d) and (e) is immediately obtained from 23.1 and 23.3.

$(e) \Rightarrow (c)$ We always have $Jac(R)M \subset Rad(M)$. The factor module $M/Jac(R)M$ is an $R/Jac(R)$-module and has radical zero by (e). This implies $Rad(M) \subset Jac(R)M$.

$(c) \Rightarrow (e)$ Set $\bar{R} = R/Jac(R)$. For every left ideal J in $\bar{R}$, we have $Rad(\bar{R}/J) = Jac(R)(\bar{R}/J) = 0$, i.e. $\bar{R}$ is left co-semisimple.

23.8 Self-projective co-semisimple and good modules.

Let M be a self-projective R-module. Then:

(1) If M is co-semisimple, then M is a generator in $\sigma[M]$.

(2) If M is co-semisimple and finitely generated, then $End_R(M)$ is left co-semisimple.

(3) If M is good and finitely generated, then $S = End_R(M)$ is left good.

Proof: (1) Every simple module in $\sigma[M]$ is M-injective and hence M-generated. A self-projective module M which generates all simple modules in $\sigma[M]$ is a generator (see 18.5).

(2) We will see later on that, for any finitely generated projective generator M in $\sigma[M]$, the functor $Hom(M,-) : \sigma[M] \to S$-MOD is an *equivalence* (§ 46). If in this case M is co-semisimple, then in S-MOD all simple modules are injective, i.e. S is left co-semisimple.

(3) $M/Rad\, M$ is a co-semisimple module and, by 22.2, $S/Jac(S) \simeq End(M/Rad(M))$. Hence $S/Jac(S)$ is left co-semisimple by (2).

23.9 Exercises.

(1) Let R be a left fully idempotent ring (see 3.15). Show:

(i) If $I \subset R$ is an ideal and N is an R/I-injective module, then N is also R-injective.

(ii) R is left co-semisimple if and only if every primitive factor ring of R is left co-semisimple.

(2) Show that for a ring R the following assertions are equivalent:

(a) R is strongly regular (see 3.11);

(b) R is a left V-ring and every (maximal) left ideal is an ideal.

(3) Show that for a ring R the following assertions are equivalent:

(a) R is a left V-ring;

(b) every left R-module is small projective (see 19.10,(8)).

(4) Let Ω denote the set of maximal left ideals in a ring R. Show that in each case the following assertions are equivalent:

(i) (a) Every simple module in R-MOD is injective with respect to exact sequences $0 \to I \to_R R$ with cyclic left ideals I (= p-injective);

(b) for every maximal submodule K of a cyclic left ideal J, $\bigcap\{L \in \Omega \mid K \subset L\} \neq \bigcap\{L \in \Omega \mid J \subset L\}$.

(ii) (a) Every simple module in R-MOD is injective with respect to exact sequences $0 \to E \to_R R$ with finitely generated left ideals E;

(b) for every maximal submodule K of a finitely generated left ideal F, $\bigcap\{L \in \Omega \mid K \subset L\} \neq \bigcap\{L \in \Omega \mid F \subset L\}$.

In case the conditions in (i) are fulfilled, every left ideal in R is idempotent.

Literature: COZZENS-FAITH, KASCH;
Anderson, Baccella [2], Boyle, Byrd, Gouchot [2,4,5], Faith [1,2], Fisher [3], Fuller [1], Garcia-Gomez [1,4], Hirano-Tominaga, Kosler, Michler-Villamayor, Năstăsescu [1], Ramamurthi-Rangaswamy [1], Rege, Roitman, Sarath-Varadarajan, Tominaga, Wisbauer [2], Würfel [2], Yue [1,2].

Chapter 5

Finiteness conditions in modules

Starting with finitely generated and finitely cogenerated modules we will consider certain finiteness conditions which are all satisfied in finite dimensional vector spaces. Hereby a universal construction turns out to be useful which generalizes coproducts, the *direct limit*. Before aiming for deeper results we introduce this notion.

24 The direct limit

1.Definition. 2.Construction. 3.Properties. 4.Direct limit of morphisms. 5.Direct systems of kernels and cokernels. 6.Direct systems of short exact sequences. 7.Direct limit of submodules. 8.Hom-functor and direct limit. 9.Φ_K monic. 10.Characterization of finitely generated modules. 11.Direct limit and tensor product. 12.Functor into the category of weakly injective modules. 13.Exercises.

Let $(\Lambda, \leq)$ be a quasi-ordered directed (to the right) set, i.e. for any two elements $i, j \in \Lambda$, there exists (at least one) $k \in \Lambda$ with $i \leq k$ and $j \leq k$.

A *direct system of R-modules* $(M_i, f_{ij})_\Lambda$ consists of

(1) a family of modules $\{M_i\}_\Lambda$ and

(2) a family of morphisms $f_{ij} : M_i \to M_j$ for all pairs (i, j) with $i \leq j$, satisfying

$$f_{ii} = id_{M_i} \quad \text{and} \quad f_{ij} f_{jk} = f_{ik} \text{ for } i \leq j \leq k.$$

A *direct system of morphisms* from $(M_i, f_{ij})_\Lambda$ into an R-module L is a family of morphisms

$$\{u_i : M_i \to L\}_\Lambda \text{ with } f_{ij}u_j = u_i \text{ whenever } i \le j.$$

24.1 Direct limit. Definition.

Let $(M_i, f_{ij})_\Lambda$ be a direct system of R-modules and M an R-module.

A direct system of morphisms $\{f_i : M_i \to M\}_\Lambda$ is said to be a *direct limit of* $(M_i, f_{ij})_\Lambda$ if, for every direct system of morphisms $\{u_i : M_i \to L\}_\Lambda$, $L \in R$-*MOD*, there is a unique morphism $u : M \to L$ which makes the following diagrams commutative for every $i \in \Lambda$

$$\begin{tikzcd} M_i \arrow[rr, "f_i"] \arrow[dr, "u_i"'] & & M \arrow[dl, "u"] \\ & L & \end{tikzcd}.$$

If $\{f_i' : M_i \to M'\}_\Lambda$ is another direct limit of $(M_i, f_{ij})_\Lambda$, then by definition there is an isomorphism $h : M \to M'$ with $f_i h = f_i'$ for $i \in \Lambda$. Hence M is uniquely determined up to isomorphism.

We write $M = \varinjlim M_i$ and $(f_i, \varinjlim M_i)$ for the direct limit.

24.2 Construction of the direct limit.

Let $(M_i, f_{ij})_\Lambda$ be a direct system of R-modules. For every pair $i \le j$ we put $M_{i,j} = M_i$ and obtain (with canonical embeddings ε_i) the following mappings:

$$\begin{array}{ccccc} M_{i,j} & \xrightarrow{f_{ij}} & M_j & \xrightarrow{\varepsilon_j} & \bigoplus_\Lambda M_k \\ M_{i,j} & \xrightarrow{id_{M_i}} & M_i & \xrightarrow{\varepsilon_i} & \bigoplus_\Lambda M_k \end{array}$$

The difference yields morphisms $f_{ij}\varepsilon_j - \varepsilon_i : M_{i,j} \to \bigoplus_\Lambda M_k$ and with the coproducts we obtain a morphism $F : \bigoplus_{i \le j} M_{i,j} \to \bigoplus_\Lambda M_k$.

Coke F together with the morphisms

$$f_i = \varepsilon_i Coke\, F : M_i \to \bigoplus_\Lambda M_k \to Coke\, F$$

form a direct limit of $(M_i, f_{ij})_\Lambda$.

Proof: Let $\{u_i : M_i \to L\}_\Lambda$ be a direct system of morphisms and $\bar{u} : \bigoplus_\Lambda M_k \to L$ with $\varepsilon_k \bar{u} = u_k$. We have $(f_{ij}\varepsilon_j - \varepsilon_i)\bar{u} = f_{ij}u_j - u_i = 0$ for $i \le j$. Hence $F\bar{u} = 0$ and the diagram

$$\begin{tikzcd} \bigoplus_{i \le j} M_{i,j} \arrow[r, "F"] & \bigoplus_\Lambda M_k \arrow[r] \arrow[d, "\bar{u}"] & Coke\, F \\ & L & \end{tikzcd}$$

can be extended to a commutative diagram by a unique $u : Coke\,F \to L$ (definition of cokernel).

Remarks: (1) Regarding the quasi-ordered (directed) set Λ as a (directed) category (see 7.3,(4)), a direct system of modules corresponds to a functor $F : \Lambda \to R\text{-}MOD$. The direct systems of morphisms are functorial morphisms (see 44.1) between F and constant functors $\Lambda \to R\text{-}MOD$. Then the direct limit is called the *colimit of the functor F*. Instead of Λ more general categories can serve as 'source' and instead of R-*MOD* other categories may be used as 'target'.

(2) For the construction of the direct limit of direct systems of R-modules the construction of the direct limit of direct systems of sets can be used (see Exercise (1)).

(3) 'Direct limits' are also called *inductive limits* or *(filtered) colimits.*

(4) In case Λ has just three elements i, j, k and $i \le j$, $i \le k$, $j \ne k$, the direct limit of a direct system of modules over Λ yields the pushout.

24.3 Properties of the direct limit.
Let $(M_i, f_{ij})_\Lambda$ be a direct system of modules with direct limit $(f_i, \varinjlim M_i)$.

(1) For $m_j \in M_j$, $j \in \Lambda$, we have $(m_j)f_j = 0$ if and only if, for some $k \ge j$, $(m_j)f_{jk} = 0$.

(2) For $m, n \in \varinjlim M_i$, there exist $k \in \Lambda$ and elements m_k, $n_k \in M_k$ with $(m_k)f_k = m$ and $(n_k)f_k = n$.

(3) If N is a finitely generated submodule of $\varinjlim M_i$, then there exist $k \in \Lambda$ with $N \subset (M_k)f_k$ $(= Im\, f_k)$.

(4) $\varinjlim M_i = \bigcup_\Lambda Im\, f_i$ $(= \sum_\Lambda Im\, f_i)$.

(5) If M is an R-module and the M_i belong to $\sigma[M]$, then $(f_i, \varinjlim M_i)$ also belongs to $\sigma[M]$.

Proof: (1) If $(m_j)f_{jk} = 0$, then also $(m_j)f_j = m_j f_{jk} f_k = 0$.

Assume on the other hand $(m_j)f_j = 0$, i.e. with the notation of 24.2,

$$m_j\varepsilon_j \in Im\,F, \quad m_j\varepsilon_j = \sum_{(i,l)\in E} m_{il}(f_{il}\varepsilon_l - \varepsilon_i), \quad m_{il} \in M_{i,l},$$

where E is a finite set of pairs $i \le l$.

Choose any $k \in \Lambda$ bigger than all the indices occurring in E and $j \le k$. For $i \le k$ the $f_{ik} : M_i \to M_k$ yield a morphism

$\varphi_k : \bigoplus_{i\le k} M_i \to M_k$ with $\varepsilon_i\varphi_k = f_{ik}$ and

$$m_j f_{jk} = m_j\varepsilon_j\varphi_k = \sum\nolimits_E m_{il}(f_{il}\varepsilon_l\varphi_k - \varepsilon_i\varphi_k) = \sum\nolimits_E m_{il}(f_{il}f_{lk} - f_{ik}) = 0.$$

(2) For $m \in \varinjlim M_i$, let $(m_{i_1}, \dots, m_{i_r})$ be a preimage of m in $\bigoplus_\Lambda M_k$ (under $Coke\, F$). For $k \geq i_1, \dots, i_r$ we get

$$m = m_{i_1} f_{i_1} + \cdots + m_{i_r} f_{i_r} = (m_{i_1} f_{i_1 k} + \cdots + m_{i_r} f_{i_r k}) f_k .$$

For $m, n \in \varinjlim M_i$ and $k,\, l \in \Lambda$, $m_k \in M_k$, $n_l \in M_l$ with $m = (m_k) f_k$, $n = (n_l) f_l$, we choose $s \geq k$, $s \geq l$ to obtain $m = (m_k f_{ks}) f_s$ and $n = (n_l f_{ls}) f_s$.

(3),(4) are consequences of (2); (5) follows from the construction.

24.4 Direct limit of morphisms.

Let $(M_i, f_{ij})_\Lambda$ and $(N_i, g_{ij})_\Lambda$ be two direct systems of R-modules over the same set Λ and $(f_i, \varinjlim M_i)$ resp. $(g_i, \varinjlim N_i)$ their direct limits.

For any family of morphisms $\{u_i : M_i \to N_i\}_\Lambda$, with $u_i g_{ij} = f_{ij} u_j$ for all indices $i \leq j$, there is a unique morphism

$$u : \varinjlim M_i \to \varinjlim N_i \,,$$

such that, for every $i \in \Lambda$, the following diagram is commutative

$$\begin{array}{ccc} M_i & \xrightarrow{u_i} & N_i \\ f_i \downarrow & & \downarrow g_i \\ \varinjlim M_i & \xrightarrow{u} & \varinjlim N_i \end{array} \quad .$$

If all the u_i are monic (epic), then u is also monic (epic).
Notation: $u = \varinjlim u_i$.

Proof: The mappings $\{u_i g_i : M_i \to \varinjlim N_i\}_\Lambda$ form a direct system of morphisms since for $i \leq j$ we get $f_{ij} u_j g_j = u_i g_{ij} g_j = u_i g_i$. Hence the existence of u follows from the defining property of the direct limit.

Consider $m \in \varinjlim M_i$ with $(m)u = 0$. By 24.3, there exist $k \in \Lambda$ and $m_k \in M_k$ with $(m_k) f_k = m$ and hence $(m_k) f_k u = (m_k) u_k g_k = 0$. Now there exists $l \geq k$ with $0 = (m_k u_k) g_{kl} = (m_k f_{kl}) u_l$. If u_l is monic, then $(m_k) f_{kl} = 0$ and also $m = (m_k) f_k = 0$. Consequently, if all $\{u_i\}_\Lambda$ are monic, then u is monic.

For $n \in \varinjlim N_i$, by 24.3, there exist $k \in \Lambda$ and $n_k \in N_k$ with $(n_k) g_k = n$. If u_k is surjective, then $n_k = (m_k) u_k$ for some $m_k \in M_k$ and $(m_k f_k) u = (m_k u_k) g_k = n$. If all the $\{u_i\}_\Lambda$ are surjective, then u is surjective.

24.5 Direct systems of kernels and cokernels.

Using the notation of 24.4, we obtain, for $i \leq j$, commutative diagrams

$$\begin{array}{ccccccc} Ke\, u_i & \longrightarrow & M_i & \xrightarrow{u_i} & N_i & \longrightarrow & Coke\, u_i \\ & & \downarrow & & \downarrow & & \\ Ke\, u_j & \longrightarrow & M_j & \xrightarrow{u_j} & N_j & \longrightarrow & Coke\, u_j \end{array} \;,$$

which can be extended by $k_{ij}: Ke\,u_i \to Ke\,u_j$ *and* $h_{ij}: Coke\,u_i \to Coke\,u_j$ *to commutative diagrams.*

It is easy to check that $(Ke\,u_i, k_{ij})_\Lambda$ *and* $(Coke\,u_i, h_{ij})_\Lambda$ *also form direct systems of R-modules.*

24.6 Direct systems of short exact sequences.

Consider direct systems of R-modules $(L_i, f_{ij})_\Lambda$, $(M_i, g_{ij})_\Lambda$ *and* $(N_i, h_{ij})_\Lambda$ *with direct limits* $(f_i, \varinjlim L_i)$, $(g_i, \varinjlim M_i)$, $(h_i, \varinjlim N_i)$ *and families of morphisms* $\{u_i\}_\Lambda$, $\{v_i\}_\Lambda$ *which make the following diagrams commutative with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_i & \xrightarrow{u_i} & M_i & \xrightarrow{v_i} & N_i & \longrightarrow & 0 \\ & & \downarrow f_{ij} & & \downarrow g_{ij} & & \downarrow h_{ij} & & \\ 0 & \longrightarrow & L_j & \xrightarrow{u_j} & M_j & \xrightarrow{v_j} & N_j & \longrightarrow & 0 \end{array} .$$

Then, with $u = \varinjlim u_i$ *and* $v = \varinjlim v_i$, *the following sequence is also exact:*

$$0 \longrightarrow \varinjlim L_i \xrightarrow{u} \varinjlim M_i \xrightarrow{v} \varinjlim N_i \longrightarrow 0 .$$

Proof: It has already been shown in 24.4 that u is monic and v is epic. $Im\,u \subset Ke\,v$ is obvious. Consider $m \in Ke\,v$. There exist $k \in \Lambda$ and $m_k \in M_k$ with $m_k g_k = m$ and $0 = mv = m_k g_k v = m_k v_k h_k$.

Now, by 24.3, we can find an $s \in \Lambda$ with $m_k g_{ks} v_s = m_k v_k h_{ks} = 0$. This implies $m_k g_{ks} = l_s u_s$ for some $l_s \in L_s$ and $l_s f_s u = l_s u_s g_s = m_k g_{ks} g_s = m_k g_k = m$. Consequently $m \in Im\,u$ and $Im\,u = Ke\,v$.

As an important special case we notice:

24.7 Direct limit of submodules.

Let M *be an R-module,* Λ *a set, and* $\{M_i\}_\Lambda$ *a family of submodules of* M *directed with respect to inclusion and with* $\bigcup_\Lambda M_i = M$.

Defining $i \leq j$ *if* $M_i \subset M_j$ *for* $i, j \in \Lambda$, *the set* Λ *becomes quasi-ordered and directed. With the inclusions* $f_{ij}: M_i \to M_j$ *for* $i \leq j$, *the family* $(M_i, f_{ij})_\Lambda$ *is a direct system of modules and* $M = \varinjlim M_i$.

In particular, every module is a direct limit of its finitely generated submodules.

Remark: We have pointed out (in 24.2) that direct limits can be defined in arbitrary categories. Even if direct limits exist in general they need not be (left) exact in the sense of 24.6, and 24.7 need not hold (for 'subobjects'). The exactness of (filtered) direct limits is an important special feature of module categories. Abelian categories with coproducts and a generator, in which direct limits are exact (in the sense of 24.6 or 24.7), are called *AB*5 *Categories* or *Grothendieck Categories*. For every R-module M the category $\sigma[M]$ is of this type.

24.8 Hom-functor and direct limit.
Let $(M_i, f_{ij})_\Lambda$ be a direct system of modules, $(f_i, \varinjlim M_i)$ its direct limit and K an R-module. With the assignments, for $i \leq j$,

$$h_{ij} := Hom(K, f_{ij}) : \; Hom(K, M_i) \to Hom(K, M_j), \quad \alpha_i \mapsto \alpha_i f_{ij},$$

we obtain a direct system of $\mathbb{Z}$-modules $(Hom(K, M_i), h_{ij})_\Lambda$ with direct limit $(h_i, \varinjlim Hom(K, M_i))$ and the assignment

$$u_i := Hom(K, f_i) : \; Hom(K, M_i) \to Hom(K, \varinjlim M_i), \quad \alpha_i \mapsto \alpha_i f_i,$$

defines a direct system of $\mathbb{Z}$-morphisms and hence a $\mathbb{Z}$-morphism

$$\Phi_K := \varinjlim u_i : \varinjlim Hom(K, M_i) \longrightarrow Hom(K, \varinjlim M_i).$$

These $\mathbb{Z}$-morphisms may be regarded as $End(K)$-morphisms.

We are interested in special properties of Φ_K, in particular we ask when Φ_K is an isomorphism, i.e. for which K the direct limit commutes with the functor $Hom(K, -)$. The answer will be given in the next section (in 25.2). Using the above notations we first show:

24.9 Φ_K monic.

If K is a finitely generated R-module, then Φ_K is monic.

Proof: Consider $\alpha \in Ke\,\Phi_K$. There exist $i \in \Lambda$ and $\alpha_i \in Hom(K, M_i)$ with $(\alpha_i)h_i = \alpha$ and $\alpha_i f_i = 0$. Since $K\alpha_i \subset Ke\,f_i$ is a finitely generated submodule of M_i, there exists $i \leq j \in \Lambda$ with $K\alpha_i f_{ij} = 0$ (by 24.3). This implies $(\alpha_i)h_{ij} = \alpha_i f_{ij} = 0$ and $(\alpha_i)h_i = 0$ in $\varinjlim Hom(K, M_i)$.

24.10 Characterization of finitely generated modules by $\varinjlim$.
An R-module K is finitely generated if and only if

$$\Phi_K : \varinjlim Hom(K, M_i) \longrightarrow Hom(K, \varinjlim M_i)$$

is an isomorphism for every direct system $(M_i, f_{ij})_\Lambda$ of modules (in $\sigma[K]$) with f_{ij} monomorphisms.

Proof: Let K be finitely generated. By 24.9, Φ_K is monic. With the f_{ij} monic, the f_i are monic. For every $\alpha \in Hom(K, \varinjlim M_i)$, the image $K\alpha$

is finitely generated and (by 24.3) $K\alpha \subset M_k f_k \simeq M_k$ for some $k \in \Lambda$. With $f_k^{-1} : M_k f_k \to M_k$ we get $\alpha f_k^{-1} \in Hom(K, M_k)$ and

$(\alpha f_k^{-1}) h_k \Phi_K = (\alpha f_k^{-1}) f_k = \alpha$, i.e. Φ_K is surjective.

On the other hand, assume Φ_K to be an isomorphism for the direct system $(K_i, f_{ij})_\Lambda$ of the finitely generated submodules $K_i \subset K$, i.e.

$$\varinjlim Hom(K, K_i) \simeq Hom(K, \varinjlim K_i) \simeq Hom(K, K).$$

By 24.3, there exist $j \in \Lambda$ and $\alpha_j \in Hom(K, K_j)$ with $\alpha_j f_j = id_K$, i.e. $K = K\alpha_j f_j = K_j f_j$. Hence K is finitely generated.

We have seen in 12.4 that the tensor product commutes with direct sums. Applying this observation we show:

24.11 Direct limit and tensor product.

Let $(M_i, f_{ij})_\Lambda$ be a direct system of left R-modules with direct limit $(f_i, \varinjlim M_i)$ and L_R a right R-module. Then

(1) $(L \otimes M_i, id \otimes f_{ij})_\Lambda$ is a direct system of $\mathbb{Z}$-modules;

(2) $\{id \otimes f_i : L \otimes M_i \to L \otimes \varinjlim M_i\}_\Lambda$ is a direct system of morphisms;

(3) $\lambda = \varinjlim id \otimes f_i : \varinjlim (L \otimes M_i) \to L \otimes \varinjlim M_i$ is an isomorphism of $\mathbb{Z}$-modules (and $End(L)$-modules).

Proof: The first two assertions are easily verified.

To show (3) recall the construction of $\varinjlim M_i$ in 24.2. With the notation used there, we have the exact sequence

$$\bigoplus_{i \leq j} M_{i,j} \longrightarrow \bigoplus_{\Lambda} M_\lambda \longrightarrow \varinjlim M_i \longrightarrow 0.$$

Applying the functor $L \otimes_R -$ we get the commutative diagram

$$\begin{array}{ccccccc}
\bigoplus_{i \leq j}(L \otimes M_{ij}) & \longrightarrow & \bigoplus_\Lambda (L \otimes M_k) & \longrightarrow & \varinjlim (L \otimes M_i) & \longrightarrow & 0 \\
\| & & \| & & \downarrow \lambda & & \\
L \otimes (\bigoplus_{i \leq j} M_{ij}) & \longrightarrow & L \otimes (\bigoplus_\Lambda M_k) & \longrightarrow & L \otimes \varinjlim M_i & \longrightarrow & 0.
\end{array}$$

The rows are exact: The first one since it defines $\varinjlim (L \otimes M_i)$. The second since $L \otimes_R -$ is right exact (see 12.8). The first two vertical mappings are isomorphisms. Hence λ is also an isomorphism.

By forming the injective hull we may assign to every module N (in $\sigma[M]$) an injective module $\widehat{N}$ (in $\sigma[M]$), and morphisms $N \to L$ can be extended to morphisms $\widehat{N} \to \widehat{L}$. Since this extension is not unique this assignment, in general, does *not* define a functor. However, using the direct limit we are able to construct a functor from $\sigma[M]$ into the (full) subcategory of (weakly) M-injective modules in $\sigma[M]$:

24.12 Functor into the category of weakly injective modules.
For any R-module M there is a functor $Q : \sigma[M] \to \sigma[M]$ with the following properties for every $N \in \sigma[M]$:

(1) There is a monomorphism $q_N : N \to Q(N)$;

(2) $Q(N)$ is weakly M-injective;

(3) for every weakly M-injective module $E \in \sigma[M]$ the functor $Hom(-,E)$ is exact with respect to $0 \longrightarrow N \xrightarrow{q_N} Q(N)$.

Proof: Put $G = M^{(\mathbb{N})}$ and let $\mathcal{M}$ denote the set of finitely generated submodules of G. For $N \in \sigma[M]$ we form the direct sums and a morphism

$$V_N := \bigoplus_{U\in\mathcal{M}} U^{(Hom(U,N))} \xrightarrow{\varepsilon_N} \bigoplus_{U\in\mathcal{M}} G^{(Hom(U,N))} =: W_N,$$

where ε_N denotes the direct sum of the inclusions $U \subset G$. The application of the mappings yields a morphism

$$\alpha_N : V_N \to N, \ \ (u_1^{\gamma_1}, \ldots, u_k^{\gamma_k}) \mapsto \sum_{i\leq k}(u_i)\gamma_i,$$

which we use to form the pushout

$$\begin{array}{ccc} V_N & \xrightarrow{\varepsilon_N} & W_N \\ {\scriptstyle \alpha_N}\downarrow & & \downarrow \\ N & \xrightarrow{q_1} & Q_1(N) \end{array}$$

with ε_N and q_1 monic. For every $g : U \to N$, $U \in \mathcal{M}$, we obtain (by restriction to the g-th summand) the commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & G \\ {\scriptstyle g}\downarrow & & \downarrow \\ N & \xrightarrow{q_1} & Q_1(N). \end{array}$$

Put $Q_o(N) := N$, $Q_{i+1}(N) := Q_1(Q_i(N))$ for $i \in \mathbb{N}$. For $i \leq j$ we define $g_{ij} : Q_i(N) \to Q_j(N)$ as the composition of the monomorphisms $q_{i+1}q_{i+2}\cdots q_j$. Then $(Q_i(N), g_{ij})_{\mathbb{N}}$ is a direct system of modules (in $\sigma[M]$) with direct limit $(g_i, \varinjlim Q_i(N))$. Regarding the monomorphisms $g_j : Q_j(N) \to \varinjlim Q_i(N)$ as inclusions, we get a monomorphism

$$q_N := q_1 : \ N \to Q(N) := \varinjlim Q_i(N).$$

(2) $Q(N)$ is weakly M-injective (Def. 16.9): For every diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & U & \longrightarrow & G = M^{(I\!N)} \\ & & \beta\downarrow & & \\ & & Q(N) & & \end{array}$$

with finitely generated U, there must be a $G \to Q(N)$ yielding a commutative diagram. Now we know $(U)\beta \subset Q_i(N)$, for some $i \in I\!N$, and – by construction of $Q_{i+1}(N)$ – the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & U & \longrightarrow & G \\ & & \beta\downarrow & & \\ & & Q_i(N) & \xrightarrow{q_{i+1}} & Q_{i+1}(N) \subset Q(N) \end{array}$$

can be extended commutatively by some $G \to Q_{i+1}(N)$. Hence $Q(N)$ is weakly M-injective.

(3) A weakly M-injective module E is injective with respect to $0 \longrightarrow V_N \xrightarrow{\varepsilon_N} W_N$ since it is injective with respect to every component. From the definition of $Q_1(N)$, by forming a pushout, we see that E is also injective with respect to $0 \longrightarrow N \xrightarrow{q_1} Q_1(N)$ and more generally with respect to

$$0 \longrightarrow Q_i(N) \xrightarrow{q_{i+1}} Q_{i+1}(N), \quad i \in I\!N.$$

Hence, for every morphism $\beta : N \to E$, we obtain a direct system of morphisms $\{u_i : Q_i(N) \to E\}$ and finally the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{q_N} & Q(N) \\ & & \beta\downarrow & \swarrow \varinjlim u_i & \\ & & E & & . \end{array}$$

It remains to show that Q defines a functor, i.e. for every morphism $f : N \to L$ we must find a morphism $Q(f) : Q(N) \to Q(L)$ such that the conditions for a functor (see 11.1) are satisfied. First $f : N \to L$ determines a map

$$f_V : \; V_N = \bigoplus U^{(Hom(U,N))} \to \bigoplus U^{(Hom(U,L))} = V_L, \quad (U,h) \mapsto (U,hf),$$

and similarly a map $f_W : W_N \to W_L$. We obtain the commutative diagram

$$\begin{array}{ccccccc}
V_N & & & \longrightarrow & & & W_N \\
& \searrow & & I & & \swarrow & \\
& & N & \longrightarrow & Q_1(N) & & \\
f_V\downarrow & & f\downarrow & & & & \downarrow f_W \\
& & L & \longrightarrow & Q_1(L) & & \\
& \nearrow & & II & & \nwarrow & \\
V_L & & & \longrightarrow & & & W_L \quad ,
\end{array}$$

with I and II the defining pushout diagrams. Since I is a pushout, there is a unique $Q_1(f) : Q_1(N) \to Q_1(L)$ extending the diagram commutatively. Repeating this argument we get a family of morphisms

$Q_i(f) : Q_i(N) \to Q_i(L)$ inducing a $Q(f) : Q(N) \to Q(L)$ (see 24.4).
Now it is easy to verify that

$$Q(-) : \sigma[M] \to \sigma[M], \quad N \mapsto Q(N), \quad f \mapsto Q(f),$$

defines a functor.

Remark: By transfinite induction, in a similar way a functor
$\bar{Q}(-) : \sigma[M] \to \sigma[M]$
may be constructed such that $\bar{Q}(N)$ is M-injective for all $N \in \sigma[M]$.

24.13 Exercises.

$(\Lambda, \leq)$ *denotes a quasi-ordered directed index set.*

(1) Let $(M_i, f_{ij})_\Lambda$ *be a direct system of R-modules and* $\dot{\bigcup}_\Lambda M_i$ *the disjoint union of the* M_i.

Two elements $x \in M_i$, $y \in M_j$ *in* $\dot{\bigcup}_\Lambda M_i$ *are called* equivalent, *written* $x \sim y$, *if there exists* $k \in \Lambda$ *with* $i, j \leq k$ *and* $(x)f_{ik} = (y)f_{jk}$. *Show:*

(i) $\sim$ *defines an equivalence relation on the set* $\dot{\bigcup}_\Lambda M_i$.

Denote the set of equivalence classes by M *(*$:= \dot{\bigcup}_\Lambda M_i / \sim$*).*

(ii) There are mappings $f_i : M_i \to M$, $m_i \mapsto [m_i]_\sim$, *such that for* $i \leq j$ *we have* $f_i = f_{ij} f_j$.

(iii) For every $x \in M$, *there exist* $i \in \Lambda$ *and* $x_i \in M_i$ *with* $x = (x_i)f_i$.

(iv) An R-module structure may be defined on M such that every f_i *is a morphism.*

(v) $\{f_i : M_i \to M\}_\Lambda$ *is a direct limit of* $(M_i, f_{ij})_\Lambda$.

(2) Let $(M_i, f_{ij})_\Lambda$ *be a direct system of modules and* $\{f_i : M_i \to M\}_\Lambda$ *its direct limit. Show:*

If $x_k, y_k \in M_k$, $k \in \Lambda$, *and* $(x_k)f_k = (y_k)f_k$, *then there exists* $l \in \Lambda$, $l \geq k$, *with* $(x_k)f_{kl} = (y_k)f_{kl}$.

(3) Let $\{M_\alpha\}_A$ *be a family of* R*-modules. Show:*

$$\bigoplus\nolimits_A M_\alpha = \varinjlim\{\bigoplus\nolimits_E M_\alpha \mid E \subset A,\ E \ finite\}.$$

(4) Find an example to show that the direct limit of splitting short exact sequences need not be a splitting sequence. (Hint: regular rings)

(5) Show that, for suitable direct systems of $\mathbb{Z}$*-modules, we have:*

$\mathbb{Z}_{p^\infty} = \varinjlim\{\mathbb{Z}_{p^k} \mid k \in \mathbb{N}\}$ *(p a prime number);*

$\mathbb{Q}/\mathbb{Z} = \varinjlim\{\mathbb{Z}_n \mid n \in \mathbb{N}\}$;

$\mathbb{Q} = \varinjlim\{\frac{1}{n}\mathbb{Z} \mid n \in \mathbb{N}\}$.

Literature: ROTMAN, SOLIAN, STENSTRÖM.

25 Finitely presented modules

1.Properties. 2.Characterization. 3.Direct limit of finitely presented modules. 4.Characterization of f.p. in R-MOD. 5.Hom-tensor relations for f.p. modules. 6.Exercises.

Let $\mathcal{C}$ be a subcategory of *R-MOD*. A module N in $\mathcal{C}$ is called *finitely presented* (for short *f.p.*) *in* $\mathcal{C}$ if

(i) N is finitely generated and

(ii) in every exact sequence $0 \to K \to L \to N \to 0$ in $\mathcal{C}$, with L finitely generated, K is also finitely generated.

A finitely generated module which is projective in $\mathcal{C}$ is finitely presented in $\mathcal{C}$ since the sequences considered split.

A module which is finitely presented in *R-MOD* is also finitely presented in every subcategory $\mathcal{C}$ of *R-MOD*. However, finitely presented modules in $\sigma[M]$ need not be finitely presented in *R-MOD*: For example, a simple module M is always finitely presented (projective) in $\sigma[M]$ but need not be finitely presented in *R-MOD*.

Similarly to projective modules, we have no general assertions about the existence of finitely presented modules in $\sigma[M]$.

25.1 Properties of f.p. modules in $\boldsymbol{\sigma[M]}$.
Let M be an R-module.

(1) If N is a finitely presented module in $\sigma[M]$, then N is isomorphic to a submodule of M^k/K, for some $k \in I\!N$ and finitely generated $K \subset M^k$.

(2) Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be an exact sequence in $\sigma[M]$. Then

(i) If N_2 is finitely presented in $\sigma[M]$ and N_1 is finitely generated, then N_3 is finitely presented in $\sigma[M]$.

(ii) If N_1 and N_3 are finitely presented, then N_2 is finitely presented in $\sigma[M]$.

(iii) A finite direct sum of modules is finitely presented in $\sigma[M]$ if and only if every summand is finitely presented in $\sigma[M]$.

Proof: (1) If N is finitely presented in $\sigma[M]$, then there is an exact sequence $0 \to K \to U \to N \to 0$, with finitely generated $U \subset M^k$, $k \in I\!N$, and K finitely generated. Hereby $N \simeq U/K$ and $U/K \subset M^k/K$.

(2)(i) Let $0 \to K \to L \to N_3 \to 0$ be an exact sequence in $\sigma[M]$, L finitely generated. Forming a pullback we obtain the following commutative diagram with exact rows and columns (see 10.3)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N_1 & = & N_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & N_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N_3 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

Since L and N_1 are finitely generated this is also true for P (see 13.9). By assumption, N_2 is finitely presented, i.e. K is finitely generated.

(ii) Let $0 \to K \to L \to N_2 \to 0$ be an exact sequence in $\sigma[M]$, L finitely generated. By forming a pullback and applying the Kernel Cokernel Lemma, we obtain the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & N_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N_3 & = & N_3 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

Since N_3 is finitely presented, P has to be finitely generated. Since N_1 is finitely presented, K has to be finitely generated.

(iii) is an immediate consequence of *(i)* and *(ii)*.

25.2 Characterization of f.p. modules in $\sigma[M]$.

For $M \in R$-MOD and $N \in \sigma[M]$ the following are equivalent:

(a) N is finitely presented in $\sigma[M]$;

(b) if $\{V_\alpha\}_A$ is a set of generators in $\sigma[M]$ with V_α finitely generated, then for any epimorphism $p : \bigoplus V_\alpha \to N$, with finite sums $\bigoplus V_\alpha$, the submodule $Ke\, p$ is finitely generated;

(c) For every direct system of modules $(M_i, f_{ij})_\Lambda$ in $\sigma[M]$,

$$\Phi_N : \varinjlim Hom(N, M_i) \to Hom(N, \varinjlim M_i)$$

is an isomorphism (see 24.8), i.e. Hom$(N,-)$ *preserves direct limits;*

(d) *N is finitely generated and, for every direct system of weakly M-injective modules $(M_i, f_{ij})_\Lambda$ in $\sigma[M]$, the map Φ_N (as in (c)) is an isomorphism.*

Proof: The implications $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ are obvious.

$(b) \Rightarrow (a)$ Let $0 \to K \to L \xrightarrow{g} N \to 0$ be exact with finitely generated $L \in \sigma[M]$. Then for a finite sum of V_α's there is an epimorphism $h : \bigoplus V_\alpha \to L$ and we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ke\,hg & \longrightarrow & \bigoplus V_\alpha & \xrightarrow{hg} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow h & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & L & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & & .
\end{array}$$

If $Ke\,hg$ is finitely generated, then this is also true for K.

$(a) \Rightarrow (c)$ By Lemma 24.9, Φ_N is monic.

Consider $\alpha \in Hom(N, \varinjlim M_i)$. Since $N\alpha$ is finitely generated, by 24.3, there exists $j \in \Lambda$ with $N\alpha \subset M_j f_j$. Forming a pullback we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{\varphi} & N & \longrightarrow & 0 \\
 & & \| & & \psi\downarrow & & \downarrow\alpha & & \\
0 & \longrightarrow & K & \longrightarrow & M_j & \xrightarrow{f_j} & M_j f_j & \longrightarrow & 0 \ .
\end{array}$$

Now choose a finitely generated submodule $\bar{P} \subset P$ for which the restriction $\bar{\varphi} = \varphi|_{\bar{P}} : \bar{P} \to N$ is still epic. Then with $Ke\,\bar{\varphi}$ also $(Ke\,\bar{\varphi})\psi$ is finitely generated. Since $(Ke\,\bar{\varphi})\psi f_j = (Ke\,\bar{\varphi})\varphi\alpha = 0$, by 24.3, there exists $j \leq k \in \Lambda$ with $(Ke\,\bar{\varphi})\psi f_{jk} = 0$ and the morphism in the first row factorizes over an $\alpha_k : N \to M_k$:

$$\begin{array}{cccc}
\bar{P} & \subset & P \xrightarrow{\psi} M_j \xrightarrow{f_{jk}} M_k \\
 & {\scriptstyle\bar{\varphi}} \searrow & \\
 & & N & .
\end{array}$$

By construction, restricted to $\bar{P}$, we have the relations

$$\bar{\varphi}\alpha_k f_k = \psi f_{jk} f_k = \psi f_j = \bar{\varphi}\alpha.$$

Since $\bar{\varphi}$ is epic, this implies $\alpha_k f_k = \alpha$ and hence α belongs to the image of Φ_N (see 24.8).

$(c) \Rightarrow (a)$ By 24.10, N is finitely generated. Consider the exact sequence $0 \to K \to L \to N \to 0$ in $\sigma[M]$ with L finitely generated. Denote by $(K_i, \varepsilon_{ij})_\Lambda$ the direct system of finitely generated submodules of K, and by $(L/K_i, \pi_{ij})_\Lambda$ the corresponding direct system of cokernels (see 24.5) with direct limit $(\pi_i, \varinjlim L/K_i)$.

The exactness of the direct limit implies $N = \varinjlim L/K_i$ and, by assumption, $\varinjlim Hom(N, L/K_i) \simeq Hom(N,N)$. Hence (by 24.3) there exist $j \in \Lambda$ and $\alpha_j \in Hom(N, L/K_j)$ with $\alpha_j \pi_j = id_N$, i.e. π_j is a retraction.

By construction, we have the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & & & Ke\,\pi_j & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & K_j & \longrightarrow & L & \longrightarrow & L/K_j & \longrightarrow & 0 \\
 & & \downarrow \varepsilon_j & & \| & & \downarrow \pi_j & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & K/K_j & & & & & & .
\end{array}$$

By the Kernel Cokernel Lemma, we derive $K/K_j \simeq Ke\,\pi_j$. Therefore K_j and K/K_j are finitely generated. This implies that K is finitely generated.

$(d) \Rightarrow (c)$ Let $(M_i, f_{ij})_\Lambda$ be a direct system of modules in $\sigma[M]$. With the functor $Q(-) : \sigma[M] \to \sigma[M]$ described in 24.12, $(Q(M_i), Q(f_{ij}))_\Lambda$ is a direct system of weakly M-injective modules. From the exact rows

$$0 \to M_i \to Q(M_i) \to Q(M_i)/M_i \to 0$$

we obtain (see 24.5, 24.6) the exact row

$$0 \to \varinjlim M_i \to \varinjlim Q(M_i) \to \varinjlim (Q(M_i)/M_i) \to 0.$$

We use this to construct the commutative diagram with exact rows (notation as in 24.8)

$$\begin{array}{ccccc}
0 \to \varinjlim Hom(N, M_i) & \to & \varinjlim Hom(N, Q(M_i)) & \to & \varinjlim Hom(N, Q(M_i)/M_i) \\
\downarrow \Phi_1 & & \downarrow \Phi_2 & & \downarrow \Phi_3 \\
0 \to Hom(N, \varinjlim M_i) & \to & Hom(N, \varinjlim Q(M_i)) & \to & Hom(N, \varinjlim Q(M_i)/M_i).
\end{array}$$

Since N is finitely generated, the maps Φ_1, Φ_2 and Φ_3 are monic. By assumption (d), Φ_2 is an isomorphism. Hence Φ_1 is also an isomorphism.

The following observation has interesting applications:

25.3 Direct limit of finitely presented modules.

Let M be an R-module. For any module N in $\sigma[M]$ the following properties are equivalent:

(a) N is generated by finitely presented modules;

(b) N is a direct limit of finitely presented modules.

Proof: This will follow from a more comprehensive assertion proved in 34.2.

By 25.3, every R-module can be written as a direct limit of finitely presented modules in R-MOD.

25.4 Characterization of f.p. modules in R-MOD.

For an R-module N the following assertions are equivalent:

(a) N is finitely presented in R-MOD;

(b) there is an exact sequence

$$0 \to K \to R^n \to N \to 0$$

for some $n \in I\!N$ and K finitely generated;

(c) there is an exact sequence

$$R^m \to R^n \to N \to 0$$

for some $m, n \in I\!N$;

(d) $Hom_R(N, -) : R\text{-}MOD \to AB$ commutes with direct limits;

(e) N is finitely generated and $Hom_R(N, -) : R\text{-}MOD \to AB$ commutes with direct limits of FP-injective (= weakly R-injective) modules;

(f) the functor $- \otimes_R N : MOD\text{-}R \to AB$ commutes with direct products (in MOD-R);

(g) for every set Λ, the canonical map $\tilde{\varphi}_N : R^\Lambda \otimes_R N \to N^\Lambda$ is bijective.

Proof: The equivalence of (a), (b) and (c) follows immediately from the definitions and 25.1.

$(a) \Leftrightarrow (d) \Leftrightarrow (e)$ is shown in 25.2.

$(c) \Leftrightarrow (f) \Leftrightarrow (g)$ is part of 12.9.

In addition to the general Hom-tensor relations in 12.12 there are special isomorphisms for finitely presented modules which we shall need later on:

25.5 Hom-tensor relations for finitely presented modules. *Let R and S be rings, L_S in MOD-S and ${}_RK_S$ an (R,S)-bimodule.*

(1) If ${}_RP$ is a left R-module and
(i) ${}_RP$ is finitely generated and projective or
(ii) ${}_RP$ is finitely presented and L_S is (K_S-) injective,
then the map

$$\begin{array}{rcl}\lambda_P & : & Hom_S(K,L) \otimes_R P \to Hom_S(Hom_R(P,K),L), \\ & & f \otimes p \mapsto [g \mapsto f((p)g)],\end{array}$$

is an isomorphism (functorial in P).

(2) Assume Q_R in MOD-R to be flat (with respect to R-MOD).
(i) If L_S is finitely generated, then the following map is monic:

$$\nu_L : Q \otimes_R Hom_S(L,K) \to Hom_S(L, Q \otimes K), \quad q \otimes h \mapsto [l \mapsto q \otimes h(l)].$$

(ii) If L_S is finitely presented, then ν_L is an isomorphism.

Proof: (1) First of all it is easy to check that λ_P is an isomorphism for $P = R$ and $P = R^k$, $k \in I\!N$.

Let $R^k \to R^n \to P \to 0$ be an exact sequence with $k, n \in I\!N$. With the functors $Hom_S(K,L) \otimes_R -$, $Hom_R(-,K)$ and $Hom_S(-,L)$ we obtain the commutative diagram with exact first row

$$\begin{array}{ccccccc} Hom(K,L) \otimes R^k & \to & Hom(K,L) \otimes R^n & \to & Hom(K,L) \otimes P & \to 0 \\ \downarrow \lambda_{R^k} & & \downarrow \lambda_{R^n} & & \downarrow \lambda_P & \\ Hom(Hom(R^k,K),L) & \to & Hom(Hom(R^n,K),L) & \to & Hom(Hom(P,K),L) & \to 0, \end{array}$$

in which λ_{R^k} and λ_{R^n} are isomorphisms.

If P is projective or L_S is (K_S-) injective, then the second row is also exact and λ_P is an isomorphism.

(2) Again it is easy to see that ν_L is an isomorphism for $L = S$ and $L = S^k$, $k \in I\!N$.

Let $S^{(\Lambda)} \to S^n \to L \to 0$ be exact, Λ an index set and $n \in I\!N$. With the functors $Q \otimes_R Hom_S(-,K)$ and $Hom_S(-, Q \otimes_R K)$ we obtain the exact commutative diagram

$$\begin{array}{cccccc} 0 \to Q \otimes Hom(L,K) & \to & Q \otimes Hom(S^n,K) & \to & Q \otimes Hom(S^{(\Lambda)},K) \\ \downarrow \nu_L & & \downarrow \nu_{S^n} & & \downarrow \nu_{S^{(\Lambda)}} \\ 0 \to Hom(L, Q \otimes K) & \to & Hom(S^n, Q \otimes K) & \to & Hom(S^{(\Lambda)}, Q \otimes K). \end{array}$$

Since ν_{S^n} is an isomorphism, ν_L has to be monic.

If L_S is finitely presented we can choose Λ to be finite. Then also $\nu_{S(\Lambda)}$ and ν_L are isomorphisms.

25.6 Exercises.

(1)(i) Find the finitely presented left ideals in the ring $\begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$.

(ii) Find a cyclic left module over $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ *which is not finitely presented.*

(2) Let $K[X_n]_{\mathbb{N}}$ *be the polynomial ring in countably many indeterminates over the field K. Show that not all simple modules over* $K[X_n]_{\mathbb{N}}$ *are finitely presented.*

(3) Let R be a ring with center C. Then R^o *(see 6.1) is also a C-algebra and we may regard R as a left* $R \otimes_C R^o$*-module (with* $(a \otimes b)c = acb$*).*

Show:

(i) The map $\mu : R \otimes_C R^o \to R,\ \ a \otimes b \mapsto ab$*, is an* $R \otimes_C R^o$*-module morphism.*

(ii) $Ke\,\mu$ *is generated as an* $R \otimes_C R^o$*-module by* $\{a \otimes 1 - 1 \otimes a \mid a \in R\}$.

(iii) If R is finitely generated as a C-algebra, then R is finitely presented as an $R \otimes_C R^o$*-module.*

Literature: STENSTRÖM; Sklyarenko [2].

26 Coherent modules and rings

1.Locally coherent modules. 2.M locally coherent in $\sigma[M]$. 3.M coherent in $\sigma[M]$. 4.Finitely presented generators and coherent modules. 5.Locally coherent modules in R-MOD. 6.Left coherent rings. 7.Examples. 8.Properties. 9.Exercises.

Let M be an R-module. A module $N \in \sigma[M]$ is called *coherent in* $\sigma[M]$ if

(i) N is finitely generated and

(ii) any finitely generated submodule of N is finitely presented in $\sigma[M]$.

If all finitely generated submodules of a module $N \in \sigma[M]$ are finitely presented (and hence coherent) in $\sigma[M]$, then N is called *locally coherent in* $\sigma[M]$.

Obviously, N is locally coherent in $\sigma[M]$ if and only if in every exact sequence $0 \to K \to L \to N$ in $\sigma[M]$, with L finitely generated, K is also finitely generated.

If $\{V_\alpha\}_A$ is a set of generators of $\sigma[M]$ with finitely generated V_α's, then in this sequence L can be chosen as a finite direct sum of V_α's (see 25.2).

Every (finitely generated) submodule of a locally coherent module is locally coherent (coherent) in $\sigma[M]$. Like 'finitely presented', '(locally) coherent' also depends on the category $\sigma[M]$.

We shall first derive general assertions and then turn to the case $\sigma[M] =$ *R-MOD.*

26.1 Properties of locally coherent modules.
Let M be an R-module and $0 \to N' \to N \to N'' \to 0$ an exact sequence in $\sigma[M]$. Then

(1) If N is locally coherent and N' finitely generated, then N'' is locally coherent in $\sigma[M]$.

(2) If N' and N'' are locally coherent, then N is also locally coherent in $\sigma[M]$.

(3) The direct sum of locally coherent modules is again locally coherent in $\sigma[M]$.

(4) If N is locally coherent in $\sigma[M]$ and K, L are finitely generated submodules of N, then $K \cap L$ is finitely generated.

(5) If $f : L \to N$ is a morphism between coherent modules L, N in $\sigma[M]$, then $Ke\, f$, $Im\, f$ and $Coke\, f$ are also coherent in $\sigma[M]$.

Proof: (1) and (2) are demonstrated with the same proofs as the corresponding assertions for finitely presented modules (see 25.1,(1) and (2)).

(3) By (2), every finite direct sum of locally coherent modules is again locally coherent in $\sigma[M]$. Since a finitely generated submodule of an infinite direct sum is already contained in a finite partial sum, every direct sum of locally coherent modules is locally coherent in $\sigma[M]$.

(4) Under the given assumptions, $N' = K + L$ is coherent in $\sigma[M]$ and there is an exact sequence

$$0 \to K \cap L \to N' \to (N'/K) \oplus (N'/L).$$

By (1) and (2), $(N'/K) \oplus (N'/L)$ is coherent and hence $K \cap L$ has to be finitely generated.

(5) essentially follows from (1).

26.2 M locally coherent in $\sigma[M]$. Properties.
Assume the R-module M to be locally coherent in $\sigma[M]$. Then

(1) Every module in $\sigma[M]$ is generated by coherent modules.

(2) Every finitely presented module is coherent in $\sigma[M]$.

(3) Every module is a direct limit of coherent modules in $\sigma[M]$.

(4) An R-module N is weakly M-injective if and only if the functor $Hom_R(-, N)$ is exact with respect to exact sequences $0 \to K \to M$ with K finitely generated.

Proof: (1) By 26.1, $M^{(\mathbb{N})}$ is locally coherent and the finitely generated submodules form a set of generators of coherent modules in $\sigma[M]$.

(2) If N is finitely presented, then by (1), there is an exact sequence

$$0 \longrightarrow K \longrightarrow \bigoplus_{i \leq k} U_i \longrightarrow N \longrightarrow 0$$

with the central expression coherent and K finitely generated. Then, by 26.1, N is also coherent in $\sigma[M]$.

(3) Because of (1), every module is a direct limit of finitely presented modules (see 25.3) which, by (2), are coherent in $\sigma[M]$.

(4) Let $Hom(-, N)$ be exact for all exact sequences $0 \to K \to M$ with K finitely generated. We show by induction that this implies that $Hom(-, N)$ is exact with respect to all exact sequences $0 \to K \to M^n$, $n \in \mathbb{N}$, K finitely generated, i.e. N is weakly M-injective (Def. 16.9):

Assume, for $n \in \mathbb{N}$, the functor $Hom(-, N)$ to be exact with respect to sequences $0 \to L \to M^{n-1}$, L finitely generated, and take a finitely generated submodule $K \subset M^n$. Forming a pullback we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & K & \longrightarrow & K/L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \longrightarrow & M & \longrightarrow & 0 \quad .
\end{array}$$

Since M is locally coherent, K/L is finitely presented and L is finitely generated. By assumption, $Hom(-,N)$ is exact with respect to the first and last column. By the Kernel Cokernel Lemma, we see that $Hom(-,N)$ is also exact with respect to the central column.

26.3 M coherent in $\sigma[M]$. Characterizations.

If the R-module M is finitely presented in $\sigma[M]$, then the following are equivalent:

(a) M is coherent in $\sigma[M]$;

(b) the direct limit of weakly M-injective modules in $\sigma[M]$ is weakly M-injective.

Proof: Let $(Q_i, f_{ij})_\Lambda$ be a direct system of R-modules in $\sigma[M]$ and $0 \to K \to M$ exact with K finitely generated. If the Q_i are weakly M-injective we obtain the commutative diagram (Φ_M as in 24.8)

$$\begin{array}{ccccc}
\varinjlim Hom(M, Q_i) & \longrightarrow & \varinjlim Hom(K, Q_i) & \longrightarrow & 0 \\
\downarrow \Phi_M & & \downarrow \Phi_K & & \\
Hom(M, \varinjlim Q_i) & \longrightarrow & Hom(K, \varinjlim Q_i) & \longrightarrow & 0
\end{array}$$

with exact first row, Φ_M an isomorphism and Φ_K monic.

$(a) \Rightarrow (b)$ If M is coherent, then K is finitely presented and, by 25.2, Φ_K is an isomorphism and hence the second row is exact. By 26.2, this implies that $\varinjlim Q_i$ is weakly M-injective.

$(b) \Rightarrow (a)$ Now assume $\varinjlim Q_i$ to be weakly M-injective, i.e. the second row in our diagram is exact. Then Φ_K is an isomorphism and $Hom(K,-)$ commutes with direct limits of weakly M-injective modules. Now we learn from 25.2 that K has to be finitely presented in $\sigma[M]$.

26.4 Finitely presented generators and coherent modules.

Let M be an R-module, U a finitely presented module in $\sigma[M]$ and $N \in \sigma[M]$. If every submodule of N is U-generated, then the following *assertions are equivalent:*

(a) N is locally coherent in $\sigma[M]$;

(b) for every $f \in Hom(U^k, N)$, $k \in I\!N$, the submodule $Ke\, f$ is finitely generated ($Im\, f$ is finitely presented);

(c) (i) for any $f \in Hom(U, N)$, the submodule $Ke\, f$ is finitely generated and (ii) the intersection of any two finitely generated submodules of N is finitely generated.

Proof: $(a) \Leftrightarrow (b)$ Under the given assumptions, for every finitely generated submodule $K \subset N$, there is an epimorphism $f : U^k \to K$, for some $k \in I\!N$.

$(a) \Rightarrow (c)$ follows from 26.1,(4) and (5).

$(c) \Rightarrow (b)$ We prove this by induction on $k \in I\!N$. The case $k = 1$ is given by (i).

Assume that, for $k \in I\!N$, all homomorphic images of U^{k-1} in N are finitely presented, and consider $g \in Hom(U^k, N)$. In the exact sequence

$$0 \longrightarrow (U^{k-1})g \cap (U)g \longrightarrow (U^{k-1})g \oplus (U)g \longrightarrow (U^k)g \longrightarrow 0$$

the central expression is finitely presented by assumption, and $(U^{k-1})g \cap (U)g$ is finitely generated because of (ii). Hence $Im\, g$ is finitely presented and $Ke\, g$ is finitely generated.

For coherence in *R-MOD* we obtain from the proof of 26.4:

26.5 Locally coherent modules in *R-MOD*. Characterizations.

For an R-module N the following assertions are equivalent:

(a) N is locally coherent in R-MOD;

(b) for every $f \in Hom(R^k, N)$, $k \in I\!N$, $Ke\, f$ is finitely generated;

(c) for every $n \in N$ the annihilator $An_R(n)$ is finitely generated and

(i) the intersection of two finitely generated submodules of N is finitely generated, or

(ii) the intersection of a cyclic with a finitely generated submodule of N is finitely generated.

A ring R is called *left (right) coherent* if ${}_RR$ (resp. R_R) is coherent in *R-MOD* (resp. *MOD-R*).

26.6 Characterizations of left coherent rings.

For a ring R the following assertions are equivalent:

(a) *R is left coherent;*

(b) *the direct limit of FP-injective (= weakly R-injective) modules is FP-injective;*

(c) *every finitely presented R-module is coherent (in R-MOD);*

(d) *every free (projective) R-module is locally coherent;*

(e) *for every $r \in R$ the annihilator $An_R(r)$ is finitely generated and*
(i) *the intersection of two finitely generated left ideals is finitely generated, or*
(ii) *the intersection of a cyclic with a finitely generated left ideal is finitely generated;*

(f) *every product of flat right R-modules is flat (w.resp. to R-MOD);*

(g) *for every set Λ, the module R_R^Λ is flat (w.resp. to R-MOD).*

Proof: $(a) \Leftrightarrow (b)$ has been shown in 26.3.

$(a) \Leftrightarrow (c)$ Every finitely presented R-module is the cokernel of a morphism $R^m \to R^n$, $m, n \in I\!N$, and hence coherent, if R is coherent (see 26.1). On the other hand, R itself is finitely presented.

$(a) \Leftrightarrow (d)$ is easily seen from 26.1.

$(a) \Leftrightarrow (e)$ has been shown in 26.5.

$(a) \Rightarrow (f)$ Let $\{N_\lambda\}_\Lambda$ be a family of flat right R-modules. We have to show that $\prod_\Lambda N_\lambda \otimes_R -$ is exact with respect to all exact sequences $0 \to I \stackrel{\varepsilon}{\to} {}_RR$ with ${}_RI$ finitely generated (see 12.16). With the canonical mappings φ (see 12.9) we obtain the commutative diagram with exact row

$$\begin{array}{ccccc} & & (\prod_\Lambda N_\lambda) \otimes_R I & \stackrel{id\otimes\varepsilon}{\longrightarrow} & (\prod_\Lambda N_\lambda) \otimes_R R \\ & & \downarrow \varphi_I & & \downarrow \varphi_R \\ 0 & \longrightarrow & \prod_\Lambda (N_\lambda \otimes I) & \longrightarrow & \prod_\Lambda (N_\lambda \otimes R) \end{array} .$$

Since ${}_RI$ is finitely presented, φ_I is an isomorphism. Therefore $id \otimes \varepsilon$ is monic and $\prod_\Lambda N_\lambda$ is flat.

$(f) \Rightarrow (g)$ is obvious.

$(g) \Rightarrow (a)$ Let I be a finitely generated left ideal of R. Then, for every index set Λ, we have the commutative diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & R^\Lambda \otimes_R I & \longrightarrow & R^\Lambda \otimes_R R \\ & & \downarrow \varphi_I & & \downarrow \varphi_R \\ 0 & \longrightarrow & I^\Lambda & \longrightarrow & R^\Lambda \end{array} .$$

From this we see that φ_I is an isomorphism and, by 12.9, ${}_RI$ is finitely presented.

26.7 Examples of left coherent rings:

(α) left noetherian rings (§ 27),

(β) rings whose finitely generated left ideals are projective (*semihereditary* rings, §§ 39,40),

(γ) regular rings (§ 37),

(δ) polynomial rings over any set of indeterminates over commutative noetherian rings.

Proof: We only have to show (δ): Let R be a commutative noetherian ring and $A = R[X_1, X_2, \dots]$ a polynomial ring over any set of indeterminates. If the number of indeterminates is finite, then A is noetherian (Hilbert Basis Theorem, see 27.6). Let I be a left ideal in A generated by $p_1, \dots, p_m \in A$. The p_i, $i \leq m$, only contain finitely many indeterminates, say $X_1, \dots, X_n$. Put $A_n = R[X_1, \dots, X_n]$. Then A_n is noetherian and all $p_i \in A_n$. Let I_n denote the ideal in A_n generated by $p_1, \dots, p_m$. In A_n-*MOD* we have an exact sequence

$$A_n^k \longrightarrow A_n^l \longrightarrow I_n \longrightarrow 0, \quad k,\, l \in I\!N.$$

Tensoring with $- \otimes_{A_n} A$ we obtain the commutative diagram with exact first row

$$\begin{array}{ccccccc}
A_n^k \otimes_{A_n} A & \longrightarrow & A_n^l \otimes_{A_n} A & \longrightarrow & I_n \otimes_{A_n} A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow\psi & & \\
A^k & \longrightarrow & A^l & \longrightarrow & I_nA & \longrightarrow & 0
\end{array} ,$$

in which the first two vertical mappings are canonical isomorphisms. Also $\psi : I_n \otimes A \to I_nA = I,\ i \otimes a \mapsto ia$, is an isomorphism since A is a free A_n-module (A may be regarded as polynomial ring over A_n). Hence the second row is also exact (see 7.19) and I is finitely presented in A-*MOD*.

26.8 Properties of left coherent rings.
Assume R to be a left coherent ring. Then

(1) For an R-module N the following assertions are equivalent:

(a) N is FP-injective (weakly R-injective);

(b) N is injective relative to $0 \to J \to R$ for finitely generated left ideals J;

(c) for every finitely generated left ideal $J \subset R$ and every $h \in Hom(J, N)$, there exists $u \in N$ with $(a)h = au$ for all $a \in J$.

(2) For any ideal I which is finitely generated as left ideal, the ring R/I is left coherent.

Proof: (1) This is just a translation of 26.2,(4) to the given situation (see Baer's Criterion 16.4).

(2) By 26.6, R/I is coherent in *R-MOD*. Hence a finitely generated left ideal in R/I is finitely presented in *R-MOD* and then, of course, finitely presented in R/I*-MOD* ($\subset$ *R-MOD*).

Not every factor ring of a left coherent ring is again a left coherent ring:

Consider $A = R[X_1, X_2, \ldots]$ with R commutative and noetherian and I the ideal of A generated by $X_1^2, X_1X_2, X_1X_3, \ldots$. A is coherent (see 26.7) but A/I is not coherent: The annihilator ideal of $X_1 + I$ in A/I is generated by $X_1 + I, X_2 + I, X_3 + I, \ldots$ and hence is not finitely generated. According to 26.6, A/I is not coherent.

26.9 Exercises.

(1) Show that for a ring R the following assertions are equivalent:

(a) R is left coherent;

(b) for every finitely presented N in MOD-R the left R-module $Hom_R(N, R)$ *is finitely generated;*

(c) a module N in MOD-R is flat if $N \otimes_R -$ *is exact with respect to exact sequences* $R^k \to R^l \to R \to L \to 0$ *in R-MOD with* $k, l \in I\!N$.

(Hint: For (b) observe 36.5; for (c) see 26.6.)

(2) Assume the R-module M to be self-projective and coherent in $\sigma[M]$ *and* $S = End({}_RM)$. *Show:*

(i) If M_S *is flat, then S is left coherent.*

(ii) If M_S *is flat and* ${}_RM$ *is weakly M-injective, then* ${}_SS$ *is FP-injective.*

(3) Show for a commutative ring R: If M and N are coherent R-modules, then $M \otimes_R N$ *and* $Hom_R(M, N)$ *are also coherent R-modules.*

Literature: STENSTRÖM; Damiano [1], Gomez-Hernandez, Hannick, Lenzing [1], Matlis [1,2], Osofsky, Sklyarenko [2], Soublin, Stenström, Xu Yan.

27 Noetherian modules and rings

1.Internal characterization. 2.Locally noetherian modules. 3.External characterization. 4.Matlis' Theorem. 5. Decomposition of injective modules. 6.Polynomial rings. 7.Exercises.

A non-empty set $\mathcal{M}$ of submodules of an R-module is called *noetherian* if it satisfies the *ascending chain condition (acc)*, i.e. if every ascending chain

$$M_1 \subset M_2 \subset \cdots \text{ of modules in } \mathcal{M}$$

becomes stationary after finitely many steps.

$\mathcal{M}$ is called *artinian* if it satisfies the *descending chain condition (dcc)*, i.e. every descending chain

$$M_1 \supset M_2 \supset \cdots \text{ of modules in } \mathcal{M}$$

becomes stationary after finitely many steps.

An R-module M is called *noetherian (artinian)* if the set of all submodules of M is noetherian. We call M *locally noetherian* if every finitely generated submodule of M is noetherian (artinian).

By definition R is a left noetherian (artinian) ring (see § 4) if and only if the module ${}_RR$ is noetherian (artinian).

27.1 Internal characterization of noetherian modules.

For any R-module M the following properties are equivalent:

(a) M is noetherian;

(b) the set of finitely generated submodules of M is noetherian;

(c) every non-empty set of (finitely generated) submodules of M has a maximal element;

(d) every submodule of M is finitely generated.

Proof: $(a) \Rightarrow (b)$ is trivial.

$(b) \Rightarrow (c)$ Let $\mathcal{U}$ be a non-empty set of (finitely generated) submodules of M. If $\mathcal{U}$ has no maximal element, then for every $U \in \mathcal{U}$ the set

$$\{U' \in \mathcal{U} \mid U' \supset U,\ U' \neq U\}$$

is not empty. Thus we obtain an infinite ascending chain of submodules.

$(c) \Rightarrow (d)$ Let N be a submodule of M. In the set of finitely generated submodules of N there is a maximal element N_o and obviously $N_o = N$.

$(d) \Rightarrow (a)$ Let $K_1 \subset K_2 \subset \cdots$ be an ascending chain of submodules of M. By assumption the submodule $\bigcup_{I\!N} K_i \subset M$ is finitely generated and all generating elements are contained in K_j for some $j \in I\!N$.

Hence $K_j = K_{j+t}$ for all $t \in I\!N$.

27.2 Properties of locally noetherian modules.

(1) Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence in R-MOD.

(i) If N is (locally) noetherian, then N' and N'' are (locally) noetherian.

(ii) If N' and N'' are noetherian, then N is noetherian.

(iii) If N' is noetherian and N'' is locally noetherian, then N is locally noetherian.

(2) The direct sum of locally noetherian modules is again locally noetherian.

Proof: $(1)(i)$ is easy to verify.

$(1)(ii)$ If N', N'' are noetherian and K is a submodule of N, we have the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K \cap N' & \longrightarrow & K & \longrightarrow & K/K \cap N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \ .
\end{array}$$

Since $K \cap N'$ and $K/K \cap N'$ are submodules of noetherian modules they are finitely generated and hence K is also finitely generated (see 13.9).

$(1)(iii)$ Let N' be noetherian, N'' locally noetherian and K a finitely generated submodule of N. Then in the above diagram $K \cap N'$ and $K/K \cap N'$ are noetherian modules. By $(1)(ii)$, K is also noetherian.

(2) By $(1)(ii)$, every finite direct sum of noetherian modules is noetherian. If N and M are locally noetherian, then $N \oplus M$ is locally noetherian: Let K be a submodule of $N \oplus M$ generated by finitely many elements $(n_1, m_1), \ldots, (n_r, m_r)$ in K (with $n_i \in N$, $m_i \in M$, $r \in I\!N$). The submodules $N' = \sum_{i \leq r} Rn_i \subset N$ and $M' = \sum_{i \leq r} Rm_i \subset M$ are noetherian by assumption and hence $N' \oplus M'$ is noetherian. Since $K \subset N' \oplus M'$, K is also noetherian.

By induction, we see that every finite direct sum of locally noetherian modules is locally noetherian. Then the corresponding assertion is true for arbitrary sums since every finitely generated submodule of it is contained in a finite partial sum.

By definition, '(locally) noetherian' is an internal property of a module M, i.e. it is independent of surrounding categories. However, there are also remarkable characterizations of this property in the category $\sigma[M]$:

27.3 External characterization of locally noetherian modules.
For an R-module M the following assertions are equivalent:

(a) *M is locally noetherian;*

(b) *$M^{(\mathbb{N})}$ is locally noetherian;*

(c) *$\sigma[M]$ has a set of generators consisting of noetherian modules;*

(d) *every finitely generated module in $\sigma[M]$ is noetherian;*

(e) *every finitely generated module is coherent in $\sigma[M]$;*

(f) *every finitely generated module is finitely presented in $\sigma[M]$;*

(g) *every module in $\sigma[M]$ is locally noetherian;*

(h) *every weakly M-injective module is M-injective;*

(i) *every direct sum of M-injective modules is M-injective;*

(j) *every countable direct sum of M-injective hulls of simple modules (in $\sigma[M]$) is M-injective;*

(k) *the direct limit of M-injective modules in $\sigma[M]$ is M-injective;*

(l) *there is a cogenerator Q in $\sigma[M]$ with $Q^{(\mathbb{N})}$ M-injective.*

Proof: $(a) \Leftrightarrow (b)$ follows from 27.2.

$(b) \Rightarrow (c)$ The finitely generated submodules of $M^{(\mathbb{N})}$ are noetherian and form a set of generators.

$(c) \Rightarrow (d)$ follows from 27.2, $(d) \Rightarrow (e) \Rightarrow (f)$ are obvious.

$(f) \Rightarrow (g)$ Let N be a finitely generated module in $\sigma[M]$ and $K \subset N$. Then N/K is finitely generated, hence finitely presented and consequently K is finitely generated, i.e. N is noetherian.

$(g) \Rightarrow (a)$ is trivial.

$(a) \Rightarrow (h)$ Let N be a finitely generated submodule of M and U a weakly M-injective R-module. Then every submodule $K \subset N$ is finitely generated and every diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & N \subset M \\ & & \downarrow & & \\ & & U & & \end{array}$$

can be extended commutatively by an $M \to U$. Hence U is N-injective for every finitely generated $N \subset M$ and, by 16.3, U is M-injective.

$(h) \Rightarrow (i)$ The direct sum of M-injective modules is always weakly M-injective (see 16.10).

$(i) \Rightarrow (j)$ is trivial.

$(j) \Rightarrow (b)$ Let K be a finitely generated submodule of $M^{(\mathbb{N})}$. We show that K satisfies the ascending chain condition for finitely generated submodules: Let $U_0 \subset U_1 \subset U_2 \subset \cdots$ be a strictly ascending chain of finitely generated submodules of K. In every U_i, $i \in I\!N$, we choose a maximal submodule $V_i \subset U_i$ with $U_{i-1} \subset V_i$ and obtain the ascending chain

$$U_0 \subset V_1 \subset U_1 \subset V_2 \subset U_2 \subset \cdots,$$

where the factors $E_i := U_i/V_i \neq 0$ are simple modules as long as $U_{i-1} \neq U_i$. With the M-injective hulls $\widehat{E}_i$ of E_i, $i \in I\!N$, and $U = \bigcup_{I\!N} U_i \subset K$ we get the commutative diagrams

$$\begin{array}{ccccc} 0 & \rightarrow & U_i/V_i & \rightarrow & U/V_i \\ & & \downarrow & \swarrow_{g_i} & \\ & & \widehat{E}_i & & \end{array}$$

and hence a family of mappings

$$f_i : U \xrightarrow{p_i} U/V_i \xrightarrow{g_i} \widehat{E}_i, \quad i \in I\!N,$$

yielding a map into the product: $\quad f : U \to \prod_{I\!N} \widehat{E}_i$.
Now any $u \in U$ is not contained in at most finitely many V_i's and hence $(u)f\pi_i = (u)f_i \neq 0$ only for finitely many $i \in I\!N$, which means $Im\, f \subset \bigoplus_{I\!N} \widehat{E}_i$. By assumption (j), this sum is M-injective and hence the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & U & \longrightarrow & K \\ & & f\downarrow & & \\ & & \bigoplus_{I\!N} \widehat{E}_i & & \end{array}$$

can be extended commutatively by an $h : K \to \bigoplus_{I\!N} \widehat{E}_i$. Since K is finitely generated, $Im\, h$ is contained in a finite partial sum, i.e.

$$(U)f \subset (K)h \subset \widehat{E}_1 \oplus \cdots \oplus \widehat{E}_r \quad \text{for some } \; r \in I\!N.$$

Then, for $k \geq r$, we must get $0 = (U)f_k = (U)p_k g_k$ and

$$0 = (U_k)f_k = (U_k/V_k)g_k = U_k/V_k,$$

implying $U_k = V_k$. Hence the sequence considered terminates at r and K is noetherian.

$(f) \Rightarrow (k)$ Let $0 \to K \to L \to N \to 0$ be an exact sequence in $\sigma[M]$ with L finitely generated (hence K, L, N are finitely presented) and $(Q_i, f_{ij})_\Lambda$ a direct system of M-injective modules in $\sigma[M]$. We obtain the following commutative diagram with exact first row (see 24.8)

$$\begin{array}{ccccccc} 0 \to & \varinjlim Hom(N, Q_i) & \to & \varinjlim Hom(L, Q_i) & \to & \varinjlim Hom(K, Q_i) & \to 0 \\ & \downarrow \Phi_N & & \downarrow \Phi_L & & \downarrow \Phi_K & \\ 0 \to & Hom(N, \varinjlim Q_i) & \to & Hom(L, \varinjlim Q_i) & \to & Hom(K, \varinjlim Q_i) & \to 0. \end{array}$$

Since Φ_N, Φ_L and Φ_K are isomorphisms (see 25.4), the second row also has to be exact. Hence $\varinjlim Q_i$ is injective with respect to all finitely generated $L \in \sigma[M]$ and therefore is M-injective (see 16.3).

$(k) \Rightarrow (i)$ Let $N \in \sigma[M]$ be a direct sum of M-injective modules $\{N_\lambda\}_\Lambda$. Then all the finite partial sums are M-injective, their direct limit is equal to N (see 24.7) and, by (k), it is M-injective.

$(i) \Rightarrow (l)$ is obvious.

$(l) \Rightarrow (a)$ will be shown in 28.4,(3).

The following result shows that in the locally noetherian case the investigation of injective modules can be reduced to indecomposable injective modules. A somewhat more general assertion (with a similar proof) will be considered in 28.6.

27.4 Matlis' Theorem.

Let M be a locally noetherian R-module. Then every injective module in $\sigma[M]$ is a direct sum of indecomposable modules with local endomorphism rings.

Proof: Assume U to be an M-injective module in $\sigma[M]$.

(i) U contains an indecomposable M-injective submodule: If U is not indecomposable, then there is a direct summand $L \neq U$. Choose a $u \in U \setminus L$ and consider the set

$$\mathcal{L}_u = \{L' \subset U \mid L' \text{ is } M\text{-injective, } u \notin L'\}.$$

$\mathcal{L}_u$ is not empty ($L \in \mathcal{L}_u$) and inductive (by inclusion) since the union of a chain of M-injective submodules again is M-injective (see 27.3). By Zorn's Lemma, there is a maximal element L_o in $\mathcal{L}_u$ and a submodule $F \subset U$ with $U = L_o \oplus F$. This F is indecomposable: Assume $F = F_1 \oplus F_2$. Then $(L_o + F_1) \cap (L_o + F_2) = L_o$ and hence $u \notin L_o + F_1$ or $u \notin L_o + F_2$. Now $u \notin L_o + F_i$ implies $L_o + F_i \in \mathcal{L}_u$ for $i = 1$ or 2. Because of the maximality of L_o, we conclude $F_1 = 0$ or $F_2 = 0$.

(ii) Now let G be a maximal direct sum of indecomposable M-injective submodules of U. G is M-injective by 27.3, i.e. $U = G \oplus H$. Then H is also M-injective and, if $H \neq 0$, by (i), it contains a non-zero indecomposable summand. This contradicts the maximality of G. Hence $U = G$ is a direct sum of indecomposable M-injective modules whose endomorphism rings are local by 19.9.

We shall see in the next theorem that the decomposition properties of injective modules described in 27.4 characterize locally noetherian categories

$\sigma[M]$. The proof for this requires relationships involving the cardinality of modules and their generating subsets:

The socle of every R-module M is a semisimple module, i.e. $Soc(M) = \bigoplus_\Lambda M_\lambda$ with simple modules $M_\lambda \neq 0$. Let $c(M)$ denote $card(\Lambda)$. By 20.5, this cardinal number is uniquely determined and obviously

$$c(M) \leq card(Soc(M)) \leq card(M).$$

If M is indecomposable and self-injective, then $Soc(M)$ is zero or simple, i.e. $c(M) = 0$ or $c(M) = 1$.

27.5 Decomposition of injective modules.

For an R-module M the following assertions are equivalent:

(a) M is locally noetherian;

(b) every injective module in $\sigma[M]$ is a direct sum of indecomposable modules;

(c) there is a cardinal number κ, such that every injective module in $\sigma[M]$ is a direct sum of modules $\{N_\lambda\}_\Lambda$ with $c(N_\lambda) \leq \kappa$;

(d) there is a cardinal number κ', such that every injective module in $\sigma[M]$ is a direct sum of modules $\{N'_\lambda\}_\Lambda$ with $card(N'_\lambda) \leq \kappa'$.

Proof: $(a) \Rightarrow (b)$ is the assertion of Matlis' Theorem 27.4.

$(b) \Rightarrow (c)$ The assertion holds for $\kappa = 1$.

$(b) \Rightarrow (d)$ Every indecomposable injective module in $\sigma[M]$ is an injective hull of a cyclic module in $\sigma[M]$. Since the totality of the isomorphism classes of cyclic modules form a set, the isomorphism classes of indecomposable injective modules also form a set $\{E_\alpha\}_A$. Then the assertion holds for $\kappa' = card(\bigcup_A E_\alpha)$.

$(d) \Rightarrow (c)$ immediately follows from the inequality

$$c(N_\lambda) \leq card(Soc(N_\lambda)) \leq card(N_\lambda).$$

$(c) \Rightarrow (a)$ By 27.3, we have to show that every countable direct sum of M-injective hulls of simple modules in $\sigma[M]$ is again M-injective.

Let $\{E_n\}_{I\!N}$ be a family of simple modules in $\sigma[M]$ and $\widehat{E}_n$ an M-injective hull of E_n, $n \in I\!N$. We take an index set Δ with $card(\Delta) \gneqq \kappa$ and put

$$E = \bigoplus\nolimits_{I\!N} E_i, \quad F = E^{(\Delta)}.$$

For the M-injective hull $\widehat{F}$ of F we get (recall $F \trianglelefteq \widehat{F}$)

$$Soc(\widehat{F}) = Soc(F) = E^{(\Delta)}.$$

By assumption (c), $\widehat{F}$ can be written as $\widehat{F} = \bigoplus_\Lambda N_\lambda$ with $c(N_\lambda) \leq \kappa$. For every $\lambda \in \Lambda$ we have, by (c), $Soc(N_\lambda) = \bigoplus_{A_\lambda} G_\alpha$, with G_α simple and $card(A_\lambda) \leq \kappa$, and hence

$$Soc(\widehat{F}) = \bigoplus\nolimits_\Lambda \bigoplus\nolimits_{A_\lambda} G_\alpha = (\bigoplus\nolimits_{I\!N} E_n)^{(\Delta)}.$$

For every $n \in I\!N$, denote

$$\Lambda(n) := \{\lambda \in \Lambda \mid \text{there exists } \alpha \in A_\lambda \text{ with } G_\alpha \simeq E_n\}.$$

Any of these sets $\Lambda(n)$ is infinite: Assume one of them to be finite, i.e. for some $n \in I\!N$ we have $\Lambda(n) = \{\lambda_1, \ldots, \lambda_k\}$, $k \in I\!N$. For the number of the summands isomorphic to E_n we get the inequality

$$card(\Delta) \leq card(A_{\lambda_1}) + \cdots + card(A_{\lambda_k}) \leq k \cdot \kappa.$$

If κ is finite, then also $k \cdot \kappa$ is finite. If κ is infinite, then $k \cdot \kappa = \kappa$ and $card(\Delta) \leq \kappa$. In both cases we have a contradiction to the choice of Δ.

The $\Lambda(n)$ being infinite, we can choose a sequence of different elements $\delta_1, \delta_2, \ldots$ with $\delta_n \in \Lambda(n)$. For every $n \in I\!N$, there is an $\alpha_n \in A_{\delta_n}$ with $G_{\alpha_n} \simeq E_n$. Since N_{δ_n} is M-injective and $G_{\alpha_n} \subset N_{\delta_n}$, the module N_{δ_n} contains an M-injective hull $\widehat{G}_{\alpha_n}$ of G_{α_n}, i.e. $N_{\delta_n} = \widehat{G}_{\alpha_n} \oplus H_n$ for some $H_n \subset N_{\delta_n}$ and

$$\bigoplus\nolimits_{I\!N} N_{\delta_n} \simeq (\bigoplus\nolimits_{I\!N} \widehat{G}_{\alpha_n}) \oplus (\bigoplus\nolimits_{I\!N} H_n) \simeq (\bigoplus\nolimits_{I\!N} \widehat{E}_n) \oplus (\bigoplus\nolimits_{I\!N} H_n).$$

Now $\{\delta_n \mid n \in I\!N\} \subset \Lambda$ and $\bigoplus_{I\!N} N_{\delta_n}$ is a direct summand of $\bigoplus_\Lambda N_\lambda = \widehat{F}$

(see 9.7). Consequently $\bigoplus_{I\!N} \widehat{E}_n$ is a direct summand of $\widehat{F}$ and hence it is M-injective.

For $M = R$ the preceding assertions 27.3 and 27.5 yield characterizations of left noetherian rings by properties of *R-MOD*. Over a noetherian ring every module is locally noetherian (see 27.3).

A variety of examples of noetherian rings can be derived from the *Hilbert Basis Theorem:*

27.6 Polynomial rings over noetherian rings.

Assume R to be a left noetherian ring. Then polynomial rings in finitely many commuting indeterminates over R are also left noetherian.

Proof: It is enough to show that $R[X]$ is left noetherian if R is left noetherian. Assume there exists an ideal I in $R[X]$ which is not finitely generated. We choose a polynomial $f_1 \in I$ with smallest degree. Then the left ideal $(f_1]$ generated by f_1 is not equal to I. Now let f_2 be a polynomial with smallest degree in $I \setminus (f_1]$. Again $(f_1, f_2] \neq I$ and we find a polynomial f_3 with smallest degree in $I \setminus (f_1, f_2]$. By recursion we obtain f_{k+1} as a polynomial with smallest degree in $I \setminus (f_1, \dots, f_k]$.

Put n_k = degree of f_k and denote by a_k the leading coefficient of f_k. By construction, $n_1 \leq n_2 \leq \cdots$ and $(a_1] \subset (a_1, a_2] \subset \cdots$ is an ascending chain of ideals in R which does not become stationary:

Assume $(a_1, \dots, a_k] = (a_1, \dots, a_{k+1}]$ for some $k \in \mathbb{N}$.
Then $a_{k+1} = \sum_{i=1}^{k} r_i a_i$ for suitable $r_i \in R$ and

$$g = f_{k+1} - \sum_{i=1}^{k} r_i X^{n_{k+1}-n_i} f_i \in I \setminus (f_1, \dots, f_k].$$

Since the coefficient of $X^{n_{k+1}}$ in g is zero, we have

$$deg(g) < deg(f_{k+1}) = n_{k+1} \ .$$

This contradicts the choice of f_{k+1}. Hence every left ideal in $R[X]$ has to be finitely generated.

27.7 Exercises.

(1) Let M be a finitely generated, self-projective R-module. Prove that End(M) is left noetherian if and only if M satisfies the ascending chain condition for M-generated submodules.

(2) Show for an R-module M: If every self-injective module in $\sigma[M]$ is M-injective, then M is co-semisimple and locally noetherian.

(3) For rings R, S and a bimodule ${}_RM_S$, the set $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *forms a ring with the usual matrix operations (see 6.7,(4)). Show:*

(i) A is right noetherian if and only if R_R, S_S and M_S are noetherian.

(ii) A is left noetherian if and only if ${}_RR$, ${}_SS$ and ${}_RM$ are noetherian.

(4) Show that for an R-module M each of the following assertions are equivalent:

(i) (a) M is noetherian;

(b) every countably generated submodule of M is finitely generated.

(ii) (a) M is semisimple;

(b) every countably generated submodule of M is a direct summand.

(5) Let R be a commutative, noetherian ring and A an R-algebra with unit. Show:

If A is finitely generated as an R-module, then A is left and right noetherian as a ring.

(6) Let M be a finitely generated module over the commutative ring R. Show that M is noetherian if and only if $R/An_R(M)$ is noetherian.

(7) Let M be an R-module which is finitely presented in $\sigma[M]$. Show that the following assertions are equivalent:
(a) M is noetherian;
(b) a module $N \in \sigma[M]$ is M-injective if every exact sequence $0 \to N \to X \to P \to 0$ in $\sigma[M]$ with P finitely presented splits.

Literature: ALBU-NĂSTĂSESCU, COZZENS-FAITH, STENSTRÖM; Albu [1], Antunes-Nicolas, Beachy-Weakley [1], Beck [1], Contessa, Fisher [1,2], Ginn, Gordon-Green, Heinzer-Lantz, van Huynh [3], Karamzadeh, Kurshan, Năstăsescu [3], Rayar [1], Renault [2], Rososhek, Shock, Smith [2], Zöschinger [7].

28 Annihilator conditions

1.Annihilators. 2.Annihilator conditions and injectivity. 3.Ascending chain condition. 4.Injectivity of direct sums. 5.$L^{(\mathbb{N})}$ as a direct summand in $L^{\mathbb{N}}$. 6.Decomposition of M with $M^{(\mathbb{N})}$ injective. 7.Exercises.

Injectivity and cogenerator properties of a module M are reflected by *annihilator conditons* in $Hom_R(N,M)_S$ and ${}_RM$:

28.1 Annihilators. Definitions and properties.
Let N and M be R-modules and $S = End_R(M)$. We denote by
$\mathcal{N}$ the set of R-submodules of N and by
$\mathcal{H}$ the set of S-submodules of $Hom_R(N,M)_S$.
For $K \in \mathcal{N}$ and $X \in \mathcal{H}$ we put:

$$\begin{aligned} An(K) &= \{f \in Hom(N,M) \mid (K)f = 0\} (\simeq Hom_R(N/K,M)) \in \mathcal{H}, \\ Ke(X) &= \bigcap \{Ke\, g \mid g \in X\} \in \mathcal{N}. \end{aligned}$$

This yields order reversing mappings

$$An : \mathcal{N} \to \mathcal{H}, \quad Ke : \mathcal{H} \to \mathcal{N}.$$

(1) An and Ke induce a bijection between the subsets

$$\begin{aligned} \mathcal{A}(N,M) &= \{An(K) \mid K \subset N\} \subset \mathcal{H} \ \text{ and} \\ \mathcal{K}(N,M) &= \{Ke(X) \mid X \subset Hom(N,M)\} \subset \mathcal{N}. \end{aligned}$$

These are called *annihilator submodules.*

(2) $Ke\, An(K) = K$ for all $K \in \mathcal{N}$, i.e. $\mathcal{N} = \mathcal{K}(N,M)$, if and only if every factor module of N is M-cogenerated.

(3) If M is N-injective, then

$$(i) \quad An(K_1 \cap K_2) = An(K_1) + An(K_2) \ \text{for all} \ K_1, K_2 \subset N.$$

(4) If M is self-injective, or N is finitely generated and M is weakly M-injective, then

(ii) for every finitely generated S-submodule $X \subset Hom(N,M)_S$ $An\, Ke(X) = X$, i.e. $X \in \mathcal{A}(N,M)$.

Proof: (1) For $K \in \mathcal{N}$ and $X \in \mathcal{H}$ we obviously have $K \subset Ke\, An(K)$ and $X \subset An\, Ke(X)$.

The first relation implies $An(K) \supset An\, Ke\, An(K)$ and, for $X = An(K)$, the second yields $An(K) \subset An\, Ke\, An(K)$. Hence $An\, Ke(-)$ is the identity on $\mathcal{A}(N, M)$.

Similarly we see that $Ke\, An(-)$ is the identity on $\mathcal{K}(N, M)$.

(2) For $K \subset N$ we have $K = \bigcap\{Ke\, f \mid f \in Hom(N, M),\ (K)f = 0\}$ if and only if N/K is M-cogenerated.

(3) Obviously $An(K_1 \cap K_2) \supset An(K_1) + An(K_2)$. The diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N/K_1 \cap K_2 & \longrightarrow & N/K_1 \oplus N/K_2 \\ & & \downarrow g & & \\ & & M & & \end{array}$$

can be extended commutatively by an $N/K_1 \oplus N/K_2 \to M$ (since M is N-injective, see 16.2).

This means $An(K_1 \cap K_2) \subset An(K_1) + An(K_2)$.

(4) Consider $X = f_1 S + \cdots + f_k S$ with $f_i \in Hom_R(N, M)$.

Then $Ke(X) = \bigcap_{i \le k} Ke\, f_i$ and, for every $g \in Hom(N/Ke(X), M)$, the diagram with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & N/Ke(X) & \xrightarrow{(f_i)} & M^k \\ & & \downarrow g & & \\ & & M & & \end{array}$$

can be extended commutatively by some $\sum s_i : M^k \to M$, $s_i \in S$, i.e. $g = \sum_{i \le k} f_i s_i \in X$ and $X = An(Ke(X))$.

It is interesting to observe that the annihilator conditions (i) and (ii) considered in 28.1 for the case $N = M$ imply some injectivity properties:

28.2 Annihilator conditions and injectivity.

Let M be an R-module, $S = End_R(M)$, and assume

(i) $An(K_1 \cap K_2) = An(K_1) + An(K_2)$ for submodules $K_1, K_2 \subset M$ and

(ii) $X = An\, Ke(X) (= Hom_R(M/Ke(X), M))$ for every finitely generated right ideal $X \subset S$.

Then $Hom(-, M)$ is exact with respect to exact sequences $0 \to U \xrightarrow{i} M$ with U finitely M-generated.

Proof: By assumption, U is a homomorphic image of M^n for some $n \in I\!N$. The proof is by induction on n.

If $n = 1$, then there is an epimorphism $f : M \to U$. Consider $g \in Hom_R(U, M)$. Since $fi \in S$ we apply (ii) to get

$$f\, Hom(U, M) = Hom(M/Ke\, f, M) = fiS,$$

i.e. $fg = fis$ for some $s \in S$. f being epic we conclude $g = is$. The situation is illustrated by the commutative diagram

$$\begin{array}{ccccc} & & M & & \\ & & f\downarrow & & \\ 0 & \to & U & \xrightarrow{i} & M \\ & & g\downarrow & \swarrow s & \\ & & M & & . \end{array}$$

Now assume the assertion is true for all $k \leq n-1$. Let $h : M^n \to U$ be epic and $g \in Hom(U, M)$. Then U can be written as $U = U_1 + U_2$ with epimorphisms $h_1 : M^{n-1} \to U_1$ and $h_2 : M \to U_2$.

By assumption, there is an $s_1 \in S$ with $(u_1)g = (u_1)s_1$ for all $u_1 \in U_1$. For the map

$$g_1 := is_1 : U \xrightarrow{i} M \xrightarrow{s_1} M$$

we get $U_1(g - g_1) = 0$. Now the diagram

$$\begin{array}{cccc} 0 & \longrightarrow & U_2 & \xrightarrow{i} & M \\ & & \downarrow g - g_1 & & \\ & & M & & \end{array}$$

can be extended commutatively by an $s_2 \in S$.

By construction, $U_1 \cap U_2 \subset Ke\, s_2$ and, by (i), we have

$$s_2 \in An(U_1 \cap U_2) = An(U_1) + An(U_2).$$

Hence $s_2 = t_1 + t_2$ with $t_1 \in An(U_1)$, $t_2 \in An(U_2)$ whereby $(u_2)s_2 = (u_2)t_1$ for all $u_2 \in U_2$.

For $s_1 + t_1 \in S$ we now show $g = i(s_1 + t_1)$:
If $u = u_1 + u_2 \in U_1 + U_2$ we have from our construction

$$\begin{array}{rcl} u(s_1 + t_1) & = & (u_1 + u_2)(s_1 + t_1) = u_1 s_1 + u_2 s_1 + u_2 t_1 \\ & = & u_1 g + u_2 s_1 + u_2(g - g_1) = (u_1 + u_2)g + u_2(s_1 - g_1) = ug. \end{array}$$

The property that, for a (weakly) M-injective module Q, any direct sum $Q^{(\Lambda)}$ is M-injective can be expressed by chain conditions for annihilators.

By the bijection between $\mathcal{A}(N, M)$ and $\mathcal{K}(N, M)$ considered in 28.1, it is clear that the ascending chain condition in one set is equivalent to the descending chain condition in the other.

If N is a noetherian module, then of course $\mathcal{K}(N, M)$ is also noetherian. The converse conclusion need not be true. However we can show:

28.3 Ascending chain condition for annihilators.
With the above notation let $\mathcal{K}(N,M)$ be noetherian. Then

(1) For every submodule $K \subset N$ there is a finitely generated submodule $K_o \subset K$ with $An(K) = An(K_o)$.

(2) If M is self-injective or N is finitely generated and M is weakly M-injective, then $Hom(N,M)_S$ has dcc on finitely generated S-submodules.

Proof: (1) We write $K = \sum_\Lambda K_\lambda$ with finitely generated $K_\lambda \subset K$ and obtain $An(K) = An(\sum_\Lambda K_\lambda) = \bigcap_\Lambda An(K_\lambda)$. Since $\mathcal{K}(N,M)$ is noetherian, we have the descending chain condition in $\mathcal{A}(N,M)$ and hence the intersection can be written with finitely many K_λ's,

$$An(K) = \bigcap\nolimits_{i\leq k} An(K_{\lambda_i}) = An(\sum\nolimits_{i\leq k} K_{\lambda_i}),$$

with $K_o = \sum_{i\leq k} K_{\lambda_i}$ finitely generated.

(2) We have seen in 28.1 that, under the given assumptions, for every finitely generated S-submodule $X \subset Hom(N,M)$, $X = An\,Ke(X)$.

28.4 Injectivity of direct sums.
Let M be an R-module.

(1) For a finitely generated R-module N the following are equivalent:
(a) $M^{(\Lambda)}$ is N-injective for every index set Λ;
(b) $M^{(\mathbb{N})}$ is N-injective;
(c) M is weakly N-injective and $\mathcal{K}(N,M)$ is noetherian.

(2) The following assertions are equivalent:
(a) $M^{(\Lambda)}$ is M-injective for every index set Λ;
(b) $M^{(\mathbb{N})}$ is M-injective;
(c) M is weakly N-injective and $\mathcal{K}(N,M)$ is noetherian for
(i) every finitely generated module N in $\sigma[M]$, or
(ii) every finitely generated submodule $N \subset M$, or
(iii) every N in a set of finitely generated generators of $\sigma[M]$.

(3) M is locally noetherian if and only if there is a cogenerator Q in $\sigma[M]$ with $Q^{(\mathbb{N})}$ M-injective.

Proof: (1) $(a) \Rightarrow (b)$ is trivial.

$(b) \Rightarrow (c)$ Let $M^{(\mathbb{N})}$ be N-injective and assume there is a strictly ascending chain $K_1 \subset K_2 \subset \cdots$ of modules in $\mathcal{K}(N,M)$. Put $K = \bigcup_{\mathbb{N}} K_i \subset N$ and choose, for every $i \in \mathbb{N}$, an $f_i \in Hom(N,M)$ with

$$f_i \in An(K_i) \setminus An(K_{i+1}).$$

Then $(K_j)f_i = 0$ for all $j \leq i$ and $(K_l)f_i \neq 0$ for all $l \geq i+1$. The product of the $\{f_i\}_{I\!N}$ yields a map

$$f = \prod\nolimits_{I\!N} f_i : K \to M^{I\!N} \quad \text{with } (K)f \subset M^{(I\!N)},$$

since every $n \in K$ lies in a K_i, $i \in I\!N$. By assumption, f can be extended to an $\tilde{f} : N \to M^{(I\!N)}$ and – since N is finitely generated – we can assume $Im\,\tilde{f} \subset M^r$ and hence $(K)f \subset M^r$ for some $r \in I\!N$.

As a consequence $(K_l)f_{r+1} = 0$ for almost all $l \in I\!N$. This is a contradiction to the choice of the f_i. Hence $\mathcal{K}(N, M)$ is noetherian.

$(c) \Rightarrow (a)$ Now assume $\mathcal{K}(N, M)$ to be noetherian, M weakly N-injective and consider the diagram with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & N \\ & & \downarrow f & & \\ & & M^{(\Lambda)} & & . \end{array}$$

By 28.3, there is a finitely generated submodule $K_o \subset K$ with $An(K_o) = An(K)$ (in $Hom_R(N, M)$).

Then $(K_o)f \subset M^k$ for a finite partial sum $M^k \subset M^{(\Lambda)}$, $k \in I\!N$. With the canonical projections $\pi_\lambda : M^{(\Lambda)} \to M$ we may assume $(K_o)f\pi_\lambda = 0$ for all $\lambda \notin \{1, \ldots, k\}$. By the choice of K_o, this also means $(K)f\pi_\lambda = 0$ for all $\lambda \notin \{1, \ldots, k\}$ and hence $Im\, f \subset M^k$.

Since M^k is weakly N-injective, there exists a morphism $g : N \to M^k$ with $f|_{K_o} = g|_{K_o}$, and $K_o(f - g) = 0$ implies $K(f - g) = 0$. Hence the above diagram is commutatively extended by g.

(2) $(a) \Rightarrow (b)$ is trivial and $(b) \Rightarrow (c)$ follows from (1) since M is N-injective for every N in $\sigma[M]$.

$(c) \Rightarrow (a)$ By (1), $M^{(I\!N)}$ is N-injective for the sets of modules described in (i), (ii) or (iii). By 16.3, this implies in each case the M-injectivity of $M^{(I\!N)}$.

(3) Let Q be an injective cogenerator in $\sigma[M]$. If M is locally noetherian, then $Q^{(I\!N)}$ is M-injective by 27.3. On the other hand, if $Q^{(I\!N)}$ is M-injective, then by (2), for every finitely generated module $N \in \sigma[M]$, the set $\mathcal{K}(N, Q)$ is noetherian. However, for a cogenerator Q, every submodule of N belongs to $\mathcal{K}(N, Q)$ and hence N is noetherian.

An interesting case with noetherian $\mathcal{K}(N, M)$ is the following:

28.5 $L^{(I\!N)}$ as a direct summand in $L^{I\!N}$.
Let M be an R-module and $L \in \sigma[M]$.

(1) If $L^{(\mathbb{N})}$ is a direct summand in $L^{\mathbb{N}}$ (product in $\sigma[M]$), then $\mathcal{K}(N,L)$ is noetherian for every finitely generated $N \in \sigma[M]$.

(2) The following assertions are equivalent:
(a) $L^{(\Lambda)}$ is M-injective for every index set Λ;
(b) $L^{(\mathbb{N})}$ is a direct summand in $L^{\mathbb{N}}$ (product in $\sigma[M]$), and L is weakly N-injective for

(i) $N = M$, or

(ii) all finitely generated submodules $N \subset M$, or

(iii) all N in a set of generators for $\sigma[M]$.

Proof: (1) Let $N \in \sigma[M]$, N finitely generated, and $K_1 \subset K_2 \subset \cdots$ a strictly ascending chain of modules in $\mathcal{K}(N,L)$. For every $i \in \mathbb{N}$ we choose an $f_i \in Hom(N,L)$ with $f_i \in An(K_i) \setminus An(K_{i+1})$ (as in the proof 28.4,(1)). Putting $L_i = L$ for $i \in \mathbb{N}$, we get, with the product in $\sigma[M]$, the mappings

$$N \longrightarrow \prod\nolimits_{\mathbb{N}} N/K_i \xrightarrow{\prod f_i} \prod\nolimits_{\mathbb{N}} L_i \longrightarrow \bigoplus\nolimits_{\mathbb{N}} L_i .$$

Since N is finitely generated, the image of this homomorphism is contained in a finite partial sum $L_1 \oplus \cdots \oplus L_k \subset \bigoplus_{\mathbb{N}} L_i$. This means $f_j = 0$ for all $j > k$, contradicting the choice of f_i. Hence the chain considered has to be finite.

(2) Because of (1), this is a consequence of 28.4,(1) since injectivity with respect to the given modules in each case implies M-injectivity.

In Matlis' Theorem we have proved the decomposition of locally noetherian injective modules. It was observed by *A. Cailleau* that such a decomposition can be found more generally for the modules studied next:

28.6 Decomposition of M with $M^{(\mathbb{N})}$ injective.
Let M be an R-module with $M^{(\mathbb{N})}$ M-injective. Then:

(1) Every direct summand of M contains an indecomposable direct summand.

(2) M is a direct sum of indecomposable summands.

Proof: (1) Let L be a direct summand of M. Then $L^{(\mathbb{N})}$ is L-injective. Consider a finitely generated module $N \in \sigma[M]$ with $Hom(N, L) \neq 0$. By 28.4, $\mathcal{K}(N, L)$ is noetherian and in particular the ascending chain condition for submodules $\{Ke\, f \mid 0 \neq f \in Hom(N, L)\}$ is satisfied.

Choose $g \in Hom(N, L)$ with $Ke\, g$ maximal in this set. Put $G = \widehat{Im\, g}$, the M-injective hull of $Im\, g$. Since L is M-injective we may regard G as a direct summand of L.

G is indecomposable: Assume $G = G_1 \oplus G_2$ to be a decomposition. Then

$$Hom(N, G) = Hom(N, G_1) \oplus Hom(N, G_2)$$

and $g = g_1 + g_2$ with $g_i \in Hom(N, G_i)$. We have $Ke\, g = Ke\, g_1 \cap Ke\, g_2$ and hence (because of the maximality of $Ke\, g$) $Ke\, g = Ke\, g_1 = Ke\, g_2$. If $g_1 \neq 0$, then $Im\, g \cap Im\, g_1 \neq 0$ ($Im\, g \trianglelefteq G$) and $0 \neq (n)g \in Im\, g_1$ for some $n \in N$. From

$$(n)g - (n)g_1 = (n)g_2 \in Im\, g_1 \cap Im\, g_2 = 0$$

we deduce $n \in Ke\, g_2$ but $n \notin Ke\, g_1$. This is a contradiction to $Ke\, g_1 = Ke\, g_2$, and hence G has to be indecomposable.

(2) This is shown in a similar way to the proof of Matlis' Theorem:
Let $\{M_\lambda\}_\Lambda$ be a maximal family of independent direct summands of M. Then the internal direct sum $\bigoplus_\Lambda M_\lambda$ is isomorphic to a direct summand of $M^{(\Lambda)}$ and hence is M-injective. Therefore $M = (\bigoplus_\Lambda M_\lambda) \oplus L$. Assume $L \neq 0$. Then by (1), there is a non-trivial indecomposable summand in L, contradicting the maximality of $\{M_\lambda\}_\Lambda$, i.e. $M = \bigoplus_\Lambda M_\lambda$.

28.7 Exercises.

(1) Let M be an R-module. Then $\mathcal{K}(R, M)$ are just the left ideals in R which annihilate subsets of M. Show that the following are equivalent:
(a) $M^{(\Lambda)}$ is R-injective for every index set Λ;
(b) $M^{(\mathbb{N})}$ is R-injective;
(c) M is FP-injective and $\mathcal{K}(R, M)$ is noetherian;
(d) M is (FP-) injective and $M^{(\mathbb{N})}$ is a direct summand in $M^{\mathbb{N}}$ (product in R-MOD).

(2) Let Q be an injective cogenerator in R-MOD. Show:
${}_RR$ is noetherian if and only if $Q^{(\mathbb{N})}$ is R-injective.

Literature: ALBU-NĂSTĂSESCU, STENSTRÖM;
Baer, Beck [1], Brodskii [2], Cailleau-Renault, Camillo [2], Faith [1], Gomez [3], Gupta-Varadarajan, Harada-Ishii, Izawa [3], Johns [1,2], Lenzing [3], Masaike [2], Megibben [2], Miller-Turnidge [1,2], Năstăsescu [3], Prest, Takeuchi,Y., Yue [3,5], Zelmanowitz [2], Zimmermann [1] .

Chapter 6

Dual finiteness conditions

29 The inverse limit

1.Definition. 2.Construction. 3.Inverse limit of morphisms. 4.Inverse systems of exact sequences. 5.Hom-functors and limits. 6.Inverse limit of submodules. 7.Linearly compact modules. 8.Properties of linearly compact modules. 9.Properties of f-linearly compact modules. 10.Characterization of finitely cogenerated modules. 11.Exercises.

The notion of an *inverse limit* is dual to the notion of a *direct limit*. Existence and some properties are obtained dual to the considerations in § 24. However, not all properties of direct limits can be dualized in module categories.

Let $(\Delta, \leq)$ be a quasi-ordered set. Occasionally it will be useful to have Δ as a directed set.

An *inverse system of R-modules* $(N_i, f_{ji})_\Delta$ consists of

(1) a family of modules $\{N_i\}_\Delta$ and

(2) a family of morphisms $f_{ji} : N_j \to N_i$, for all pairs (j, i) with $i \leq j$, satisfying

$$f_{ii} = id_{N_i} \quad \text{and} \quad f_{kj} f_{ji} = f_{ki} \text{ for } i \leq j \leq k.$$

An *inverse system of morphisms* of an R-module L into $(N_i, f_{ji})_\Delta$ is a family of morphisms

$$\{v_i : L \to N_i\}_\Delta \;\text{ with }\; v_j f_{ji} = v_i \text{ for } i \leq j.$$

29.1 Inverse limit. Definition.

Let $(N_i, f_{ji})_\Delta$ be an inverse system of R-modules and N an R-module. An inverse system of morphisms $\{f_i : N \to N_i\}_\Delta$ is called the *inverse limit of* $(N_i, f_{ji})_\Delta$ if, for every inverse system of morphisms $\{v_i : L \to N_i\}_\Delta$, $L \in R\text{-}MOD$, there is a unique morphism $v : L \to N$ making the following diagram commutative, for every $i \in \Delta$,

$$\begin{array}{ccccc} N & & \xrightarrow{f_i} & & N_i \\ & {}^{v}\nwarrow & & \nearrow_{v_i} & \\ & & L & & \end{array} \quad .$$

If $\{f_i' : N' \to N_i\}_\Delta$ is also an inverse limit of $(N_i, f_{ji})_\Delta$, then the definition implies the existence of an isomorphism $f' : N' \to N$ with $f'f_i = f_i'$ for all $i \in \Delta$. Hence N is unique up to isomorphism.

We usually write $N = \varprojlim N_i$ and $(f_i, \varprojlim N_i)$ for the inverse limit.

29.2 Construction of the inverse limit.

Let $(N_i, f_{ji})_\Delta$ be an inverse system of R-modules. For every pair $i \leq j$ in $\Delta \times \Delta$ we put $N_{j,i} = N_i$ and (with the canonical projections π_j) we obtain the mappings

$$\begin{array}{ccccc} \prod_\Delta N_k & \xrightarrow{\pi_j} & N_j & \xrightarrow{f_{ji}} & N_{j,i} \\ \prod_\Delta N_k & \xrightarrow{\pi_i} & N_i & \xrightarrow{id} & N_{j,i} \end{array} \quad .$$

The difference between these yields morphisms $\pi_j f_{ji} - \pi_i : \prod_\Delta N_k \to N_{j,i}$, and forming the product we get $F : \prod_\Delta N_k \to \prod_{i \leq j} N_{j,i}$.

$Ke\,F$ together with the mappings

$$f_i : Ke\,F \hookrightarrow \prod\nolimits_\Delta N_k \xrightarrow{\pi_i} N_i$$

forms an inverse limit of $(N_i, f_{ji})_\Delta$ *and*

$$Ke\,F = \{(n_k)_\Delta \in \prod\nolimits_\Delta N_k \mid n_j f_{ji} = n_i \textit{ for all } i \leq j\}.$$

Proof: Let $\{v_i : L \to N_i\}_\Delta$ be an inverse system of morphisms and $v : L \to \prod_\Delta N_k$ with $v_k = v\pi_k$, $k \in \Delta$. Since $v(\pi_j f_{ji} - \pi_i) = v_j f_{ji} - v_i = 0$ for $i \leq j$, we get $vF = 0$ and $Im\,v \subset Ke\,F$.

Hence $v : L \to Ke\,F$ is the desired morphism.

The presentation of $Ke\,F$ follows from the definition of F.

Remarks: (1) The interpretation of direct limits as functors (see Remark (1) after 24.2) similarly applies to inverse limits.

(2) The construction of the inverse limit in 29.2 is possible in all categories with products and kernels. In many cases the inverse limit can be interpreted as a subset of the cartesian product (with the given properties) and the canonical projections (e.g. for groups, rings, sets, etc.).

(3) In particular, every inverse system of modules in $\sigma[M]$ ($M \in R$-*MOD*) has an inverse limit in $\sigma[M]$.

(4) Instead of 'inverse limit' the notions *projective limit* or just *limit* are also used.

(5) In case Δ has just three elements i, j, k with $i \neq j$, $k < i$, $k < j$, the inverse limit of an inverse system over Δ yields the pullback (to the given morphisms).

(6) For the quasi-ordered set $(\Delta, =)$ (not directed) the direct product $\prod_\Delta N_i$ is equal to the inverse limit $\varprojlim N_i$.

29.3 Inverse limit of morphisms.
Let $(N_i, f_{ji})_\Delta$ and $(L_i, g_{ji})_\Delta$ be two inverse systems of R-modules over Δ with inverse limits $(\pi_i, \varprojlim N_i)$ and $(\pi'_i, \varprojlim L_i)$.

If $\{v_i : N_i \to L_i\}_\Delta$ is a family of morphisms with $v_j g_{ji} = f_{ji} v_i$ for all indices $i \leq j$, then there is a unique morphism

$$v : \varprojlim N_i \to \varprojlim L_i,$$

such that, for every $j \in \Delta$, the following diagram is commutative

$$\begin{array}{ccc} \varprojlim N_i & \xrightarrow{v} & \varprojlim L_i \\ \downarrow \pi_j & & \downarrow \pi'_j \\ N_j & \xrightarrow{v_j} & L_j \end{array} .$$

If all v_j are monic (isomorphisms), then v is monic (an isomorphism).

We write $v = \varprojlim v_i$.

Proof: The mappings $\{\pi_j v_j : \varprojlim N_i \to L_j\}_\Delta$ form an inverse system of morphisms since $\pi_j v_j g_{ji} = \pi_j f_{ji} v_i = \pi_i v_i$ for any $i \leq j$. Hence the existence of v is a consequence of the universal property of $\varprojlim L_i$.

Consider $(n_i)_\Delta \in Ke\, v$. Then $0 = (n_i)_\Delta v \pi'_j = (n_j) v_j$ for every $j \in \Delta$, i.e. all $n_j = 0$ if the v_j are monic and $Ke\, v = 0$. If all v_i are isomorphisms, then $\prod_\Delta v_i : \prod_\Delta N_i \to \prod_\Delta L_i$ is an isomorphism and v is obtained by the restriction of $\prod_\Delta v_i$ to the submodules $\varprojlim N_i$ and $\varprojlim L_i$.

Observe that for surjective v_i's the inverse limit $\varprojlim v_i$ need not be surjective.

To the family of morphisms $\{v_i : N \to L_i\}_\Delta$ given in 29.3 we may construct further inverse systems with the families $\{Ke\, v_i\}_\Delta$ and $\{Coke\ v_i\}_\Delta$ (see 24.5). Of particular interest is the following situation:

29.4 Inverse systems of exact sequences.
Let $(K_i, f_{ji})_\Delta$, $(L_i, g_{ji})_\Delta$ and $(N_i, h_{ji})_\Delta$ be inverse systems of modules with inverse limits $(f_i, \varprojlim K_i)$, $(g_i, \varprojlim L_i)$ resp. $(h_i, \varprojlim N_i)$.

Assume $\{u_i\}_\Delta$, $\{v_i\}_\Delta$ to be families of morphisms with the following diagrams commutative for $i \leq j$ and the rows exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_j & \xrightarrow{u_j} & L_j & \xrightarrow{v_j} & N_j \\ & & \downarrow f_{ji} & & \downarrow g_{ji} & & \downarrow h_{ji} \\ 0 & \longrightarrow & K_i & \xrightarrow{u_i} & L_i & \xrightarrow{v_i} & N_i \end{array} .$$

Then with $u = \varprojlim u_i$ and $v = \varprojlim v_i$ the following sequence is also exact:

$$0 \longrightarrow \varprojlim K_i \xrightarrow{u} \varprojlim L_i \xrightarrow{v} \varprojlim N_i.$$

Proof: We already know from 29.3 that u is monic. Also from 29.3 we have the commutativity of the diagram

$$\begin{array}{ccccc} \varprojlim K_i & \xrightarrow{u} & \varprojlim L_i & \xrightarrow{v} & \varprojlim N_i \\ \downarrow f_j & & \downarrow g_j & & \downarrow h_j \\ K_j & \xrightarrow{u_j} & L_j & \xrightarrow{v_j} & N_j \end{array} .$$

From this we see $uvh_j = f_j u_j v_j = 0$ for all $j \in \Delta$ implying $uv = 0$.

Consider $(l_i)_\Delta \in Ke\, v$. Then $l_j v_j = 0$ for every $j \in \Delta$, i.e. $l_j = (k_j)u_j$ and hence $(l_i)_\Delta \in Im\, u$.

Since the functor $Hom(L, -)$ preserves products and kernels, it also preserves inverse limits. In contrast, the covariant functor $Hom(-, L)$ converts direct limits to inverse limits:

29.5 Hom-functors and limits. *Let L be an R-module.*

(1) For an inverse system of R-modules $(N_i, f_{ji})_\Delta$ the morphisms ($i \leq j$)

$$Hom_R(L, f_{ji}) : Hom_R(L, N_j) \to Hom_R(L, N_i)$$

yield an inverse system of $\mathbb{Z}$-modules and the canonical map

$$Hom_R(L, \varprojlim N_i) \to \varprojlim Hom_R(L, N_i)$$

is an isomorphism.

(2) For a direct system of R-modules $(M_i, g_{ij})_\Delta$ the morphisms $(i \le j)$

$$Hom_R(g_{ij}, L) : Hom_R(N_j, L) \to Hom_R(N_i, L)$$

yield an inverse system of $\mathbb{Z}$-modules and the canonical map

$$Hom_R(\varinjlim M_i, L) \to \varprojlim Hom_R(M_i, L)$$

is an isomorphism.

Here Δ need not be directed.

These relations hold in any category in which the corresponding constructions are possible.

Let N be an R-module. A family $\{N_i\}_\Delta$ of submodules is called *inverse* or *downwards filtered* if the intersection of two of its modules again contains a module in $\{N_i\}_\Delta$. Defining

$$i \le j \text{ if } N_j \subset N_i \text{ for } i,\, j \in \Delta,$$

Δ becomes a quasi-ordered directed set.

With the inclusions $e_{ji} : N_j \to N_i$, for $i \le j$, the family $(N_i, e_{ji})_\Delta$ is an inverse system of modules.

Also the factor modules N/N_i with the canonical projections

$$p_{ji} : N/N_j \to N/N_i,\ x + N_j \mapsto x + N_i \text{ for } i \le j,$$

form an inverse system (see 24.5) and with the canonical projections $p_i : N \to N/N_i$ we obtain:

29.6 Inverse limit of submodules.
Assume $\{N_i\}_\Delta$ to be an inverse family of submodules of the R-module M. Then, with the above notation, $\bigcap_\Delta N_i \simeq \varprojlim N_i$ and the following sequence is exact:

$$0 \longrightarrow \bigcap_\Delta N_i \longrightarrow N \xrightarrow{\varprojlim p_i} \varprojlim N/N_i.$$

Proof: The inclusions $e_j : \bigcap_\Delta N_i \to N_j$ form an inverse system of morphisms and hence there is a map $\bigcap_\Delta N_i \to \varprojlim N_i$ which is monic by 29.3. For $(n_l)_\Delta \in \varprojlim N_i$ and $i,\, j \in \Delta$, there is a $k \ge i,\, j$ such that

$$n_i = n_k e_{ki} = n_k, \quad n_j = n_k e_{kj} = n_k.$$

Hence all $n_i = n_j \in \bigcap_\Delta N_i$ and the map is surjective.

The exactness of the sequence follows from 29.4.

Concerning the problem when inverse limits are right exact we show:

29.7 Linearly compact modules. Characterizations.
For an R-module M the following assertions are equivalent:

(a) *for every inverse family $\{M_i\}_\Delta$ with $M_i \subset M$, the map $\varprojlim p_i : M \to \varprojlim M/M_i$ is epic;*

(b) *for every inverse family $\{M_i\}_\Delta$ with $M_i \subset M$ and M/M_i finitely cogenerated, the map $\varprojlim p_i : M \to \varprojlim M/M_i$ is epic;*

(c) *if, for a family of cosets $\{x_i + M_i\}_\Delta$, $x_i \in M$, and submodules $M_i \subset M$ (with M/M_i finitely cogenerated), the intersection of any finitely many of these cosets is not empty, then also $\bigcap_\Delta (x_i + M_i) \neq \emptyset$.*

A module M satisfying these conditions is called *linearly compact*.

Proof: For the non-empty intersection of finitely many cosets we get $\bigcap_{i=1}^k (x_i + M_i) = y + \bigcap_{i=1}^k M_i$ for a suitable $y \in M$. Hence the family given in (c) can be replaced by the family of finite intersections of cosets, i.e., without restriction, in (c) we may also assume $\{M_i\}_\Delta$ to be an inverse family. The elements of $\varprojlim M/M_i$ are just the families $\{x_i + M_i\}_\Delta$ with $x_j + M_i = x_i + M_i$ for $M_j \subset M_i$, i.e.

$$x_j + M_j \subset x_i + M_i \text{ for } M_j \subset M_i.$$

$(a) \Rightarrow (c)$ If for $\{x_i + M_i\}_\Delta$ the intersection of any finite set of cosets is not empty, then in particular for $M_j \subset M_i$,

$$\emptyset \neq (x_j + M_j) \cap (x_i + M_i) = y + M_j \text{ for some } y \in x_i + M_i$$

and hence $x_j + M_j \subset x_i + M_i$. Therefore $\{x_i + M_i\}_\Delta$ belongs to $\varprojlim M/M_i$ and, by (a), there exists $x \in M$ with $x + M_i = x_i + M_i$ for all $i \in \Delta$. This means $x \in \bigcap_\Delta (x_i + M_i) \neq \emptyset$.

$(c) \Rightarrow (a)$ Now assume $\{x_i + M_i\}_\Delta$ to be an element of $\varprojlim M/M_i$, i.e. $x_j + M_j \subset x_i + M_i$ for $M_j \subset M_i$.

$\{M_i\}_\Delta$ being an inverse family, for any finite subset $i_1, \ldots, i_r \subset \Delta$ there exists $k \in \Delta$ with $M_k \subset M_{i_1} \cap \cdots \cap M_{i_r}$ and hence

$$x_k + M_k \subset \bigcap_{s=1}^r (x_{i_s} + M_{i_s}) \neq \emptyset.$$

By (c), we now find an $x \in \bigcap_\Delta (x_i + M_i)$, i.e. $x + M_i = x_i + M_i$ for all $i \in \Delta$. Hence $\varprojlim p_i$ is epic.

$(c) \Leftrightarrow (b)$ Every module M_i is of the form $M_i = \bigcap M_{i,j}$, with finitely cogenerated $M/M_{i,j}$ (see 14.9), and the family $\{x_i + M_i\}_\Delta$ can obviously be replaced by $\{x_i + M_{i,j}\}_{\Delta'}$.

29.8 Properties of linearly compact modules.
Let N be a submodule of the R-module M.

(1) Assume N to be linearly compact and $\{M_i\}_\Delta$ to be an inverse family of submodules of M. Then

$$N + \bigcap\nolimits_\Delta M_i = \bigcap\nolimits_\Delta (N + M_i)\,.$$

(2) M is linearly compact if and only if N and M/N are linearly compact.

(3) Assume M to be linearly compact. Then
(i) there is no non-trivial decomposition of M as an infinite direct sum;
(ii) $M/Rad\,M$ is semisimple and finitely generated;
(iii) every finitely generated module in $\sigma[M]$ is linearly compact.

Proof: (1) The families of submodules $\{N \cap M_i\}_\Delta$ in N and $\{(N + M_i)/N\}_\Delta$ in M/N are inverse. By 29.6, we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigcap_\Delta(N \cap M_i) & \longrightarrow & \bigcap_\Delta M_i & \xrightarrow{f} & \bigcap_\Delta(N + M_i/N) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow{p} & M/N & \longrightarrow & 0 \\
 & & \downarrow p_N & & \downarrow p_M & & \downarrow p_{\overline{M}} & & \\
0 & \longrightarrow & \varprojlim N/N \cap M_i & \longrightarrow & \varprojlim M/M_i & \longrightarrow & \varprojlim M/N + M_i & & .
\end{array}$$

If p_N is epic, then, by the Kernel Cokernel Lemma, f is also epic, i.e.

$$Im\ f = (\bigcap\nolimits_\Delta M_i) + N/N = \bigcap\nolimits_\Delta (N + M_i/N) = (\bigcap\nolimits_\Delta N + M_i)/N,$$

and hence $N + \bigcap_\Delta M_i = \bigcap_\Delta (N + M_i)$.

(2) From the above diagram we see:
Assume M to be linearly compact. Then p_M and

$$pp_{\overline{M}} = \varprojlim\{p_i : M \to M/(N + M_i)\}$$

are epic. Hence $p_{\overline{M}}$ is epic and M/N is linearly compact.
In case all $M_i \subset N$, p_N is also epic and N is linearly compact.

If N and M/N are linearly compact, then p_N and $p_{\overline{M}}$ are epic. Then p_M is epic and M is linearly compact.

(3)(i) Assume $M = \bigoplus_\Lambda M_\lambda$ to be a linearly compact module and define

$$N_\lambda = \bigoplus\nolimits_{\mu \neq \lambda} M_\mu \text{ for every } \lambda \in \Lambda.$$

Choose $0 \neq x_\lambda \in M_\lambda$ and consider the cosets $\{x_\lambda + N_\lambda\}_\Lambda$.

Any finite set of these cosets have non-empty intersection since, for $x_1 + N_1, \ldots, x_r + N_r$, we get $x_1 + \cdots + x_r \in \bigcap_{i=1}^r (x_i + N_i)$.

Consequently $\bigcap_\Lambda (x_\lambda + N_\lambda) \neq \emptyset$. However, for an element x in this intersection we have $x + N_\lambda = x_\lambda + N_\lambda$ for all $\lambda \in \Lambda$. Since this can only happen for finitely many λ's, the set Λ has to be finite.

(ii) By (2), $M/Rad\, M$ is linearly compact. We shall see in 41.10 that it is *supplemented* and hence semisimple by 41.2. Then $M/Rad\, M$ is finitely generated by (i).

(iii) is a consequence of (2) .

Modifying the conditions for linearly compact modules we define:

An R-module M is called *f-linearly compact* if, for every inverse family $\{M_i\}_\Delta$ of finitely generated submodules $M_i \subset M$, $\bigcap_\Delta M_i$ is also finitely generated and $\varprojlim p_i : M \to \varprojlim M/M_i$ is epic.

Similarly to 29.7 these modules can also be characterized by corresponding intersection properties of cosets. Observing that in coherent modules the intersection of two finitely generated submodules is again finitely generated, we obtain from the proof of 29.8:

29.9 Properties of f-linearly compact modules.

Let N be a finitely generated submodule of the R-module M where M is coherent in $\sigma[M]$.

(1) If N is f-linearly compact and $\{M_i\}_\Delta$ is an inverse family of finitely generated submodules of M, then

$$N + \bigcap\nolimits_\Delta M_i = \bigcap\nolimits_\Delta (N + M_i)\,.$$

(2) M is f-linearly compact if and only if N and M/N are f-linearly compact.

A first relationship between inverse limits and (co-) finiteness conditions is observed in a further

29.10 Characterization of finitely cogenerated modules.

An R-module M is finitely cogenerated if and only if, for every inverse system $\{M_i\}_\Delta$ of submodules $0 \neq M_i \subset M$, there is a non-zero submodule $K \subset M$ with $K \subset M_i$ for all $i \in \Lambda$.

Proof: By 21.3, M is finitely cogenerated if and only if $Soc(M)$ is finitely generated and essential in M.

$\Rightarrow$ If M is finitely cogenerated, then for every $i \in \Delta$ we have $Soc(M_i)$ non-zero and finitely generated. Hence there is an $i_o \in \Delta$ for which $Soc(M_{i_o})$ contains a minimal number of simple summands. Then, for every $i \in \Delta$, there must be a $k \in \Delta$ with $M_k \subset M_i \cap M_{i_o}$, in particular $Soc(M_k) \subset Soc(M_{i_o})$. By the choice of i_o, this means $Soc(M_k) = Soc(M_{i_o})$ and hence $Soc(M_{i_o}) \subset M_i$ for all $i \in \Delta$.

$\Leftarrow$ Assume $0 \neq N \subset M$ and let $\mathcal{N}$ be the set of all non-zero submodules of N with a quasi-order defined by reversing inclusion. By assumption, every chain (linearly ordered subset) $\mathcal{N}'$ in $\mathcal{N}$ has an upper bound ($= \bigcap$ the elements in $\mathcal{N}'$), i.e. $\mathcal{N}$ is an inductive quasi-ordered set and, by Zorn's Lemma, there is a maximal element in $\mathcal{N}$. This is a minimal submodule $E \subset N$. Hence $0 \neq E \subset N \cap Soc(M)$ and $Soc(M)$ is essential in M.

Assume $Soc(M)$ not to be finitely generated. Then $Soc(M)$ contains a countable direct sum $\bigoplus E_i$ of simple submodules. The partial sums $\{\bigoplus_{n \leq i} E_i\}_{I\!N}$ form a non-trivial inverse system of submodules of M whose intersection is zero and hence does not contain a non-zero submodule, contradicting our assumption. Therefore $Soc(M)$ must be finitely generated.

29.11 Exercises.

(1) Let $\{M_\alpha\}_A$ be a family of R-modules. Show:
$\prod_A M_\alpha = \varprojlim\{\prod_E M_\alpha \mid E \subset A,\ E\ finite\}$.

(2) Show that, for a suitable inverse system of $Z\!\!Z$-modules,
$End_{Z\!\!Z}(Z\!\!Z_{p^\infty}) \simeq \varprojlim\{Z\!\!Z_{p^k} \mid k \in I\!N\}$ *(= p-adic integers).*
Hint: 24.13,(5).

(3) For an R-module M, let $\{W_i\}_\Delta$ denote the family of essential submodules of M and $\{K_i\}_\Lambda$ the family of superfluous submodules of M.

Determine $\varprojlim W_i$ and $\varinjlim K_i$.

(4) For an R-module M, let $F : \sigma[M] \to AB$ be a left exact covariant functor. Show that F preserves products if and only if F preserves inverse limits.

(5) In 29.4 consider the case $\Delta = I\!N$ and assume every v_j to be epic.

Show: If the $f_{j+1,j} : K_{j+1} \to K_j$ *are epic for every* $j \in \mathbb{N}$*, then the sequence*

$$0 \to \varprojlim K_i \to \varprojlim L_i \to \varprojlim N_i \to 0$$

is exact.

(6) Show that ${}_{\mathbb{Z}}\mathbb{Z}$ *is not linearly compact.*

(7) Let P be a right module over the ring R. Prove: P is finitely generated and projective (in MOD-R) if and only if $P \otimes_R -$ *commutes with inverse limits in R-MOD.*

(8) Let P and M be left modules over the ring R, $S = End(P)$ *and* ${}_S Hom_R(P, M)$ *linearly compact as S-module.*

Show that P is M-projective if and only if $Hom_R(P, -)$ *is exact with respect to exact sequences*

$0 \to K \to M \to M/K \to 0$,

with M/K *finitely cogenerated.*

Literature: NĂSTĂSESCU, SOLIAN; Dikranjan-Orsatti, van Huynh [2], Menini [1,2], Oberst-Schneider, Müller [1], Onodera [2,6], Orsatti-Roselli, Sandomierski, Takeuchi [4], Vámos [1], Zöschinger [5].

30 Finitely copresented modules

1.Characterization. 2.Properties. 3.$\varprojlim$ of finitely copresented modules. 4.Pseudo co-coherent modules. 5.Finiteness and dual conditions. 6.R commutative. 7.Exercises.

Let M be an R-module. We call a module $X \in \sigma[M]$ *finitely copresented in* $\sigma[M]$ if

(i) X is finitely cogenerated and

(ii) in every exact sequence $0 \to X \to L \to N \to 0$ in $\sigma[M]$ with L finitely cogenerated, N is also finitely cogenerated.

Similar to 'finitely presented', 'finitely copresented' also depends on the category referred to ($\sigma[M]$, R-MOD). As a consequence of properties stated below we anticipate:

30.1 Characterization. *A module $X \in \sigma[M]$ is finitely copresented in $\sigma[M]$ if and only if its M-injective hull $\widehat{X}$ and the factor module $\widehat{X}/X$ are finitely cogenerated.*

In particular, finitely cogenerated injective modules are also finitely copresented in $\sigma[M]$ and hence every finitely cogenerated module is a submodule of a finitely copresented module.

30.2 Properties of finitely copresented modules in $\sigma[M]$.
Let M be an R-module and $0 \to X_1 \to X_2 \to X_3 \to 0$ an exact sequence in $\sigma[M]$. Then

(1) If X_2 is finitely copresented in $\sigma[M]$ and X_3 is finitely cogenerated, then X_1 is also finitely copresented in $\sigma[M]$.

(2) If X_1 and X_3 are finitely copresented, then X_2 is also finitely copresented in $\sigma[M]$.

(3) A finite direct sum of modules is finitely copresented in $\sigma[M]$ if and only if every summand is finitely copresented.

(4) If Y and Z are finitely copresented submodules of the finitely copresented module X, then their intersection $Y \cap Z$ is also finitely copresented (in $\sigma[M]$).

Proof: (1) is seen dually to the proof of 25.1:

With the exact sequence $0 \to X_1 \to N \to L \to 0$ we obtain with a pushout the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & X_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & & .
\end{array}$$

If N is finitely cogenerated, then P and L are finitely cogenerated.

Moreover, (2) and (3) are seen dually to 25.1. (4) follows (by (1) and (3)) from the exactness of the sequence $0 \to Y \cap Z \to X \to X/Y \oplus X/Z$.

30.3 $\varprojlim$ of finitely copresented modules.
Every finitely cogenerated module in $\sigma[M]$, $M \in R$-MOD, is an inverse limit of finitely copresented modules in $\sigma[M]$.

Proof: If $L \in \sigma[M]$ is finitely cogenerated, then its M-injective hull $\widehat{L}$ is finitely copresented in $\sigma[M]$.

For $x \in \widehat{L} \setminus L$ there is a submodule $K_x \subset \widehat{L}$ which is maximal with respect to $L \subset K_x$ and $x \notin K_x$. Since $\widehat{L}/K$ is cocyclic and hence finitely cogenerated (see 14.8), the module K_x is finitely copresented. The finite intersections of the modules $\{K_x \mid x \in \widehat{L} \setminus L\}$ form an inverse system of finitely copresented modules with $\varprojlim K_x = \bigcap K_x = L$.

We call an R-module $X \in \sigma[M]$ *pseudo co-coherent in* $\sigma[M]$ if every finitely cogenerated factor module of X is finitely copresented. X is called *co-coherent* if it is pseudo co-coherent and finitely cogenerated.

30.4 Properties of pseudo co-coherent modules.
Let M be an R-module and $0 \to X_1 \to X_2 \to X_3 \to 0$ an exact sequence in $\sigma[M]$. Then

(1) If X_2 is pseudo co-coherent and X_3 is finitely cogenerated, then X_1 is pseudo co-coherent in $\sigma[M]$.

(2) If X_1 and X_3 are pseudo co-coherent, then X_2 is also pseudo co-coherent in $\sigma[M]$.

(3) A finite direct sum of pseudo co-coherent modules is again pseudo co-coherent in $\sigma[M]$.

(4) Assume X to be pseudo co-coherent and Y, Z to be finitely cogenerated factor modules of X. If

$$\begin{array}{ccc} X & \rightarrow & Y \\ \downarrow & & \downarrow \\ Z & \rightarrow & P \end{array}$$

is a pushout diagram, then P is finitely cogenerated.

(5) If $f : X \rightarrow Y$ is a morphism between co-coherent modules, then $Ke\, f$, $Im\, f$ and $Coke\, f$ are co-coherent in $\sigma[M]$.

Proof: (1) Let $X_1 \rightarrow Y$ be epic and Y finitely cogenerated.
Forming a pushout we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & P & \longrightarrow & X_3 & \longrightarrow & 0 \end{array} .$$

Since Y and X_3 are finitely cogenerated, P is finitely cogenerated and – by assumption – finitely copresented. Because of 30.2,(1), Y is also finitely copresented.

(2) Let $X_2 \rightarrow Z$ be epic and Z finitely cogenerated. By forming a pushout we get the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & Z & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} .$$

Here K is finitely copresented and P is finitely cogenerated, hence finitely copresented and Z is also finitely copresented.

(3) Of course, this follows immediately from (2).

(4) The given diagram can be extended to the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & Z & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} .$$

Hereby K is pseudo co-coherent by (1), and hence L is finitely copresented. Therefore P is finitely cogenerated.

(5) follows from the preceding observations.

30.5 Finiteness and dual conditions.
Let R and T be rings and ${}_RQ_T$ an (R,T)-bimodule.

(1) If Q_T is finitely cogenerated and ${}_RN$ is a finitely generated left R-module, then $Hom_R(N,Q)_T$ is a finitely cogenerated right T-module .

(2) If Q_T is finitely copresented in $\sigma[Q_T]$ and ${}_RN$ is a finitely presented left R-module, then $Hom_R(N,Q)_T$ is finitely copresented in $\sigma[Q_T]$.

(3) If ${}_RQ$ cogenerates the factor modules of an R-module ${}_RN$ and $Hom_R(N,Q)_T$ is finitely cogenerated, then N is finitely generated.

(4) Let ${}_RQ$ be an injective cogenerator in $\sigma[M]$, $M \in R$-MOD and Q_T finitely cogenerated. If, for $N \in \sigma[M]$, the module $Hom_R(N,Q)_T$ is finitely copresented in $\sigma[Q_T]$, then N is finitely presented in $\sigma[M]$.

(5) If ${}_RQ$ is an injective cogenerator in R-MOD and Q_T is co-coherent in $\sigma[Q_T]$, then R is left coherent.

Proof: (1) and (2) are obtained applying $Hom_R(-,Q)$ to the exact sequences $R^k \to N \to 0$ resp. $R^l \to R^k \to N \to 0$.

(3) Let $Hom_R(N,Q)_T$ be finitely cogenerated and $\{N_i\}_\Lambda$ the family of finitely generated submodules of N. Then $N = \varinjlim N_i$. Considering the inverse limit of the exact sequences

$$0 \to Hom_R(N/N_i,Q) \to Hom_R(N,Q) \to Hom_R(N_i,Q)\,,$$

we get $\varprojlim Hom_R(N/N_i,Q) = 0$ (see 29.4). $Hom_R(N,Q)_T$ is finitely cogenerated and from the characterization of these modules in 29.10 we conclude $Hom_R(N/N_k,Q) = 0$ for some $k \in \Lambda$. Q being a cogenerator, this implies $N/N_k = 0$ and N $(= N_k)$ is finitely generated.

(4) If $Hom_R(N,Q)_T$ is finitely copresented, then, by (3), N is finitely generated. From an exact sequence

$$0 \to K \to L \to N \to 0$$

with L finitely generated we obtain the exact sequence in T-MOD

$$0 \to Hom_R(N,Q) \to Hom_R(L,Q) \to Hom_R(K,Q) \to 0.$$

By (1), $Hom_R(L,Q)$ is finitely cogenerated and hence $Hom_R(K,Q)$ has to be finitely cogenerated. Then by (3), K is finitely generated and N is finitely presented in $\sigma[M]$.

(5) For any finitely generated left ideal $I \subset R$ we have the exact sequence

$$0 \to Hom_R(R/I,Q)_T \to Q_T \to Hom_R(I,Q)_T \to 0\,.$$

By (2), $Hom_R(R/I, Q)_T$ is finitely copresented. Hence $Hom_R(I, Q)_T$ is finitely cogenerated and therefore finitely presented by the assumption on Q_T. According to (4), I is finitely presented.

As an application we consider the following special case:

30.6 *R* commutative.
Let R be a commutative ring with $R/Jac(R)$ semisimple and Q the minimal cogenerator in R-MOD. Then

(1) An R-module N is finitely generated (finitely presented) if and only if $Hom_R(N, Q)$ is finitely cogenerated (finitely copresented) as an R-module.

(2) If ${}_RQ$ is co-coherent, then R is coherent.

Proof: There are only finitely many non-isomorphic simple R-modules. Hence the minimal cogenerator Q (= the direct sum of injective hulls of simple modules) is finitely copresented.

30.7 Exercises.

(1) Let R be a commutative ring with $R/Jac(R)$ semisimple. Show:
If injective hulls of simple R-modules are coherent, then R is coherent.

(2) Let us call an R-module X codefined *if its R-injective hull $E(X)$ is a direct sum of injective hulls of simple modules. X is called* copresented *if X and $E(X)/X$ are codefined (Salles [1]).*

Consider an exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ in R-MOD. Show:
(i) If X_2 is codefined, then X_1 is also codefined.
(ii) If X_1 and X_3 are codefined, then X_2 is also codefined.
(iii) If X_1 is copresented and X_2 codefined, then X_3 is codefined.
(iv) If X_2 is copresented and X_3 codefined, then X_1 is copresented.
(v) If X_1 and X_3 are copresented, then X_2 is also copresented.

Literature: Couchot [1], Hiremath [3], Salles [1].

31 Artinian and co-noetherian modules

1.Artinian modules. 2.Locally artinian modules. 3.Finiteness conditions for semisimple modules. 4.Left artinian rings. 5.Modules over artinian rings. 6.Co-noetherian modules. 7.Co-semisimple modules. 8.dcc for cyclic submodules. 9.dcc for cyclic left ideals. 10.End(M) of modules with dcc for cyclic submodules. 11.End(M) of artinian modules. 12.End(M) of modules with acc for annihilators. 13.Powers of endomorphisms. 14.End(M) of artinian uniform M. 15.Exercises.

An R-module M is called *artinian* if its submodules satisfy the *descending chain condition (dcc)*, i.e. every descending chain $M_1 \supset M_2 \supset \cdots$ of submodules becomes stationary after finitely many steps (see § 27).

M is called *locally artinian* if every finitely generated submodule of M is artinian.

A ring R is left artinian (see § 4) if and only if ${}_RR$ is an artinian module. Dual to characterizations of noetherian modules in 27.1 we have:

31.1 Characterization of artinian modules.

For an R-module M the following properties are equivalent:

(a) M is artinian;

(b) every non-empty set of submodules of M has a minimal element;

(c) every factor module of M is finitely cogenerated;

(d) M is linearly compact and every factor module $\neq 0$ has non-zero socle.

Proof: $(a) \Rightarrow (b)$ is dual to $(b) \Rightarrow (c)$ in 27.1.

$(b) \Rightarrow (c)$ Every inverse system of submodules ($\neq 0$) in M contains a minimal element ($\neq 0$). Hence, by 29.10, M is finitely cogenerated. Property (b) is obviously inherited by factor modules.

$(c) \Rightarrow (a)$ Let $M_1 \supset M_2 \supset M_3 \supset \cdots$ be a descending chain of submodules and $N = \bigcap_{\mathbb{N}} M_i$. Since M/N is finitely cogenerated we must get $N = M_k$ for some $k \in \mathbb{N}$, i.e. $M_{k+l} = M_k$ for all $l \in \mathbb{N}$.

$(b) \Rightarrow (d)$ Any inverse system of submodules of M is in fact finite.

$(d) \Rightarrow (c)$ will be shown in 41.10.

31.2 Properties of locally artinian modules.

(1) Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence in R-MOD.

(i) If N is (locally) artinian, then N' and N'' are (locally) artinian.

(ii) If N' and N'' are artinian, then N is also artinian.

(iii) If N' is artinian and N'' is locally artinian, then N is locally artinian.

(2) Any direct sum of locally artinian modules is locally artinian.

(3) If M is a locally artinian R-module, then
(i) every finitely generated module in $\sigma[M]$ is artinian.
(ii) $M/Rad\, M$ is a semisimple module.

Proof: (1) (i) is easy to see.

(ii) If N' and N'' are artinian modules and L is a factor module of N, we obtain by forming a pushout the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

By assumption, K and P are finitely cogenerated and, by 21.4, L is also finitely cogenerated. Hence N is artinian (see 31.1).

(iii) Let K be a finitely generated submodule of N. By the given assumptions, $K \cap N'$ and $K/(K \cap N') \subset N''$ are artinian modules. According to (ii), K is also artinian.

(2) Using (1), the assertion is obtained with the same proof as for the related statement for locally noetherian modules (see 27.2,(2)).

(3)(i) Every finitely generated module in $\sigma[M]$ is factor module of a finitely generated submodule of $M^{(\mathbb{N})}$. By (2), $M^{(\mathbb{N})}$ is locally artinian.

(ii) Every finitely generated submodule of $M/Rad\, M$ is finitely cogenerated and is cogenerated by simple modules, and hence is semisimple. Therefore $M/Rad\, M$ is a sum of simple modules.

For semisimple modules the finiteness conditions just introduced are equivalent. We complete 20.8 by

31.3 Finiteness conditions for semisimple modules.

For a semisimple R-module M the following are equivalent:
(a) M is finitely generated;
(b) M is finitely copresented in $\sigma[M]$;
(c) M is (co-)coherent in $\sigma[M]$;
(d) M is artinian;
(e) M is noetherian;
(f) M is artinian and noetherian;
(g) M is linearly compact.

Proof: The equivalences from (a) to (f) easily follow from 20.8, the definitions, and properties of semisimple modules (see § 20).

$(a) \Leftrightarrow (g)$ is obtained from 29.8.

31.4 Characterization of left artinian rings.

For a ring R the following statements are equivalent:

(a) ${}_RR$ is artinian (R is left artinian);

(b) every finitely generated (cyclic) left R-module is finitely cogenerated;

(c) (i) $R/Jac\,R$ is a left semisimple ring,

(ii) $Jac\,R$ is nilpotent, and

(iii) ${}_RR$ is noetherian.

Proof: $(a) \Leftrightarrow (b)$ is clear by 31.1 and 31.2.

$(a) \Rightarrow (c)$ (i) has already been shown in 31.2.

(ii) Put $J = Jac\,R$. The descending chain of ideals $J \supset J^2 \supset J^3 \supset \cdots$ has to become stationary, i.e. $J^n = J^{n+1}$ for some $n \in I\!N$.

Assume $J^n \neq 0$. Then the set of left ideals

$$\mathcal{J} = \{I \subset_R R \mid J^n I \neq 0\}$$

is not empty ($J \in \mathcal{J}$). By 31.1, it contains a minimal element I_o. For $a \in I_o$ with $J^n a \neq 0$ we get $Ja \subset Ra \subset I_o$ and $J^n(Ja) = J^{n+1}a = J^n a \neq 0$. By the minimality of I_o, this implies $Ja = Ra$. This is a contradiction to the Nakayama Lemma 21.13, and we conclude $J^n = 0$.

(iii) Since $J^n = 0$, we may consider J^{n-1} as an R/J-module. Therefore J^{n-1} is artinian semisimple and hence noetherian. In the exact sequence

$$0 \longrightarrow J^{n-1} \longrightarrow J^{n-2} \longrightarrow J^{n-2}/J^{n-1} \longrightarrow 0\,,$$

J^{n-1} is noetherian and J^{n-2}/J^{n-1} is an artinian R/J-module and noetherian. By induction we see that J and $J^0 = R$ are also noetherian.

$(c) \Rightarrow (a)$ Under the assumptions (i), (ii) the above reasoning can be repeated interchanging 'artinian' and 'noetherian'.

31.5 Properties of modules over artinian rings.

Let M be a module over a left artinian ring R. Then

(1)(i) $Soc(M) = \{m \in M \mid Jac(R)m = 0\}$ and is essential in M;

(ii) $Rad(M) = Jac(R)M$ and is superfluous in M;

(iii) $R/An(M) \in \sigma[M]$, i.e. $\sigma[M] = R/An(M)$-MOD.

(2) The following properties of M are equivalent:

(a) M is finitely generated;

(b) M is noetherian;

(c) M is artinian;
(d) $M/Rad(M)$ is finitely generated.

Proof: (1)(i) We always have $Jac(R)Soc(M) = 0$. If $Jac(R)m = 0$ for some $m \in M$, then $Jac(R)Rm = 0$ and Rm is an $R/Jac(R)$-module and hence semisimple, implying $m \in Soc(M)$.

Every submodule $N \subset M$ contains a cyclic, hence artinian, submodule. Thus a non-zero simple submodule is contained in N and $Soc(M) \trianglelefteq M$.

(ii) R being a good ring we have $Rad(M) = Jac(R)M$. Put $J = Jac(R)$. Assume $J^n = 0$ and consider a submodule $K \subset M$ with $JM + K = M$. Multiplying with J we obtain $J^2M + JK = JM$, then $J^2M + JK + K = M$ and finally $K = J^nM + K = M$.

(iii) $R/An(M)$ is finitely cogenerated and M-cogenerated. This implies $R/An(M) \subset M^k$ for some $k \in \mathbb{N}$.

(2) (a) $\Leftrightarrow$ (d) follows from $Rad\,M \ll M$ (see 19.6).

(a) $\Leftrightarrow$ (b) is obtained from 31.4 (${}_RR$ is noetherian).

(a) $\Rightarrow$ (c) $\Rightarrow$ (d) is clear by 31.2.

In contrast to artinian rings, artinian modules need not be noetherian, nor even finitely generated. For example, the injective hulls of simple $\mathbb{Z}$-modules ($\mathbb{Z}_{p^\infty}$, see 17.13) are artinian but not finitely generated. Hence every finitely cogenerated $\mathbb{Z}$-module is artinian. This property is dual to external characterizations of noetherian modules in 27.3 and can be described in the following way:

31.6 Co-noetherian modules. Characterizations.

An R-module M is called co-noetherian *if it satisfies the following equivalent conditions:*
(a) Every finitely cogenerated module is finitely copresented in $\sigma[M]$;
(b) every finitely cogenerated module is co-coherent in $\sigma[M]$;
(c) every finitely cogenerated module in $\sigma[M]$ is artinian;
(d) injective hulls of simple modules in $\sigma[M]$ are artinian;
(e) $\sigma[M]$ has a set of cogenerators consisting of artinian modules.

Proof: The equivalence of (a), (b) and (c) easily follows from the definitions and 31.1.

(c) $\Rightarrow$ (d) $\Rightarrow$ (e) are trivial.

(e) $\Rightarrow$ (c) Every finitely cogenerated module is a submodule of a finite direct sum of modules from the set of cogenerators and hence is artinian.

Co-noetherian modules are only artinian if they are finitely cogenerated. Examples of non-artinian co-noetherian modules are (non-semisimple) co-semisimple modules M; these are characterized by the fact that in their category $\sigma[M]$, all finitely cogenerated modules are injective (hence finitely copresented, see 23.1).

With these newly introduced notions we obtain

31.7 Characterizations of co-semisimple modules.

For an R-module M the following assertions are equivalent:

(a) M is co-semisimple;

(b) every finitely copresented module in $\sigma[M]$ is semisimple;

(c) every finitely copresented module in $\sigma[M]$ is M-injective.

Proof: $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$ are trivial.

Let K be a finitely cogenerated module in $\sigma[M]$ and $\widehat{K}$ its M-injective hull. Then $\widehat{K}$ is finitely copresented.

By (b), $\widehat{K}$ is semisimple and hence $\widehat{K} = K$, i.e. $(b) \Rightarrow (a)$.

Assume (c) and suppose $K \neq \widehat{K}$. Choose an $n \in \widehat{K} \setminus K$. Let L be a submodule of $\widehat{K}$ maximal with respect to $K \subset L$ and $n \notin L$. Then $\widehat{K}/L$ is finitely cogenerated (cocyclic, see 14.8) and L is finitely copresented (see 30.1). By (c), this implies $L = \widehat{K}$ and $n \in L$ contradicting the choice of L. Hence $(c) \Rightarrow (a)$.

In contrast to the situation for the ascending chain condition, the descending chain condition for *finitely generated* submodules does not imply the descending chain condition for all submodules.

We say a subset $I \subset R$ *acts t-nilpotently on* M if, for every sequence $a_1, a_2, \ldots$ of elements in I and $m \in M$, we get

$$a_i a_{i-1} \cdots a_1 m = 0$$

for some $i \in I\!N$ (depending on m).

I is called *left t-nilpotent* if it acts t-nilpotently on $_RR$.

A module is called *(amply) supplemented* if every submodule has (ample) supplements (see § 41).

31.8 Descending chain condition for cyclic submodules.
Let the R-module M satisfy dcc for cyclic submodules. Then

(1) M also satifies dcc for finitely generated submodules.

(2) Every non-zero module in $\sigma[M]$ has a simple submodule.

(3) Every finitely generated submodule of M is (amply) supplemented.

(4) $M/Rad\, M$ is semisimple.

(5) If M is coherent, then M is f-linearly compact.

(6) Jac(R) acts t-nilpotently on M.

(7) If ${}_RM$ is faithful and is finitely generated over $End({}_RM)$, then Jac(R) is left t-nilpotent.

Proof: (1) The set of submodules of M satisfying dcc for finitely generated submodules obviously is inductive with respect to inclusion. By Zorn's Lemma, there is a maximal element $L \subset M$ in this set.

Assume $L \neq M$. By assumption, in the family of cyclic modules $\{Rm \mid m \in M \setminus L\}$ we find a minimal element Rx. Let us show that $L + Rx$ also satisfies dcc for finitely generated submodules. This will contradict the maximality of L and the assumption $L \neq M$.

Consider a descending chain $L_1 \supset L_2 \supset \cdots$ of finitely generated submodules of $L + Rx$. If, for some $i \in I\!N$, we have $L_i \subset L$, then the chain is finite (by the choice of L). Assume $L_i \not\subset L$ for every $i \in I\!N$. We prove:

(*) *Every L_i contains an element $x+x_i$ with $x_i \in L$, i.e. $L+L_i = L+Rx$.*

Any $y \in L_i \setminus L$ can be written as $y = wx + z$ with $w \in R$, $z \in L$. Then, by the choice of Rx, $Rwx \neq Rx$ would imply $L + Rwx = L$, i.e. $wx \in L$. Hence $Rwx = Rx$ and $x = rwx$ for some $r \in R$, $0 \neq ry = x + rz \in L_i$. Therefore we may choose $y_i \in L_i$ such that the Ry_i are minimal with respect to $Ry_i + L = Rx + L$. For these we prove:

(**) *For any submodule $K \subset L$ with $L_i = Ry_i + K$, we have*
$L_i = Ry_{i+1} + K$.

Because $y_{i+1} \in L_{i+1} \subset L_i$, we get $y_{i+1} + K = ry_i + K$, $r \in R$.
Since $K \subset L$ and $Ry_{i+1} + L = Rx + L$, we conclude $Rry_i + L = Rx + L$. By the choice of y_i, this means $Rry_i = Ry_i$, i.e. $sry_i = y_i$ for some $s \in R$. From this we derive $sy_{i+1} + K = y_i + K$, i.e. $L_i = Ry_{i+1} + K$.

L_1 being a finitely generated submodule of $L + Rx$, there is a finitely generated $L_1' \subset L$ with $L_1 = Ry_1 + L_1'$, implying $L_1 = Ry_2 + L_1'$ by (**). Because of $L_2 \subset L_1$, there is a finitely generated submodule $L_2' \subset L_1'$ with $L_2 = Ry_2 + L_2'$. Continuing in this way we obtain a descending chain of finitely generated submodules $L_n' \subset L$ with $L_n = Ry_n + L_n' = Ry_{n+1} + L_n'$.

By definition of L, the chain $\{L'_n\}_{\mathbb{N}}$ becomes stationary and hence also the chain $\{L_n\}_{\mathbb{N}}$.

(2) It is obvious that non-zero factor modules of M and M-generated modules have simple submodules. Now the assertion follows from the fact that every module in $\sigma[M]$ is essential in an M-generated module.

(3) Let $N \subset M$ be finitely generated and $K \subset N$. By (1), there exists a finitely generated submodule $L \subset N$ which is minimal with respect to $K + L = N$. This is a supplement of K in N.

(4) By (3), every finitely generated submodule of $M/Rad(M)$ is supplemented and hence semisimple (see 41.2). Thus $M/Rad(M)$ is semisimple.

(5) Every inverse family of finitely generated submodules of M has a minimal element.

(6) Let $a_1, a_2, \ldots$ be a sequence of elements in $Jac(R)$ and $m \in M$. The descending chain of submodules $Ra_1m \supset Ra_2a_1m \supset Ra_3a_2a_1m \supset \cdots$ becomes stationary and hence, for some $i \in \mathbb{N}$, we have

$$Ra_ia_{i-1}\cdots a_1m = Ra_{i+1}a_i\cdots a_1m \subset Jac(R)a_ia_{i-1}\cdots a_1m.$$

By the Nakayama Lemma, this means $a_ia_{i-1}\cdots a_1m = 0$.

(7) In view of 15.3 and 15.4, this follows immediately from (6) .

As a corollary we notice:

31.9 Descending chain condition for cyclic left ideals.
Assume the ring R to satisfy dcc for cyclic left ideals. Then

(1) $R/Jac(R)$ is left semisimple and $Jac(R)$ is left t-nilpotent.

(2) Every module in R-MOD has a simple submodule and R does not contain an infinite set of orthogonal idempotents.

Proof: (1) and part of (2) have been shown in 31.8. An infinite set of orthogonal idempotents $\{e_i\}_{\mathbb{N}}$ in R would lead to the construction of an infinite descending chain of cyclic left ideals
$R(1 - e_1) \supset R(1 - e_1 - e_2) \supset \cdots$.

We shall encounter these rings again later on as *right perfect* rings. Then we will show that (1) and (2) in 31.9 are in fact equivalent to the descending chain condition for cyclic left ideals (see 43.9).

For the interconnection between finiteness conditions in M and $End_R(M)$ weakened projectivity and injectivity properties play an important part:

We call an R-module M *semi-projective* if, for any submodule $N \subset M$, every diagram with exact row

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow g & & \\ M & \xrightarrow{f} & N & \longrightarrow & 0 \end{array}$$

can be extended by an $h : M \to M$ with $hf = g$.

Obviously, M is semi-projective if and only if $Sf = Hom_R(M, Mf)$ for every $f \in End_R(M) = S$.

For example, a self-projective module is also semi-projective.

31.10 *End(M)* of modules with dcc for cyclic submodules.
Let M be a finitely generated, semi-projective R-module satisfying dcc for cyclic submodules. Then $End_R(M)$ satifies dcc for cyclic left ideals.

Proof: Put $S = End_R(M)$. A descending chain of cyclic left ideals $Sf_1 \supset Sf_2 \supset \cdots$ yields a descending chain of finitely generated submodules $Mf_1 \supset Mf_2 \supset \cdots$. By assumption, this chain becomes stationary after a finite number of steps. Since $Sf_i = Hom(M, Mf_i)$, this is also true for the cyclic left ideals.

31.11 *End(M)* of artinian modules.
Let M be an artinian module and $S = End_R(M)$.

(1) If M is semi-projective, then $S/Jac(S)$ is left semisimple and $Jac(S)$ is nilpotent.

(2) If M is finitely generated and self-projective, then S is left artinian and M satisfies the ascending chain condition for M-generated submodules.

(3) If M is self-injective, then S is right noetherian.

(4) If M is self-injective and self-projective, then S is right artinian.

Proof: (1) With the proof of 31.10 we obtain that $S/Jac(S)$ is left semisimple and $Jac(S)$ is left t-nilpotent (see 31.9). For $J = Jac(S)$ the descending chain of R-submodules $MJ \supset MJ^2 \supset MJ^3 \supset \cdots$ has to become stationary after finitely many steps. Hence we get, for some $n \in I\!N$ and $B = J^n$, that $MB = MB^2$.

Assume J not to be nilpotent. Then this is also true for B and hence there exists $c' \in B$ with $Bc' \neq 0$. Let Mc denote a minimal element in the set $\{Mc' \subset M \mid c' \in B,\ Bc' \neq 0\}$. Since $0 \neq MBc = MBBc$, there exists $d \in Bc \subset B$ with $Bd \neq 0$ and $Md \subset MBc \subset Mc$. By minimality of Mc, this means $Md = Mc$ and hence $Mbc = Mc$ for some $b \in B$.

M being semi-projective, there exists $f \in S$ with $fbc = c$, i.e. $c = \bar{b}c$ for $fb =: \bar{b} \in B \subset J$. Since $\bar{b}$ is nilpotent, this implies $c = 0$, contradicting the choice of c. Therefore J has to be nilpotent.

(2) Under the given assumptions, for every left ideal $I \subset S$, we have $I = Hom(M, MI)$. As in the proof of 31.10 we conclude that ${}_SS$ is artinian. Now every M-generated submodule of M is of the form MI with $I \subset {}_SS$. Since ${}_SS$ is noetherian (by 31.4), we deduce the ascending chain condition for these submodules.

(3) By 28.1, we have $I = Hom(M/Ke\,I, M)$ for every finitely generated right ideal $I \subset S_S$. Therefore the descending chain condition for submodules of type $Ke(I)$ yields the ascending chain condition for finitely generated right ideals $I \subset S_S$, i.e. S_S is noetherian.

(4) By (1), $S/Jac(S)$ is semisimple and $Jac(S)$ nilpotent. Since ${}_SS$ is noetherian by (3), the assertion follows from 31.4.

Dualising some of the preceding arguments we obtain statements about the endomorphism rings of noetherian modules with weakened injectivity conditions:

We call an R-module M *semi-injective* if, for any factor module N of M, every diagram with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{k} & M \\ & & \downarrow g & & \\ & & M & & \end{array}$$

can be extended by an $h : M \to M$ with $kh = g$. This is obviously the case if and only if, for every $f \in End_R(M) = S$, $fS = \{g \in S \,|\, (Ke\,f)g = 0\}$ $(\simeq Hom_R(M/Ke\,f, M))$.

Recall that, for subsets $I \subset End_R(M)$, the modules

$$Ke\,I = \bigcap\{Ke\,f \mid f \in I\}$$

are named *annihilator submodules* in § 28.

A ring R is called *semiprimary* if $R/Jac(R)$ is left semisimple and $Jac(R)$ is nilpotent.

31.12 *End(M)* of modules with acc for annihilators.

Let M be a semi-injective R-module with acc for annihilator submodules. Then $End_R(M)$ is semiprimary.

Proof: Consider a descending chain $f_1S \supset f_2S \supset \cdots$ of cyclic right ideals in $S = End_R(M)$.

The ascending chain of submodules $Ke\, f_1 \subset Ke\, f_2 \subset \cdots$ becomes stationary after finitely many steps and hence this is also true for the chain $f_1S \supset f_2S \supset \cdots$. Therefore S satisfies dcc for cyclic (finitely generated) right ideals (see also 28.3) and, by 31.9, $S/Jac(S)$ is right semisimple and $Jac(S)$ is a nil ideal (right t-nilpotent).

For $J = Jac(S)$, the ascending chain $Ke\, J \subset Ke\, J^2 \subset \cdots$ becomes stationary after finitely many steps. Hence $Ke\, J^n = Ke\, J^{2n}$ for some $n \in I\!N$, i.e. for $B = J^n \subset Jac(S)$ we get $Ke\, B = Ke\, B^2$.

Assume $Jac(S)$ is not nilpotent. Then $B^2 \neq 0$ and the non-empty set

$$\{Ke\, g' \subset M \mid g' \in B \text{ and } g'B \neq 0\}$$

has a maximal element $Ke\, g$, $g \in B$. The relation $gBB = 0$ would imply $Im\, g \subset Ke\, B^2 = Ke\, B$ and hence $gB = 0$, contradicting the choice of g. Therefore we can find an $h \in B$ with $ghB \neq 0$. However, since $Ke\, g \subset Ke\, gh$ the maximality of $Ke\, g$ implies $Ke\, g = Ke\, gh$. Recalling that M is semi-injective, this implies $gS = ghS$, i.e. $g = ghs$ for some $s \in S$. Since $hs \in B \subset Jac(S)$ and so is nilpotent, this means $g = 0$, a contradiction. Thus $Jac(S)$ has to be nilpotent.

For an R-module M and $f \in End_R(M)$, the powers of f, f^i with $i \in I\!N$, belong to $End_R(M)$ and we have chains of submodules

$$Im\, f \supset Im\, f^2 \supset Im\, f^3 \supset \cdots \quad \text{and} \quad Ke\, f \subset Ke\, f^2 \subset Ke\, f^3 \subset \cdots .$$

31.13 Powers of endomorphisms.
Let M be an R-module and $f \in End_R(M)$.

(1) Assume M is artinian, or M is finitely generated with dcc on cyclic submodules. Then there exists $n \in I\!N$ with
$Im\, f^n + Ke\, f^n = M$. If f is monic, then f is an isomorphism.

(2) Assume M is noetherian. Then there exists $n \in I\!N$ with
$Im\, f^n \cap Ke\, f^n = 0$. If f is epic, then f is an isomorphism.

(3) Fitting's Lemma: *Assume M is artinian and noetherian. Then there exists $n \in I\!N$ with $M = Im\, f^n \oplus Ke\, f^n$ and the following are equivalent:*
(a) f is monic;
(b) f is epic;
(c) f is an isomorphism.

Proof: (1) The descending chain $Im\, f \supset Im\, f^2 \supset \cdots$ becomes stationary after a finite number of steps and we can find an $n \in I\!N$ with

$Im\, f^n = Im\, f^{2n}$.
For $x \in M$, we have $(x)f^n \in Im\, f^{2n}$, i.e. $(x)f^n = (y)f^{2n}$ for some $y \in M$, and

$$x = (y)f^n + (x - (y)f^n) \in Im\, f^n + Ke\, f^n.$$

If $Ke\, f = 0$, we see that f^n - and hence f - has to be epic.

(2) Now the chain $Ke\, f \subset Ke\, f^2 \subset \cdots$ becomes stationary after finitely many steps, i.e. $Ke\, f^n = Ke\, f^{2n}$ for some $n \in I\!N$. For $x \in Im\, f^n \cap Ke\, f^n$ there exists $y \in M$ with $(y)f^n = x$. Since $0 = (x)f^n = (y)f^{2n}$, this implies $x = (y)f^n = 0$.

If f is epic, then $M = (M)f = (M)f^n$ and $Ke\, f = 0$.

(3) is an immediate consequence of (1) and (2).

31.14 *End(M)* of artinian uniform modules.

Let M be an artinian uniform module (all non-zero submodules essential). Then $End_R(M)$ is a local ring.

Proof: (compare 19.9) For $f \in End(M)$ we have $Ke\, f \cap Ke\,(1-f) = 0$. Since M is uniform, either f or $1-f$ has to be monic and hence an isomorphism by 31.13,(1). Thus $End(M)$ is a local ring.

31.15 Exercises.

(1) Let R be a left artinian ring. Show:
(i) For an ideal $I \subset R$, the ring R/I is primitive if and only if I is maximal.
(ii) If $E_1, \ldots, E_k$ is a set of representatives of simple R-modules, then
$\{Re(R, E_i)\}_{i \le k}$ is the set of maximal ideals in R (for reject see 14.5),
$Jac(R) = \bigcap_{i \le k} Re(R, E_i)$ and $R/Re(R, E_i) \simeq Tr(E_i, R/Jac(R))$.

(2) Show that the following are equivalent for an R-module M:
(a) The set of direct summands of M satisfies acc;
(b) the set of direct summands of M satisfies dcc;
(c) End(M) contains no infinite set of non-zero orthogonal idempotents.

(3) Show for a ring R:
If $_RR$ is semi-injective and noetherian, then $_RR$ is artinian.

(4) Show that for a ring R the following assertions are equivalent:
(a) every factor ring of R is left finitely cogenerated;
(b) for every $M \in R$-MOD, we have $\sigma[M] = R/An(M)$-MOD;
(c) every self-injective $M \in R$-MOD is finitely generated as a module over End(M). (Hint: see exercise 17.15,(12).)

(5) Let M be an artinian or noetherian R-module. Show:
$M^n \simeq M^k$ *for* $n, k \in \mathbb{N}$ *if and only if* $n = k$.

(6) Show that the following are equivalent for a $\mathbb{Z}$*-module M:*
(a) M is locally artinian;
(b) M has dcc for cyclic submodules;
(c) M has essential socle;
(d) M is a torsion module (see 15.10).

Literature: CHATTERS-HAJARNAVIS, COHN, KERTÉSZ;
Albu [1,2], Anh [3], Bueso-Jara, Contessa, Fakhruddin [2], Fisher [1,2], Garcia-Gomez [4], Gueridon [1,2], Gupta,R.N., Gupta-Varadarajan, Hein, Heinzer-Lantz, van Huynh [1,3,4], Inoue [2], Ishikawa, Menini [3], Năstăsescu [2,3,4], Rangaswamy [2], Rant, Satyanarayana, Shock, Smith [2], Takeuchi [2], Würfel [2], Xu Yonghua, Yue [6], Zöschinger [7].

32 Modules of finite length

1.Refinement of normal series. 2. Zassenhaus' Lemma. 3.Modules with composition series. 4.Modules of finite length. 5.Locally finite modules. 6.Loewy series. 7.Artinian self-generators. 8.$\sigma[M]$ with artinian generator. 9.Exercises.

Let M be an R-module. A finite chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M,\ k \in \mathbb{N},$$

is called a *normal series of M*. The number k is said to be the *length of the normal series* and the factor modules M_i/M_{i-1}, $1 \leq i \leq k$ are called its *factors.*

A further normal series $0 = N_0 \subset N_1 \subset \cdots \subset N_n = M$, $n \in \mathbb{N}$, is said to be a *refinement* of the above normal series if it contains all modules M_i. We call the two normal series *isomorphic* if they have same length, i.e. if $k = n$, and there is a permutation π of $\{1, \ldots, k\}$ with $M_i/M_{i-1} \simeq N_{\pi(i)}/N_{\pi(i)-1}$.

32.1 Refinement of normal series (Schreier).
Any two normal series of an R-module M have isomorphic refinements.

Proof: Consider two normal series of M

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M \text{ and } 0 = N_0 \subset N_1 \subset \cdots \subset N_n = M.$$

Between M_i and M_{i+1}, $0 \leq i \leq k-1$, we include the chain of modules $M_{i,j} = M_i + (M_{i+1} \cap N_j)$, $0 \leq j \leq n$, to obtain

$$M_i = M_{i,0} \subset M_{i,1} \subset \cdots \subset M_{i,n} = M_{i+1}.$$

Similarly, between N_j and N_{j+1} we include the chain $N_{j,i} = N_j + (N_{j+1} \cap M_i)$, $0 \leq i \leq k$.

The chains $\{M_{i,j}\}$ and $\{N_{j,i}\}$ obviously are normal series of M with length kn and are refinements of $\{M_i\}$ resp. $\{N_j\}$. They are isomorphic since

$$M_{i,j+1}/M_{i,j} \simeq N_{j,i+1}/N_{j,i} \text{ for } 0 \leq i \leq k-1,\ 0 \leq j \leq m-1.$$

This isomorphisms are derived from the

32.2 Zassenhaus' Lemma.
Assume $K' \subset K \subset M$ and $L' \subset L \subset M$ to be submodules of the R-module M. Then

$$[K' + (K \cap L)]/[K' + (K \cap L')] = [L' + (K \cap L)]/[L' + (K' \cap L)].$$

Proof: Applying the equality (modularity condition)

$$[K' + (K \cap L')] \cap (K \cap L) = (K' \cap L) + (K \cap L')$$

and isomorphism theorems, we obtain the exact commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & K' \cap L + K \cap L' & \to & K \cap L & \to & K \cap L / K' \cap L + K \cap L' & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & K' + K \cap L' & \to & K' + K \cap L & \to & K' + K \cap L / K' + K \cap L' & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & K'/K' \cap L & \to & K'/K' \cap L & \to & 0 & \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & & .
\end{array}$$

From this we see that the left side of the required isomorphism is isomorphic to $K \cap L/(K' \cap L + K \cap L')$. Interchanging K and L we observe that the right side is also isomorphic to the same module.

Example of normal series:
In the $\mathbb{Z}$-module $\mathbb{Z}/12\mathbb{Z}$ we have the normal series

$0 \subset 2\mathbb{Z}/12\mathbb{Z} \subset \mathbb{Z}/12\mathbb{Z}$ and $0 \subset 3\mathbb{Z}/12\mathbb{Z} \subset \mathbb{Z}/12\mathbb{Z}$.

As refinements we obtain

$0 \subset 6\mathbb{Z}/12\mathbb{Z} \subset 2\mathbb{Z}/12\mathbb{Z} \subset \mathbb{Z}/12\mathbb{Z}$ and
$0 \subset 6\mathbb{Z}/12\mathbb{Z} \subset 3\mathbb{Z}/12\mathbb{Z} \subset \mathbb{Z}/12\mathbb{Z}$.

The refinements are isomorphic, all factors are simple $\mathbb{Z}$-modules.

A normal series $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ is called a *composition series of M* if all factors M_i/M_{i-1} are simple modules.

32.3 Modules with composition series. *Let M be an R-module with composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$. Then*

(1) Every normal series of M (with non-trivial factors) can be refined to a composition series.

(2) Every composition series of M is isomorphic to the above series, in particular has the same length k.

Proof: (1) By 32.1, there are isomorphic refinements to any normal series and the given composition series. However, a composition series has no proper refinement. Hence the desired refinement is isomorphic to the composition series and hence is itself a composition series.

(2) is an immediate consequence of (1).

A module M which has a composition series is called a *module of finite length*. The length of a composition series of M is said to be the *length of M*. Notation: $lg(M)$.

If M is a finite dimensional vector space over a field K with basis $\{m_1, \dots, m_k\}$, then

$$0 \subset Km_1 \subset Km_1 + Km_2 \subset \dots \subset Km_1 + \dots + Km_k = M$$

is a composition series of M (with factors isomorphic to ${}_KK$). In this case the length of M is equal to the dimension of M.

Similarly we see that the length of a finitely generated, semisimple module is equal to the number of simple summands in a decomposition.

32.4 Properties of modules of finite length.
Let M be an R-module.

(1) M has finite length if and only if M is artinian and noetherian.

(2) If $0 \to K \to M \to M/K \to 0$ is exact, then M has finite length if and only if K and M/K both have finite length, and in this case

$$lg(M) = lg(K) + lg(M/K).$$

(3) Assume M has finite length. Then
(i) every finitely generated module in $\sigma[M]$ has finite length;
(ii) there are only finitely many non-isomorphic simple modules in $\sigma[M]$;
(iii) if M is indecomposable, then $End_R(M)$ is a local ring;
(iv) $End_R(M)$ is a semiprimary ring.

Proof: (1) $\Rightarrow$ Assume $lg(M) = k$. Consider any properly ascending chain $M_1 \subset M_2 \subset \cdots$ of submodules of M. Then we can find a normal series of M of the length $k+1$. By 32.3, this can be refined to a composition series of M whose length would be $\geq k+1$. This contradicts 32.3 and hence the chain has to become stationary after k steps and M is noetherian.

Similarly we see that a properly descending chain has at most k different members and M is artinian.

$\Leftarrow$ Now assume M is artinian and noetherian. Choose a maximal submodule $M_1 \subset M$. Since this is finitely generated, it contains a maximal submodule $M_2 \subset M_1$. Continuing in this way we find a descending chain $M \supset M_1 \supset M_2 \supset \cdots$ with simple factors M_i/M_{i+1}. Since this chain has to terminate after finitely many steps (M is artinian), we obtain a (finite) composition series of M.

(2) The first assertion follows (by (1)) from the corresponding properties of artinian and noetherian modules. The normal series $0 \subset K \subset M$ can be refined to a composition series

$$0 = K_0 \subset K_1 \subset \cdots \subset K_r = K \subset \cdots \subset K_k = M$$

with $k = lg(M)$. Hence K has a composition series of length $r = lg(K)$ and in M/K we get a composition series

$$0 \subset K_{r+1}/K \subset K_{r+2}/K \subset \cdots \subset K_k/K = M/K$$

of length $lg(M/K) = k - r$.

(3)(i) If M is artinian and noetherian, then this is true for all finitely generated modules in $\sigma[M]$ (see 27.3, 31.2).

(ii) The simple modules in $\sigma[M]$ are factor modules of submodules of M (see proof of 18.5), i.e. factors of a composition series of M. By 32.3, there are only finitely many such factors.

(iii) Consider $f \in End_R(M)$. If $Ke\, f = 0$, then f is invertible by 31.13.
Assume $Ke\, f \neq 0$. Since M is indecomposable we conclude from 31.13 (Fitting's Lemma) that f is nilpotent and $1 - f$ is invertible (see 21.9). Hence $End_R(M)$ is local (see 19.8).

(iv) This is a special case of the Harada-Sai Lemma 54.1.

We say that an R-module M has *locally finite length* or *M is locally finite* if every finitely generated submodule of M has finite length, i.e. is artinian and noetherian.

The following observation will be useful:

Any direct sum of locally finite modules is locally finite.

Proof: Clearly M is locally finite if and only if M is locally artinian and locally noetherian (see 32.4). Hence the assertion follows from 27.2,(2) and 31.2,(2).

32.5 Locally finite modules. Characterizations.

For an R-module M the following assertions are equivalent:

(a) M is locally finite;

(b) $M^{(\mathbb{N})}$ is locally finite;

(c) every finitely generated module in $\sigma[M]$ has finite length;

(d) every injective module in $\sigma[M]$ is a direct sum of finitely cogenerated modules;

(e) every injective module in $\sigma[M]$ is a direct sum of injective hulls of simple modules.

Proof: $(a) \Leftrightarrow (b)$ was shown above, $(b) \Leftrightarrow (c)$ is clear.

$(c) \Rightarrow (d) \Leftrightarrow (e)$ In particular, M is locally noetherian and, by 27.4, every injective module in $\sigma[M]$ is a direct sum of indecomposable injective modules. These contain simple modules and consequently are injective hulls of simple modules and are finitely cogenerated.

$(d) \Rightarrow (c)$ Finitely cogenerated modules are direct sums of indecomposable modules (see 21.3). Hence, by 27.5, M is locally noetherian. Consider a finitely generated $N \in \sigma[M]$ with M-injective hull $\widehat{N} = \bigoplus_\Lambda U_\lambda$, U_λ finitely cogenerated. Then N is contained in a finite partial sum of the U_λ and hence is finitely cogenerated. This also applies to every factor module of N, i.e. N is artinian (see 31.1).

We have seen in 31.4 that left artinian rings have finite length. On the other hand, we have pointed out that artinian modules need not be finitely generated (e.g. $\mathbb{Z}_{p^\infty}$). Now we may ask if, for an artinian module M, the category $\sigma[M]$ is locally finite. This is the case if there is an artinian generator in $\sigma[M]$. To prove this we look at a special chain of submodules which is of general interest:

In an R-module M, consider the ascending chain of submodules

$$0 = L_0(M) \subset L_1(M) \cdots \subset L_\alpha(M) \subset \cdots ,$$

indexed over all ordinals $\alpha \geq 0$, defined by

$L_1(M) = Soc(M)$,

$L_{\alpha+1}(M)/L_\alpha(M) = Soc(M/L_\alpha(M))$,

$L_\alpha(M) = \sum_{\beta<\alpha} L_\beta(M)$ when α is a limit ordinal,

which is called the ascending *Loewy series* of M.

32.6 Properties of Loewy series.

Let M be an R-module with Loewy series $\{L_\alpha(M)\}_{\alpha\geq 0}$.

(1) For submodules $N \subset M$ we have $L_i(N) = L_i(M) \cap N$, for $i \in \mathbb{N}$.

(2) Any artinian submodule N of M with $N \subset L_n(M)$, for some $n \in \mathbb{N}$, has finite length; if $N \not\subset L_{n-1}(M)$, then $lg(N) \geq n$.

(3) Assume M is artinian. Then:

(i) For every $i \in \mathbb{N}$, $L_i(M)$ is a submodule of finite length.

(ii) If R is commutative, then $M = \bigcup_{\mathbb{N}} L_i(M)$.

(4) If $f : M \to N$ is a morphism of artinian modules and $\{L_i(N)\}_{\mathbb{N}}$ the beginning of the Loewy series of N, then $(L_i(M))f \subset L_i(N)$ for all $i \in \mathbb{N}$.

(5) There is a least ordinal γ with $L_\gamma(M) = L_{\gamma+1}(M)$, called the Loewy length *of M.*

(6) $M = L_\gamma(M)$ for some ordinal γ if and only if every non-zero factor module of M has non-zero socle. Then M is called semi-artinian*.*

Proof: (1) is seen by induction on $i \in I\!N$: We know

$$L_1(N) = Soc(N) = Soc(M) \cap N = L_1(M) \cap N\,.$$

Assume $L_i(N) = L_i(M) \cap N$ and hence $L_i(N) = L_i(M) \cap L_{i+1}(N)$. From the isomorphism

$$L_{i+1}(N) + L_i(M)/L_i(M) \simeq L_{i+1}(N)/L_{i+1}(N) \cap L_i(M) = L_{i+1}(N)/L_i(N)$$

we conclude $L_{i+1}(N) + L_i(M) \subset L_{i+1}(M)$ and hence

$$L_{i+1}(N) = L_{i+1}(N) + (L_i(M) \cap N) \subset L_{i+1}(M) \cap N\,.$$

The equality $(L_{i+1}(M) \cap N)/L_i(N) = (L_{i+1}(M) \cap N)/(L_i(M) \cap N)$ shows that these modules are isomorphic to submodules of $L_{i+1}(M)/L_i(M)$ and therefore semisimple. This implies $L_{i+1}(M) \cap N \subset L_{i+1}(N)$.

By transfinite induction it can be shown that (1) holds in fact for any ordinal i.

(2) If N is artinian, then $L_1(N) = Soc(N)$ is finitely generated and, moreover, the modules $L_{i+1}(N)/L_i(N)$ are finitely generated and semisimple. Hence all $L_i(N)$ have finite length. Besides $L_n(N) = L_n(M) \cap N = N$.

In case $N \not\subset L_{n-1}(M)$

$$0 \subset L_1(N) \subset L_2(N) \subset \cdots \subset L_{n-1}(N) \subset N$$

is a strictly ascending chain which can be refined to a composition series of length $\geq n$.

(3) (i) is obvious.

(ii) For $x \in M$ the module Rx is artinian and $R/An(x)$ is an artinian – hence noetherian – ring (see 31.4). Therefore Rx is noetherian and the chain of submodules $\{Rx \cap L_i(M)\}_{I\!N}$ becomes stationary after finitely many steps, i.e.

$$Rx \cap L_n(M) = Rx \cap L_{n+1}(M) \text{ for some } n \in I\!N.$$

Assume $x \notin L_n(M)$. Then there exists $r \in R$ with $rx \in L_{n+1}(M)$ and $rx \notin L_n(M)$ (since $Soc\,[M/L_n(M)] \trianglelefteq M/L_n(M)$).

However, $Rx \cap L_n(M) = Rx \cap L_{n+1}(M)$ implies $rx \in L_n(M)$.

(4) Starting with $(Soc\, M)f \subset Soc\, N = L_1(N)$ this is shown by induction on $i \in I\!N$.

(5) This is clear since the submodules form a set.

(6) The Loewy series only becomes stationary at γ if $M/L_\gamma(M)$ has zero socle.

32.7 Artinian self-generators.
Let U be an artinian module and M an R-module. Assume the submodules of M to be U-generated. Then

(1) The Loewy series $L_1(M) \subset L_2(M) \subset \cdots$ becomes stationary after finitely many steps.

(2) If M is artinian, then M is noetherian.

(3) If U is a self-generator, then U is noetherian.

Proof: (1) Assume the Loewy series to be strictly ascending. Since U generates the submodules of M, we can find a family of morphisms

$$f_n : U \to M,\ (U)f_n \subset L_n(M),\ (U)f_n \not\subset L_{n-1}(M).$$

The finite intersections of the submodules $Ke\, f_n \subset U$ form an inverse family. U being artinian, this family has a minimal element U_o with $U_o \subset Ke\, f_n$ for all $n \in I\!N$ and $lg(U/U_o) = k \in I\!N$. From this we have

$$lg((U)f_n)) = lg(U/Ke\, f_n) \leq lg(U/U_o) = k.$$

On the other hand, we know from 32.6, by the choice of the f_n, that $lg((U)f_n) \geq n$ for all $n \in I\!N$, a contradiction.

(2) By (1), we have $L_r(M) = L_{r+1}(M)$ for some $r \in I\!N$. For artinian M this implies $L_r(M) = M$: Otherwise the artinian module $M/L_r(M) \neq 0$ has a non-zero socle which means $L_{r+1}(M) \neq L_r(M)$.

(3) is a special case of (2).

32.8 $\sigma[M]$ with artinian generator.
Let M be an R-module and U an artinian generator in $\sigma[M]$. Then

(1) U is noetherian (of finite length), and $End_R(U)$ is semiprimary.

(2) All artinian modules in $\sigma[M]$ have finite length.

(3) All finitely generated modules in $\sigma[M]$ have finite length.

(4) There are only finitely many non-isomorphic simple modules in $\sigma[M]$.

(5) There is a finitely generated projective generator P in $\sigma[M]$ and hence $\sigma[M]$ is equivalent to the category $End_R(P)$-MOD.

(6) For every $N \in \sigma[M]$, there is an artinian N-projective generator in $\sigma[N]$.

Proof: (1) and (2) follow immediately from 32.7 and 32.4.

(3) By (1), $\sigma[M]$ is a locally finite category (see 32.5).

(4) Since $\sigma[M] = \sigma[U]$, this follows from 32.4.

(5) Since $End(U)$ is semiprimary by 32.4, the first statement follows from a more general result in 51.13. The resulting equivalence is described in 46.2.

(6) Put $K = Re(P, N) = \bigcap\{Ke\,f \mid f \in Hom_R(P, N)\}$, with P as in (5). Since N is P-generated, it is also P/K-generated and hence $N \in \sigma[P/K]$.

On the other hand, P/K is N-cogenerated and as an artinian module it is contained in a finite sum N^k, i.e. $P/K \in \sigma[N]$. P/K generates all simple modules in $\sigma[P/K] = \sigma[N]$. Since K is a fully invariant submodule of P, the factor module P/K is self-projective (see 18.2) and therefore a projective generator in $\sigma[N]$ (see 18.5).

32.9 Exercises.

(1) Let $M_1, \ldots, M_k$ be submodules of the R-module M. Show that the following statements are equivalent:
(a) M/M_i has finite length for all $i = 1, \ldots, k$;
(b) $M/\bigcap_{i\leq k} M_i$ has finite length.

(2) Let $0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$ be an exact sequence of modules of finite length. Show $\sum_{k=1}^{n}(-1)^k lg(M_k) = 0$.

(3) Let I be an ideal in the ring R. Assume that I/I^2 is finitely generated as a left module. Show:
(i) I^k/I^{k+1} is finitely generated for every $k \in I\!N$.
(ii) If R/I is left semisimple, then I^k/I^{k+1} is a left R-module of finite length for every $k \in I\!N$.

(4) Let M be a noetherian R-module. We form the submodules
$M_0 = M$ and $M_{k+1} = Rad(M_k)$ for $k \in I\!N$.
Show: If $R/Jac(R)$ is semisimple, then
(i) M_k/M_{k+1} has finite length for every $k \in I\!N$.
(ii) If L is a submodule with $M_{k+1} \subset L \subset M$ and $lg(M/L) \leq k \in I\!N$, then $M_k \subset L$.
(iii) If M is a self-cogenerator, then M is artinian.

(5) Let R be a commutative local ring and E an injective hull of the simple R-module $R/Jac(R)$. Show:
(i) For every R-module M of finite length, M and $Hom_R(M, E)$ have the same length and the canonical map
$\Phi_M : M \to Hom_R(Hom_R(M, E), E)$ *(see 45.10)*
is an isomorphism.
(ii) If R is artinian, then $R \simeq End({}_RE)$ and $lg(R) = lg(E)$.
(Hint: Induction on $lg(M)$.)

(6) Let R be a commutative ring and M an artinian uniform R-module (every non-zero submodule essential).
Show that M is self-injective and a cogenerator in $\sigma[M]$.
(Hint: Form the Loewy series $L_i(M)$, show that M is $L_i(M)$-injective. Observe that $R/An(L_i(M))$ is artinian local and use exercise (5).)

(7) Let R be a commutative ring and assume M, N are R-modules with $lg(M) \le m$ and $lg(N) \le n$ for $m, n \in I\!N$. Prove

$$lg({}_RHom_R(M, N)) \le mn.$$

Literature: ALBU-NĂSTĂSESCU, NĂSTĂSESCU;
Facchini [1], Ginn, Gómez Pardo [3], Gupta-Singh, Izawa [1,3], Năstăsescu [2], Rege-Varadarajan, Roux [5], Schulz [2], Shores.

Chapter 7

Pure sequences and derived notions

In this chapter we shall introduce the notion of pure exact sequences with respect to a class $\mathcal{P}$ of modules, generalizing splitting sequences. We already know that injective, semisimple and projective modules may be characterized by the splitting of certain exact sequences. Similarly we shall consider modules distinguished by the $\mathcal{P}$-purity of certain short exact sequences (see § 35). Choosing in R-MOD as $\mathcal{P}$ the class of all finitely presented R-modules we obtain the usual notion of purity in R-MOD.

33 $\mathcal{P}$-pure sequences, pure projective modules

1.Definitions. 2.Composition of $\mathcal{P}$-pure morphisms. 3.$\mathcal{P}$-pure submodules. 4.Pushout and pullback with $\mathcal{P}$-pure morphisms. 5.Existence of $\mathcal{P}$-pure epimorphisms. 6.$\mathcal{P}$-pure projective modules. 7.$\mathcal{P}$-pure injective modules. 8.Direct limit of $\mathcal{P}$-pure sequences. 9.Examples. 10.Exercises.

33.1 Definitions. Let $\mathcal{P}$ be a non-empty class of modules in $\sigma[M]$, $M \in R$-MOD. An exact sequence $0 \to K \xrightarrow{f} L \xrightarrow{g} N \to 0$ in $\sigma[M]$ is called *$\mathcal{P}$-pure in* $\sigma[M]$, if every module P in $\mathcal{P}$ is projective with respect to this sequence, i.e. if every diagram

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & K & \xrightarrow{f} & L & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

can be extended commutatively by a morphism $P \to L$. Equivalently, we may demand the sequence

$$0 \longrightarrow Hom(P,K) \longrightarrow Hom(P,L) \longrightarrow Hom(P,N) \longrightarrow 0$$

to be exact, or also only $Hom(P,g)$ to be epic for every $P \in \mathcal{P}$.

In this case we call f a *$\mathcal{P}$-pure monomorphism*, g a *$\mathcal{P}$-pure epimorphism* and $Im\, f = (K)f$ a *$\mathcal{P}$-pure submodule* of L.

The properties of $\mathcal{P}$-pure sequences, of course, strongly depend on the choice of the class $\mathcal{P}$. For example, if $\mathcal{P}$ consists only of projective modules, then every short exact sequence is $\mathcal{P}$-pure. On the other hand, a splitting short exact sequence is $\mathcal{P}$-pure for every class $\mathcal{P}$.

The following classes (sets) $\mathcal{P}$ of modules are of interest:
– finitely presented modules (in $\sigma[M]$),
– cyclic modules (in $\sigma[M]$),
– finitely presented, cyclic modules in $\sigma[M]$,
– factor modules R/I, with ${}_RI$ finitely generated or cyclic.

In case $\mathcal{P}$ consists of all finitely presented modules (in $\sigma[M]$), instead of *$\mathcal{P}$-pure* we just say *pure (in $\sigma[M]$)*, similarly *pure submodule* etc. This case will be studied in detail in § 34.

33.2 Composition of $\mathcal{P}$-pure morphisms.
Let M be an R-module, $\mathcal{P}$ a class of modules in $\sigma[M]$ and $f: K \to L$, $g: L \to N$ morphisms in $\sigma[M]$.

(1)(i) If f and g are $\mathcal{P}$-pure epimorphisms, then fg is also a $\mathcal{P}$-pure epimorphism.
(ii) If fg is a $\mathcal{P}$-pure epimorphism, then g is a $\mathcal{P}$-pure epimorphism.

(2)(i) If f and g are $\mathcal{P}$-pure monomorphisms, then fg is a $\mathcal{P}$-pure monomorphism.
(ii) If fg is a $\mathcal{P}$-pure monomorphism, then f is a $\mathcal{P}$-pure monomorphism.

Proof: (1) If f and g are $\mathcal{P}$-pure epimorphisms, then, for every $P \in \mathcal{P}$, the morphisms $Hom(P,f)$ and $Hom(P,g)$ are epic. Then also $Hom(P,fg) = Hom(P,f)Hom(P,g)$ is epic, i.e. fg is $\mathcal{P}$-pure. On the other hand, if $Hom(P,fg)$ is epic, then this is also true for $Hom(P,g)$.

(2) With the monomorphisms f, g we form the diagram

$$\begin{array}{ccc} K & = & K \\ \downarrow f & & \downarrow fg \\ L & \xrightarrow{g} & N \end{array} \quad ,$$

which can be completed by kernels and cokernels in a canonical way. Applying the functor $Hom(P,-)$, with $P \in \mathcal{P}$, we obtain the commutative diagram

$$\begin{array}{ccccccc}
 & Hom(P,K) & = & Hom(P,K) & & & \\
 & \downarrow {\scriptstyle Hom(P,f)} & & \downarrow {\scriptstyle Hom(P,fg)} & & & \\
0 \to & Hom(P,L) & \stackrel{Hom(P,g)}{\to} & Hom(P,N) & \to & Hom(P,N/L) & \to 0 \\
 & \downarrow & & \downarrow & & \| & \\
0 \to & Hom(P,L/K) & \to & Hom(P,N/K) & \to & Hom(P,N/L) & \to 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & & .
\end{array}$$

Because of the $\mathcal{P}$-purity of f and g, the first column and the second row are exact. By the Kernel Cokernel Lemma, the middle column is also exact, i.e. fg is a $\mathcal{P}$-pure monomorphism.

Now assume fg to be $\mathcal{P}$-pure. Then in the above diagram, the middle column is exact. Again by the Kernel Cokernel Lemma, we conclude that the first column has to be exact, i.e. f is $\mathcal{P}$-pure (for this the second row need not be exact at the end).

Applying these results to the canonical embeddings we obtain:

33.3 $\mathcal{P}$-pure submodules.
Let M be an R-module, $\mathcal{P}$ a class of modules in $\sigma[M]$ and $K \subset L \subset N$ modules in $\sigma[M]$.

(1) If K is $\mathcal{P}$-pure in L and L is $\mathcal{P}$-pure in N, then K is $\mathcal{P}$-pure in N.

(2) If K is $\mathcal{P}$-pure in N, then K is also $\mathcal{P}$-pure in L.

(3) If L is $\mathcal{P}$-pure in N, then L/K is $\mathcal{P}$-pure in N/K.

(4) If K is $\mathcal{P}$-pure in N and L/K is $\mathcal{P}$-pure in N/K, then L is $\mathcal{P}$-pure in N.

(5) If K is $\mathcal{P}$-pure in N, then there is a bijection between the $\mathcal{P}$-pure submodules of N containing K and the $\mathcal{P}$-pure submodules of N/K.

Proof: (1) and (2) are assertions of 33.2.
(3), (4) This can be seen from the diagram in the proof of 33.2,(2).

33.4 Pushout and pullback with $\mathcal{P}$-pure morphisms.
Let M be an R-module and $\mathcal{P}$ a class of modules in $\sigma[M]$. Consider the following commutative diagram in $\sigma[M]$:

$$\begin{array}{ccc}
K & \xrightarrow{f_1} & L_1 \\
{\scriptstyle f_2}\downarrow & & \downarrow {\scriptstyle g_1} \\
L_2 & \xrightarrow{g_2} & N \quad .
\end{array}$$

(1) If the square is a pullback and g_2 is a $\mathcal{P}$-pure epimorphism, then f_1 is also a $\mathcal{P}$-pure epimorphism.

(2) If the square is a pushout and f_1 is a $\mathcal{P}$-pure monomorphism, then g_2 is also a $\mathcal{P}$-pure monomorphism.

Proof: (1) Let the square be a pullback and g_2 a $\mathcal{P}$-pure epimorphism. For $P \in \mathcal{P}$ and $h_1 : P \to L_1$ there exists $h_2 : P \to L_2$ with $h_1 g_1 = h_2 g_2$. Then we can find an $h : P \to K$ with $hf_1 = h_1$ (and $hf_2 = h_2$, pullback).

(2) Assume the square to be a pushout and f_1 to be a $\mathcal{P}$-pure monomorphism. We have the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f_1} & L_1 & \xrightarrow{p_1} & X & \longrightarrow & 0 \\ & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} & & \| & & \\ 0 & \longrightarrow & L_2 & \xrightarrow{g_2} & N & \xrightarrow{p_2} & X & \longrightarrow & 0 \end{array} .$$

For every $h : P \to X$, $P \in \mathcal{P}$, there exists $l : P \to L_1$ with $h = lp_1$, and hence $h = (lg_1)p_2$. Therefore p_2 is a $\mathcal{P}$-pure epimorphism and g_2 is a $\mathcal{P}$-pure monomorphism.

33.5 Existence of $\mathcal{P}$-pure epimorphisms.
Let M be an R-module, $\mathcal{P}$ a class of modules in $\sigma[M]$ and $N \in \sigma[M]$. Assume there is a set $\mathcal{P}_o$ of representatives for $\mathcal{P}$ (i.e. every $P \in \mathcal{P}$ is isomorphic to an element in $\mathcal{P}_o$) and N is generated by $\mathcal{P}$. Then there exists a $\mathcal{P}$-pure epimorphism $\widetilde{P} \to N$, with $\widetilde{P}$ a direct sum of modules in $\mathcal{P}_o$.

Proof: For $P \in \mathcal{P}_o$, put $\Lambda_P = Hom(P, N)$ and consider the canonical mappings

$$\varphi_P : P^{(\Lambda_P)} \longrightarrow N, \quad (p_f)_{\Lambda_P} \mapsto \sum (p_f)f, \quad f \in \Lambda_P.$$

These can be extended to

$$\varphi = \sum\nolimits_{P\in\mathcal{P}_o} \varphi_P : \bigoplus\nolimits_{P\in\mathcal{P}_o} P^{(\Lambda_P)} \longrightarrow N.$$

Since N is also $\mathcal{P}_o$-generated, φ is an epimorphism. Every module in $\mathcal{P}$ is isomorphic to some $P \in \mathcal{P}_o$. To prove that φ is $\mathcal{P}$-pure it suffices to show that, for every $P' \in \mathcal{P}_o$ and $f \in Hom(P', N)$, the diagram

$$\begin{array}{ccc} & & P' \\ & & \downarrow{\scriptstyle f} \\ \bigoplus_{P\in\mathcal{P}_o} P^{(\Lambda_P)} & \longrightarrow & N \end{array}$$

can be extended commutatively by some $g : P' \to \bigoplus_{P\in\mathcal{P}_o} P^{(\Lambda_P)}$. Since the pair (P', f) corresponds to a summand in the direct sum, we may take the corresponding injection as g.

Remark: With the notation of 33.5, of course '$\mathcal{P}$-pure' is equivalent to '$\mathcal{P}_o$-pure'. If there exists a set of representatives for $\mathcal{P}$, we may, without restriction, assume $\mathcal{P}$ itself to be a set.

An R-module $X \in \sigma[M]$ is called *$\mathcal{P}$-pure projective* if X is projective with respect to every $\mathcal{P}$-pure sequence in $\sigma[M]$, i.e. $Hom(X, -)$ is exact with respect to $\mathcal{P}$-pure sequences in $\sigma[M]$.

It is easy to see (with the same argument as for projective modules) that direct sums and direct summands of $\mathcal{P}$-pure projective modules are again $\mathcal{P}$-pure projective. Of course, all $P \in \mathcal{P}$ are $\mathcal{P}$-pure projective.

33.6 $\mathcal{P}$-pure projective modules. Characterizations.

Let M be an R-module, $\mathcal{P}$ a class of modules in $\sigma[M]$ and $X \in \sigma[M]$. The following assertions are equivalent:

(a) X is $\mathcal{P}$-pure projective;

(b) every $\mathcal{P}$-pure sequence $0 \to K \to L \to X \to 0$ in $\sigma[M]$ splits.

If $\mathcal{P}$ is a set and X is $\mathcal{P}$-generated (and finitely generated), then (a),(b) are also equivalent to:

(c) X is a direct summand of a (finite) direct sum of modules in $\mathcal{P}$.

Proof: $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (a)$ Assume $0 \to K \to L \to N \to 0$ to be a $\mathcal{P}$-pure sequence in $\sigma[M]$ and $f : X \to N$ a morphism. Forming a pullback, we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & Q & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \ .
\end{array}$$

The first row is $\mathcal{P}$-pure, by 33.4, and hence splits by (b). This yields the desired morphism $X \to L$.

$(b) \Rightarrow (c)$ By 33.5, there is a $\mathcal{P}$-pure epimorphism $\widetilde{P} \to X$, with $\widetilde{P}$ a direct sum of modules in $\mathcal{P}$. Because of (b), this epimorphism splits and X is a direct summand of $\widetilde{P}$.

$(c) \Rightarrow (a)$ we already noted above. The assertion for finitely generated X is evident.

An R-module $Y \in \sigma[M]$ is called *$\mathcal{P}$-pure injective* if Y is injective with respect to every $\mathcal{P}$-pure sequence in $\sigma[M]$, i.e. $Hom(-, Y)$ is exact with respect to $\mathcal{P}$-pure sequences.

Direct products and direct summands of $\mathcal{P}$-pure injective modules are again $\mathcal{P}$-pure injective.

33.7 $\mathcal{P}$-pure injective modules. Characterization.
Let M be an R-module and $\mathcal{P}$ a class of R-modules. For $Y \in \sigma[M]$, the following assertions are equivalent:
(a) Y is $\mathcal{P}$-pure injective;
(b) every $\mathcal{P}$-pure sequence $0 \to Y \to L \to N \to 0$ in $\sigma[M]$ splits.

Proof: Dual to 33.6.

The importance of finitely presented modules in investigating $\mathcal{P}$-purity is mainly based on the following observations:

33.8 Direct limit of $\mathcal{P}$-pure sequences.
Let M be an R-module and $0 \to K_i \to L_i \to N_i \to 0$ a direct system of exact sequences in $\sigma[M]$ (with index set Λ, see 24.6). Assume $\mathcal{P}$ to be a class of finitely presented modules in $\sigma[M]$. Then:

(1) If the given sequences are $\mathcal{P}$-pure, then the sequence

$$0 \longrightarrow \varinjlim K_i \xrightarrow{u} \varinjlim L_i \xrightarrow{v} \varinjlim N_i \longrightarrow 0$$

is also $\mathcal{P}$-pure in $\sigma[M]$.

(2) The direct limit of $\mathcal{P}$-pure submodules of a module in $\sigma[M]$ is a $\mathcal{P}$-pure submodule.

(3) The set of $\mathcal{P}$-pure submodules of a module in $\sigma[M]$ is inductive with respect to inclusion.

Proof: (1) We know, from 24.6, that the limit sequence is exact and it remains to show that $Hom(P, v)$ is epic for every $P \in \mathcal{P}$. In the commutative diagram

$$\begin{array}{ccccc}
\varinjlim Hom(P, L_i) & \longrightarrow & \varinjlim Hom(P, N_i) & \longrightarrow & 0 \\
\downarrow \Phi_P & & \downarrow \Phi'_P & & \\
Hom(P, \varinjlim L_i) & \xrightarrow{Hom(P,v)} & Hom(P, \varinjlim N_i) & &
\end{array}$$

the first row is exact. Since P is finitely presented, Φ_P and Φ'_P are isomorphisms, by 25.2, and hence $Hom(P, v)$ is epic.

(2) and (3) are immediate consequences of (1).

33.9 Examples of $\mathcal{P}$-pure sequences.
Let M be an R-module and $\mathcal{P}$ a class of finitely presented modules in $\sigma[M]$.

(1) For every family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M]$, the canonical embedding $\bigoplus_\Lambda N_\lambda \to \prod_\Lambda^M N_\lambda$ (product in $\sigma[M]$) is $\mathcal{P}$-pure.

(2) For every direct system $(N_i, f_{ij})_\Lambda$ of modules in $\sigma[M]$, the canonical epimorphism $\bigoplus_\Lambda N_i \to \varinjlim N_i$ is $\mathcal{P}$-pure (see 24.2).

Proof: (1) For every finite subset $\Lambda' \subset \Lambda$, the submodule $\bigoplus_{\Lambda'} N_\lambda$ is a direct summand - and hence a $\mathcal{P}$-pure submodule - of $\prod_\Lambda^M N_\lambda$. $\bigoplus_\Lambda N_\lambda$ is the direct limit of these $\mathcal{P}$-pure submodules and hence also a $\mathcal{P}$-pure submodule by 33.8.

(2) For every P in $\mathcal{P}$, we have, with obvious mappings, the commutative diagram with exact first row

$$\begin{array}{ccccc}
\bigoplus_\Lambda Hom(P, N_i) & \longrightarrow & \varinjlim Hom(P, N_i) & \longrightarrow & 0 \\
\downarrow \Phi_P & & \downarrow \Phi'_P & & \\
Hom(P, \bigoplus_\Lambda N_i) & \longrightarrow & Hom(P, \varinjlim N_i) & & ,
\end{array}$$

with Φ_P and Φ'_P isomorphisms. Hence the lower morphism is epic.

33.10 Exercises.

(1) Let K, L be submodules of the R-module N and $\mathcal{P}$ a class of R-modules. Show: If $K \cap L$ and $K + L$ are $\mathcal{P}$-pure in N, then K and L are also $\mathcal{P}$-pure in N.

(2) Let $\mathcal{I}$ be the class of all injective modules in $\sigma[M]$, $M \in$ R-MOD. Show that the following assertions are equivalent:
(a) M is locally noetherian;
(b) every $\mathcal{I}$-generated, $\mathcal{I}$-pure projective module is M-injective;
(c) for every $\mathcal{I}$-generated N, there exists an $\mathcal{I}$-pure epimorphism $E \to N$, with E M-injective.

Literature: FUCHS, MISHINA-SKORNJAKOV; Azumaya [3], Choudhury, Choudhury-Tewari, Crivei [1,2,3], Enochs [4], Fakhruddin [1,2], Generalov [1,2], Marubayashi, Naudé-Naudé, Naudé-Naudé-Pretorius, Onishi, Rangaswamy [1,3], Rege-Varadarajan, de la Rosa-Fuchs, Rychkov, Salles [2], Simson [1,2], Sklyarenko [1,2], Talwar.

34 Purity in $\sigma[M]$, R-MOD and $\mathbb{Z}$-MOD

1.Pure projective modules. 2.Pure epimorphisms and direct limits. 3.Systems of equations. 4.Pure injective modules. 5.Pure sequences in R-MOD. 6.Pure extensions in R-MOD. 7.Further characterizations of pure sequences. 8.Relatively divisible submodules. 9.Sequences exact under $R/J \otimes -$. 10.Purity over commutative rings. 11.Pure injective modules over commutative R. 12.Purity in $\mathbb{Z}$-MOD. 13.Pure injective $\mathbb{Z}$-modules. 14.Properties of $\mathbb{Z}_n$-MOD. 15.Exercises.

In this paragraph we study the purity determined by the class $\mathcal{P}$ of all finitely presented modules. As already announced, instead of '$\mathcal{P}$-pure' we simply say 'pure' (in $\sigma[M]$). Observe that, in general, we have no assertions about the existence of finitely presented modules in $\sigma[M]$. In case there are no non-zero finitely presented modules in $\sigma[M]$, every exact sequence is pure in $\sigma[M]$. By 25.1,(1), the class of all finitely presented modules in $\sigma[M]$ has a set of representatives, and it is closed with respect to forming finite direct sums and factor modules by finitely generated submodules (see 25.1). This has some remarkable consequences for pure projective modules:

34.1 Pure projective modules. Properties.
Let M,P be R-modules and assume P is generated by finitely presented modules in $\sigma[M]$. If P is pure projective in $\sigma[M]$, then:

(1) P is a direct summand of a direct sum of finitely presented modules.

(2) For every finitely generated submodule K of P, the factor module P/K is pure projective.

(3) Every finitely generated pure submodule of P is a direct summand.

Proof: (1) follows from 33.6.

(2) First consider $P = \bigoplus_\Lambda P_\lambda$ with finitely presented modules P_λ. Any finitely generated submodule K is contained in a finite partial sum $\bigoplus_{\Lambda'} P_\lambda$ and we have the commutative exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & \bigoplus_{\Lambda'} P_\lambda \arrow[r] \arrow[d] & \bigoplus_{\Lambda'} P_\lambda / K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d] & P/K \arrow[r] \arrow[d] & 0 \\
 & & P'' \arrow[r, equal] & P'' &
\end{tikzcd},$$

with $P'' = \bigoplus_{\Lambda\setminus\Lambda'} P_\lambda$. Since the last column splits, P/K is a direct sum of the pure projective module P'' and the finitely presented module $\bigoplus_{\Lambda'} P_\lambda \,/\, K$. Hence P/K is pure projective.

Now assume P to be a direct summand of $\bigoplus_\Lambda P_\lambda$, with finitely presented modules P_λ, and K to be a finitely generated submodule of P. Then P/K is a direct summand of the pure projective module $\bigoplus_\Lambda P_\lambda \,/\, K$, i.e. it is also pure projective.

(3) follows from (2).

34.2 Pure epimorphisms and direct limits.
Let M be an R-module, $N \in \sigma[M]$ and $\varphi : \bigoplus_\Lambda P_\lambda \to N$ an epimorphism, with finitely presented P_λ in $\sigma[M]$. Then:

(1) N is a direct limit of finitely presented modules in $\sigma[M]$.

(2) If φ is a pure epimorphism, then N is a direct limit of finite direct sums of the modules P_λ, $\lambda \in \Lambda$.

(3) Every pure exact sequence $0 \to K \to L \to N \to 0$ in $\sigma[M]$ is a direct limit of splitting sequences.

Proof: (1) For every P_λ, we choose a family $\{P_{\lambda,i} \mid i \in \mathbb{N}\}$ of modules $P_{\lambda,i} = P_\lambda$. This yields splitting epimorphisms

$$\bigoplus\nolimits_{i\in\mathbb{N}} P_{\lambda,i} \to P_\lambda \quad \text{and} \quad \pi : \bigoplus\nolimits_{\Lambda\times\mathbb{N}} P_{\lambda,i} \to \bigoplus\nolimits_{\Lambda} P_\lambda .$$

Put $P := \bigoplus_{\Lambda\times\mathbb{N}} P_{\lambda,i}$ and form the exact sequence

$$0 \longrightarrow K \longrightarrow P \xrightarrow{\pi\varphi} N \longrightarrow 0\,.$$

In the set $\mathcal{M}$ of all pairs (U, E) with

E is a finite subset of $\Lambda \times \mathbb{N}$,
U is a finitely generated submodule of $K \cap \bigoplus_E P_{\lambda,i}$,

we define a quasi-order by

$$(U, E) < (U', E') \ \text{ if and only if } \ U \subset U',\ E \subset E'.$$

Obviously this makes $\mathcal{M}$ directed (to the right). The relevant modules $\{U\}$ and $\{\bigoplus_E P_{\lambda,i}\}$ form direct systems of submodules of K, resp. P, with $\varinjlim U = K$ and $\varinjlim \bigoplus_E P_{\lambda,i} = P$.

With the corresponding inclusions we have the following commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & \bigoplus_E P_{\lambda,i} & \xrightarrow{p} & \bigoplus_E P_{\lambda,i}/U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f_{U,E} & & \\
0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{\pi\varphi} & N & \longrightarrow & 0 \quad .
\end{array}$$

Here, the $f_{U,E}$ are uniquely determined by the cokernel property. With the canonical morphisms for $(U,E) < (U',E')$, the finitely presented modules $\{\bigoplus_E P_{\lambda,i}/U\}_{U\times E}$ are turned into a direct system with $\varinjlim \bigoplus_E P_{\lambda,i}/U = N$ (since $\varinjlim$ is exact, see 33.8).

Observe that so far we only used $P_{\lambda,1}$ from $\{P_{\lambda,i} \mid i \in I\!N\}$.

(2) If φ is a pure epimorphism, the lower row is pure. We are going to show that, for every (U,E), there is a (V,G) in $\mathcal{M}$ with $(U,E) < (V,G)$ such that $\bigoplus_G P_{\lambda,i}/V$ is a finite direct sum of modules P_λ. Then the $\bigoplus_G P_{\lambda,i}/V$ form a direct (partial) system with same limit N (see 24.3,(4)).

For the above diagram we find $g : \bigoplus_E P_{\lambda,i}/U \to P$ with $g\pi\varphi = f_{U,E}$. Since $Im\, g$ is finitely generated, there is a finite subset $E' \subset \Lambda \times I\!N$ with $Im\, g \subset \bigoplus_{E'} P_{\lambda,i} =: F$. This leads to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U' & \longrightarrow & \bigoplus_E P_{\lambda,i} & \xrightarrow{pg} & F & & \\
 & & \downarrow & & \downarrow & & \downarrow \pi\varphi & & \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \quad ,
\end{array}$$

with $U \subset U'$. Now we refer to the fact that, for every P_λ, there are infinitely many copies of $P_{\lambda,i}$ in P: We choose a subset $E'' \subset \Lambda \times I\!N$ by replacing in every $(\lambda, i) \in E'$ the i by an $i' \in I\!N$ with the property $(\lambda, i') \notin E$ and for $i \neq j$ also $i' \neq j'$. Then $E \cap E'' = \emptyset$ and the bijection between E' and E'' yields the commutative diagram with an isomorphism h

$$\begin{array}{ccc}
\bigoplus_{E''} P_{\lambda,i} & \xrightarrow{h} & \bigoplus_{E'} P_{\lambda,i} = F \\
\downarrow & & \downarrow \pi\varphi \\
P & \xrightarrow{\pi\varphi} & N \quad .
\end{array}$$

Combining this with the above diagram and putting $G = E \bigcup E''$, we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & \bigoplus_G P_{\lambda,i} & \xrightarrow{(pg,h)} & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \pi\varphi & & \\
0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{\pi\varphi} & N & \longrightarrow & 0 \quad .
\end{array}$$

Since F is finitely presented, V is finitely generated. Hence (V,G) is in $\mathcal{M}$ and obviously $(U,E) < (V,G)$.

(3) Assume $N = \varinjlim N_\lambda$, with finitely presented N_λ in $\sigma[M]$, $\lambda \in \Lambda$. With the canonical mappings $N_\lambda \to N$ we obtain, by forming a pullback, the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L_\lambda & \longrightarrow & N_\lambda & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \end{array} .$$

The first row is pure, by 33.4, and hence splits. The family $\{L_\lambda\}_\Lambda$ can be turned into a direct system in a canonical way and the lower row is the direct limit of the splitting upper rows.

Another interpretation of purity is obtained with the following notions:

34.3 Systems of equations. Definitions.
Let M be an R-module and $X \in \sigma[M]$. A diagram with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{\varepsilon} & P \\ & & \downarrow f & & \\ & & X & & \end{array} ,$$

with P a direct sum of finitely presented modules, is called a *system of equations over* X (in $\sigma[M]$). If K and P are finitely generated, then this is called a *finite system of equations*. We say the system is *solvable (with solution h)* if there exists a morphism $h : P \to X$ with $f = \varepsilon h$.

If, for a monomorphism $\mu : X \to Y$, there exists $g : P \to Y$ with $f\mu = \varepsilon g$, then g is said to be a *solution of the system in* Y. We also say the system is *solvable in* Y.

If P' is a partial sum of P, and K' a submodule of K with $(K')\varepsilon \subset P'$, then the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K' & \xrightarrow{\varepsilon|_{K'}} & P' \\ & & \downarrow f|_{K'} & & \\ & & X & & \end{array}$$

is said to be a *partial system* of the given system.

A system of equations is called *finitely solvable*, if every finite partial system of it is solvable.

Solvability of systems of equations.
Let M be an R-module with $X \in \sigma[M]$.

(1) A system of equations over X is finitely solvable if and only if it is solvable in some pure extension $X \to Y$.

(2) If $X \to Y$ is a pure monomorphism, then every finite system of equations over X which is solvable in Y is already solvable in X.

Proof:

(1) $\Rightarrow$ Assume the system $\begin{array}{ccccc} 0 & \to & K & \xrightarrow{\varepsilon} & P \\ & & \downarrow f & & \\ & & X & & \end{array}$ to be finitely solvable.

For a finite partial system $\begin{array}{ccccc} 0 & \to & K_i & \to & P_i \\ & & \downarrow f|_{K_i} & & \\ & & X & & \end{array}$

we form the pushout

$$\begin{array}{ccc} K_i & \xrightarrow{\varepsilon_i} & P_i \\ \downarrow f|_{K_i} & & \downarrow \\ X & \xrightarrow{\mu_i} & Q_i \end{array} .$$

By the finite solvability, there is an $h_i : P_i \to X$ with $\varepsilon_i h_i = f|_{K_i}$. The pushout property now implies that μ_i splits. All finite partial systems form a directed set (with respect to inclusion) in an obvious way. Since $\varinjlim K_i = K$ and $\varinjlim P_i = P$, we obtain the diagram, with the lower row pure by 33.8,

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & P \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & \varinjlim Q_i \end{array} .$$

$\Leftarrow$ If the system of equations $\begin{array}{ccccc} 0 & \to & K & \to & P \\ & & \downarrow f & & \\ & & X & & \end{array}$

is solvable in a pure extension $X \to Y$ by $g : P \to Y$, we have, for every finite partial system $0 \to K_i \to P_i$, the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \xrightarrow{\varepsilon_i} & P_i & \longrightarrow & P_i/K_i & \longrightarrow & 0 \\ & & \downarrow f|_{K_i} & & \downarrow g|_{P_i} & & \downarrow h & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{p} & Y/X & \longrightarrow & 0 \end{array} .$$

Since the lower row is pure and P_i/K_i is finitely presented, there is a morphism $\beta : P_i/K_i \to Y$ with $\beta p = h$. By the Homotopy Lemma 7.16, we also obtain some $\gamma : P_i \to X$ with $\varepsilon_i \gamma = f|_{K_i}$.

(2) is a consequence of (1).

34.4 Characterization of pure injective modules.

Let M be an R-module. Consider the following assertions for $X \in \sigma[M]$:

(i) X is pure injective;

(ii) every finitely solvable system of equations over X is solvable in X.

(1) Then $(i) \Rightarrow (ii)$ always holds.

(2) If the finitely presented modules in $\sigma[M]$ form a generating set, then $(ii) \Rightarrow (i)$ also holds.

Proof: (1) $(i) \Rightarrow (ii)$ We have shown above that every finitely solvable system of equations over X is solvable in a pure extension of X. However, X is a direct summand in such an extension.

(2) $(ii) \Rightarrow (i)$ We show that every pure sequence $0 \to X \to Y$ splits. Since Y is generated by finitely presented modules, there is an epimorphism $P \to Y$ with $P = \bigoplus_\Lambda P_\lambda$, P_λ finitely presented. Hence we have the commutative exact diagram (pushout)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & X/Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & X/Y & \longrightarrow & 0 \end{array} .$$

The system of equations $\begin{array}{ccccc} 0 & \to & K & \to & P \\ & & \downarrow & & \\ & & X & & \end{array}$

is finitely solvable in X (see 34.3) and hence solvable in X. Now we obtain, by 7.16, a morphism $X/Y \to Y$ which makes the lower row splitting.

Besides the characterizations already seen, pure sequences in *R-MOD* can also be described using the tensor product.

Denote by $\overline{\mathbb{Q}} = \mathbb{Q}/\mathbb{Z}$ a (minimal) injective cogenerator in $\mathbb{Z}$-*MOD*.

A left R-module ${}_RN$ may be regarded as a bimodule ${}_RN_{\mathbb{Z}}$, and we consider $Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}})$ in the usual way as a right R-module:

$$fr(n) = f(rn) \text{ for } f \in Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}}),\ r \in R,\ n \in N.$$

34.5 Pure sequences in *R-MOD*. Characterizations.

For a short exact sequence

$$(*) \quad 0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$$

in R-MOD the following assertions are equivalent:

(a) The sequence $(*)$ is pure in R-MOD (Def. 33.1);

(b) the sequence $0 \to F \otimes_R K \to F \otimes_R L \to F \otimes_R N \to 0$ is exact for

(i) all finitely presented right R-modules F, or
(ii) all right R-modules F;

(c) the sequence $0 \to Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(L,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}}) \to 0$
(i) remains exact under $-\otimes_R P$*, P finitely presented, or*
(ii) splits in MOD-R;

(d) every finite system of equations over K which is solvable in L is already solvable in K;

(e) for every commutative diagram

$$\begin{array}{ccccc} & & R^n & \xrightarrow{g} & R^k \\ & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & L \end{array}$$

with $n,\, k \in \mathbb{N}$*, there exists* $h : R^k \to K$ *with* $f = gh$*;*

(f) if for any $n, k \in \mathbb{N}$*, the system of equations*

$$\sum\nolimits_{j=1}^{k} a_{ij} X_j = m_i,\ i = 1, \ldots, n,\ a_{ij} \in R,\ m_i \in K,$$

has a solution $x_1, \ldots, x_k \in L$*, then it also has a solution in K;*

(g) the sequence $(*)$ *is a direct limit of splitting sequences.*

Proof: The equivalence of (i) and (ii) in (b) follows from the facts that every R-module is a direct limit of finitely presented modules and that direct limits commute with tensor products.

$(b) \Rightarrow (c)$ Let F be a right R-module and assume

$$0 \to F \otimes_R K \to F \otimes_R L \to F \otimes_R N \to 0$$

to be exact. With the functor $Hom_{\mathbb{Z}}(-,\overline{\mathbb{Q}})$ and the canonical isomorphisms 12.12 we obtain the commutative diagram with exact first row

$$\begin{array}{ccccccc} 0 \to & Hom(F \otimes N,\overline{\mathbb{Q}}) & \to & Hom(F \otimes L,\overline{\mathbb{Q}}) & \to & Hom(F \otimes K,\overline{\mathbb{Q}}) & \to 0 \\ & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & \\ 0 \to & Hom(F, Hom(N,\overline{\mathbb{Q}})) & \to & Hom(F, Hom(L,\overline{\mathbb{Q}})) & \to & Hom(F, Hom(K,\overline{\mathbb{Q}})) & \to 0. \end{array}$$

Then the lower row has also to be exact. This means that every right R-module F is projective with respect to the sequence

$$(**) \quad 0 \to Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(L,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}}) \to 0\,.$$

Hence this sequence splits in *MOD-R*. Then, of course, it remains exact under tensor functors.

$(c.ii) \Rightarrow (b)$ Reverse the above argument (use 14.6,(e)).

$(c.i) \Leftrightarrow (a)$ Assume $(c.i)$. Tensoring the sequence $(**)$ with a finitely presented R-module ${}_RP$ we obtain, with the (functorial) isomorphisms from 25.5, the commutative diagram with first row exact

$$\begin{array}{ccccccc}
0 \to & Hom(N,\overline{\mathbb{Q}}) \otimes P & \to & Hom(L,\overline{\mathbb{Q}}) \otimes P & \to & Hom(K,\overline{\mathbb{Q}}) \otimes P & \to 0 \\
& \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & \\
0 \to & Hom(Hom(P,N),\overline{\mathbb{Q}}) & \to & Hom(Hom(P,L),\overline{\mathbb{Q}}) & \to & Hom(Hom(P,K),\overline{\mathbb{Q}}) & \to 0.
\end{array}$$

Then the lower row also is exact. Since the cogenerator $\overline{\mathbb{Q}}$ reflects exact sequences (see 14.6), the sequence

$$0 \longrightarrow Hom_R(P,K) \longrightarrow Hom_R(P,L) \longrightarrow Hom_R(P,N) \longrightarrow 0$$

must be exact and hence $(*)$ is pure. This argument is reversible.

$(c.i) \Rightarrow (b)$ The implication $(b) \Rightarrow (c)$ is also true for right modules. Hence we obtain, from $(c.i)$, that the sequence $(**)$ remains exact under $Hom_R(F,-)$, for finitely presented $F \in MOD\text{-}R$. From the diagram in the proof of $(b) \Rightarrow (c)$, we now conclude that $(b.i)$ holds.

$(a) \Rightarrow (d)$ follows from 34.3.

$(d) \Rightarrow (a)$ Let P be a finitely presented left R-module and $u : P \to N$ a morphism. For some $n \in \mathbb{N}$ we can construct a commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ke\,g & \longrightarrow & R^n & \xrightarrow{g} & P & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow u & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \ .
\end{array}$$

Since $Ke\,g$ is finitely generated, by (d) and the Homotopy Lemma 7.16, we find morphisms $R^n \to K$ and $P \to L$ as desired.

$(a) \Leftrightarrow (e)$ is a consequence of the Homotopy Lemma 7.16.

$(e) \Leftrightarrow (f)$ The morphism g (in (e)) is described by an (n,k)-matrix (a_{ij}) and f is determined by the $m_i \in K$. A solution of the equations in L yields a morphism $R^k \to L$. By this the equivalence of (e) and (f) is obvious.

$(a) \Leftrightarrow (g)$ is derived from 34.2 and 33.8.

The functor $Hom_{\mathbb{Z}}(-,\overline{\mathbb{Q}})$ also facilitates simple proofs for assertions about pure injective R-modules. Let us apply this functor twice to an R-module ${}_RN$ and define the evaluation morphism

$$\varphi_N : \ {}_RN \longrightarrow Hom_{\mathbb{Z}}(Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}}),\overline{\mathbb{Q}}), \ \ n \mapsto [\alpha \to \alpha(n)]\,.$$

This is an R-homomorphism with $[(n)\varphi_N]\alpha = \alpha(n)$ and

$$Ke\,\varphi_N = \{n \in N \mid \alpha(n) = 0 \text{ for all } \alpha \in Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}})\}\,.$$

Since $\overline{\mathbb{Q}}$ cogenerates $N_{\mathbb{Z}}$, this means $Ke\,\varphi_N = 0$, i.e. φ_N is monic.

34.6 Pure extensions in *R-MOD*.

(1) For every R-module ${}_RN$, the map $\varphi_N : {}_RN \to Hom_{\mathbb{Z}}(Hom(N,\overline{\mathbb{Q}}),\overline{\mathbb{Q}})$ is a pure monomorphism.

(2) For every right R-module F_R, the left R-module $Hom_{\mathbb{Z}}(F,\overline{\mathbb{Q}})$ is pure injective. If F_R is free (flat), then $Hom_{\mathbb{Z}}(F,\overline{\mathbb{Q}})$ is R-injective.

(3) Every R-module is a pure submodule of a pure injective R-module.

(4) An R-module N is pure injective if and only if φ_N splits.

Proof: (1) For a finitely presented right R-module F, we obtain the commutative diagram

$$\begin{array}{ccc}
F \otimes_R N & \xrightarrow{id\otimes\varphi_N} & F \otimes_R Hom_{\mathbb{Z}}(Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}}),\overline{\mathbb{Q}}) \\
 & & \downarrow\simeq \\
\| & & Hom_{\mathbb{Z}}(Hom_R(F, Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}})),\overline{\mathbb{Q}}) \\
 & & \downarrow\simeq \\
F \otimes_R N & \xrightarrow{\varphi_{F\otimes N}} & Hom_{\mathbb{Z}}(Hom_{\mathbb{Z}}(F \otimes_R N,\overline{\mathbb{Q}}),\overline{\mathbb{Q}})
\end{array}$$

with isomorphisms on the right side (see 25.5, 12.12). Since $\varphi_{F\otimes N}$ is monic, this is also true for $id \otimes \varphi_N$. Hence φ_N is a pure monomorphism by 34.5.

(2) Let $0 \to K \to L \to N \to 0$ be an exact sequence in R-MOD and F a right R-module. With the functor $Hom_R(-, Hom_{\mathbb{Z}}(F,\overline{\mathbb{Q}}))$ and canonical isomorphisms we obtain the commutative diagram

$$\begin{array}{ccccccc}
0 \to Hom(N, Hom(F,\overline{\mathbb{Q}})) & \to & Hom(L, Hom(F,\overline{\mathbb{Q}})) & \to & Hom(K, Hom(F,\overline{\mathbb{Q}})) \to 0 \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\
0 \to \quad Hom(F \otimes_R N,\overline{\mathbb{Q}}) & \to & Hom(F \otimes_R L,\overline{\mathbb{Q}}) & \to & Hom(F \otimes_R K,\overline{\mathbb{Q}}) \to 0.
\end{array}$$

If the given sequence is pure, the lower row is exact. Then the upper row is also exact and $Hom_{\mathbb{Z}}(F,\overline{\mathbb{Q}})$ is pure injective.

If F_R is flat (w.r. to R-MOD, see 12.16), then the lower and the upper row are again exact and $Hom_{\mathbb{Z}}(F,\overline{\mathbb{Q}})$ is R-injective.

(3) and (4) are immediately derived from (1), (2).

As a consequence of the preceding proof, we obtain a

34.7 Further characterizations of pure sequences in *R-MOD*.
For an exact sequence

$$(*) \qquad 0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$$

in R-MOD, the following statements are equivalent:
(a) The sequence $(*)$ *is pure;*
(b) every pure injective left R-module is injective with respect to $(*)$*;*
(c) $Hom_{\mathbb{Z}}(F,\overline{\mathbb{Q}})$ *is injective with respect to* $(*)$ *for any right module* F_R*;*
(d) $K^{**} = Hom_{\mathbb{Z}}(Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}}),\overline{\mathbb{Q}})$ *is injective with respect to* $(*)$*.*

Proof: $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ are obvious, $(b) \Rightarrow (c)$ follows from 34.6.

$(d) \Rightarrow (a)$ The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow \varphi_K & & & & & & \\ & & K^{**} & & & & & & \end{array}$$

can be extended commutatively by $g : L \to K^{**}$. Since $\varphi_K = fg$ is a pure monomorphism, by 33.2, f is also a pure monomorphism.

Let us now investigate $\mathcal{P}$-pure submodules for the class $\mathcal{P}$ of cyclic modules of the form R/Rr, $r \in R$. The modules R/Rr are finitely presented and form a set of generators in *R-MOD*.

34.8 Relatively divisible submodules.
For an exact sequence

$$(*) \qquad 0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$$

in R-MOD and $r \in R$*, the following assertions are equivalent:*
(a) R/Rr *is projective with respect to* $(*)$*;*
(b) the functor $R/rR \otimes_R -$ *is exact with respect to* $(*)$*;*
(c) $rK = K \cap rL$*.*

If these conditions are satisfied for every $r \in R$, then K is called a *relatively divisible* submodule of L and $(*)$ an *RD-pure* sequence.

Proof: $(a) \Rightarrow (c)$ The relation $rK \subset K \cap rL$ always holds. For $l \in L$ with $rl \in K$, define a morphism $f_l : R \to L$, $a \mapsto al$. Form the exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Rr & \xrightarrow{\varepsilon} & R & \longrightarrow & R/Rr & \longrightarrow & 0 \\ & & & & \downarrow f_l & & & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \end{array} .$$

Since $Im\,\varepsilon f_l \subset K$, we may extend this diagram commutatively with morphisms $g : Rr \to K$ and $R/Rr \to N$. By assumption, we obtain a morphism $R/Rr \to L$ and finally (Homotopy Lemma) an $h : R \to K$ with $g = \varepsilon h$. From this we obtain $(r)g = (r)\varepsilon h = r((1)h)$ with $(1)h = k \in K$, and
$rl = (r)\varepsilon f_l = (r)g = rk$, i.e. $rl \in rK$. This implies $K \cap rL \subset rK$.

$(c) \Rightarrow (a)$ Assume the following commutative exact diagram to be given

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Rr & \longrightarrow & R & \longrightarrow & R/Rr & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \end{array} .$$

Then $(r)f = r(1)f \in rL \cap K$ and there exists $k \in K$ with $rk = r(1)f$. Defining $R \to K$ by $a \mapsto ak$, we obtain a morphism which yields the desired morphism $R/rR \to L$ (Homotopy Lemma).

$(b) \Leftrightarrow (c)$ For $r \in R$, we obtain, with the isomorphisms from 12.11, the following commutative diagram with exact lower row in $\mathbb{Z}$-*MOD*

$$\begin{array}{ccccccc} & & & & R/rR \otimes K & \longrightarrow & R/rR \otimes L \\ & & & & \downarrow\simeq & & \downarrow\simeq \\ 0 & \longrightarrow & rL \cap K/rK & \longrightarrow & K/rK & \longrightarrow & L/rL \end{array} .$$

Then the morphism in the upper row is monic if and only if $rL \cap K = rK$.

The argument used for $(b) \Leftrightarrow (c)$ obviously remains valid if the cyclic right ideal rR is replaced by any (finitely generated) right ideal J. Since R/J is a direct limit of R/J''s, with finitely generated $J' \subset {}_RR$, we have:

34.9 Sequences exact under $R/J \otimes -$.
For an exact sequence

$$(*) \quad 0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$$

in R-MOD, the following assertions are equivalent:
(a) the sequence $0 \to R/J \otimes_R K \to R/J \otimes_R L \to R/J \otimes_R N \to 0$ *is exact for every (finitely generated) right ideal* $J \subset R$;
(b) for every (finitely generated) right ideal $J \subset R$, $JK = K \cap JL$.

Over a commutative ring R, every R-module may be considered as (R, R)-bimodule and the purity of an exact sequence can be tested with an injective cogenerator in *R-MOD* (instead of $\mathbb{Z}$-*MOD*):

34.10 Purity over commutative rings.
Let R be a commutative ring and Q an injective cogenerator in R-MOD.

(1) An exact sequence $0 \to K \to L \to N \to 0$ *in R-MOD is pure if and only if* $0 \to Hom_R(N, Q) \to Hom_R(L, Q) \to Hom_R(K, Q) \to 0$ *is a pure (splitting) sequence.*

(2) For every R-module N, the canonical map

$$\varphi_N : N \to Hom_R(Hom_R(N, Q), Q)$$

is a pure monomorphism and $Hom_R(N, Q)$ *is pure injective.*

(3) Every R-module N is a pure submodule of a product of finitely copresented, pure injective R-modules.

Proof: (1) For commutative rings R, in 34.5 the injective $\mathbb{Z}$-cogenerator $\overline{\mathbb{Q}}$ can be replaced by ${}_RQ$.

(2) This is seen using the proof of 34.6, again replacing $\overline{\mathbb{Q}}$ by Q.

(3) Let $\{E_\lambda\}_\Lambda$ be a family of injective hulls of the simple R-modules and put $Q = \prod_\Lambda E_\lambda$. By (2), N is a pure submodule of $Hom_R(K, Q)$ with $K = Hom_R(N, Q)$. Writing K as a direct limit of finitely presented R-modules $\{K_i\}_I$, we obtain a pure epimorphism $\bigoplus_I K_i \to \varinjlim K_i = K$ (see 33.9). By (1), we obtain a pure monomorphism

$$Hom(K, Q) \longrightarrow Hom(\bigoplus\nolimits_I K_i, Q) \simeq \prod\nolimits_I \prod\nolimits_\Lambda Hom(K_i, E_\lambda).$$

Let $R^k \to R^n \to K_i \to 0$, with $k, n \in \mathbb{N}$, be a representation of K_i. Applying the functor $Hom_R(-, E_\lambda)$ we obtain the exact sequence

$$0 \longrightarrow Hom_R(K_i, E_\lambda) \longrightarrow E_\lambda^n \longrightarrow E_\lambda^k \,.$$

Hence $Hom_R(K_i, E_\lambda)$ is finitely copresented (see § 30) and – as a direct summand of $Hom_R(K_i, Q)$ – pure injective. By construction, N is a pure submodule of the product of these modules.

34.11 Pure injective modules over commutative R.

Let R be a commutative ring, Q an injective cogenerator in R-MOD and N an R-module. Then:

(1) The following assertions are equivalent:

(a) N is pure injective;

(b) N is a direct summand of a module $Hom_R(K, Q)$ *with* $K \in$ *R-MOD;*

(c) N is a direct summand of a product of finitely copresented, pure injective R-modules.

(2) If N is linearly compact, then N is pure injective.

Proof: (1) According to 33.7, N is pure injective if and only if (as a pure submodule) it is always a direct summand. Hence the assertions follow from 34.10.

(2) From the proof of 47.8,(1), we see that, for any linearly compact N, the canonical map $N \to Hom_R(Hom_R(N,Q),Q)$ is an isomorphism.

The notion of purity in module categories has been developed from the corresponding notion for $\mathbb{Z}$-modules (abelian groups). The special properties of the ring $\mathbb{Z}$ allow various other characterization of purity. It is interesting to present these classical results within our framework:

34.12 Purity in $\mathbb{Z}$-MOD. Characterizations.
For an exact sequence

$$(*) \quad 0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$$

of $\mathbb{Z}$-modules the following assertions are equivalent:
(a) The sequence $(*)$ is $\mathcal{P}$-pure for the class $\mathcal{P}$ of
(i) all finitely generated $\mathbb{Z}$-modules,
(ii) all cyclic $\mathbb{Z}$-modules;
(b) the sequence $(*)$ is a direct limit of splitting sequences;
(c) the sequence $0 \to F \otimes_{\mathbb{Z}} K \to F \otimes_{\mathbb{Z}} L \to F \otimes_{\mathbb{Z}} N \to 0$ is
(i) exact for all (finitely generated) $\mathbb{Z}$-modules F, or
(ii) pure exact for all $\mathbb{Z}$-modules F;
(d) for every $n \in I\!N$, the sequence $0 \to K/nK \to L/nL \to N/nN \to 0$ is
(i) exact, or
(ii) exact and splitting;
(e) for every $n \in I\!N$, $nK = nL \cap K$;
(f) if, for $n \in I\!N$ and $k \in K$, the equation $nx = k$ is solvable in L, then it is already solvable in K;
(g) $0 \to Hom_{\mathbb{Z}}(N,G) \to Hom_{\mathbb{Z}}(L,G) \to Hom_{\mathbb{Z}}(K,G) \to 0$ is exact for
(i) all pure injective $\mathbb{Z}$-modules G, or
(ii) all cocyclic $\mathbb{Z}$-modules G, or
(iii) all finite (cyclic) abelian groups G.

Proof: $(a.i) \Leftrightarrow (a.ii)$ In $\mathbb{Z}$-MOD, all finitely generated modules are finitely presented and finite direct sums of cyclic groups.

$(a) \Leftrightarrow (b)$ and $(a) \Leftrightarrow (c.i)$ are contained in 34.5.

$(c.i) \Leftrightarrow (c.ii)$ follows from the associativity of the tensor product.

$(c) \Rightarrow (d)$ The sequence $0 \to K/nK \to L/nL \to N/nN \to 0$ is obtained from $(*)$ by tensoring with $\mathbb{Z}/n\mathbb{Z}$ and hence is pure exact.

By a structure theorem for abelian groups, every $\mathbb{Z}$-module X, with $nX = 0$ for some $n \in \mathbb{N}$, is a direct sum of cyclic $\mathbb{Z}$-modules (see also 56.11). Therefore N/nN is pure projective and the sequence splits.

$(d) \Rightarrow (c)$ It follows from (d) that $(*)$ remains exact under $\mathbb{Z}/n\mathbb{Z} \otimes -$. Since a finitely generated $\mathbb{Z}$-module F is a direct sum of cyclic modules, the sequence $(*)$ remains exact under $F \otimes -$.

$(d) \Leftrightarrow (e)$ has been shown in 34.8.

$(e) \Leftrightarrow (f)$ can easily be verified directly. It is a special case of our considerations of systems of equations at the beginning of § 34.

$(a) \Leftrightarrow (g.i)$ follows from 34.7.

$(g.ii) \Rightarrow (f)$ Consider $n \in \mathbb{N}$, $k \in K$ and assume the equation $nx = k$ to be solvable in L, i.e. $nl = k$ for some $l \in L$, but not solvable in K. Then $k \notin nK$ and we choose a submodule $U \subset K$ maximal with respect to $nK \subset U$ and $k \notin U$. The factor module K/U is finitely cogenerated (cocyclic, see 14.9), and, by assumption, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & L \\ & & \downarrow p & & \\ & & K/U & & \end{array}$$

can be extended commutatively by some $g : L \to K/U$. For this we obtain

$$(k)p = (nl)g = n \cdot (l)g \in n \cdot K/U = 0,$$

implying $k \in U$, a contradiction. So the given equation is solvable in K.

$(g.iii) \Rightarrow (g.ii)$ Any cocyclic $\mathbb{Z}$-module is a submodule of some $\mathbb{Z}_{p^\infty}$. $\mathbb{Z}_{p^\infty}$ is injective and every proper submodule is finite (see 17.13).

The implication $(a) \Rightarrow (g.ii)$, $(g.iii)$ follows from

34.13 Pure injective $\mathbb{Z}$-modules.

(1) A $\mathbb{Z}$-module is pure injective if and only if it is isomorphic to a direct summand of a module $Hom_{\mathbb{Z}}(F,\overline{\mathbb{Q}})$, with some $F \in \mathbb{Z}$-MOD.

(2) $\mathbb{Z}$-modules K, with $nK = 0$ for some $n \in \mathbb{N}$, are pure injective.

(3) Cocyclic and finitely cogenerated $\mathbb{Z}$-modules are pure injective.

Proof: (1) is a consequence of 34.6.

(2) We have to show that a pure exact sequence $0 \to K \to L \to N \to 0$ splits if $nK = 0$. With 34.12,(d), we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & L/nL & \longrightarrow & N/nN & \longrightarrow & 0 \end{array} ,$$

in which the lower row splits. Hence the upper row also splits.

(3) Cocyclic $\mathbb{Z}$-modules are either injective ($\simeq \mathbb{Z}_{p^\infty}$) or finite, i.e. they are pure injective. Finitely cogenerated $\mathbb{Z}$-modules are finite direct sums of cocyclic modules.

An exact sequence $0 \to K \to L \to N \to 0$ *in* $\mathbb{Z}_n$*-MOD is pure in* $\mathbb{Z}_n$*-MOD if and only if it is pure in* $\mathbb{Z}$*-MOD.*

This follows from the fact that, in this case, tensoring with a $\mathbb{Z}$-module $F_{\mathbb{Z}}$ can be achieved by tensoring with $(F \otimes_{\mathbb{Z}} \mathbb{Z}_n) \otimes_{\mathbb{Z}_n} -$.

Again referring to the theorem that every $\mathbb{Z}$-module L with $nL = 0$ is a direct sum of cyclic $\mathbb{Z}$-modules, we obtain:

34.14 Properties of $\mathbb{Z}_n$-MOD.

For $n \in \mathbb{N}$ *and* $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ *we have:*

(1) every $\mathbb{Z}_n$*-module is pure projective;*
(2) every $\mathbb{Z}_n$*-module is pure injective;*
(3) every pure exact $\mathbb{Z}_n$*-sequence splits;*
(4) every $\mathbb{Z}_n$*-module is a direct sum of finitely presented modules;*
(5) every indecomposable $\mathbb{Z}_n$*-module is finitely presented.*

By the way, the properties considered in 34.14 are equivalent to each other for any ring. They determine interesting classes of rings and modules (see Exercise (1) and § 53).

34.15 Exercises.

(1) Show that for an R-module M the following are equivalent:
(a) Every module in $\sigma[M]$ *is pure projective;*
(b) every module in $\sigma[M]$ *is pure injective;*
(c) every pure exact sequence in $\sigma[M]$ *splits.*
Modules with these properties are called pure semisimple.

(2) Show for two $\mathbb{Z}$*-modules* $K \subset L$*:*
(i) If the factor module L/K *is torsion free, then K is pure in L.*
(ii) If L is torsion free, then:
(α) *K is pure in L if and only if* L/K *is torsion free;*
(β) *the intersection of pure submodules in L is also pure in L.*

(3) For a prime number p we form, in $\mathbb{Z}$*-MOD,* $\mathcal{P}_p = \{\mathbb{Z}_{p^k} \mid k \in \mathbb{N}\}$. *Instead of '*$\mathcal{P}_p$*-pure' we just say 'p-pure'.*

Let K be a submodule of the $\mathbb{Z}$*-module L. Show:*
(i) The following assertions are equivalent:
(a) K is p-pure in L;

(b) $p^k K = K \cap p^k L$ *for every* $k \in I\!N$;

(c) the inclusion $0 \to K \to L$ *remains exact under* $\mathbb{Z}_{p^k} \otimes -$, $k \in I\!N$.

(ii) K is pure in L if and only if K is p-pure in L, for every prime number p.

(iii) If $K \subset p(L)$ *(= p-component of L, see 15.10), then:*
K is pure in L if and only if K is p-pure in L.

(4) In $\mathbb{Z}$*-MOD consider* $\mathcal{E} = \{\mathbb{Z}_p \mid p \text{ prime number}\}$. *Let K be a submodule of the* $\mathbb{Z}$*-module L. Show:*

(i) The following assertions are equivalent:

(a) K is $\mathcal{E}$*-pure in L;*

(b) $pK = K \cap pL$ *for every prime number p;*

(c) $0 \to K/pK \to L/pL$ *is exact (and splits) for every prime number p;*

(d) $0 \to pK \to pL \to p \cdot L/K \to 0$ *is exact for every prime number p.*

$\mathcal{E}$*-pure sequences are also called* neat exact.

(ii) If L is torsion free, then K is pure in L if and only if K is $\mathcal{E}$*-pure in L.*

(iii) For every prime number p, $\mathbb{Z}_p$ *is* $\mathcal{E}$*-pure injective.*

(iv) A $\mathbb{Z}$*-module is* $\mathcal{E}$*-pure projective if and only if it is a direct sum of a semisimple* $\mathbb{Z}$*-module and a free* $\mathbb{Z}$*-module.*

(5) Let M be a self-injective R-module and ${}_RU_S$ *an (R,S)-bimodule with* ${}_RU \in \sigma[M]$. *Show that* ${}_S Hom_R(U, M)$ *is pure injective in S-MOD.*
Hint: Hom-Tensor-Relation 12.12.

Literature: FUCHS, MISHINA-SKORNJAKOV; Azumaya [3], Couchot [2,4], Crivei [3], Döman-Hauptfleisch, Enochs [1,4], Facchini [2,3], Fakhrudin [2], Fuchs-Hauptfleisch, Héaulme, Hunter, Jøndrup [1], Jøndrup-Trosborg, Lenzing [4], Naudé-Naudé, Naudé-Naudé-Pretorius, Onishi, Rangaswamy [3], Rychkov, Salles [2], Simson [1,2], Singh-Talwar, Sklyarenko [1,2], Stenström, Zimmermann, Zimmermann-Huisgen-Zimmermann.

35 Absolutely pure modules

1.Absolutely $\mathcal{P}$-pure modules. 2.Properties. 3.Weakly M-injective modules. 4.Absolutely pure modules. 5.Pure factor modules of weakly M-injective modules. 6.Locally coherent M. 7.Locally noetherian M. 8.Absolutely pure modules in R-MOD. 9.Coherent rings. 10.Exercises.

Let $(*)\ \ 0 \to K \to L \to N \to 0$ be an exact sequence in $\sigma[M]$.
We are familiar with the following characterizations:

(1) K is *injective* in $\sigma[M]$ if and only if every sequence $(*)$ splits;
(2) N is *projective* in $\sigma[M]$ if and only if every sequence $(*)$ splits;
(3) L is *semisimple* if and only if every sequence $(*)$ splits.

Similarly one may ask for which modules the above exact sequences are not *splitting* but ($\mathcal{P}$-)*pure*. The next three paragraphs are dedicated to this problem.

Let $\mathcal{P}$ be a non-empty class of modules in $\sigma[M]$, $M \in R$-*MOD*. A module $K \in \sigma[M]$ is called *absolutely $\mathcal{P}$-pure*, if every exact sequence of the type $(*)$ is $\mathcal{P}$-pure in $\sigma[M]$.

In case $\mathcal{P}$ consists of *all* finitely presented modules in $\sigma[M]$ we just say *absolutely pure* instead of *absolutely $\mathcal{P}$-pure*.

Of course, injective modules in $\sigma[M]$ are absolutely $\mathcal{P}$-pure for any $\mathcal{P}$.

35.1 Characterizations of absolutely $\mathcal{P}$-pure modules.

Let M be an R-module. For any $K \in \sigma[M]$ the following are equivalent:

(a) K is absolutely $\mathcal{P}$-pure (in $\sigma[M]$);

(b) every exact sequence $0 \to K \to L \to P \to 0$ in $\sigma[M]$ with $P \in \mathcal{P}$ splits;

(c) K is injective with respect to any exact sequence $0 \to U \to V \to P \to 0$ in $\sigma[M]$ with $P \in \mathcal{P}$ (or P $\mathcal{P}$-pure projective);

(d) K is a $\mathcal{P}$-pure submodule of an absolutely $\mathcal{P}$-pure module in $\sigma[M]$.

Proof: $(a) \Rightarrow (b)$ Every $P \in \mathcal{P}$ is $\mathcal{P}$-pure projective.

$(b) \Rightarrow (c)$ Assume the sequence in (c) and $\alpha : U \to K$ to be given. Forming a pushout we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow\alpha & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & P & \longrightarrow & 0 \end{array} .$$

If (b) holds, the lower row splits and we obtain the desired morphism $V \to K$, i.e. (c) holds.

$(c) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (a)$ Let $0 \to K \to L \to N \to 0$ be an exact sequence in $\sigma[M]$, $P \in \mathcal{P}$ and $\beta : P \to N$ a morphism. Forming a pullback we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow\beta & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \end{array} .$$

Since the first row splits, we obtain a morphism $P \to L$ with the desired properties.

$(a) \Rightarrow (d)$ K is a $\mathcal{P}$-pure submodule of its M-injective hull.

$(d) \Rightarrow (b)$ Let the sequence in (b) be given and assume K to be a $\mathcal{P}$-pure submodule of an absolutely $\mathcal{P}$-pure module $V \in \sigma[M]$. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & K & \longrightarrow & V & \longrightarrow & V/K & \longrightarrow & 0 \end{array}$$

can be extended commutatively by morphisms $L \to V$ (observe $(a) \Leftrightarrow (c)$) and $P \to V/K$. The lower sequence being $\mathcal{P}$-pure, we obtain morphisms $P \to V$ and (Homotopy Lemma) $L \to K$ which make the first row split.

35.2 Properties of absolutely $\mathcal{P}$-pure modules.
Let M be an R-module and $\mathcal{P}$ a non-empty class of modules in $\sigma[M]$.

(1) A product of modules in $\sigma[M]$ is absolutely $\mathcal{P}$-pure if and only if every factor is absolutely $\mathcal{P}$-pure.

(2) Suppose the modules in $\mathcal{P}$ are finitely presented in $\sigma[M]$. Then any direct sum of absolutely $\mathcal{P}$-pure modules is again absolutely $\mathcal{P}$-pure.

(3) If $0 \to N' \to N \to N'' \to 0$ is an exact sequence in $\sigma[M]$ with N' and N'' absolutely $\mathcal{P}$-pure, then N is also absolutely $\mathcal{P}$-pure.

Proof: (1) follows from the characterization 35.1,(c).

(2) Under the given assumption, the direct sum is a $\mathcal{P}$-pure submodule of the direct product (see 33.9), and the assertion follows from 35.1,(d).

(3) Let N' and N'' be absolutely $\mathcal{P}$-pure and $N \to L$ a monomorphism.

Forming a pushout we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & L & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L/N & = & L/N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

Apply the functor $Hom(P,-)$ with $P \in \mathcal{P}$. It is exact with respect to the upper rows and the left and right columns. Then it is also exact with respect to the central column, i.e. $N \to L$ is a $\mathcal{P}$-pure monomorphism.

35.3 Weakly M-injective modules *are absolutely pure in* $\sigma[M]$.

Proof: Let K be a weakly M-injective module (see 16.9) in $\sigma[M]$ and $0 \to K \to L \xrightarrow{g} P \to 0$ an exact sequence with P finitely presented in $\sigma[M]$. Consider a finitely generated submodule $L' \subset L$ with $(L')g = P$ and an epimorphism $h : U \to L'$ with finitely generated $U \subset M^{(\mathbb{N})}$. We obtain the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ke\,hg & \longrightarrow & U & \xrightarrow{hg} & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & P & \longrightarrow & 0 \quad ,
\end{array}$$

with the finitely generated module $Ke\,hg \subset U \subset M^{(\mathbb{N})}$. K being a weakly M-injective module, it is injective with respect to the first row. We can extend the diagram commutatively with $U \to K$ and then with $P \to L$ to make the lower row split. Therefore K is absolutely pure by 35.1,(b).

35.4 Absolutely pure modules for M finitely presented.

If the R-module M is a submodule of a direct sum of finitely presented modules in $\sigma[M]$, then, for $K \in \sigma[M]$, the following are equivalent:

(a) K is weakly M-injective;

(b) K is absolutely pure in $\sigma[M]$.

Proof: $(a) \Rightarrow (b)$ has been shown in 35.3.

$(b) \Rightarrow (a)$ If M is a submodule of a direct sum P of finitely presented modules, then, for every finitely generated submodule $U \subset M^{(\mathbb{N})} \subset P^{(\mathbb{N})}$, the factor module $P^{(\mathbb{N})}/U$ is pure projective (see 34.1).

According to 35.1, an absolutely pure module K is injective relative to

$$0 \to U \to P^{(\mathbb{N})} \to P^{(\mathbb{N})}/U \to 0\,,$$

and hence injective with respect to $0 \to U \to M^{(\mathbb{N})}$, i.e. K is weakly M-injective.

The following observation extends the assertions of 26.3:

35.5 Pure factor modules of weakly M-injective modules.
Assume the R-module M to be locally coherent in $\sigma[M]$. Then:

(1) Every factor module of a weakly M-injective module by a pure submodule is weakly M-injective.

(2) Direct limits of weakly M-injective modules are weakly M-injective.

Proof: (1) Let L be a weakly M-injective module, $L \to N \to 0$ a pure epimorphism and K a finitely generated submodule of M. By 26.2, it is enough to show that N is injective with respect to $0 \to K \to M$. Since K is finitely presented, a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & M \\ & & \downarrow & & \\ L & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

can be extended commutatively by some $K \to L$ and then (L being weakly injective) by some $M \to L$. This yields the desired morphism $M \to N$.

(2) Let $(L_i, f_{ij})_\Lambda$ be a direct system of weakly M-injective modules. Then $\bigoplus_\Lambda L_i$ is weakly M-injective by 16.10, the canonical epimorphism $\bigoplus_\Lambda L_i \to \varinjlim L_i$ is pure (see 33.9), and the assertion follows from (1).

The two preceding statements now yield:

35.6 Absolutely pure modules and locally coherent M.
If the R-module M is a submodule of a direct sum of finitely presented modules in $\sigma[M]$, then the following statements are equivalent:

(a) M is locally coherent in $\sigma[M]$;

(b) every finitely presented module is coherent in $\sigma[M]$;

(c) every factor module of an absolutely pure (= weakly M-injective) module by a pure submodule is absolutely pure in $\sigma[M]$;

(d) direct limits of absolutely pure modules are absolutely pure in $\sigma[M]$.

Proof: $(a) \Leftrightarrow (b)$ is clear by 26.1 and 26.2.

$(b) \Rightarrow (c) \Rightarrow (d)$ follows from 35.5, since 'absolutely pure' and 'weakly M-injective' are equivalent by 35.4.

$(d) \Rightarrow (b)$ Let K be a finitely generated submodule of a finitely presented module P and $(Q_i, f_{ij})_\Lambda$ a direct system of absolutely pure modules in $\sigma[M]$. Then we have the commutative exact diagram (see proof of 26.3)

$$\begin{array}{ccccc} \varinjlim Hom(P,Q_i) & \longrightarrow & \varinjlim Hom(K,Q_i) & \longrightarrow & 0 \\ \downarrow \Phi_P & & \downarrow \Phi_K & & \\ Hom(P,\varinjlim Q_i) & \longrightarrow & Hom(K,\varinjlim Q_i) & \longrightarrow & 0 \end{array} .$$

Since Φ_P is an isomorphism and Φ_K is monic, Φ_K is also an isomorphism. Hence K is finitely presented by 25.2.

As a supplement to the external characterization of locally noetherian modules in 27.3 we have:

35.7 Absolutely pure modules and locally noetherian *M*.

For an R-module M the following assertions are equivalent:

(a) M is locally noetherian;

(b) every absolutely pure module is injective in $\sigma[M]$.

Proof: $(a) \Rightarrow (b)$ Assume (a). Then, by 35.1, every absolutely pure module in $\sigma[M]$ is K-injective for every finitely generated submodule $K \subset M$, and hence M-injective (see 16.3).

$(b) \Rightarrow (a)$ By 35.4, every weakly M-injective module in $\sigma[M]$ is absolutely pure, hence M-injective. Now the assertion follows from 27.3.

For $M = R$ the preceding assertions yield:

35.8 Absolutely pure modules in *R-MOD*.

For an R-module K the following statements are equivalent:

(a) K is absolutely pure;

(b) K is weakly R-injective (= FP-injective);

(c) K is injective with respect to exact sequences $0 \to U \to V \to P \to 0$ *in R-MOD with P finitely presented;*

(d) K is a pure submodule of an (FP-) injective R-module;

(e) for every exact sequence $0 \to K \to L \to N \to 0$ *in R-MOD, the sequence*

$$0 \to Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(L,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}}) \to 0$$

is pure (or splitting) in MOD-R.

If ${}_RR$ *is coherent, then (a)-(e) are also equivalent to:*

(f) $Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}})_R$ *is flat with respect to R-MOD.*

Proof: For the equivalence of (a), (b), (c) and (d) see 35.1 and 35.4.

$(a) \Leftrightarrow (e)$ follows from the characterizations of pure sequences in 34.5.

(f) just means that the functor $Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}}) \otimes_R -$ is exact with respect to exact sequences $0 \to I \to {}_RR$, with I finitely generated (see 12.16). Since I is finitely presented, we obtain, with $\overline{\mathbb{Q}} = \mathbb{Q}/\mathbb{Z}$ and isomorphisms from 25.5, the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}}) \otimes_R I & \longrightarrow & Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}}) \otimes_R R \\ & & \downarrow\simeq & & \downarrow\simeq \\ 0 & \longrightarrow & Hom_{\mathbb{Z}}(Hom_R(I,K),\overline{\mathbb{Q}}) & \longrightarrow & Hom_{\mathbb{Z}}(Hom_R(R,K),\overline{\mathbb{Q}}) \end{array} .$$

$(b) \Rightarrow (f)$ If K is FP-injective, the lower row is exact (by 26.2) and hence also the upper row is exact.

$(f) \Rightarrow (b)$ Now the exactness of the upper row implies that the lower row is also exact. Since $\overline{\mathbb{Q}}$ is a cogenerator, this yields the exactness of
$Hom_R(R,K) \to Hom_R(I,K) \to 0$, i.e. K is FP-injective.

The implication $(f) \Rightarrow (e)$ can also be obtained from the fact that the exactness of $Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}}) \otimes_R -$ implies the purity of the sequence in (e). This will be seen in 36.5. It is also used in the last part of our next proof:

35.9 Absolutely pure modules and coherent rings.

For a ring R the following statements in R-MOD are equivalent:

(a) ${}_RR$ *is coherent;*

(b) for every absolutely pure module $K \in$ *R-MOD,* $Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}})_R$ *is flat;*

(c) every factor module of an absolutely pure module by a pure submodule is absolutely pure;

(d) direct limits of absolutely pure modules are absolutely pure.

Proof: The equivalence of (a), (c) and (d) has been shown in 35.6.

$(a) \Rightarrow (b)$ is contained in 35.8.

$(b) \Rightarrow (c)$ Let $0 \to A \to B \to C \to 0$ be a pure sequence in R-MOD and B absolutely pure. The sequence

$$0 \to Hom_{\mathbb{Z}}(C,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(B,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(A,\overline{\mathbb{Q}}) \to 0$$

splits by 34.5. Hence, $Hom_{\mathbb{Z}}(C,\overline{\mathbb{Q}})$ is a direct summand of the flat module $Hom_{\mathbb{Z}}(B,\overline{\mathbb{Q}})$, and therefore it is also flat with respect to R-MOD.

We shall see in 36.5 that, as a consequence, for every exact sequence $0 \to C \to U \to V \to 0$ in R-MOD the sequence

$$0 \to Hom_{\mathbb{Z}}(V,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(U,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(C,\overline{\mathbb{Q}}) \to 0$$

is pure in *MOD-R*. By 35.8,(e), this implies that C is absolutely pure.

35.10 Exercises.

(1) Show that for an R-module K the following are equivalent:

(a) K is absolutely pure in R-MOD;

(b) every morphism $K \to Q$, with pure injective Q, can be factorized over a product of (indecomposable) injective R-modules.

(2) Let R be a left coherent ring and $K \in$ R-MOD. Show that K is absolutely pure if and only if every exact sequence

$0 \to K \to N \to R/I \to 0$ *in R-MOD,*

with finitely generated left ideal $I \subset R$, splits.
Hint: 26.2.

Literature: Enochs [1], Couchon [3,7], Megibben [1], Stenström, Würfel [1], Xu Yan.

36 Flat modules

1. $\mathcal{P}$-flat modules. 2.Flat modules generated by projectives. 3.Pure submodules of projective modules. 4.Projective and flat modules. 5.Flat modules in R-MOD. 6.Pure submodules of flat modules. 7.Flat modules and non zero divisors. 8.Exercises.

Let M be an R-module and $\mathcal{P}$ a non-empty class of modules in $\sigma[M]$. A module $N \in \sigma[M]$ is called *$\mathcal{P}$-flat in $\sigma[M]$* if every exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$$

in $\sigma[M]$ is $\mathcal{P}$-pure. In case $\mathcal{P}$ consists of all finitely presented modules in $\sigma[M]$ instead of $\mathcal{P}$-flat we just say *flat in $\sigma[M]$*.

Obviously, a module $N \in \sigma[M]$ is $\mathcal{P}$-flat and $\mathcal{P}$-pure projective if and only if it is projective in $\sigma[M]$. In particular, finitely presented flat modules in $\sigma[M]$ are projective in $\sigma[M]$.

36.1 Properties of $\mathcal{P}$-flat modules.
With the above notation we have:

(1) Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence in $\sigma[M]$.
(i) If the sequence is $\mathcal{P}$-pure and N is $\mathcal{P}$-flat, then N'' is $\mathcal{P}$-flat.
(ii) If N' and N'' are $\mathcal{P}$-flat, then N is $\mathcal{P}$-flat.

(2) Finite direct sums of $\mathcal{P}$-flat modules are $\mathcal{P}$-flat in $\sigma[M]$.

(3) Assume the modules in $\mathcal{P}$ to be finitely generated. Then any direct sum of $\mathcal{P}$-flat modules is again $\mathcal{P}$-flat.

(4) Assume the modules in $\mathcal{P}$ to be finitely presented. Then direct limits of $\mathcal{P}$-flat modules are $\mathcal{P}$-flat.

Proof: (1)(i) For an epimorphism $g : L \to N''$, we form the pullback

$$\begin{array}{ccccc} Q & \longrightarrow & L & \longrightarrow & 0 \\ \downarrow & & \downarrow g & & \\ N & \longrightarrow & N'' & \longrightarrow & 0 \end{array} .$$

For $P \in \mathcal{P}$, we obtain the commutative exact diagram

$$\begin{array}{ccccc} Hom(P,Q) & \longrightarrow & Hom(P,L) & \longrightarrow & 0 \\ \downarrow & & \downarrow Hom(P,g) & & \\ Hom(P,N) & \longrightarrow & Hom(P,N'') & \longrightarrow & 0 \\ \downarrow & & & & \\ 0 & & & & . \end{array}$$

From this we see that $Hom(P, g)$ is epic, i.e. g is a $\mathcal{P}$-pure epimorphism and N'' is $\mathcal{P}$-flat.

(ii) Let N', N'' be $\mathcal{P}$-flat and $h : L \to N$ epic. We have the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ke\, hf & \longrightarrow & L & \xrightarrow{hf} & N'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow h & & \| & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \xrightarrow{f} & N'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & . \end{array}$$

Applying the functor $Hom(P, -)$ with $P \in \mathcal{P}$, we obtain a diagram from which we can see that $Hom(P, h)$ is epic.

(2) follows from (1) by induction.

(3) Assume N to be a direct sum of $\mathcal{P}$-flat modules and $h : P \to N$ a morphism with $P \in \mathcal{P}$. Then $(P)h = Im\, h$ is finitely generated and hence contained in a finite partial sum of N which is $\mathcal{P}$-flat by (2). From this we readily see that N is also $\mathcal{P}$-flat.

(4) For a direct system $(V_i, f_{ij})_\Lambda$ of flat modules in $\sigma[M]$, we have a $\mathcal{P}$-pure epimorphism $\oplus_\Lambda V_i \to \varinjlim V_i$ (see 33.9). Since $\oplus_\Lambda V_i$ is $\mathcal{P}$-flat by (3), we conclude from $(1)(i)$ that $\varinjlim V_i$ is also $\mathcal{P}$-flat.

36.2 Flat modules generated by projectives.

Let M be an R-module and $N \in \sigma[M]$. Assume N to be generated by finitely generated projective modules in $\sigma[M]$. Then the following assertions are equivalent:

(a) N is flat in $\sigma[M]$;

(b) N is a direct limit of finitely generated projective modules in $\sigma[M]$.

Proof: $(a) \Rightarrow (b)$ By assumption we have an exact sequence

$$\bigoplus\nolimits_\Lambda P_\lambda \to N \to 0 \text{ in } \sigma[M],$$

with P_λ finitely generated and projective. If N is flat, then the sequence is pure and, by 34.2, N is a direct limit of finite direct sums of the P_λ's.

$(b) \Rightarrow (a)$ follows from 36.1,(4).

Direct summands are never superfluous submodules. A similar assertion holds for pure submodules in the following case:

36.3 Pure submodules of projective modules.

Let M be an R-module and P a projective module in $\sigma[M]$ generated by finitely presented modules.

If $U \subset P$ is a pure and superfluous submodule, then $U = 0$.

Proof: We show that every finitely generated submodule $K \subset U$ is zero. Consider the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{p_K} & P/K & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow h & & \\
0 & \longrightarrow & U & \longrightarrow & P & \xrightarrow{p_U} & P/U & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & & .
\end{array}$$

Since P/K is pure projective by 34.1, there exists an $\alpha : P/K \to P$ with $\alpha p_U = h$. Because $Ke\, p_U = U \ll P$, this α is epic. Hence $p_K\alpha$ is also epic and splits since P is projective. Now we have

$$Ke\, p_K\alpha \subset Ke\, p_K\alpha p_U = Ke\, p_K h = Ke\, p_U = U,$$

i.e. $Ke\, p_K\alpha$ is superfluous and a direct summand in P, and hence zero. This implies $K \subset Ke\, p_K\alpha = 0$.

36.4 Projective and flat modules.

(1) Let M, N be R-modules and assume N is flat in $\sigma[M]$ and generated by finitely presented modules in $\sigma[M]$.

If $f : P \to N$ is a projective hull of N in $\sigma[M]$, then $P \simeq N$.

(2) For a flat module N in $\sigma[M]$, the following are equivalent:

(a) every flat factor module of N is projective in $\sigma[M]$;

(b) every indecomposable flat factor module of N is projective in $\sigma[M]$;

(c) every flat factor module of N has a direct summand which is projective in $\sigma[M]$.

Proof: (1) Let $\{L_i\}_\Lambda$ be a family of finitely presented modules in $\sigma[M]$ and $h : \oplus_\Lambda L_i \to N$ an epimorphism. Since $\oplus_\Lambda L_i$ is pure projective, there exists $g : \oplus_\Lambda L_i \to P$ with $gf = h$. $Ke\, f$ superfluous in P implies that g is epic. Now we see from 36.3 that $Ke\, f$, as a pure and superfluous submodule of P, has to be zero.

(2) Observe that, according to 36.1, flat factor modules of N are just factor modules by pure submodules.

$(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (c)$ For $0 \neq a \in N$, the set of pure submodules of N not containing a is inductive (with respect to inclusion). Hence, by Zorn's Lemma, it contains a maximal element $K \subset N$. N/K is indecomposable:
Assume $N/K = N_1/K \oplus N_2/K$ with $K \subset N_1$, $N_2 \subset N$ and $N_1 \cap N_2 = K$. Since $K \subset N$ is pure and the direct summands N_1/K and N_2/K are pure in N/K, by 33.3, N_1 and N_2 are also pure in N.

Now $a \notin K = N_1 \cap N_2$ implies $a \notin N_1$ or $a \notin N_2$. By the maximality of K, this means $N_1 = K$ or $N_2 = K$. Hence N/K is indecomposable.

By assumption (b), the flat module N/K is projective in $\sigma[M]$ and hence isomorphic to a direct summand of N.

The same arguments apply for every flat factor module of N.

$(c) \Rightarrow (a)$ Consider the set of independent families $\{P'_\lambda\}_\Lambda$ of projective submodules $P'_\lambda \subset N$ for which the internal sum $\bigoplus_\Lambda P'_\lambda$ is a pure submodule of N. This set is inductive with respect to inclusion and so, by Zorn's Lemma, it contains a maximal family $\{P_\lambda\}_\Lambda$. We show $P := \bigoplus_\Lambda P_\lambda = N$.

Assume $P \neq N$. Then N/P has a projective direct summand L/P with $P \subset L \subset N$ and hence $L = P \oplus Q$ for some projective $Q \simeq L/P$. Since $P \subset N$ and $L/P \subset N/P$ are pure submodules, L is also pure in N (see 33.3). This contradicts the maximality of P. Therefore $N = P$ and hence it is projective.

The same argument applies to flat factor modules of N.

In *R-MOD*, flat modules may be described in many different ways. In particular, we now obtain the connection with the notion *flat with respect to MOD-R* in 12.16:

36.5 Flat modules in *R-MOD*. Characterizations.

For an R-module $_RN$ the following assertions are equivalent:

(a) $_RN$ is flat in R-MOD (Def. before 36.1);

(b) the functor $-\otimes_R N$: MOD-R $\to$ AB is exact (see 12.16);

(c) $_RN$ is a direct limit of (finitely generated) projective (free) R-modules;

(d) there is a pure exact sequence $0 \to K \to F \to N \to 0$ in R-MOD with F flat (projective, free);

(e) $Hom_{\mathbb{Z}}(N, \bar{\mathbb{Q}})$ is (FP-) injective in MOD-R;

(f) for every finitely presented left R-module P, the canonical morphism $\nu_N : Hom_R(P,R) \otimes_R N \to Hom_R(P,N)$ is epic (see 25.5);

(g) every exact sequence $0 \to U \to V \to N \to 0$ in R-MOD, with pure injective U, splits;

(h) N is projective with respect to exact sequences $0 \to U \to V \to W \to 0$ *in R-MOD, with U pure injective.*

Proof: $(a) \Leftrightarrow (c)$ follows from 36.2.

$(c) \Rightarrow (b)$ For projective R-modules F_i, the functor $-\otimes_R F_i$ is exact. Since the tensor product commutes with direct limits (see 24.11), $-\otimes_R \varinjlim F_i$ is also exact.

$(b) \Rightarrow (f)$ follows with Hom-Tensor-Relations from 25.5.

$(f) \Rightarrow (a)$ If $g : L \to N$ is an epimorphism, $Hom(P, g)$ also has to be epic for every finitely presented R-module P: In the commutative diagram

$$\begin{array}{ccc} Hom(P,R)\otimes_R L & \xrightarrow{id\otimes g} & Hom(P,R)\otimes_R N \\ \downarrow \nu_L & & \downarrow \nu_N \\ Hom(P,L) & \xrightarrow{Hom(P,g)} & Hom(P,N) \end{array} \quad ,$$

the morphisms $id \otimes g$ and (by (f)) ν_N are epic. Hence $Hom(P, g)$ is also epic.

$(a) \Leftrightarrow (d)$ follows from 36.1 and the fact that every module in *R-MOD* is a factor module of a free module.

$(e) \Rightarrow (a)$ is a consequence of the characterization of pure sequences in *R-MOD* in 34.5, and of 17.14.

$(d) \Leftrightarrow (e)$ The sequence $0 \to K \to F \to N \to 0$ is pure if and only if

$$0 \to Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(F,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}}) \to 0$$

splits in *MOD-R*. Since $Hom_{\mathbb{Z}}(F,\overline{\mathbb{Q}})$ is R-injective for any free F (see 34.6), this is the case if and only if $Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}})$ is injective.

$Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}})$ is always pure injective (see 34.6). Hence *absolutely pure* (= FP-injective) and *injective* are equivalent for this module.

$(a) \Rightarrow (g)$ is obvious.

$(g), (h) \Rightarrow (a)$ Let $0 \to K \xrightarrow{f} L \to N \to 0$ be exact and $\gamma : K \to \widetilde{K}$ a pure monomorphism with pure injective $\widetilde{K}$ (see 34.6). Forming a pushout we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & N & \longrightarrow & 0 \\ & & \gamma\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \widetilde{K} & \longrightarrow & Q & \longrightarrow & N & \longrightarrow & 0 \end{array} .$$

Now (g) and (h) imply the existence of some $\delta : L \to \widetilde{K}$ with $\gamma = f\delta$. Then f is pure by 33.2, and hence N is flat.

$(g) \Rightarrow (h)$ If the sequence in (h) and $f: N \to W$ are given, we obtain with a pullback the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \downarrow & & {\scriptstyle f}\downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \quad , \end{array}$$

in which (by (g)) the first row splits.

The properties of flat modules now obtained lead to new characterizations of pure submodules of flat modules and hence also to further characterizations of flat modules themselves:

36.6 Pure submodules of flat modules.

For a short exact sequence $(*)\quad 0 \to K \to F \to N \to 0$ *in R-MOD with flat (or free) module F, the following assertions are equivalent:*

(a) $(*)$ *is pure;*

(b) N is flat;

(c) for every (finitely generated) right ideal $I \subset R_R$,

(i) $(*)$ *remains exact under* $R/I \otimes_R -$, *or*

(ii) $IK = K \cap IF$.

Proof: $(a) \Leftrightarrow (b)$ follows from 36.5, and $(a) \Rightarrow (c)(i)$ from 34.5.

$(c)(i) \Leftrightarrow (c)(ii)$ has been observed in 34.9.

$(c)(ii) \Rightarrow (b)$ By 12.16 and 36.5, it suffices to show that the canonical map $\mu_I : I \otimes_R N \to IN$ is monic for all finitely generated right ideals I. From $(*)$ we form the exact commutative diagram

$$\begin{array}{ccccccccc} & & I \otimes K & \longrightarrow & I \otimes F & \longrightarrow & I \otimes N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu'_I} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mu_I} & & \\ 0 & \longrightarrow & K \cap IF & \longrightarrow & IF & \longrightarrow & IN & \longrightarrow & 0 \quad . \end{array}$$

Because of $(c)(ii)$, the map $\mu'_I : I \otimes K \to IK \subset K \cap IF$ is surjective, and hence μ_I is monic by the Kernel Cokernel Lemma.

Finally let us make a remark about non zero divisors and flat modules which gives a description of flat $\mathbb{Z}$-modules:

36.7 Flat modules and non zero divisors.

(1) Let ${}_RN$ be a flat R-module and $s \in R$.
If $sr \neq 0$ for all $0 \neq r \in R$, then also $sn \neq 0$ for all $0 \neq n \in N$.

(2) Let R be a ring without zero divisors and assume that every finitely generated right ideal of R is cyclic. Then an R-module N is flat if and only if $rn \neq 0$ for all $0 \neq r \in R$ and $0 \neq n \in N$ (N is said to be torsion free*).*

(3) A $\mathbb{Z}$-module is flat if and only if it is torsion free (i.e. if it has no non-zero elements of finite order).

Proof: (1) Under the given assumptions, the map $R \to sR$, $r \mapsto sr$, is monic. Then $R \otimes N \to sR \otimes N \simeq sN$ is also monic, i.e. $sn \neq 0$ for all $0 \neq n \in N$.

(2) By (1), flat modules are torsion free. On the other hand, for a torsion free R-module N the map $sR \otimes N \to sN$ is always monic. Hence by 12.16, under the given conditions, N is flat.

(3) follows from (2).

36.8 Exercises.

(1) Let $0 \to U' \to U \to U'' \to 0$ be an exact sequence in R-MOD and F_R a flat module in MOD-R. Show:

If U' and U'' are F-flat modules, then U is also F-flat (see 12.15).

(2) Let $0 \to K \to F \to N \to 0$ be an exact sequence in R-MOD with free module F. Show that the following assertions are equivalent:

(a) N is flat;

(b) for every $a \in K$, there exists $f : F \to K$ with $(a)f = a$;

(c) for any finitely many $a_1, \ldots, a_n \in K$ there exists $f : F \to K$ with $(a_i)f = a_i$ for $i = 1, \ldots, n$.

(3) Show that for an R-module N the following are equivalent:

(a) N is flat (in R-MOD);

(b) every morphism $f : P \to N$, with finitely presented P, can be factorized via a finitely generated free (projective) R-module;

(c) every exact diagram

$$\begin{array}{ccccc} P & \to & N & & \\ & & \downarrow & & \\ M & \to & L & \to & 0 \end{array}$$

in R-MOD, with finitely presented P, can be extended commutatively with a morphism $P \to M$.

(4) For an R-module ${}_RM$ and $\overline{\mathbb{Q}} = \mathbb{Q}/\mathbb{Z}$, put $M^* = Hom_{\mathbb{Z}}(M, \overline{\mathbb{Q}})$. Show that an R-module ${}_RN$ is M^*-flat if and only if N^* is M^*-injective.
Hint: Hom-Tensor-Relation 12.12.

(5) Let R be a left coherent ring, M in R-MOD and $\overline{\mathbb{Q}} = \mathbb{Q}/\mathbb{Z}$. Show that the following assertions are equivalent:
(a) ${}_RM$ is FP-injective;
(b) $Hom_{\mathbb{Z}}(M,\overline{\mathbb{Q}})$ is flat in MOD-R.

(6) Let ${}_RM$ be a faithful R-module. Prove that the following assertions are equivalent:
(a) ${}_RM$ is flat (in R-MOD);
(b) ${}_RM$ is $Hom_{\mathbb{Z}}(M,D)$-flat for every injective $\mathbb{Z}$-module D;
(c) $Hom_{\mathbb{Z}}(M,D)$ is self-injective for every injective $\mathbb{Z}$-module D.

(7) Let ${}_RM$ be a self-injective R-module and $S = End(M)$. Show:
If M_S is flat in MOD-S, then ${}_SS$ is injective in S-MOD.

(8) Let M be in R-MOD. Denote by $\mathcal{F}$ the set of left ideals $J \subset R$ with $R/J \in \sigma[M]$. For a right R-module N_R we form the exact sequences

$(*)$ $0 \to K \to L \to N \to 0$ *in MOD-R.*

Show that for N_R the following assertions are equivalent:
(a) For every J in $\mathcal{F}$, $-_R \otimes R/J$ is exact with respect to all sequences $(*)$;
(b) for every $P \in \sigma[M]$, $- \otimes_R P$ is exact with respect to all sequences $(*)$;
(c) $Hom_{\mathbb{Z}}(N,\mathbb{Q}/\mathbb{Z})$ is injective with respect to exact sequences $0 \to J \to R$ with $J \in \mathcal{F}$ (see 16.12,(5));
(d) $KJ = K \cap LJ$ for all sequences $(*)$ with $J \in \mathcal{F}$;
(e) the canonical map $N \otimes_R J \to NJ$ is an isomorphism for all $J \in \mathcal{F}$;
(f) $N \otimes_R -$ is exact with respect to all exact sequences $0 \to A \to B$ in R-MOD with B/A in $\sigma[M]$.

(9) Let R be a subring of the ring S containing the unit of S. Show:
(i) If ${}_RN$ is a flat R-module, then $S \otimes_R N$ is a flat S-module.
(ii) A flat R-module N is projective if and only if there exists an exact sequence $0 \to K \to P \to N \to 0$ in R-MOD with P projective and $S \otimes_R K$ finitely generated as an S-module.

(10) Let R be an integral domain. Show that every finitely generated, flat R-module is projective. Hint: Exercise (9).

(11) Let us call an R-module N semi-flat, *if every exact diagram*

$$\begin{array}{ccccc} P & \to & N & & \\ & & \downarrow & & \\ M & \to & L & \to & 0 \end{array}$$

in R-MOD, with P finitely presented and L injective, can be extended commutatively by a morphism $P \to M$ (see Exercise (3)). Show:

(i) The direct sum of a family of R-modules is semi-flat if and only if every summand is semi-flat.

(ii) If $0 \to N' \to N \to N'' \to 0$ is a pure exact sequence with N semi-flat, then N' and N'' are also semi-flat.

(iii) A finitely presented R-module N is semi-flat if and only if it is a submodule of a free module.

(iv) Injective, semi-flat modules are flat.

Literature: Azumaya [3], Choudhury, Colby-Rutter [2], Döman-Hauptfleisch, Enochs [3], Gomez [1], Gouguenheim [1,2], Hauptfleisch-Döman, Hill [1], Jøndrup [2], Jothilingam, Nishida [1], Ramamurthi [3,4], Salles [1].

37 Regular modules and rings

1.$\mathcal{P}$-regular modules. 2.M $\mathcal{P}$-regular in $\sigma[M]$. 3.Regular modules. 4.M regular in $\sigma[M]$. 5.Locally noetherian regular modules. 6.Regular rings. 7.Regular endomorphism rings. 8.Projective regular modules. 9.Matrix rings over regular rings. 10.Co-semisimple and regular modules over commutative rings. 11.Projective regular modules over commutative rings. 12.Exercises.

Let M be an R-module and $\mathcal{P}$ a non-empty class of modules in $\sigma[M]$.

A module L in $\sigma[M]$ is called *$\mathcal{P}$-regular in $\sigma[M]$*, if every exact sequence $0 \to K \to L \to N \to 0$ in $\sigma[M]$ is $\mathcal{P}$-pure.

Evidently, L is $\mathcal{P}$-regular if and only if every P in $\mathcal{P}$ is L-projective.

In case $\mathcal{P}$ consists of all finitely presented modules in $\sigma[M]$, instead of $\mathcal{P}$-regular we just say *regular in $\sigma[M]$*.

Obviously, an R-module in $\sigma[M]$ is semisimple if and only if it is $\mathcal{P}$-regular with respect to all non-empty classes $\mathcal{P}$ (in $\sigma[M]$).

37.1 Properties of $\mathcal{P}$-regular modules.
With the above notation we have:

(1) Let $0 \to L' \to L \to L'' \to 0$ be an exact sequence in $\sigma[M]$.

(i) If L is $\mathcal{P}$-regular, then L' and L'' are also $\mathcal{P}$-regular.

(ii) If the sequence is $\mathcal{P}$-pure and L' and L'' are $\mathcal{P}$-regular, then L is $\mathcal{P}$-regular.

(2) A finite direct sum of $\mathcal{P}$-regular modules is $\mathcal{P}$-regular.

(3) Assume the modules in $\mathcal{P}$ to be finitely generated. Then direct sums and direct limits of $\mathcal{P}$-regular modules are again $\mathcal{P}$-regular.

(4) A $\mathcal{P}$-flat module in $\sigma[M]$ is $\mathcal{P}$-regular if and only if all its factor modules are $\mathcal{P}$-flat.

(5) An absolutely $\mathcal{P}$-pure module is $\mathcal{P}$-regular if and only if all its submodules are absolutely $\mathcal{P}$-pure.

Proof: (1)(i) If every P in $\mathcal{P}$ is L-projective, then it is L'- and L''-projective (see 18.2).

(ii) For every epimorphism $g : L \to N$ we can form the commutative exact diagram

$$\begin{array}{ccccccccc}
0 \rightarrow & L' & \rightarrow & L & \rightarrow & L'' & \rightarrow 0 \\
 & \downarrow & & \downarrow g & & \downarrow & \\
0 \rightarrow & N' & \rightarrow & N & \rightarrow & N'' & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & .
\end{array}$$

For $P \in \mathcal{P}$, we apply the functor $Hom_R(P, -)$. If the sequence in (1) is $\mathcal{P}$-pure and L', L'' are $\mathcal{P}$-regular, then we see from the resulting diagram that $Hom(P, g)$ is epic. Hence L is $\mathcal{P}$-regular.

(2), (3) If $P \in \mathcal{P}$ is L_i-projective for $i = 1, \ldots, k$, then it is also $\oplus_{i \leq k} L_i$-projective. In case P is finitely generated the corresponding assertion also holds for infinite (direct) sums (see 18.2). The direct limit is a factor module of a direct sum of $\mathcal{P}$-regular modules and hence also $\mathcal{P}$-regular.

(4), resp. (5), result from 36.1, resp. 35.1.

Of special interest is the case when M itself is $\mathcal{P}$-regular. We have:

37.2 M $\mathcal{P}$-regular in $\sigma[M]$. Characterizations.

Let M be an R-module and $\mathcal{P}$ a non-empty class of finitely generated modules in $\sigma[M]$. Then the following assertions are equivalent:

(a) M is $\mathcal{P}$-regular in $\sigma[M]$;

(b) every finitely generated submodule of M is $\mathcal{P}$-regular;

(c) every module in $\sigma[M]$ is $\mathcal{P}$-regular;

(d) every short exact sequence in $\sigma[M]$ is $\mathcal{P}$-pure;

(e) every module in $\sigma[M]$ is $\mathcal{P}$-flat;

(f) every factor module of M is $\mathcal{P}$-flat;

(g) every module in $\sigma[M]$ is absolutely $\mathcal{P}$-pure;

(h) every submodule of M is absolutely $\mathcal{P}$-pure;

(i) every module in $\mathcal{P}$ is projective in $\sigma[M]$;

(j) every $\mathcal{P}$-pure projective module is projective in $\sigma[M]$.

Proof: The equivalence of (a), (b) and (c) is obtained from 37.1 (M is generated by its finitely generated submodules). The remaining equivalences follow fairly immediately from the definitions.

Let us point out that 37.2 yields a description of semisimple modules if for $\mathcal{P}$ the class (set) of all finitely generated (cyclic, simple) modules in $\sigma[M]$ is taken (see 20.3).

Taking for $\mathcal{P}$ all finitely presented modules in $\sigma[M]$, we obtain:

37.3 Regular modules. Characterizations.

For a module L in $\sigma[M]$, $M \in R$-MOD, the following are equivalent:

(a) L is regular in $\sigma[M]$;

(b) every finitely presented module in $\sigma[M]$ is L-projective;

(c) every finitely generated submodule of L is pure in L;

(d) every finitely generated submodule of L is regular in $\sigma[M]$.

If L is a direct summand of a direct sum of finitely presented modules in $\sigma[M]$, then (a)-(d) are also equivalent to:

(e) every finitely generated submodule of L is a direct summand of L.

Proof: $(a) \Leftrightarrow (b)$ is just the definition of regular in $\sigma[M]$.
$(a) \Leftrightarrow (c)$ The direct limit of pure submodules is pure (see 33.8).
$(a) \Leftrightarrow (d)$ The direct limit of regular modules is regular (see 37.1).
$(c) \Leftrightarrow (e)$ follows from 34.1.

Besides the general description of modules M being $\mathcal{P}$-regular in $\sigma[M]$ given in 37.2, we now have:

37.4 M regular in $\sigma[M]$. Characterizations.

Assume the R-module M to be a submodule of a direct sum of finitely presented modules in $\sigma[M]$. Then the following assertions are equivalent:

(a) M is regular in $\sigma[M]$;

(b) every finitely generated submodule of M (or $M^{(\mathbb{N})}$) is a direct summand;

(c) every finitely generated submodule of a finitely presented module in $\sigma[M]$ is a direct summand;

(d) every R-module (in $\sigma[M]$) is weakly M-injective.

Proof: Let P be a direct sum of finitely presented modules in $\sigma[M]$ and $M \subset P$.

$(a) \Rightarrow (b)$ For a finitely generated submodule $K \subset M$, the factor module P/K is pure projective (by 34.1) and hence projective (see 37.2). This implies that K is a direct summand in P, and in M.

$(b) \Rightarrow (a) \Rightarrow (c)$ has been shown already in 37.2 and 37.3.

$(c) \Rightarrow (a)$ With (c), every finitely presented module in $\sigma[M]$ is regular. Now P is a direct sum of such modules and hence is regular. Then the submodule M of P is also regular.

$(b) \Leftrightarrow (d)$ is easily seen.

From the properties just seen we easily derive:

37.5 Locally noetherian regular modules.

For an R-module M the following assertions are equivalent:

(a) M is locally noetherian and regular in $\sigma[M]$;

(b) M is semisimple.

A ring R is called *left regular*, if ${}_RR$ is a regular module in R-*MOD*. By 37.4, these are just the rings in which finitely generated left ideals are direct summands. Already in 3.10 we have seen that this is a characterization of (von Neumann) regular rings. They can now be described by

37.6 External characterizations of regular rings.

For a ring R, the following assertions are equivalent:

(a) R is left regular (regular in R-MOD);

(b) R is (von Neumann) regular;

(c) every left R-module is regular;

(d) every (cyclic) left R-module is flat;

(e) every left R-module is absolutely pure (= FP-injective);

(f) every finitely presented left R-module is projective;

(g) every factor module R/I, with $I \subset {}_RR$ finitely generated (or cyclic), is projective;

(h) every pure injective left R-module is injective;

(i) R is right regular (regular in MOD-R).

(c)–(h) are also true for right modules.

Proof: The equivalences of (a) to (f) and $(a) \Leftrightarrow (i)$ follow from 3.10, 37.2 and 37.4.

$(a) \Rightarrow (g)$, (h) is obvious.

$(g) \Rightarrow (a)$ If R/Rr is projective, then Rr is a direct summand in R. By 3.10, this implies that R is regular.

$(h) \Leftrightarrow (a)$ By 34.7, (h) implies that every short exact sequence in R-*MOD* is pure.

Remark: The assertions in (g) allow the following equivalent definition of regular rings: ${}_RR$ is $\mathcal{P}$-regular with respect to the class $\mathcal{P}$

– of all finitely presented cyclic modules, or

– of all modules R/Rr with $r \in R$.

We have seen in 3.9 that the endomorphism ring of a vector space is regular. Applying the same arguments we may formulate more generally:

37.7 Regular endomorphism rings.
Let M be an R-module and $S = End_R(M)$.

(1) For $f \in S$, the following properties are equivalent:
(a) There exists $g \in S$ with $fgf = f$;
(b) Ke f and Im f are direct summands of M.

(2) S is regular if and only if Im f and Ke f are direct summands of M for every $f \in S$.

(3) If S is regular, then every finitely M-generated submodule of M is a direct summand in M.

Proof: (1) $(a) \Rightarrow (b)$ Assume $g \in S$ with $fgf = f$.
The sequence $0 \to Ke\ f \to M \to Im\ f \to 0$ splits since, for all $m \in M$, $(m)fgf = (m)f$, i.e. $gf = id_{Im\,f}$.

The sequence $0 \to Im\ f \xrightarrow{i} M$ splits since $(m)fgf = (m)f$ implies $i(gf) = id_{Im\,f}$.

$(b) \Rightarrow (a)$ Since the inclusion $Im\ f \subset M$ splits, there is an $\alpha : M \to Im\ f$ with $(m)f\alpha = (m)f$ for all $m \in M$.

$Ke\ f$ being a direct summand, $Ke\ f \to M \to Im\ f$ splits, and there exists $\beta : Im\ f \to M$ with $(m)f\beta f = (m)f$ (see 8.3). By construction, we have $f(\alpha\beta)f = f$ with $\alpha\beta \in S$, since

$$(m)f(\alpha\beta)f = (m)f\beta f = (m)f \text{ for all } m \in M.$$

(2) follows immediately from (1).

(3) If K is a finitely M-generated submodule of M, there is a morphism $M^k \to M$ with $Im\ f = K$. Since $M \subset M^k$ we may consider f as an element of $End(M^k) \simeq S^{(k,k)}$. As we shall see in 37.9, $S^{(k,k)}$ is a regular ring and so $Im\ f$ is a direct summand in M^k and hence also in M.

37.8 Projective regular modules.
Let the R-module M be finitely generated, projective and regular in $\sigma[M]$. Then:
(1) M is a projective generator in $\sigma[M]$;
(2) $End_R(M)$ is a regular ring.

Proof: (1) Since finitely generated submodules of M are direct summands (by 37.4), M is a self-generator. Projective self-generators M are generators in $\sigma[M]$ by 18.5.

(2) For $f \in End_R(M)$, $Im\ f$ is a finitely generated submodule of M, hence a direct summand and projective. Then also $Ke\ f$ is a direct summand and the assertion follows from 37.7.

As an application of 37.8 we state:

37.9 Matrix rings over regular rings.

For a ring R the following assertions are equivalent:
(a) R is regular;
(b) $End_R(P)$ is regular for finitely generated, projective R-modules P;
(c) the matrix ring $R^{(n,n)}$ is regular for some (every) $n \in \mathbb{N}$.

Proof: $(a) \Rightarrow (b)$ By 37.6, P is regular and, by 37.8, $End_R(P)$ is a regular ring.

$(b) \Rightarrow (c)$ Matrix rings are endomorphism rings of free modules.

$(c) \Rightarrow (a)$ If, for example, $R^{(2,2)}$ is a regular ring, then this is also true for $\binom{10}{00} R^{(2,2)} \binom{10}{00} \simeq R$.

Observe that, for a regular R, the endomorphism ring of $R^{(\mathbb{N})}$ need not be regular unless R is left semisimple (see 43.4, 43.9).

It is obvious that a factor ring of a regular ring is again regular. Also, a regular ring has no small submodules, i.e. its (Jacobson) radical is zero. However, for a left ideal $I \subset R$ of a regular ring, in general $Rad_R(R/I)$ need not be zero, i.e. R is not necessarily co-semisimple. For commutative rings we have seen in 23.5 that *regular* and *co-semisimple (left V-ring)* are equivalent properties. Let us find out to which extent this is true for modules over commutative rings:

37.10 Co-semisimple modules over commutative rings.

Let M be a module over a commutative ring R. Then:

(1) If M is co-semisimple, then M is regular in $\sigma[M]$.

(2) If M is regular in $\sigma[M]$, then every finitely presented module in $\sigma[M]$ is co-semisimple.

(3) Assume in $\sigma[M]$ that there is a direct sum P of finitely presented modules with $\sigma[M] = \sigma[P]$. Then the following are equivalent:
(a) M is co-semisimple;
(b) M is regular in $\sigma[M]$.

Proof: (1) Let M be co-semisimple and P a finitely presented module in $\sigma[M]$. We show that every exact sequence

$$(*) \quad 0 \longrightarrow K \longrightarrow L \longrightarrow P \longrightarrow 0$$

in $\sigma[M]$, with finitely generated L, splits. Then, by 18.3, P is projective in $\sigma[M]$ and M is regular (see 37.3).

Consider $(*)$ as a sequence in $\sigma[L] = \overline{R}$-*MOD* with $\overline{R} = R/An(L)$. Since L is co-semisimple, this is also true for $\overline{R}$. Hence $\overline{R}$ is regular (see 23.5). Therefore in $\sigma[L]$ every finitely presented module is L-projective, and $(*)$ splits.

(2) Let M be regular in $\sigma[M]$ and P finitely presented in $\sigma[M]$. For the factor ring $\overline{R} = R/An(P)$ we know $\overline{R} \subset P^k$, $k \in I\!N$. Then, by 37.4, every finitely generated left ideal of $\overline{R}$ is a direct summand in $\overline{R}$. Hence $\overline{R}$ is regular. Now 23.5 implies that every simple module in $\overline{R}$-*MOD* $= \sigma[P]$ is P-injective, i.e. P is co-semisimple.

(3) $(a) \Rightarrow (b)$ has been shown in (1).

$(b) \Rightarrow (a)$ If M is regular, then, by (2), P is a direct sum of co-semisimple modules, and hence co-semisimple. Then M is co-semisimple by 23.1.

Observe that the condition in (3) is satisfied if M is finitely generated (then $\sigma[M] = R/An(M)$-*MOD*).

Applying our knowledge about regular endomorphism rings we obtain:

37.11 Projective regular modules over commutative rings.
For a finitely generated, self-projective module M over a commutative ring R, the following assertions are equivalent:

(a) M is regular in $\sigma[M]$;

(b) M is co-semisimple;

(c) $End_R(M)$ is a regular ring;

(d) $End_R(M)$ is left co-semisimple;

(e) $\overline{R} = R/An_R(M)$ is a regular ring.

Proof: By 18.11, M is a generator in $\sigma[M] = \overline{R}$-*MOD*.

$(a) \Leftrightarrow (b)$ follows from 37.10.

$(a) \Rightarrow (c)$ was shown in 37.8, $(b) \Rightarrow (d)$ in 23.8.

$(c) \Rightarrow (a)$ is obtained from 37.7,(3).

$(d) \Rightarrow (b)$ With the given properties of M, the functor

$Hom_R(M, -) : \sigma[M] \to End_R(M)\text{-}MOD$

is an equivalence (see § 46). Then (d) implies that all simple modules in $\sigma[M]$ are M-injective.

$(a) \Leftrightarrow (e)$ is trivial since $\sigma[M] = \overline{R}$-*MOD*.

37.12 Exercises.

(1) Show that, for a ring R, the following assertions are equivalent :
(a) R is left fully idempotent (see 3.15);
(b) for every ideal $I \subset R$, R/I is flat in MOD-R;
(c) every ideal I is flat in MOD-R;
(d) every ideal I is a pure submodule of R_R.

(2) Let R be a regular ring. Show:
(i) Every countably generated left ideal in R is projective.
(ii) Every countably generated submodule of a projective R-module is projective. Hint: see 8.9.

(3) Let R be a regular ring. Show: If every finitely generated faithful R-module is a generator in R-MOD, then R is biregular (see 3.18,(6)).

(4) Let R be a ring with center C which is finitely generated as a C-algebra. Show that the following assertions are equivalent:
(a) R is biregular (see 3.18,(6));
(b) R is regular in $R \otimes_C R^o$-MOD;
(c) R is projective in $R \otimes_C R^o$-MOD (Azumaya algebra) and C is regular;
(d) R is a generator in $R \otimes_C R^o$-MOD and C is regular.
Hint: Exercise 25.6,(3).

(5) Let R be a commutative, regular ring. Show that the following assertions are equivalent:
(a) R is self-injective;
(b) every finitely generated, faithful module is a generator in R-MOD;
(c) for every finitely generated, faithful R-module N, the trace $Tr(N, R)$ is finitely generated.

(6) Show that, for a $\mathbb{Z}$-module M, the following are equivalent:
(a) M is regular in $\sigma[M]$;
(b) M is co-semisimple;
(c) M is semisimple;
(d) M is regular in $\mathbb{Z}$-MOD.

(7) (i) Show that, for an R-module ${}_RM$, the following are equivalent:
(a) For every $m \in M$ there is an $f \in Hom_R(M, R)$ with $((m)f)m = m$;
(b) every cyclic submodule of M is a direct summand and R-projective;
(c) every finitely generated submodule of M is a direct summand and R-projective.
Modules with these properties are called Z-regular *(see Zelmanowitz).*

(ii) Verify the following assertions:

(α) Every Z-regular module is flat.

(β) Every countably generated, Z-regular R-module is projective and a direct sum of finitely generated modules.

(8) Show that for a projective R-module ${}_RP$, the following are equivalent:

(a) P is Z-regular (Exercise (7));

(b) P is regular in $\sigma[P]$;

(c) P is regular in R-MOD.

(9) Show that for a ring R, the following assertions are equivalent:

(a) Every R-module is Z-regular (Exercise (7));

(b) every regular module in R-MOD is Z-regular;

(c) R is left semisimple.

(10) Let R be a ring with every simple left R-module flat (left SF ring). Show:

(i) Every maximal left ideal is a pure submodule of R.

(ii) Every maximal left ideal in R is a flat R-module.

(iii) For every ideal $I \subset R$, R/I is a left SF ring.

(iv) If R is semiperfect (see § 42), then ${}_RR$ is semisimple.

Literature: GOODEARL; Ahsan-Ibrahim, Armendariz, Baccella [1,2], Chandran, Cheatham-Enochs [2], Choudhury-Tewari, Faith [4], Fieldhouse [1], Finkelstein, Fisher [3], Fisher-Snider, Fontana, Goodearl [2], Gupta V., Hauptfleisch-Roos, Hirano, Hirano-Tominaga, Kishimoto-Tominaga, Kobayashi, Lajos, Mabuchi, Maoulaoui, Menal [1], Nicholson [3,4], O'Meara, Oshiro, Page [1,2], Ramamurthi [1,2,3,], Rangaswamy-Vanaja [1], Raphael, Rege, Renault [1], Sai, Singh-Jain, Tiwary-Pandeya, Tominaga, Tuganbaev [3], Ware, Wisbauer [1], Yue [3,4,6,7,], Zelmanowitz [1].

38 Copure sequences and derived notions

1.Definitions. 2.Properties. 3.Existence of $\mathcal{Q}$-copure monomorphisms. 4.$\mathcal{Q}$-copure injective modules. 5.$\mathcal{Q}$-copure projective modules. 6.Absolutely $\mathcal{Q}$-copure modules. 7.$\mathcal{Q}$-coflat modules. 8.Properties of $\mathcal{Q}$-coflat modules. 9.$\mathcal{Q}$-coregular modules. 10.M $\mathcal{Q}$-coregular in $\sigma[M]$. 11.Co-semisimple modules. 12. $\mathcal{Q}_c$-copure sequences. 13.Exercises.

$\mathcal{P}$-pure sequences have been introduced in § 33 as short exact sequences with respect to which modules in a given class $\mathcal{P}$ are projective. Dually we now formulate:

38.1 Definitions. Let $\mathcal{Q}$ be a non-empty class of modules in $\sigma[M]$, M in R-MOD. An exact sequence

$$0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} N \longrightarrow 0$$

in $\sigma[M]$ is called *$\mathcal{Q}$-copure in $\sigma[M]$* if every module Q in $\mathcal{Q}$ is injective with respect to this sequence, i.e. if every diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f} & L & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & Q & & & & & & \end{array}$$

can be extended commutatively by a morphism $L \to Q$.

Equivalently, the following sequence has to be exact

$$0 \to Hom(N, Q) \to Hom(L, Q) \to Hom(K, Q) \to 0 .$$

We then call f a *$\mathcal{Q}$-copure monomorphism*, g a *$\mathcal{Q}$-copure epimorphism* and $Im\, f = (K)f$ a *$\mathcal{Q}$-copure submodule* of L.

The fundamental properties of $\mathcal{Q}$-copure sequences coincide to a great extent with those of $\mathcal{P}$-pure sequences. In fact, we have seen in 34.7 that e.g. the pure sequences in R-MOD may be characterized as $\mathcal{Q}$-copure sequences with $\mathcal{Q}$ the class of pure injective modules.

Let us first state some basic properties whose proofs are obtained usually by dualizing the corresponding situations for pure sequences. The assertions 33.2, 33.3 and 33.4 for $\mathcal{P}$-pure morphisms are also valid here:

38.2 Properties of $\mathcal{Q}$-copure morphisms.
Let M be an R-module and $\mathcal{Q}$ a class of modules in $\sigma[M]$.

(1) For morphisms $f : K \to L$, $g : L \to N$ in $\sigma[M]$, we have:
(i) If f and g are $\mathcal{Q}$-copure epimorphisms (monomorphisms), then fg is a $\mathcal{Q}$-copure epimorphism (monomorphism).
(ii) If fg is a $\mathcal{Q}$-copure epimorphism, then this is also true for g.
(iii) If fg is a $\mathcal{Q}$-copure monomorphism, then this is also true for f.

(2) (i) Under pullbacks, $\mathcal{Q}$-copure epimorphisms are lifted to $\mathcal{Q}$-copure epimorphisms.
(ii) Under pushouts, $\mathcal{Q}$-copure monomorphisms again become $\mathcal{Q}$-copure monomorphisms (see 33.4).

Proof: Dual to the proofs of 33.2 and 33.4.

Dualizing 33.5 we now obtain:

38.3 Existence of $\mathcal{Q}$-copure monomorphisms.
Let M be an R-module and $\mathcal{Q}$ a set of modules in $\sigma[M]$. Assume N to be cogenerated by $\mathcal{Q}$. Then there exists a $\mathcal{Q}$-copure monomorphism $N \to Q$ with Q a direct product of modules in $\mathcal{Q}$.

An R-module X in $\sigma[M]$ is called *$\mathcal{Q}$-copure projective (injective)* if X is projective (injective) with respect to every $\mathcal{Q}$-copure sequence in $\sigma[M]$, i.e. $Hom(X, -)$ ($Hom(-, X)$) is exact with respect to $\mathcal{Q}$-copure sequences.

In this case, of course, the $\mathcal{Q}$-copure injective modules are determined by $\mathcal{Q}$ and dually to 33.6 we have:

38.4 $\mathcal{Q}$-copure injective modules. Characterizations.
Let M be an R-module and $\mathcal{Q}$ a set of modules in $\sigma[M]$. For a module X in $\sigma[M]$, the following assertions are equivalent:
(a) X is $\mathcal{Q}$-copure injective;
(b) every $\mathcal{Q}$-copure sequence $0 \to X \to L \to N \to 0$ in $\sigma[M]$ splits.

If X is cogenerated by $\mathcal{Q}$ then (a),(b) are equivalent to:
(c) X is a direct summand of a direct product of modules in $\mathcal{Q}$.

In general, little can be said about $\mathcal{Q}$-copure projective modules. Of course, all projective modules are of this type, and dually to 33.7 we state:

38.5 $\mathcal{Q}$-copure projective modules.
Let M be an R-module, $\mathcal{Q}$ a class of R-modules. For X in $\sigma[M]$, the following assertions are equivalent:

(a) X is $\mathcal{Q}$-copure projective.

(b) every $\mathcal{Q}$-copure sequence $0 \to K \to L \to X \to 0$ in $\sigma[M]$ splits.

Let $(*)$ $0 \to K \to L \to N \to 0$ be an exact sequence in $\sigma[M]$. The module K is called *absolutely $\mathcal{Q}$-copure* (*N $\mathcal{Q}$-coflat*, *L $\mathcal{Q}$-coregular*), if every sequence of the type $(*)$ is $\mathcal{Q}$-copure.

38.6 Properties of absolutely $\mathcal{Q}$-copure modules.
Let M be an R-module and $\mathcal{Q}$ a class of modules in $\sigma[M]$.

(1) Consider an exact sequence $0 \to K' \to K \to K'' \to 0$ in $\sigma[M]$.

(i) If the sequence is $\mathcal{Q}$-copure and K is absolutely $\mathcal{Q}$-copure, then K' is also absolutely $\mathcal{Q}$-copure.

(ii) If K' and K'' are absolutely $\mathcal{Q}$-copure, then K is absolutely $\mathcal{Q}$-copure.

(2) Every finite direct sum of absolutely $\mathcal{Q}$-copure modules is absolutely $\mathcal{Q}$-copure.

The proof is dual to the first part of 36.1.

38.7 Characterization of $\mathcal{Q}$-coflat modules.
Let M be an R-module and $\mathcal{Q}$ a set of modules in $\sigma[M]$. For a module N in $\sigma[M]$, the following properties are equivalent:

(a) N is $\mathcal{Q}$-coflat (in $\sigma[M]$);

(b) every exact sequence $0 \to Q \to L \to N \to 0$ in $\sigma[M]$, with Q in $\mathcal{Q}$, (or Q $\mathcal{Q}$-copure injective) splits;

(c) N is projective with respect to exact sequences $0 \to Q \to V \to W \to 0$ in $\sigma[M]$ with Q in $\mathcal{Q}$ (or Q $\mathcal{Q}$-copure injective);

(d) N is a $\mathcal{Q}$-copure factor module of a $\mathcal{Q}$-coflat module in $\sigma[M]$.

Noting 38.3, the proof is dual to arguments in 35.1. Also compare the characterization of flat modules in *R-MOD* (see 36.5).

38.8 Properties of $\mathcal{Q}$-coflat modules.

(1) A direct sum of modules in $\sigma[M]$ is $\mathcal{Q}$-coflat if and only if every summand is $\mathcal{Q}$-coflat.

(2) If $0 \to N' \to N \to N'' \to 0$ is an exact sequence in $\sigma[M]$ and N', N'' are $\mathcal{Q}$-coflat, then N is also $\mathcal{Q}$-coflat.

Proof: (1) follows directly from 38.7, (2) is obtained dually to 35.2,(3).

By the above definition, a module L is $\mathcal{Q}$-coregular if and only if every module in $\mathcal{Q}$ is L-injective. Referring to our observations about L-injective modules in 16.2 we obtain (dually to 37.1):

38.9 Properties of $\mathcal{Q}$-coregular modules.

(1) Let $0 \to L' \to L \to L'' \to 0$ be an exact sequence in $\sigma[M]$.

(i) If L is $\mathcal{Q}$-coregular, then L' and L'' are $\mathcal{Q}$-coregular.

(ii) If the sequence is $\mathcal{Q}$-copure and L', L'' are $\mathcal{Q}$-coregular, then L is also $\mathcal{Q}$-coregular.

(2) Direct sums and direct limits of $\mathcal{Q}$-coregular modules are $\mathcal{Q}$-coregular.

With this we obtain (dually to 37.2):

38.10 M $\mathcal{Q}$-coregular in $\sigma[M]$. Characterizations.

For an R-module M and a set $\mathcal{Q}$ of modules in $\sigma[M]$, the following assertions are equivalent:

(a) M is $\mathcal{Q}$-coregular in $\sigma[M]$;

(b) every (finitely generated) module in $\sigma[M]$ is $\mathcal{Q}$-coregular;

(c) every short exact sequence in $\sigma[M]$ is $\mathcal{Q}$-copure;

(d) every module in $\sigma[M]$ is $\mathcal{Q}$-coflat;

(e) every module in $\sigma[M]$ is absolutely $\mathcal{Q}$-copure;

(f) every module in $\mathcal{Q}$ is injective in $\sigma[M]$.

Again we obtain characterizations of semisimple modules for special classes $\mathcal{Q}$, e.g., if we choose $\mathcal{Q}$ to consist of all submodules of M.

Taking for $\mathcal{Q}$ all simple modules in $\sigma[M]$, then 38.10 describes the co-semisimple modules M (see 23.1). We are going to look at this case in more detail. For this consider copurity with respect to some special classes of modules. For an R-module M denote by

- $\mathcal{Q}_s$ the class of all simple R-modules,
- $\mathcal{Q}_c$ the class of all cocyclic modules,
- $\mathcal{Q}_f$ the class of all finitely cogenerated modules,
- $\mathcal{Q}_p$ the class of all finitely copresented modules

all in $\sigma[M]$. Each of these classes has a representing set which can be chosen as a set of submodules of $K^{(\mathbb{N})}$ for a cogenerator K of $\sigma[M]$. All these classes determine the same coregularity:

38.11 Further characterizations of co-semisimple modules.

For an R-module M the following properties are equivalent:

(a) *M is co-semisimple;*

(b) *M is $\mathcal{Q}_s$-coregular;*

(c) *M is $\mathcal{Q}_c$-coregular;*

(d) *M is $\mathcal{Q}_f$-coregular;*

(e) *M is $\mathcal{Q}_p$-coregular.*

Proof: $(a) \Leftrightarrow (b)$ is just the definition of co-semisimple.

$(a) \Rightarrow (d)$ was shown in 23.1.

$(d) \Rightarrow (c) \Rightarrow (a)$ and $(d) \Rightarrow (e)$ are trivial.

$(e) \Rightarrow (a)$ is an assertion of 31.7.

Of interest are, of course, the relations between copure and pure sequences. In this direction we prove the next result. We use the above notation.

38.12 Properties of $\mathcal{Q}_c$-copure sequences.

(1) If $0 \to K \to L \to N \to 0$ is a $\mathcal{Q}_c$-copure sequence in $\sigma[M]$, then, for every left ideal $I \subset R$, $IK = K \cap IL$.

(2) Assume that in the ring R every right ideal is also a left ideal. Then $\mathcal{Q}_c$-copure sequences remain exact under $R/J \otimes_R -$ for every (right) ideal $J \subset R$.

Proof: (1) (Compare proof $(g)(ii) \Rightarrow (f)$ of 34.12.)
We always have $IK \subset K \cap IL$. Assume there exists $a \in K \cap IL$ with $a \notin IK$. Choose a submodule $U \subset K$ maximal with respect to $IK \subset U$ and $a \notin U$ (Zorn's Lemma). Then K/U is cocyclic (see 14.9). Hence the canonical projection $p : K \to K/U$ can be extended to $h : L \to K/U$. Now we may assume $a = il$ with $i \in I$, $l \in L$, leading to the contradiction

$$0 \neq (a)p = (a)h = (il)h = i((l)h) \subset I \cdot K/U = 0.$$

Hence $IK = K \cap IL$.

(2) In view of (1), this follows from 34.9.

38.13 Exercises.

(1) Let $\mathcal{Q}$ be a class of modules in R-MOD and $K \in$ R-MOD. Show that the following assertions are equivalent:

(a) *K is absolutely $\mathcal{Q}$-copure;*

(b) *every morphism $f : K \to Q$ with $Q \in \mathcal{Q}$ can be factorized via a product of (indecomposable) injective modules in R-MOD.*

(2) Let M be an R-module and $\mathcal{Q}$ a class of finitely copresented modules in $\sigma[M]$. Show:

(i) If $\{N_\lambda\}_\Lambda$ is a family of modules in $\sigma[M]$ with $\mathcal{Q}$-copure submodules $U_\lambda \subset N_\lambda$, then $\bigoplus_\Lambda U_\lambda$ is a $\mathcal{Q}$-copure submodule of $\bigoplus_\Lambda N_\lambda$.

(ii) The following assertions are equivalent in $\sigma[M]$:
(a) The direct sum of absolutely $\mathcal{Q}$-copure modules is absolutely $\mathcal{Q}$-copure;
(b) the direct sum of injective modules is absolutely $\mathcal{Q}$-copure.

(3) Let R be a commutative ring and $\mathcal{Q}$ the class of finitely copresented R-modules. Show:
(i) Every $\mathcal{Q}$-copure exact sequence in R-MOD is pure exact.
(ii) Every absolutely $\mathcal{Q}$-copure module in R-MOD is absolutely pure.

(4) For a ring R, let $\mathcal{Q}$ be the class of finitely cogenerated modules in R-MOD. Show that, for R-modules $K \subset L$, the following are equivalent:
(a) K is a $\mathcal{Q}$-copure submodule of L;
(b) if, for a submodule $U \subset K$, the factor module K/U is finitely cogenerated, then K/U is a direct summand in L/U.

(5) Let R be a commutative, co-noetherian ring and $\mathcal{Q}$ the class of finitely cogenerated R-modules. Show that every short pure exact sequence in R-MOD is also $\mathcal{Q}$-copure.

(6) Let R be a commutative, co-noetherian ring. Show:
(i) Every finitely cogenerated R-module is pure injective.
(ii) If $\mathcal{Q}$ is the class of finitely cogenerated modules, then every pure exact sequence in R-MOD is $\mathcal{Q}$-copure.
(iii) An R-module is pure injective if and only if it is a direct summand of a direct sum of cocyclic modules.

(7) Let $\mathcal{Q} = \{\mathbb{Z}_n | 0 \neq n \in \mathbb{N}\}$ be the set of finite cyclic $\mathbb{Z}$-modules. Show that a short exact sequence in $\mathbb{Z}$-MOD is $\mathcal{Q}$-copure if and only if it is pure.

Literature: Couchot [4], Hiremath [4,5], Rangaswamy [3], Yahya-Al-Daffa [1].

Chapter 8

Modules described by means of projectivity

39 (Semi)hereditary modules and rings

1.Definitions. 2.Injective factor and projective submodules. 3.Semihereditary modules. 4.Characterization. 5.M semihereditary in $\sigma[M]$. 6.Hereditary modules. 7.Properties. 8.M hereditary in $\sigma[M]$. 9.Locally noetherian hereditary modules. 10.Left PP-endomorphism rings. 11.Right PP-endomorphism rings. 12.Modules with flat submodules. 13.Left semihereditary rings. 14.$End(M)$ with semihereditary M. 15.M hereditary in $\sigma[M]$. 16.Left hereditary rings. 17.Exercises.

In this paragraph we shall investigate modules and rings with certain submodules projective.

39.1 Definitions. Let M be an R-module. $P \in \sigma[M]$ is called
– *hereditary* in $\sigma[M]$ if every submodule of P is projective in $\sigma[M]$,
– *semihereditary* in $\sigma[M]$ if every finitely generated submodule of P is projective in $\sigma[M]$.

We call a ring R
– *left hereditary* if $_RR$ is hereditary in *R-MOD*,
– *left semihereditary* if $_RR$ is semihereditary in *R-MOD*.

Obviously, submodules of (semi) hereditary modules P are again (semi) hereditary in $\sigma[M]$. Of course, these properties of P depend on the surrounding category $\sigma[M]$. For example, every semisimple module P is hereditary in $\sigma[P]$ but need not be hereditary in *R-MOD*. Regular modules P in $\sigma[M]$

which are finitely generated and M-projective give examples of semihereditary modules in $\sigma[M]$ (see 37.3). On the other hand $\mathbb{Z}$, as a $\mathbb{Z}$-module, is (semi) hereditary but not regular.

For the investigation of these modules we need a technical lemma:

39.2 Injective factor and projective submodules.
Let P,Q be two R-modules.

(1) Consider the statements:

(i) Q is weakly P-injective, and every finitely generated submodule of $P^{(\mathbb{N})}$ is Q-projective,

(ii) every factor module of Q is weakly P-injective.

Then $(i) \Rightarrow (ii)$.

Moreover, if P is Q-projective, then $(ii) \Rightarrow (i)$ also holds.

(2) Consider the statements:

(i) Q is P-injective, and every submodule of P is Q-projective,

(ii) every factor module of Q is P-injective.

Then $(i) \Rightarrow (ii)$.

If P is Q-projective, then $(ii) \Rightarrow (i)$ also holds.

Proof: (1) Let L be a finitely generated submodule of $P^{(\mathbb{N})}$, V a factor module of Q and $f : L \to V$. We have the following exact diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{i} & P^{(\mathbb{N})} \\ & & \downarrow f & & \\ Q & \xrightarrow{p} & V & \longrightarrow & 0 \end{array} .$$

$(i) \Rightarrow (ii)$ Since L is Q-projective, there exists $g : L \to Q$, and then (since Q is weakly P-injective) an $h : P^{(\mathbb{N})} \to Q$ which complete the diagram commutatively. Hence $f = gp = i(hp)$, i.e. V is weakly P-injective.

$(ii) \Rightarrow (i)$ If P is Q-projective, this holds also for $P^{(\mathbb{N})}$ and the assertion can be derived from the same diagram.

(2) can be obtained by a slight modification of the proof of (1).

39.3 Properties of semihereditary modules.
Let M be an R-module and P a module in $\sigma[M]$.

(1) If $P = \bigoplus_\Lambda P_\lambda$, with modules P_λ semihereditary in $\sigma[M]$, then

(i) P is semihereditary in $\sigma[M]$;

(ii) every finitely generated submodule of P is isomorphic to a direct sum of submodules of the P_λ, $\lambda \in \Lambda$,

(iii) every projective module in $\sigma[P]$ is a direct sum of finitely generated submodules of the P_λ, $\lambda \in \Lambda$.

(2) If P is semihereditary in $\sigma[M]$, then

(i) every factor module of a weakly P-injective module in $\sigma[M]$ is weakly P-injective;

(ii) the finitely generated submodules of P form a generator set in $\sigma[P]$.

Proof: (1)(*i*) It is sufficient to show that the direct sum of two semihereditary modules P_1, P_2 in $\sigma[M]$ is again semihereditary:

Let K be a finitely generated submodule of $P_1 \oplus P_2$. With $K_1 = K \cap P_1$, we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & K & \longrightarrow & K/K_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus P_2 & \longrightarrow & P_2 & \longrightarrow & 0 \end{array}.$$

Being a finitely generated submodule of P_2, the module K/K_1 is M-projective and the first row splits. Then K_1 is also finitely generated, hence M-projective, and $K \simeq K_1 \oplus (K/K_1)$ is M-projective, too.

(*ii*) can be obtained by induction from the proof of (*i*).

(*iii*) Because of (*ii*), the finitely generated submodules of the P_λ form a generator set for $\sigma[P]$. A projective module $\widetilde{P}$ in $\sigma[P]$ is therefore a direct summand of a direct sum L of finitely generated submodules of suitable P_λ. Since these are semihereditary in $\sigma[M]$, the module L is also semihereditary in $\sigma[M]$ by (*i*).

Let $L = \widetilde{P} \oplus K$ with suitable $K \subset L$. By Kaplansky's Theorem 8.10, $\widetilde{P}$ is a direct sum of countably generated modules. Hence we may assume $\widetilde{P}$ to be countably generated. Then $\widetilde{P}$ is a direct sum of finitely generated modules if and only if every finitely generated submodule $U \subset \widetilde{P}$ is contained in a finitely generated direct summand of $\widetilde{P}$ (see 8.9). Now U is certainly contained in a finite partial sum L_1 of L, hence $U \subset \widetilde{P} \cap L_1$. We have

$$L_1/(\widetilde{P} \cap L_1) \simeq (\widetilde{P} + L_1)/\tilde{P} \subset L/\widetilde{P} \simeq K.$$

Therefore $L_1/(\widetilde{P} \cap L_1)$ is isomorphic to a finitely generated submodule of the semihereditary module $K \subset L$, and hence is M-projective. Then $\widetilde{P} \cap L_1$ is a finitely generated direct summand of L_1, thus of L and $\widetilde{P}$, and $\widetilde{P}$ is a direct sum of finitely generated modules.

Being a projective module in $\sigma[P]$, $\widetilde{P}$ is a submodule of a suitable sum $P^{(I)} \simeq \bigoplus_{\Lambda'} P_\lambda$. Hence $\widetilde{P}$ is a direct sum of finitely generated submodules of $P^{(I)}$. By (*ii*), these are direct sums of submodules of the P_λ.

(2)(*i*) Let Q be a weakly P-injective module in $\sigma[M]$. If P is semihereditary, then, by (1)(*i*), every finitely generated submodule of $P^{(\mathbb{N})}$ is

M-projective, and hence Q-projective too. Then, by 39.2,(1), every factor module of Q is weakly P-injective.

(ii) Finitely generated submodules of $P^{(\mathbb{N})}$ form a generator set in $\sigma[P]$. By (1)(ii), they are isomorphic to a direct sum of submodules of P.

39.4 Characterization of semihereditary modules in $\sigma[M]$.

For a projective module P in $\sigma[M]$, the following are equivalent:

(a) P is semihereditary in $\sigma[M]$;

(b) every projective module in $\sigma[P]$ is semihereditary in $\sigma[M]$;

(c) every factor module of a weakly P-injective module in $\sigma[M]$ is weakly P-injective;

(d) factor modules of the M-injective hull $\widehat{M}$ of M are weakly P-injective.

Proof: (a) $\Leftrightarrow$ (b) Every projective module in $\sigma[P]$ is a submodule of a direct sum $P^{(\Lambda)}$, and hence, by 39.3, semihereditary in $\sigma[M]$.

(a) $\Rightarrow$ (c) was shown in 39.3, (c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (a) By 39.2, we conclude from (d), that every finitely generated submodule of P is $\widehat{M}$-projective, and hence M-projective too.

39.5 M semihereditary in $\sigma[M]$. Characterizations.

If M is projective in $\sigma[M]$, then the following are equivalent:

(a) M is semihereditary in $\sigma[M]$;

(b) every projective module in $\sigma[M]$ is semihereditary in $\sigma[M]$;

(c) every factor module of a weakly M-injective module in $\sigma[M]$ is weakly M-injective;

(d) factor modules of the M-injective hull $\widehat{M}$ of M are weakly M-injective.

If M is finitely generated, then (a) - (d) are equivalent to:

(e) every factor module of an absolutely pure module is absolutely pure in $\sigma[M]$.

Proof: The first equivalences follow from 39.4.

(c) $\Leftrightarrow$ (e) is clear since in this case 'absolutely pure' and 'weakly M-injective' are the same properties (see 35.4).

39.6 Characterization of hereditary modules.

For a projective module P in $\sigma[M]$, the following are equivalent:

(a) P is hereditary in $\sigma[M]$;

(b) every factor module of a P-injective module in $\sigma[M]$ is P-injective;

(c) $P^{(\Lambda)}$ is hereditary in $\sigma[M]$, for every index set Λ;

(d) every projective module in $\sigma[P]$ is hereditary in $\sigma[M]$.

Proof: $(a) \Leftrightarrow (b)$ follows from 39.2,(2) by observing that every module in $\sigma[M]$ is a submodule of an injective module.

$(c) \Leftrightarrow (d)$ is clear since the projective modules in $\sigma[P]$ are submodules of direct sums $P^{(\Lambda)}$.

$(c) \Rightarrow (a)$ is clear and $(a) \Rightarrow (c)$ follows from the following

39.7 Properties of hereditary modules.

Let $M \in R$-MOD and $\{P_\lambda\}_\Lambda$ be a family of hereditary modules in $\sigma[M]$. Then:

(1) $P = \bigoplus_\Lambda P_\lambda$ is hereditary in $\sigma[M]$.

(2) Every submodule of P is isomorphic to a direct sum of submodules of the P_λ, $\lambda \in \Lambda$.

Proof: (1) follows from (2) but can also be proved directly: Let Q be a P-injective module in $\sigma[M]$. Then Q is P_λ-injective for all $\lambda \in \Lambda$, and, by 39.6, every factor module V of Q is P_λ-injective, too. By 16.2, V is also P-injective and P is hereditary because of 39.6.

(2) Here we need the Well Ordering Principle: Let a well ordering $\leq$ on the index set Λ be given. We define

$$Q_\lambda = \bigoplus_{\mu<\lambda} P_\mu, \quad \overline{Q}_\lambda = \bigoplus_{\mu\leq\lambda} P_\mu .$$

For every submodule $K \subset P$ and the restriction π'_λ of the canonical projection $\pi_\lambda : P \to P_\lambda$, we have the exact sequence

$$0 \longrightarrow K \cap Q_\lambda \longrightarrow K \cap \overline{Q}_\lambda \xrightarrow{\pi'_\lambda} P_\lambda \,.$$

This sequence splits since, by assumption, $Im\ \pi'_\lambda \subset P_\lambda$ is projective, i.e.

$$K \cap \overline{Q}_\lambda = (K \cap Q_\lambda) \oplus N_\lambda \ \text{ with } \ N_\lambda \simeq Im\ \pi'_\lambda \,.$$

We show $K = \bigoplus_\Lambda N_\lambda$. It is clear that the $\{N_\lambda\}_\Lambda$ form an independent family of submodules of K. Assume $K \neq \bigoplus_\Lambda N_\lambda$.

For every $k \in K$, there is a smallest index $\rho(k) \in \Lambda$ with $k \in \overline{Q}_{\rho(k)}$. The set $\{\rho(k) \mid k \in K,\ k \notin \bigoplus_\Lambda N_\lambda\} \subset \Lambda$ is not empty and therefore contains a smallest element ρ^*.

Now choose $k \in K$ with $\rho(k) = \rho^*$, $k \notin \bigoplus_\Lambda N_\lambda$. We have $k \in K \cap \overline{Q}_{\rho^*}$, hence $k = k_{\rho^*} + n_{\rho^*}$ with $k_{\rho^*} \in K \cap Q_{\rho^*}$ and $n_{\rho^*} \in N_{\rho^*}$. This means $k_{\rho^*} = k - n_{\rho^*} \in K$ and $k_{\rho^*} \notin \bigoplus_\Lambda N_\lambda$ since $k \notin \bigoplus_\Lambda N_\lambda$. Therefore $\rho(k_{\rho^*}) < \rho^*$, a contradiction to the choice of ρ^*. Hence $K = \bigoplus_\Lambda N_\lambda$.

39.8 M **hereditary in** $\sigma[M]$**. Characterizations.**

If M is projective in $\sigma[M]$, then the following are equivalent:

(a) M is hereditary in $\sigma[M]$;

(b) every finitely generated (cyclic) submodule of M is hereditary in $\sigma[M]$;

(c) every projective module in $\sigma[M]$ is hereditary in $\sigma[M]$;

(d) every factor module of an M-injective module in $\sigma[M]$ is M-injective.

Proof: $(b) \Rightarrow (a)$ M is generated by its cyclic submodules and hence – by projectivity – is isomorphic to a direct summand of a direct sum of its cyclic submodules. These are hereditary by 39.7, and therefore M is also hereditary in $\sigma[M]$.

The remaining implications result from 39.6.

Of course, for noetherian modules 'hereditary' and 'semihereditary' are identical properties. Somewhat more generally we obtain:

39.9 Locally noetherian hereditary modules.

Let M be a locally noetherian R-module and $\widehat{M}$ its M-injective hull. If M is projective in $\sigma[M]$, then the following statements are equivalent:

(a) M is hereditary in $\sigma[M]$;

(b) M is semihereditary in $\sigma[M]$;

(c) every factor module of $\widehat{M}$ is M-injective;

(d) every factor module of an indecomposable M-injective module in $\sigma[M]$ is M-injective.

Proof: $(a) \Rightarrow (b)$ is clear.

$(b) \Leftrightarrow (c)$ and $(b) \Rightarrow (d)$ follow from 39.5, since for locally noetherian modules M, 'weakly M-injective' and 'M-injective' are equivalent.

$(b) \Rightarrow (a)$ Every finitely generated submodule of M is noetherian, and hence hereditary in $\sigma[M]$. Thus the assertion follows from 39.8,(b).

$(d) \Rightarrow (b)$ Let U be an indecomposable, M-injective module in $\sigma[M]$. By 39.2, we conclude from (d) that every finitely generated submodule K of M is U-projective. Now Matlis' Theorem 27.4 tells us that, under the given assumptions, every injective module Q is a direct sum of indecomposable modules $\{U_\lambda\}_\Lambda$. K is U_λ-projective and, by 18.2, Q-projective. Hence K is projective in $\sigma[M]$.

We call a ring R a *left PP-ring* if every cyclic left ideal of R is projective (*p*rincipal ideals *p*rojective). This is equivalent to the fact that, for every $a \in R$, the map $\varphi : R \to Ra$, $r \mapsto ra$, splits, i.e. $Ke\, \varphi = An_R(a)$ is a direct summand in R and is hence generated by an idempotent.

Right PP-rings are defined and characterized in a similar way.

These notions are of interest for the investigation of endomorphism rings of (semi) hereditary modules. The assertions in 37.7 about regular endomorphism rings can now be extended in the following way:

39.10 Left PP-endomorphism rings.
Let M be an R-module, $S = End_R(M)$ and $f \in S$.

(1) If $Ke\ f$ is a direct summand in M, then the left ideal $Sf \subset S$ is projective in S-MOD.

(2) If Sf is projective, then $Tr(M, Ke\ f)$ is a direct summand in M.

(3) If every M-cyclic submodule of M is M-projective, then S is a left PP-ring.

(4) If M is a self-generator or M_S is flat, and if S is a left PP-ring, then, for every $f \in S$, the kernel $Ke\ f$ is a direct summand in M.

Proof: (1) If $Ke\ f$ is a direct summand in M, then there is an idempotent $e \in S$ with $Ke\ f = Me$. For the S-homomorphism $\varphi : S \to Sf$, $s \mapsto sf$, we have $Ke\ \varphi = Se$: First, $ef = 0$ implies $Se \subset Ke\ \varphi$.

On the other hand, for $t \in Ke\ \varphi$, $Mtf = 0$ always holds, implying $Mt \subset Ke\ f = Me$ and $t = te \in Se$.

Therefore $Ke\ \varphi$ is a direct summand in S and Sf is projective.

(2) Let Sf be projective and e an idempotent in S which generates the kernel of $\varphi : S \to Sf$, $s \mapsto sf$. We show $Tr(M, Kef) = Me$.

From $Hom(M, Kef) \subset Ke\ \varphi = Se$ we get

$$Tr(M, Kef) = M Hom(M, Ke\ f) \subset Me.$$

Since $ef = 0$, we conclude $Me \subset Tr(M, Ke\ f)$.

(3) and (4) are immediate consequences of (1) resp. (2) (see 15.9).

39.11 Right PP-endomorphism rings.
Let M be an R-module, $S = End_R(M)$ and $f \in S$.

(1) If $Im\ f$ is a direct summand in M, then the right ideal $fS \subset S$ is projective in MOD-S.

(2) If fS is projective, then the submodule $K = \bigcap\{Ke\ g \mid g \in S,\ Im\ f \subset Ke\ g\}$ is a direct summand in M.

(3) If every M-cyclic submodule of M is M-injective, or M is finitely generated and every M-cyclic submodule of M is weakly M-injective, then S is a right PP-ring.

(4) If M is a self-cogenerator and S a right PP-ring, then, for every $f \in S$, the image $Im\ f$ is a direct summand.

Proof: (1) If $Im\ f$ is a direct summand, then there is an idempotent $e \in S$ with $Im\ f = Mf = Me$. For the S-homomorphism $\psi : S \to fS$, $s \mapsto fs$, we have $Ke\ \psi = (1-e)S$:

By $Mf(1-e) = Me(1-e) = 0$, we have $(1-e)S \subset Ke\ \psi$.
On the other hand, for $u \in Ke\ \psi$, $Meu = Mfu = 0$ always holds, thus $eu = 0$ and hence $u \in (1-e)S$.

(2) Let fS be projective and e an idempotent in S which generates the kernel of $\psi : S \to fS$, $s \mapsto fs$. We show $M(1-e) = K$.

Since $fe = 0$, we have, of course, $K \subset Ke\ e = M(1-e)$.
On the other hand, for $g \in S$ with $Im\ f \subset Ke\ g$, we have $g \in eS$, and hence $M(1-e) = Ke\ e \subset Ke\ g$. Thus $M(1-e) \subset K$.

(3) follows directly from (1).

(4) If $M/Im\ f$ is cogenerated by M, then we have – with the notation of (2) – $Im\ f = K$.

In a semihereditary module in $\sigma[M]$, every submodule is a direct limit of projective modules and hence is flat in $\sigma[M]$. This property characterizes the following class of modules:

39.12 Modules and rings with flat submodules.
(1) For an R-module M, the following assertions are equivalent:

(a) Every (finitely generated) submodule of M is flat in $\sigma[M]$;
(b) M is flat in $\sigma[M]$ and every submodule of a flat module in $\sigma[M]$ is flat in $\sigma[M]$.

(2) For a ring R the following assertions are equivalent:

(a) Every (finitely generated) left ideal of R is flat (in R-MOD);
(b) every (finitely generated) right ideal of R is flat (in MOD-R);
(c) in R-MOD the submodules of flat modules are flat;
(d) in MOD-R the submodules of flat modules are flat.

Proof: (1) If the finitely generated submodules of a module are flat, then every submodule is a direct limit of flat modules and hence is flat in $\sigma[M]$.

$(b) \Rightarrow (a)$ is clear.

$(a) \Rightarrow (b)$ If the submodules of L, N in $\sigma[M]$ are flat in $\sigma[M]$, then this also holds for $L \oplus N$: Assume $K \subset L \oplus N$. Then $K \cap L \subset L$ and $K/(K \cap L) \subset N$ are flat modules, and, by 36.1, K is also flat in $\sigma[M]$.

By induction, we see that all finitely generated submodules of a direct sum $M^{(\Lambda)}$ are flat in $\sigma[M]$.

Now let V be a flat module in $\sigma[M]$ and $U \subset V$. Then there is a (flat) submodule $P \subset M^{(\Lambda)}$, with suitable Λ, and an epimorphism $p : P \to V$. Forming a pullback, we obtain the exact commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Ke\,p & \longrightarrow & Q & \longrightarrow & U & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Ke\,p & \longrightarrow & P & \xrightarrow{p} & V & \longrightarrow & 0 \quad . \end{array}$$

Here Q is a submodule of $P \subset M^{(\Lambda)}$ and hence is flat. By 33.4, the first row is pure in $\sigma[M]$. From 36.1, we know that U is also flat in $\sigma[M]$.

(2) $(a) \Leftrightarrow (c)$ and $(b) \Leftrightarrow (d)$ follow from (1).

$(a) \Leftrightarrow (b)$ A right ideal $K \subset R$ is flat (in *MOD-R*) if and only if, for every left ideal $L \subset R$, the canonical map $K \otimes_R L \to KL$ is injective (see 12.16 and 36.5).

Flat left ideals are similarly characterized.

For semihereditary rings we can combine these results to obtain:

39.13 Left semihereditary rings. Characterizations.

(1) For a ring R the following statements are equivalent:

(a) R is left semihereditary;

(b) every projective module in R-MOD is semihereditary;

(c) every factor module of an FP-injective module is FP-injective;

(d) every factor module of the injective hull $_R\widehat{R}$ *of R is FP-injective;*

(e) for every $n \in \mathbb{N}$*, the matrix ring* $R^{(n,n)}$ *is a left PP-ring;*

(f) for (every) $n \in \mathbb{N}$*, the matrix ring* $R^{(n,n)}$ *is left semihereditary;*

(g) every module in MOD-R which is cogenerated by R_R *is flat.*

(2) If R is left semihereditary, then every projective module in R-MOD is isomorphic to a direct sum of finitely generated left ideals of R.

Proof: (1) The equivalences of (a) to (d) follow from 39.4 ('FP-injective' is 'absoluty pure' in *R-MOD*).

$(a) \Rightarrow (e)$ With R left semihereditary, every sum R^n is also semihereditary and, by 39.10, $End(R^n) \simeq R^{(n,n)}$ is a left PP-ring.

$(e) \Rightarrow (a)$ Let K be a left ideal in R generated by k elements. Then there is a homomorphism $f : R^k \to R$ with $Im\ f = K$. Since $R \subset R^k$ the map f can be regarded as an element of $End(R^k)$. Given (e) we have, by 39.10, that $Ke\ f$ is a direct summand in R^k. Hence $K = Im\ f$ is projective and R is semihereditary.

$(a) \Rightarrow (f)$ Let $P = R^n$ for some $n \in I\!N$. If (a) holds, then P and P^k are semihereditary for every $k \in I\!N$, and $End(P^k) \simeq End(P)^{(k,k)}$ is a left PP-ring by 39.10. Because of the equivalence $(a) \Leftrightarrow (e)$ already shown, $End(P)$ is left semihereditary.

$(f) \Rightarrow (a)$ can be shown as $(e) \Rightarrow (a)$, observing the fact that R^n is a generator in *R-MOD* (see also 39.13,(2)).

$(a) \Rightarrow (g)$ Because R is a left semihereditary ring, ${}_RR$ is coherent. Hence, by 26.6, every product R_R^Λ, Λ an index set, is flat in *MOD-R*. By 39.12, submodules of flat modules are flat (in *MOD-R*). This proves (g).

$(g) \Rightarrow (a)$ Since all products R_R^Λ are flat, ${}_RR$ is coherent (see 26.6). Since right ideals of R are flat, left ideals of R are also flat (see 39.12). Thus the finitely generated left ideals are finitely presented (hence pure projective) and flat, i.e. they are projective.

(2) This assertion follows from 39.3.

39.14 Endomorphism rings of semihereditary modules.
Let M be a finitely generated R-module and $S = End_R(M)$.

(1) Assume M to be semihereditary in $\sigma[M]$. Then
(i) S is left semihereditary;
(ii) if M is weakly M-injective, then S is (von Neumann) regular.

(2) If M is a self-generator and M-projective, and if S is left semihereditary, then M is semihereditary in $\sigma[M]$.

Proof: (1)(i) Since M^n is semihereditary in $\sigma[M]$ for every $n \in I\!N$, by 39.10, $End(M^n) \simeq S^{(n,n)}$ is a left PP-ring. Then, by 39.13, S is left semihereditary.

(ii) By 39.5, every factor module of M^n is also weakly M-injective. From 39.11 we get that $S^{(n,n)}$ is a right PP-ring. Similarly to 39.13 we conclude now that S is right semihereditary. Hence, for $f \in S$, the image $Im\ f$ and the kernel $Ke\ f$ are direct summands, i.e. S is regular (see 37.7).

(2) For every finitely generated submodule $K \subset M$ there is a homomorphism $f : M^n \to M$ with $Im\ f = K$. Regarding f as an endomorphism of M^n we see that $Ke\ f$ is a direct summand of M^n by 39.10. Hence K is projective.

From the preceding results we obtain

39.15 Further characterization of M hereditary in $\sigma[M]$.

Assume M is projective in $\sigma[M]$. Then the following are equivalent:

(a) M is hereditary in $\sigma[M]$;

(b) for every injective (cogenerator) module Q in $\sigma[M]$ the ring $End_R(Q)$ is right semihereditary (a right PP-ring).

Proof: $(a) \Rightarrow (b)$ If Q is an injective cogenerator in $\sigma[M]$, the same is true for Q^k, $k \in I\!N$. Since every factor module of an M-injective module is again M-injective (see 39.8), the assertion follows from 39.11 and 39.13.

$(b) \Rightarrow (a)$ By 39.8, it is sufficient to show that every factor module of an injective module $Q \in \sigma[M]$ is again injective: Let $f : Q \to V$ be an epimorphism and Q' an injective cogenerator in $\sigma[M]$ which contains V. With the canonical projections, resp. injections, we have the following endomorphism of the injective cogenerator $Q \oplus Q'$

$$u \,:\, Q \oplus Q' \to Q \xrightarrow{f} V \to Q' \to Q \oplus Q'.$$

By definition $Im\ u \simeq Im\ f = V$, and, by 39.11, $Im\ u$ is a direct summand in $Q \oplus Q'$. Hence V is injective in $\sigma[M]$.

Finally we sum up for hereditary rings:

39.16 Left hereditary rings. Characterizations.

For a ring R the following statements are equivalent:

(a) R is left hereditary;

(b) every projective module in R-MOD is hereditary;

(c) every factor module of an injective module in R-MOD is injective;

(d) for every free (projective) $P \in R$-MOD, $End_R(P)$ is a left PP-ring;

(e) for every finitely generated free (projective) module $P \in R$-MOD, $End_R(P)$ is a left hereditary ring;

(f) for every injective (cogenerator) module Q in R-MOD, $End_R(Q)$ is right semihereditary.

If ${}_RR$ is noetherian, then (a) - (f) are equivalent to:

(g) Every factor module of the injective hull ${}_R\widehat{R}$ of R is injective;

(h) every factor module of an indecomposable injective R-module is again injective.

Proof: The equivalences between (a), (b), (c), (g) and (h) result from 39.8 and 39.9.

$(b) \Rightarrow (d)$ follows from 39.10.

$(d) \Rightarrow (a)$ For every left ideal $K \subset R$, there is a free R-module P and a homomorphism $f : P \to R$ with $Im\ f = K$. Regarding R as a submodule of

P, the map f is an element of the left PP-ring $End(P)$. By 39.10, $Ke\ f$ is a direct summand in P, and hence K is projective.

$(a) \Rightarrow (e)$ is similar to $(a) \Rightarrow (f)$ in 39.13.

$(a) \Rightarrow (f)$ follows from 39.15.

Important examples of hereditary rings are (upper) triangular matrix rings over a field. They are right and left hereditary.

In general 'hereditary' is not a left-right symmetric property. For example, the matrix ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right hereditary but not left hereditary.

39.17 Exercises.

(1) Prove that for a ring R the following statements are equivalent:

(a) Every left ideal of R is flat;

(b) if N in R-MOD is pure injective with injective hull $\widehat{N}$, then $\widehat{N}/N$ is also injective.

(2) Prove: For a left artinian ring R the following are equivalent:

(a) Every factor ring of R is left hereditary;

(b) every submodule of a self-projective R-module is self-projective;

(c) every factor module of a self-injective R-module is self-injective;

(d) R is left hereditary and $Jac(R)^2 = 0$;

(e) the simple R-modules are injective or projective in R-MOD.

(3) An R-module M has SSP *(Summand Sum Property) if the sum of two direct summands of M is again a direct summand of M.*

Prove (see Garcia):

(i) If $N \oplus L$ is a module with SSP and $f \in Hom_R(L, N)$, then $Im\ f$ is a direct summand of N.

(ii) $_RR$ is semisimple if and only if every projective module in R-MOD has SSP.

(iii) $_RR$ is left hereditary if and only if every injective module in R-MOD has SSP.

(iv) $_RR$ is von Neumann regular if and only if every finitely generated, projective R-module has SSP.

(4) An R-module N is called p-injective *if $Hom_R(-, N)$ is exact relative to exact sequences $0 \to I \to {}_RR$ with cyclic (principal) left ideals $I \subset R$.*

Prove:

(i) The following statements are equivalent:

(a) R is a left PP-ring;

(b) every factor module of a p-injective module is p-injective;

(c) every factor module of an injective R-module is p-injective.

(ii) If every simple R-module is p-injective, then R is left fully idempotent.

(5) Prove that in $\sigma[\mathbb{Q}/\mathbb{Z}]$ (= the category of $\mathbb{Z}$-torsion modules) the factor modules of injective modules are injective (but there are no projective modules in $\sigma[\mathbb{Q}/\mathbb{Z}]$, see 18.12).

*(6) Show that every $\mathbb{Z}$-module M has a decomposition $M = D \oplus C$, with D an injective (divisible) $\mathbb{Z}$-module and C a $\mathbb{Z}$-module not containing a non-zero injective submodule (*a reduced *module, e.g. FUCHS)*

Literature: CHATTERS-HAJARNAVIS, FUCHS; Bergman, Boyle, Chatters, Couchot [5], Faith [2], Fontana, Fuelberth-Kuzmanovich, Fuller [1], Garcia, Garcia-Gomez [5], Grigorjan, Harada [2,3], Hirano-Hongan, Hill [2,3,4,5], Jain-Singh,S. [2], Kosler, Lenzing [2], Miller-Turnidge [2], Page [3], Raynaud-Gruson, Shannon, Shrikhande, Singh [1,2], Smith [1], Szeto [1], Talwar, Tuganbaev [10,11], Wisbauer [3], Yue [2].

40 Semihereditary and hereditary domains

1.Projective ideals. 2.Factor modules by projective ideals. 3.Finitely generated torsion free modules. 4.Prüfer rings. 5.Dedekind rings. 6.Properties. 7.Exercises.

Semihereditary and hereditary rings occured first in number theoretical investigations. The rings studied there are subrings of fields, i.e. commutative and without zero divisors ((integral) domains). In this section we want to derive some assertions about this class of rings (Prüfer and Dedekind rings) which are accessible by our methods.

An important observation is that in integral domains, projective ideals can be characterized by their behavior in the related quotient field:

40.1 Projective ideals in integral domains.

Let R be an integral domain with quotient field Q. For a non-zero ideal $I \subset R$ the following assertions are equivalent:

(a) I is projective (in R-MOD);

(b) there are elements $a_1, \ldots, a_n \in I$, $q_1, \ldots, q_n \in Q$, with $Iq_i \subset R$ for every $i \le n$ and $\sum_{i\le n} a_i q_i = 1$.

In this case, $\sum_{i\le n} Iq_i = R$ and $\sum_{i\le n} Ra_i = I$.

I is also called an *invertible* ideal. It is finitely generated.

Proof: $(a) \Rightarrow (b)$ If I is R-projective, then there is a dual basis (see 18.6), i.e. there are elements $\{a_\lambda \in I\}_\Lambda$ and $\{f_\lambda \in Hom_R(I,R)\}_\Lambda$ with the properties: For every $a \in I$

(i) $(a)f_\lambda \neq 0$ only for finitely many $\lambda \in \Lambda$, and

(ii) $a = \sum (a) f_\lambda a_\lambda$.

For non-zero b, b' in I, $(bb')f_\lambda = b(b')f_\lambda = b'(b)f_\lambda$ and we obtain in Q

$$\frac{(b)f_\lambda}{b} = \frac{(b')f_\lambda}{b'}.$$

Setting $q_\lambda = \frac{(b)f_\lambda}{b} \in Q$, $0 \neq b \in I$, we know from (i) that only finitely many of the q_λ are non-zero, let's say $q_1, \ldots, q_n$.

For every $0 \neq b \in I$, we observe $bq_i = b\frac{(b)f_i}{b} = (b)f_i \in R$. Thus $Iq_i \subset R$, and from (ii) we obtain

$$b = \sum\nolimits_{i\le n} (b) f_i a_i = \sum\nolimits_{i\le n} bq_i a_i = b(\sum\nolimits_{i\le n} q_i a_i).$$

Cancelling b, we have $1 = \sum_{i\le n} a_i q_i$.

$(b) \Rightarrow (a)$ Assume I satisfies the conditions given in (b). We define $f_i : I \to R$ by $a \mapsto aq_i \in R$. Then for every $a \in I$

$$\sum_{i\leq n}(a)f_i a_i = \sum_{i\leq n} aq_i a_i = a\sum_{i\leq n} a_i q_i = a,$$

i.e. $\{a_i\}$ and $\{f_i\}$ form a dual basis and I is projective.

A module M over an integral domain R is said to be *divisible* if $rM = M$ for all non-zero $r \in R$ (see 16.6).

We call an R-module ${}_RM$ *cyclically presented*, if it is isomorphic to a factor module of R by a cyclic left ideal, i.e. $M \simeq R/Rr$, $r \in R$.

The characterization of projective ideals in integral domains given above admits the following description of

40.2 Factor modules by projective ideals.
Let I be a projective ideal in an integral domain R. Then:

(1) The factor module R/I is a direct summand of a direct sum of cyclically presented R-modules.

(2) Every divisible R-module N is injective relative to $0 \to I \to R$, i.e. $Hom(R, N) \to Hom(I, N) \to 0$ is exact.

Proof: (1) Let $\mathcal{P}_c$ be the class of cyclically presented R-modules. We show that R/I is $\mathcal{P}_c$-pure projective:
Assume $0 \to A \to B \to C \to 0$ to be a $\mathcal{P}_c$-pure exact sequence and let $g : R/I \to C$ be a homomorphism. In a canonical way we construct the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow & & \downarrow g & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array} .$$

For every $a \in I$, the factor module R/Ra is projective relative to the lower sequence. Hence there exists $\eta : R \to A$ (Homotopy Lemma) with $\eta|_{Ra} = h|_{Ra}$, and we find an element $d_a \in A$ with $(b)h = bd_a$ for all $b \in Ra$.

By 40.1, for the projective ideal I, there exist elements $a_1, \ldots, a_n \in I$ and $q_1, \ldots, q_n$ in the quotient field Q of R with $Iq_i \subset R$ and $\sum a_i q_i = 1$. For these a_i choose $d_{a_i} \in A$ with the properties noted above and set $d = \sum a_i d_{a_i} q_i$. Then we have, for every $b \in I$,

$$(b)h = (\sum a_i q_i b)h = \sum q_i b(a_i)h = \sum q_i b a_i d_{a_i} = bd\,.$$

Hence $h : I \to A$ can be extended to $R \to A$, $r \mapsto rd$. By the Homotopy Lemma we obtain a morphism $R/I \to B$ completing the above diagram in the desired way.

Now apply 33.6 for the class $\mathcal{P}_c$.

(2) For a divisible module N and $0 \neq r \in R$, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & Rr & \longrightarrow & R \\ & & \downarrow f & & \\ & & N & & \end{array}$$

can be extended commutatively by a morphism $R \to N$ (there is an $n \in R$ with $(a)h = rn$). Now the assertion can also be seen from the above proof.

A module N over an integral domain R is called *torsion free* if $rn \neq 0$ for all non-zero elements $r \in R$ and $n \in N$.

For the following proofs we need:

40.3 Finitely generated torsion free modules.

Assume R to be an integral domain with quotient field Q. Then every finitely generated, torsion free R-module is isomorphic to a submodule of a finite direct sum R^n, $n \in \mathbb{N}$.

Proof: Let N be a finitely generated, torsion free R-module. Then the injective hull $\widehat{N}$ of N is also torsion free (the elements $m \in \widehat{N}$ with $rm = 0$ for some $r \in R$ form a submodule). $\widehat{N}$ is divisible (see 16.6) and hence can be turned into a Q-vector space: For $\frac{r}{s} \in Q$ and $n \in \widehat{N}$ we choose $n' \in \widehat{N}$ with $n = sn'$ and define $\frac{r}{s}n = rn'$.

The generating elements $n_1, \ldots, n_k$ of N are contained in a finite dimensional Q-subspace of $\widehat{N}$. Let $v_1, \ldots, v_t$ be a Q-basis of $\widehat{N}$. Then

$$n_i = \sum\nolimits_j q_{ji} v_j \text{ with } q_{ji} \in Q$$

for every $i \leq k$. Choosing $s \in R$ with $q'_{ji} = sq_{ji} \in R$ we see that

$$n_i = \sum\nolimits_j q'_{ij} \frac{1}{s} v_j \in \sum\nolimits_j R \frac{1}{s} v_j \simeq R^t.$$

A semihereditary integral domain is called a *Prüfer ring.* Our techniques allow us to prove the following

40.4 Characterization of Prüfer rings.

For an integral domain R, the following properties are equivalent:

(a) *R is a Prüfer ring;*

(b) *every divisible R-module is FP-injective (absolutely pure);*

(c) *every ideal in R is flat;*

(d) *every finitely generated torsion free R-module is projective (flat);*

(e) *every torsion free R-module is flat;*

(f) *the tensor product of two torsion free R-modules is torsion free;*

(g) *the tensor product of two ideals of R is torsion free;*

(h) *every finitely presented cyclic R-module is a direct summand of a direct sum of cyclically presented modules.*

Proof: $(a) \Rightarrow (b)$ By 40.2, over a semihereditary ring R every divisible module M is injective relative to $0 \to I \to R$ with I finitely generated. Since R is coherent we see, from 26.8, that M is FP-injective.

$(b) \Rightarrow (a)$ Factor modules of divisible (injective) modules are divisible. Hence we derive from (b) that the factor modules of injective modules are FP-injective. Then R is semihereditary by 39.13.

$(a) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (e)$ By 40.3, every finitely generated, torsion free R-module is a submodule of R^n, $n \in \mathbb{N}$. If the ideals of R are flat, then, by 39.12, all these modules are flat. Hence every torsion free module being a direct limit of (finitely generated) flat modules is also flat.

$(a) \Rightarrow (d)$ Since every free module is semihereditary (by 39.3,(1)), the assertion follows again from 40.3.

$(d) \Rightarrow (c)$ is clear.

$(e) \Rightarrow (a)$ Of course, over an integral domain every module cogenerated by R is torsion free. Hence the implication follows from 39.13.

$(e) \Rightarrow (f)$, (g) The tensor product of two flat modules is again flat (see 36.5) and hence torsion free.

$(g) \Rightarrow (c)$ According to 12.16 we have to prove that, for each pair of ideals I, J of R, the map $\mu : I \otimes_R J \to J$, $a \otimes b \mapsto ab$, is injective.

For $u = \sum a_i \otimes b_i \in I \otimes J$ and $0 \neq c \in I$ we have

$$cu = \sum ca_i \otimes b_i = c \otimes \sum a_i b_i = c \otimes \mu(u).$$

Hence from $\mu(u) = 0$ we get $cu = 0$, and therefore $u = 0$ since $I \otimes_R J$ is torsion free.

$(a) \Rightarrow (h)$ We see from 40.2 that the factor modules R/I, with finitely generated ideals $I \subset R$, are direct summands of direct sums of cyclically presented modules.

$(h) \Rightarrow (e)$ Let N be a torsion free R-module and consider an exact sequence $0 \to K \to F \to N \to 0$ with F a free R-module. Then K is a relatively divisible submodule of F:

For non-zero $r \in R$ assume $k \in rF \cap K$. Then $k = rf$ for some $f \in F$ and $r(f + K) = K$. Since F/K is torsion free this means $f \in K$. Hence $rF \cap K = rK$.

According to 34.8, $R/Rr \otimes_R -$ is exact with respect to the above sequence. Because of (h), $R/I \otimes_R -$ is also exact with respect to this sequence if $I \subset R$ is a finitely generated ideal. Now we conclude from 36.6 that N is a flat R-module.

Remark: It can be shown that over a Prüfer ring every finitely presented module is a direct summand of a direct sum of cyclically presented modules.

A hereditary integral domain is called a *Dedekind ring*.

40.5 Characterization of Dedekind rings.

For an integral domain R, the following statements are equivalent:

(a) R is a Dedekind ring;

(b) R is noetherian and a Prüfer ring;

(c) every divisible R-module is injective;

(d) every cyclic R-module is a direct summand of a direct sum of cyclically presented modules.

Proof: $(a) \Leftrightarrow (b)$ By 40.1, in an integral domain every projective ideal is finitely generated.

$(b) \Leftrightarrow (c)$ The FP-injective modules are injective if and only if R is noetherian. Hence the assertion follows from 40.4.

$(b) \Leftrightarrow (d)$ The cyclic R-modules are finitely presented if and only if R is noetherian. The rest follows again from 40.4.

Finally we want to display some results about Dedekind rings demonstrating the importance of these rings in number theory:

40.6 Properties of Dedekind rings.

Let R be a Dedekind ring. Then:

(1) Every non-zero prime ideal in R is maximal.

(2) Every ideal in R is a product of prime ideals.

(3) For every ideal $I \neq R$, we have $R/I \simeq \prod_{i\leq n} R/P_i^{k_i}$ with P_i distinct prime ideals in R, $k_i \in I\!N$;

(4) For every prime ideal $P \subset R$ and $k \in I\!N$, there is a unique composition series $0 \subset P^{k-1}/P^k \subset \cdots \subset P/P^k \subset R/P^k$ in R/P^k.

Proof: (1) Let Q be the quotient field of R, I a non-zero prime ideal, and M a maximal ideal with $I \subset M \subset R$. Choose $a_1, \dots, a_n \in M$, $q_1, \dots, q_n \in Q$ with $Mq_i \subset R$ and $\sum a_i q_i = 1$ (see 40.1).

Then for $M' = \sum Rq_i$, we have $M'M = R$ and hence $(IM')M = IR = I$. From this we obtain (notice $IM' \subset R$) $IM' \subset I$ or $M \subset I$. Multiplying with M, fhe first inequality yields $I = IM$. Since I is a finitely generated R-module this contradicts an observation in 18.9. Thus $M = I$ and I is maximal.

(2) Assume the set of ideals, which cannot be represented as a product of maximal ideals, to be non-empty. Since R is noetherian, there is a maximal element J in this set (J need not be a maximal ideal). Let M be a maximal ideal in R with $J \subset M$.

With the same notation as in (1), we see $J \subset M'J$ (since $R \subset M'$) and $J \neq M'J$ (otherwise $JM = J$). Thus the ideal $M'J \subset R$ is properly larger than J and hence representable as a product of maximal ideals in R. Then this also holds for $J = M(M'J)$, a contradiction to the choice of J.

(3) By (2), $I = P_1^{k_1} \dots P_n^{k_n}$ with P_i different prime ideals and $k_i \in I\!N$. By (1), $P_i + P_j = R$ for each $i \neq j$. Hence $P_1^{k_1} + P_j^{k_j} = R$ for $j > 1$ and

$$R = P_1^{k_1} + P_2^{k_2} \dots P_n^{k_n}.$$

For ideals A, B in R with $A + B = R$ we always have

$$A \cap B = (A \cap B)(A + B) \subset AB, \text{ i.e. } A \cap B = AB.$$

From the relation above we now derive

$$P_1^{k_1} \cap (P_2^{k_2} \cdots P_n^{k_n}) = P_1^{k_1} \cdots P_n^{k_n} \text{ and}$$

$$P_1^{k_1} \cap \cdots \cap P_n^{k_n} = P_1^{k_1} \cdots P_n^{k_n}.$$

The Chinese Remainder Theorem 9.13 then implies $R/I \simeq \prod_{i\leq n} R/P_i^{k_i}$.

(4) Any ideal in R/P^k can be written as I/P^k for some ideal $I \subset R$ with $P^k \subset I$. Assume in the representation of I by (2) there is a prime ideal $Q \neq P$. Then $P^k \subset Q$ holds and hence $R = P^k + Q \subset Q$, a contradiction.

Remark: By using divisibility in rings it can also be shown:

An integral domain, in which every ideal is a product of prime ideals, is a Dedekind ring.

40.7 Exercises

(1) Prove that, for an integral domain R, the following statements are equivalent:
(a) R is a Prüfer ring;
(b) every relatively divisible submodule of an R-module is a pure submodule;
(c) every pure injective R-module is injective relative to RD-pure sequences (see 34.8).

(2) Let R be a Dedekind ring. Prove that, for any exact sequence

$$(*) \quad 0 \to K \to L \to N \to 0$$

in R-MOD, the following statements are equivalent:
(a) $Hom(P, -)$ is exact relative to $(*)$ for finitely presented R-modules (i.e. $(*)$ is pure in R-MOD);
(b) $Hom(-, Q)$ is exact relative to $(*)$ for finitely cogenerated R-modules Q;
(c) $0 \to IK \to IL \to IN \to 0$ is exact for every ideal $I \subset R$;
(d) $0 \to K/IK \to L/IL \to N/IN \to 0$ is exact for every ideal $I \subset R$.

Literature: ROTMAN; Anderson-Pascual, Facchini [3], Hiremath [2], Lucas, Naudé-Naudé, Rangaswamy-Vanaja [2], Renault-Autunes, Tuganbaev [1,2], Ukegawa, Warfield, Wilson, Yahya-Al-Daffa [2,3], Zöschinger [1,2,3,6].

41 Supplemented modules

1.Supplements. 2.Supplemented modules. 3.f-supplemented modules. 4.Hollow and local modules. 5.Sums of hollow modules. 6.Supplemented modules.Characterizations. 7.Amply supplemented modules. 8.Supplements of intersections. 9.Characterization of amply supplemented modules. 10.Linearly compact modules and supplements. 11.Submodules lying above direct summands. 12.Supplements as direct summands. 13.Finitely generated submodules and direct summands. 14.Properties of π-projective modules. 15.Supplemented π-projective modules. 16.Properties of supplemented π-projective modules. 17.Decompositions of supplemented π-projective modules. 18.Direct projective modules. 19.Endomorphism ring of direct projective modules. 20. π-injective modules. 21.Characterization of π-injective modules. 22.Endomorphism ring of direct injective modules. 23.Exercises.

Let U be a submodule of the R-module M. A submodule $V \subset M$ is called a *supplement* or *addition complement of U in M* if V is a minimal element in the set of submodules $L \subset M$ with $U + L = M$.

V is a supplement of U if and only if $U+V = M$ and $U \cap V$ is superfluous in V ($U \cap V \ll V$, see § 19):

If V is a supplement of U and $X \subset V$ with $(U \cap V) + X = V$, then we have $M = U + V = U + (U \cap V) + X = U + X$, hence $X = V$ by the minimality of V. Thus $U \cap V \ll V$.

On the other hand, let $U + V = M$ and $U \cap V \ll V$. For $Y \subset V$ with $U+Y = M$, we have $V = M \cap V = (U \cap V)+Y$ (modular law), thus $V = Y$. Hence V is minimal in the desired sense.

41.1 Properties of supplements.

Let U, V be submodules of the R-module M. Assume V to be a supplement of U. Then:

(1) If $W + V = M$ for some $W \subset U$, then V is a supplement of W.

(2) If M is finitely generated, then V is also finitely generated.

(3) If U is a maximal submodule of M, then V is cyclic, and $U \cap V = Rad(V)$ is a (the unique) maximal submodule of V.

(4) If $K \ll M$, then V is a supplement of $U + K$.

(5) For $K \ll M$ we have $K \cap V \ll V$ and so $Rad(V) = V \cap Rad(M)$.

(6) If $Rad(M) \ll M$, then U is contained in a maximal submodule of M.

(7) For $L \subset U$, $(V+L)/L$ is a supplement of U/L in M/L.

(8) If $Rad(M) \ll M$ or $Rad(M) \subset U$, and if $p : M \to M/Rad(M)$ is the canonical projection, then $M/Rad(M) = Up \oplus Vp$.

Proof: (1) follows directly from the definition of V.

(2) Let M be finitely generated. Since $U + V = M$, there is a finitely generated submodule $V' \subset V$ with $U + V' = M$. By the minimality of V, this means $V' = V$.

(3) Similarly to (2), we see that V is cyclic. Since $V/(U \cap V) \simeq M/U$, $U \cap V$ is a maximal submodule and $U \cap V \supset Rad(V)$. Since $U \cap V \ll V$, we have $U \cap V \subset Rad(V)$ and hence $U \cap V = Rad(V)$.

(4) If $K \ll M$, then, for $X \subset V$ with $U + K + X = M$, $U + X = M$, hence $X = V$.

(5) Let $K \ll M$ and $X \subset V$ with $(K \cap V) + X = V$. Then

$$M = U + V = U + (K \cap V) + X = U + X,$$

and therefore $X = V$, i.e. $K \cap V \ll V$. This yields $V \cap Rad(M) \subset Rad(V)$. Since $Rad(V) \subset V \cap Rad(M)$ always holds, we get the desired equality.

(6) For $U \subset Rad(M) \neq M$ the assertion is clear. If $U \not\subset Rad(M)$, then, by (5), $Rad(V) = V \cap Rad(M) \neq V$, i.e. there is a maximal submodule V' in V. Then $M/(U+V') \simeq V/V'$, and hence $U+V'$ is a maximal submodule in M.

(7) For $L \subset U$, we have $U \cap (V + L) = U \cap V + L$ (modularity) and

$$(U/L) \cap [(V+L)/L] = [(U \cap V) + L]/L.$$

Since $U \cap V \ll V$, it follows that $[(U \cap V) + L]/L \ll (V+L)/L$ (image of a superfluous submodule, see 19.3,(4)).

Now the assertion follows from $(U/L) + [(V+L)/L] = M/L$.

(8) If $Rad(M) \subset U$, then, by (7), $Up \cap Vp \ll Vp$. Hence also $Up \cap Vp \ll M/Rad(M)$, and therefore $Up \cap Vp = 0$.

If $Rad(M) \ll M$, then, by (4), V is a supplement of $U + Rad(M)$.

An R-module M is called *supplemented* if every submodule of M has a supplement in M.

If every finitely generated submodule of M has a supplement in M, then we call M *finitely supplemented* or *f-supplemented*.

Let us note that in the literature different terminology for this notion is used by different authors (see remarks preceding 41.7).

41.2 Properties of supplemented modules.
Let M be an R-module.

(1) Let M_1, U be submodules of M with M_1 supplemented. If there is a supplement for $M_1 + U$ in M, then U also has a supplement in M.

(2) If $M = M_1 + M_2$, with M_1, M_2 supplemented modules, then M is also supplemented.

(3) If M is supplemented, then:
(i) Every finitely M-generated module is supplemented;
(ii) $M/Rad(M)$ is semisimple.

Proof: (1) Let X be a supplement of M_1+U in M and Y a supplement of $(X+U)\cap M_1$ in M_1. Then Y is a supplement of $X+U$ in M:

$M = X + M_1 + U = X + U + Y + [(X+U)\cap M_1] = X + U + Y$ and

$Y \cap (X+U) = Y \cap ((X+U)\cap M_1) \ll Y$.

Since $Y + U \subset M_1 + U$, the module X is also a supplement of $Y+U$ (see 41.1,(1)), i.e. $X \cap (Y+U) \ll X$. Considering elements, we can see

$$(X+Y)\cap U \subset [X \cap (Y+U)] + [Y \cap (X+U)].$$

This implies $(X+Y)\cap U \ll X+Y$, i.e. $X+Y$ is a supplement of U.

(2) Let $U \subset M$. Since $M_1 + M_2 + U = M$ trivially has a supplement, we see from (1) that there are supplements for $M_2 + U$ and U in M.

(3) (*i*) From (2) we derive that every finite (direct) sum of supplemented modules is supplemented. From 41.1,(7) we learn that every factor module of a supplemented module is again supplemented.

(*ii*) Since $M/Rad(M)$ contains no superfluous submodules, every submodule is a direct summand, i.e. $M/Rad(M)$ is semisimple.

41.3 Properties of f-supplemented modules.

(1) Let M be an R-module and $M = M_1 + M_2$, with M_1, M_2 finitely generated and finitely supplemented. Assume

(i) M to be coherent in $\sigma[M]$, or

(ii) M to be self-projective and $M_1 \cap M_2 = 0$.

Then M is f-supplemented.

(2) Let M be an f-supplemented R-module.
(i) If $L \subset M$ is a finitely generated or a superfluous submodule, then M/L is also f-supplemented.
(ii) If $Rad(M) \ll M$, then finitely generated submodules of $M/Rad(M)$ are direct summands.
(iii) Assume M to be finitely generated and M-projective or to be coherent

in $\sigma[M]$. Then for any finitely generated or superfluous submodule $K \subset M^n$, with $n \in \mathbb{N}$, the factor module M^n/K is f-supplemented.

Proof: (1) Let U be a finitely generated submodule of M and assume that X is a supplement of $M_1 + U$ in M. If M is coherent in $\sigma[M]$, then $M_1 \cap (X+U)$, as an intersection of finitely generated submodules, is finitely generated (see 26.1). If M is self-projective and $M_1 \cap M_2 = 0$, then

$$(X+U)/[M_1 \cap (X+U)] \simeq M/M_1 \simeq M_2$$

is also M-projective. Then $M_1 \cap (X+U)$ is a direct summand of $X+U$, hence is also finitely generated.

With this observation we are able to carry out the proof of 41.2, (1) and (2) for f-supplemented modules M_1, M_2.

(2) (i) follows from (4) and (6) in (41.1).

(ii) By (i), $M/Rad(M)$ is f-supplemented. Since there are no superfluous submodules in $M/Rad(M)$, every finitely generated submodule is a direct summand.

(iii) By (1), the finite sums M^n are f-supplemented and, by (i), the given factor modules are also f-supplemented.

It is easy to confirm that a (von Neumann) regular ring which is not semisimple is f-supplemented but not supplemented.

Investigating supplemented modules it is interesting to look at certain extreme situations. Dual to the modules in which all non-zero submodules are large we consider the following cases:

We call a non-zero R-module M *hollow* if every proper submodule is superfluous in M. Factor modules of hollow modules are again hollow.

If M has a *largest* submodule, i.e. a proper submodule which contains all other proper submodules, then M is called a *local* module. It is obvious that a largest submodule has to be equal to the radical of M and that in this case $Rad(M) \ll M$.

A ring is a *local ring* if and only if $_RR$ (or R_R) is a local module.

For example, the supplement of a maximal submodule in a module is a local module (see 41.1,(3)).

$Rad(M) = M$ holds if and only if all finitely generated submodules of M are superfluous in M. This implies M to be f-supplemented but need not imply M to be hollow. Local modules are supplemented.

41.4 Hollow and local modules. Properties.
Let M be a non-zero R-module. Then:

(1) M is hollow if and only if every non-zero factor module of M is indecomposable.

(2) The following assertions are equivalent:
(a) M is hollow and $Rad(M) \neq M$;
(b) M is hollow and cyclic (or finitely generated);
(c) M is local.

(3) If $P \to M$ is a projective cover for M in $\sigma[M]$, then the following are equivalent:
(a) M is hollow (local);
(b) P is hollow (local);
(c) P is indecomposable and supplemented;
(d) $End(_RP)$ is a local ring.

Proof: (1) If M is hollow, then every factor module is hollow and hence indecomposable. On the other hand, assume every factor module of M be indecomposable. If U, V are proper submodules with $U + V = M$, then $M/(U \cap V) \simeq M/U \oplus M/V$ (see 9.12), contradicting our assumption.

(2) $(a) \Rightarrow (b)$ $Rad(M) \ll M$ and $M/Rad(M)$ simple imply that M is cyclic.

$(b) \Rightarrow (c) \Rightarrow (a)$ are obvious.

(3) $(a) \Rightarrow (b)$ Let $\pi : P \to M$ be a projective cover and $P = U + V$ for two submodules $U, V \subset P$. Then

$$(U + V)\pi = [U + Ke\,(\pi)]/Ke\,(\pi) + [V + Ke(\pi)]/Ke\,(\pi) = M\,,$$

and therefore $U + Ke(\pi) = P$ or $V + Ke(\pi) = P$, hence $U = P$ or $V = P$, i.e. P is hollow.

Since $P \neq Rad(P)$ for projective modules P in $\sigma[M]$ (by 22.3), we see that P is local.

$(b) \Leftrightarrow (d)$ was shown in 19.7. $(b) \Rightarrow (a)$ and $(b) \Rightarrow (c)$ are clear.

$(c) \Rightarrow (b)$ We shall see later on that in supplemented projective modules (proper) supplements are (proper) direct summands (see 41.16).

Representing modules as a sum of submodules, the following property (known for interior direct sums) turns out to be of interest:

If $M = \sum_\Lambda M_\lambda$, then this sum is called *irredundant* if, for every $\lambda_0 \in \Lambda$, $\sum_{\lambda \neq \lambda_0} M_\lambda \neq M$ holds.

41.5 Sums of hollow modules. Characterizations.

For an R-module M the following assertions are equivalent:

(a) M is a sum of hollow submodules and $Rad(M) \ll M$*;*

(b) every proper submodule of M is contained in a maximal one, and

(i) every maximal submodule has a supplement in M, or

(ii) every submodule $K \subset M$*, with* M/K *is finitely generated, has a supplement in M;*

(c) M is an irredundant sum of local modules and $Rad(M) \ll M$*.*

Proof: $(a) \Leftrightarrow (c)$ Let $M = \sum_\Lambda L_\lambda$ with hollow submodules $L_\lambda \subset M$. Then $M/Rad(M) = \sum_\Lambda (L_\lambda + Rad(M))/Rad(M)$. Since

$Rad(L_\lambda) \subset L_\lambda \cap Rad(M)$ and

$L_\lambda + Rad(M)/Rad(M) \simeq L_\lambda/(L_\lambda \cap Rad(M))$,

these factors are simple or zero. We obtain a representation

$$M/Rad(M) = \bigoplus\nolimits_{\Lambda'} (L_\lambda + Rad(M))/Rad(M),$$

and (since $Rad(M) \ll M$) an irredundant sum $M = \sum_{\Lambda'} L_\lambda$ with local modules L_λ, $\lambda \in \Lambda' \subset \Lambda$.

$(c) \Rightarrow (b)$ Obviously $M/Rad(M)$ is semisimple. Since $Rad(M) \ll M$, by 21.6, every submodule is contained in a maximal submodule.

Assume $K \subset M$ with M/K finitely generated. Then there are finitely many local submodules $L_1, \ldots, L_n$ with $M = K + L_1 + \cdots + L_n$. Being a finite sum of supplemented modules, $L_1 + \cdots + L_n$ is supplemented. Then, by 41.2, K has a supplement in M.

$(b)(i) \Rightarrow (a)$ Let H be the sum of all hollow submodules of M and assume $H \neq M$. Then there is a maximal submodule $N \subset M$ with $H \subset N$ and a supplement L of N. By 41.1, L is local (hollow) and the choice of H implies $L \subset H \subset N$, a contradiction. Hence $H = M$. Since every submodule of M is contained in a maximal one, by 21.6, $Rad(M) \ll M$ holds.

As a corollary of 41.5 we state (notice 21.6,(7)):

41.6 Supplemented modules. Characterizations.

(1) For a finitely generated module M, the following are equivalent:

(a) M is supplemented;

(b) every maximal submodule of M has a supplement in M;

(c) M is a sum of hollow submodules;

(d) M is an irredundant (finite) sum of local submodules.

(2) If M is supplemented and $Rad(M) \ll M$*, then M is an irredundant sum of local modules.*

We say a submodule U of the R-module M has *ample supplements in* M if, for every $V \subset M$ with $U + V = M$, there is a supplement V' of U with $V' \subset V$.

If every (finitely generated) submodule of M has ample supplements in M, then we call M *amply (finitely) supplemented*.

Let us remark that amply supplemented modules as defined here are called *genügend komplementiert* in the german version of this book, *modules with property* (P_2) in Varadarajan, *supplemented* in Golan [2] and *(R-) perfect* in Miyashita, J. Fac. Sci. Hokkaido Univ. 19 (1966).

41.7 Properties of amply supplemented modules.
Let M be an amply supplemented R-module. Then:

(1) Every supplement of a submodule of M is an amply supplemented module.

(2) Direct summands and factor modules of M are amply supplemented.

(3) $M = \sum_\Lambda L_\lambda + K$ where $\sum_\Lambda L_\lambda$ is an irredundant sum of local modules L_λ and $K = Rad(K)$.

If $M/Rad(M)$ is finitely generated, then the sum is finite.

Proof: (1) Let V be a supplement of $U \subset M$ and $V = X + Y$, thus $M = U + X + Y$. Then there is a supplement Y' of $U + X$ in M with $Y' \subset Y$. We get $X \cap Y' \subset (U + X) \cap Y' \ll Y'$, and $M = U + X + Y'$ implies $X + Y' = V$, so Y' is a supplement of X in V.

(2) For direct summands the assertion follows from (1), for factor modules from 41.1.

(3) Let M' be a supplement of $Rad(M)$ in M. Then, by 41.1,(5),

$$Rad(M') = M' \cap Rad(M) \ll M'.$$

By (1), M' is amply supplemented. By 41.6,(2), there is an irredundant representation $M' = \sum_\Lambda L_\lambda$ with local L_λ. For a supplement K of M' in M with $K \subset Rad(M)$, we have $Rad(K) = K \cap Rad(M) = K$. This yields the desired representation of M.

The final assertion follows from $M = \sum_\Lambda L_\lambda + Rad(M)$.

41.8 Supplements of intersections.
Let M be an R-module and $M = U_1 + U_2$.

If the submodules U_1, U_2 have ample supplements in M, then $U_1 \cap U_2$ has also ample supplements in M.

Proof: Let $V \subset M$ with $(U_1 \cap U_2) + V = M$. Then $U_1 + (U_2 \cap V) = M$ and $U_2 + (U_1 \cap V) = M$ also hold. Therefore there is a supplement V_2' of U_1

in M with $V_2' \subset U_2 \cap V$ and a supplement V_1' of U_2 with $V_1' \subset U_1 \cap V$. By construction we have, for $V_1' + V_2' \subset V$, the relations

$(U_1 \cap U_2) + (V_1' + V_2') = M$ and

$(V_1' + V_2') \cap (U_1 \cap U_2) = (V_1' \cap U_2) + (V_2' \cap U_1) \ll V_1' + V_2'$.

41.9 Characterization of amply supplemented modules.

For an R-module M the following properties are equivalent:

(a) M is amply supplemented;

(b) every submodule $U \subset M$ is of the form $U = X+Y$, with X supplemented and $Y \ll M$;

(c) for every submodule $U \subset M$, there is a supplemented submodule $X \subset U$ with $U/X \ll M/X$.

If M is finitely generated, then (a)-(c) are also equivalent to:

(d) Every maximal submodule has ample supplements in M.

Proof: $(a) \Rightarrow (b)$ Let V be a supplement of U in M and X a supplement of V in M with $X \subset U$. Then we have $U \cap V \ll M$ and

$$U = (X + V) \cap U = X + (U \cap V),$$

where X is supplemented by 41.7.

$(b) \Rightarrow (c)$ If $U = X + Y$, with X supplemented and $Y \ll M$, then, of course, $Y/(X \cap Y) \simeq U/X \ll M/X$.

$(c) \Rightarrow (a)$ If $U + V = M$ and if X is a supplemented submodule of V with $V/X \ll M/X$, then $U + X = M$ holds. For a supplement V' of $U \cap X$ in X, we have

$$M = U + (U \cap X) + V' = U + V' \text{ and } U \cap V' = (U \cap X) \cap V' \ll V',$$

i.e. $V' \subset V$ is a supplement of U in M.

$(a) \Rightarrow (d)$ is clear.

$(d) \Rightarrow (a)$ If M is finitely generated and all maximal submodules have supplements, then, by 41.6, M is supplemented and $M/Rad(M)$ is semisimple. Then, for $U \subset M$, the factor module $M/(U + Rad(M))$ is semisimple and $U + Rad(M)$ is an intersection of finitely many maximal submodules.

From 41.8 we derive that $U + Rad(M)$ has ample supplements. Since $Rad(M) \ll M$ this is also true for U.

Interesting examples for the situation described in 41.9 are provided by linearly compact modules introduced in § 29:

41.10 Linearly compact modules and supplements.
Let M be a non-zero R-module.

(1) If U is a linearly compact submodule of M, then U has ample supplements in M.

(2) Assume M to be linearly compact. Then:
(i) M is amply supplemented.
(ii) M is noetherian if and only if $Rad(U) \neq U$ for every non-zero submodule $U \subset M$.
(iii) M is artinian if and only if $Soc(L) \neq 0$ for every non-zero factor module L of M.

Proof: (1) Let U be linearly compact and $V \subset M$ with $U + V = M$. For an inverse family of submodules $\{V_\lambda\}_\Lambda$ of V, with $U + V_\lambda = M$, we have, by 29.8,

$$U + \bigcap\nolimits_\Lambda V_\lambda = \bigcap\nolimits_\Lambda (U + V_\lambda) = M.$$

Hence the set $\{V' \subset V \,|\, U + V' = M\}$ is inductive (downwards) and therefore has a minimal element by Zorn's Lemma.

(2) Let M be linearly compact. (i) follows from (1) since every submodule of M is linearly compact.

(ii) If M is noetherian every submodule is finitely generated and the assertion is clear. Now assume $U \neq Rad(U)$ for all non-zero $U \subset M$. Since $M/Rad\,M$ is finitely generated (see 29.8,(3)), M can be written as a finite sum of local modules by 41.7, and hence is finitely generated. In the same way we obtain that every submodule of M is finitely generated.

(iii) The assertion for artinian modules is clear. Assume $Soc(L) \neq 0$ for every non-zero factor module L of M. $Soc(M)$ is linearly compact and hence finitely generated. We show that $Soc(M)$ is essential in M:

Assume $U \cap Soc(M) = 0$ for $U \subset M$. Then we choose a $V \subset M$ which is maximal with respect to $U \cap V = 0$ and $Soc(M) \subset V$. With the canonical mappings $U \to M \to M/V$ we may regard U as an essential submodule of M/V. By assumption, we have
$Soc(U) = U \cap Soc(M/V) \neq 0$, hence also $U \cap Soc(M) = Soc(U) \neq 0$, contradicting our assumption. Therefore $Soc(M)$ is finitely generated and essential in M, i.e. M is finitely cogenerated (see 21.3).

Since this holds for every factor module of M, M is artinian (see 31.1).

By definition, supplements are generalizations of direct summands. We may ask when are supplements direct summands? To prepare the answer we show:

41.11 Submodules lying above direct summands.

Let U be a submodule of the R-module M. The following are equivalent:

(a) *There is a decomposition $M = X \oplus X'$, with $X \subset U$ and $X' \cap U \ll X'$;*

(b) *there is an idempotent $e \in End(M)$ with $Me \subset U$ and $U(1-e) \ll M(1-e)$;*

(c) *there is a direct summand X of M with $X \subset U$, $U = X + Y$ and $Y \ll M$;*

(d) *there is a direct summand X of M with $X \subset U$ and $U/X \ll M/X$;*

(e) *U has a supplement V in M such that $U \cap V$ is a direct summand in U.*

In this case we say U lies above a direct summand of *M.*

Proof: $(a) \Leftrightarrow (b)$ For a decomposition $M = X \oplus X'$, there is an idempotent $e \in End(M)$ with $Me = X$ and $M(1-e) = X'$. Because of $X \subset U$, we have $U \cap M(1-e) = U(1-e)$.

$(a) \Rightarrow (c)$ With the assumptions in (a) and by the modular law, we have $U = X + (U \cap X')$ and $U \cap X' \ll M$.

$(c) \Rightarrow (d)$ From (c) we derive $Y/(X \cap Y) \simeq U/X \ll M/X$.

$(d) \Rightarrow (c)$ If $M = X \oplus X'$ and $U/X \ll M/X$, then $U = X + (U \cap X')$ and $U \cap X' \simeq U/X \ll M/X \simeq X'$, hence $U \cap X' \ll M$.

$(c) \Rightarrow (a)$ If $M = X \oplus X'$, then X' is a supplement of X and hence a supplement of $X + Y$ (see 41.1), i.e. $U \cap X' = (X + Y) \cap X' \ll X'$.

$(a) \Rightarrow (e)$ With the notation in (a), $U = X \oplus (U \cap X')$ and X' is a supplement of U.

$(e) \Rightarrow (a)$ Let V be a supplement of U with $U = X \oplus (U \cap V)$ for a suitable $X \subset U$. We have $M = U + V = X + (U \cap V) + V = X + V$ and $X \cap V = 0$, i.e. X is a direct summand in M.

The connection of the property of submodules just introduced to preceding observations appears in:

41.12 Supplements as direct summands.

For an R-module M the following assertions are equivalent:

(a) *M is amply supplemented and every supplement submodule is a direct summand;*

(b) *every submodule of M lies above a direct summand;*

(c) *(i) every non-superfluous submodule of M contains a non-zero direct summand of M, and*
(ii) every submodule of M contains a maximal direct summand of M.

Proof: $(a) \Rightarrow (b)$ For $U \subset M$, let V be a supplement in M and X a supplement of V in M with $X \subset U$. Then $M = X \oplus X'$ for a suitable direct summand $X' \subset M$. Since $U \cap V \ll M$, this X' is a supplement of $X + (U \cap V) = U$ (see 41.1), and hence $U \cap X' \ll X'$.

$(b) \Rightarrow (a)$ If (b) holds, then M is obviously supplemented (see 41.11), and every submodule $U \subset M$ is of the form $U = X + Y$, with X a direct summand of M and $Y \ll M$. Since X is again supplemented it follows, from 41.9, that M is amply supplemented. Now we see, from the proof of $(e) \Rightarrow (a)$ in 41.11, that supplements are direct summands.

$(b) \Rightarrow (c)$ Let $U \subset M$ and $M = X \oplus X'$ with $X \subset U$ and $U \cap X' \ll X'$. If U is not superfluous in M, then $X' \neq M$ and hence $X \neq 0$. For a direct summand X_1 of M with $X \subset X_1 \subset U$, we have $X_1 = X \oplus (X_1 \cap X')$. Since $X_1 \cap X' \subset U \cap X' \ll M$, we obtain $X_1 \cap X' = 0$ and $X = X_1$.

$(c) \Rightarrow (b)$ Let $U \subset M$ and assume X to be a maximal direct summand of M with $X \subset U$ and $M = X \oplus X'$. If $U \cap X'$ is not superfluous in X', then, by $(c.i)$, there is a non-zero direct summand N of M with $N \subset U \cap X'$. Then the sum $X \oplus N$ is a direct summand in M, contradicting the choice of X. Thus we have $U \cap X' \ll X'$.

For example, hollow modules satisfy the conditions in 41.12. Demanding the properties in 41.12 only for *finitely generated* (instead of *all*) submodules, then modules M with $M = Rad(M)$ satisfy these new conditions. We describe this situation in

41.13 Finitely generated submodules and direct summands.
For an R-module M the following assertions are equivalent:

(a) Every finitely generated submodule lies above a direct summand;

(b) every cyclic submodule lies above a direct summand;

(c) (i) every finitely generated submodule $L \subset M$ with $L \not\subset Rad(M)$ contains a non-zero direct summand of M, and
(ii) every finitely generated submodule of M contains a maximal direct summand of M.

If M is finitely generated, then (a)-(c) are equivalent to:

(d) M is amply finitely supplemented and every supplement is a direct summand.

Proof: The equivalence of (a), (c) and (d) are seen with the corresponding proofs of 41.12. $(a) \Rightarrow (b)$ is clear.

$(b) \Rightarrow (a)$ The proof is obtained by induction on the number of generating elements of the submodules of M. The assertion in (b) provides the basis.

Assume the assertion to be proved for submodules with $n-1$ generating elements and consider $U = Ru_1 + \cdots + Ru_n$. We choose an idempotent

$e \in End(M)$ with $Me \subset Ru_n$ and

$$Ru_n \cap M(1-e) = Ru_n(1-e) \ll M.$$

Now we form $K = \sum_{i \leq n} Ru_i(1-e)$. From $Ue \subset Ru_n \subset U$ we obtain the relation $U = U(1-e) + Ue = K + Ru_n$.

By induction hypothesis, we find an idempotent $f \in End(M)$ with

$$Mf \subset K \quad \text{and} \quad K \cap M(1-f) = K(1-f) \ll M.$$

From $Mf \subset K \subset M(1-e)$, we deduce $f(1-e) = f$, i.e. $fe = 0$, and hence $g = e + f - ef$ is idempotent. We have $Mg \subset Mf + Me \subset K + Ru_n = U$ and

$$U \cap M(1-g) = U(1-e)(1-f) \subset K(1-f) + Ru_n(1-e)(1-f) \ll M.$$

Combined with a weak projectivity condition, supplemented modules admit quite nice structure theorems:

We call an R-module M *π-projective* (or *co-continuous*) if for every two submodules U, V of M with $U + V = M$ there exists $f \in End(M)$ with

$$Im\,(f) \subset U \quad \text{and} \quad Im\,(1-f) \subset V.$$

This is obviously true if and only if the epimorphism

$$U \oplus V \to M, \quad (u, v) \mapsto u + v,$$

splits. From this we see that every self-projective module is also π-projective. Hollow (local) modules trivially have this property. The importance of this notion for our investigations is seen from

41.14 Properties of π-projective modules.
Assume M to be a π-projective R-module. Then:

(1) Every direct summand of M is π-projective.

(2) If U and V are mutual supplements in M, then $U \cap V = 0$ and $M = U \oplus V$.

(3) If $M = U + V$ and U is a direct summand in M, then there exists $V' \subset V$ with $M = U \oplus V'$.

(4) If $M = U \oplus V$, then V is U-projective (and U is V-projective).

(5) If $M = U \oplus V$ with $U \simeq V$, then M is self-projective.

(6) If $M = U + V$ and U, V are direct summands in M, then $U \cap V$ is also a direct summand in M.

For every R-module M the properties (3) *and* (4) *are equivalent, and* (4) $\Rightarrow$ (5) *and* (4) $\Rightarrow$ (6) *hold.*

Proof: (1) Consider $e^2 = e \in End(M)$, i.e. $M = Me \oplus M(1-e)$. Assume $Me = X + Y$. Then $M = X + (Y + M(1-e))$, and there exists $f \in End(M)$ with $Im\,(f) \subset X$ and $Im\,(1-f) \subset Y + M(1-e)$.

Therefore we may regard fe and $1 - fe$ as endomorphisms of Me and

$$Im\,(fe) \subset X, \quad Im\,(1-fe) = Im\,((1-f)e) \subset Y.$$

(2) If U, V are mutual supplements, then we have $U \cap V \ll U$ and $U \cap V \ll V$, hence

$$\{(u,-u) \mid u \in U \cap V\} \subset (U \cap V, 0) + (0, U \cap V) \ll U \oplus V.$$

This is the kernel of the homomorphism $U \oplus V \to M$, $(u,v) \mapsto u + v$, which splits by assumption. Thus $U \cap V = 0$ has to hold.

(3) Let $M = U + V$ and $M = U \oplus X$ for a suitable $X \subset M$. We choose $f \in End(M)$ with $Im\,(f) \subset V$ and $Im\,(1-f) \subset U$. Obviously $(U)f \subset U$ and $M = (U + X)f + M(1-f) = U + (X)f$. We prove $U \cap (X)f = 0$.

Assume $u = (x)f$, with $u \in U$ and $x \in X$. Then $x - u = x(1-f) \in U$ and hence $x \in U \cap X = 0$.

(3) $\Rightarrow$ (4) Let $M = U \oplus V$, $p : U \to W$ be an epimorphism and $f : V \to W$. We form

$$P = \{u - v \in M \mid u \in U,\, v \in V \quad \text{and} \quad (u)p = (v)f\}.$$

Since p is epic, $M = U + P$. Therefore, by (3), $M = U \oplus P'$ with $P' \subset P$. Let $e : M \to U$ be the projection with respect to this decomposition. This yields a homomorphism $V \to M \to U$.

Since $V(1-e) \subset P' \subset P$ we have, for every $v \in V$, $v - (v)e \in P$, and hence $(v)f = (ve)p$, i.e. $f = ep$. Therefore V is U-projective.

(4) $\Rightarrow$ (3) Let $M = U + V$ and $M = U \oplus X$, with X U-projective. With canonical mappings, we obtain the diagram

$$\begin{array}{ccccc} & & & & X \\ & & & & \downarrow \\ U & \longrightarrow & M & \longrightarrow & M/V \end{array} ,$$

which can be commutatively extended by an $f : X \to U$. This means $x + V = (x)f + V$ for every $x \in X$ and hence $X(1-f) \subset V$. Now we have $M = U + X \subset U + X(1-f)$.

We show $U \cap X(1-f) = 0$: Assume $u = (x)(1-f)$, for $u \in U$, $x \in X$. This yields $x = u - (x)f \in U \cap X = 0$.

(4) $\Rightarrow$ (5) is clear.

(3) $\Rightarrow$ (6) Let U, V be direct summands of M and $U + V = M$. By (3), we may choose $V' \subset V$ and $U' \subset U$ with $M = U \oplus V'$ and $M = U' \oplus V$. From this we obtain by modularity, $M = (U \cap V) + (U' + V')$ and

$(U \cap V) \cap (U' + V') \subset (U' \cap V) + (U \cap V') = 0$.

41.15 Supplemented π-projective modules.

For an R-module M the following assertions are equivalent:

(a) M is supplemented and π-projective;

(b) (i) M is amply supplemented, and
(ii) the intersection of mutual supplements is zero;

(c) (i) every submodule of M lies above a direct summand, and
(ii) if U, V are direct summands of M with $M = U + V$, then $U \cap V$ is also a direct summand of M;

(d) for every two submodules U, V of M with $U + V = M$ there is an idempotent $e \in End(M)$ with
$Me \subset U$, $M(1-e) \subset V$ and $U(1-e) \ll M(1-e)$.

Proof: $(a) \Rightarrow (b)$ (i) Let $M = U + V$ and X be a supplement of U in M. For an $f \in End(M)$ with $Im\,(f) \subset V$ and $Im\,(1-f) \subset U$ we have

$$Uf \subset U,\; M = U + (X)f \text{ and } (U \cap X)f = U \cap (X)f$$

(from $u = (x)f$ we derive $x - u = (x)(1-f) \in U$ and $x \in U$). Since $U \cap X \ll X$, we also have $U \cap (X)f \ll (X)f$, i.e. $(X)f$ is a supplement of U with $(X)f \subset V$. Hence M is amply supplemented.

(ii) follows from 41.14,(2).

$(b) \Rightarrow (c)$ We first conclude from (b) that every supplement is a direct summand in M. Hence, by 41.12, every submodule lies above a direct summand.

(ii) Let $M = U + V$ and U be a direct summand in M. We choose a supplement V' of U in M with $V' \subset V$ and a supplement U' of V' in M with $U' \subset U$. Then $U \cap V' \ll V'$ and – by $(b.ii)$ – $M = U' \oplus V'$. But this means $U = U' \oplus (U \cap V')$. Hence $(U \cap V')$ is a direct summand in M implying $U \cap V' = 0$. Consequently $M = U \oplus V'$ with $V' \subset V$, i.e. we have proved (3) of 41.14, and the assertion follows from 41.14,(3) $\Rightarrow$ (6).

$(c) \Rightarrow (d)$ By 41.12, M is amply supplemented and every supplement is a direct summand in M. If $M = U + V$, then we find – similar to the argument in $(b) \Rightarrow (c)$ – a decomposition $M = U' \oplus V'$ with $U \cap V' \ll V'$. If $e : M \to U'$ is the related projection, then $Me \subset U$, $M(1-e) \subset V' \subset V$ and $U(1-e) = U \cap V' \ll M(1-e)$.

$(d) \Rightarrow (a)$ With the notation in (d), $M(1-e)$ is a supplement of U and M is π-projective.

41.16 Supplemented π-projective modules. Properties.
Let M be a supplemented π-projective R-module and $S = End(M)$. Then:

(1) Every direct summand of M is supplemented and π-projective, and every supplement submodule of M is a direct summand.

(2) Let e be an idempotent in S and N a direct summand of M.
If $N(1-e) \ll M(1-e)$, then $N \cap M(1-e) = 0$ and $N \oplus M(1-e)$ is a direct summand in M.

(3) If $\{N_\lambda\}_\Lambda$ is a family of direct summands of M, directed with respect to inclusion, then $\bigcup_\Lambda N_\lambda$ is also a direct summand in M.

(4) For every $0 \neq a \in M$, there is a decomposition $M = M_1 \oplus M_2$ with M_2 hollow and $a \notin M_1$.

(5) If N and H are direct summands of M and H is hollow, then
(i) $N \cap H = 0$ and $N \oplus H$ is a direct summand of M, or
(ii) $N + H = N \oplus K$ with $K \ll M$ and H is isomorphic to a direct summand of N.

Proof: (1) The first assertion is clear since both properties are inherited by direct summands, the second follows from 41.14,(2).

(2) For $Ne \subset Me$ there is a decomposition $Me = U \oplus V$ with $U \subset Ne$ and $Ne \cap V \ll V$. For the projection $p : U \oplus V \oplus M(1-e) \to V$, we have $Np = Nep = Ne \cap V \ll V$. Now $p + (1-e)$ is the identity on $V \oplus M(1-e)$ and hence

$$N \cap (V \oplus M(1-e)) \subset Np + N(1-e) \ll M \ .$$

Since $U \subset Ne \subset N + M(1-e)$ we derive $M = N + (V \oplus M(1-e))$. By 41.15, the intersection $N \cap (V \oplus M(1-e))$ has to be a direct summand in M and hence is zero.

(3) Assume $\{N_\lambda\}_\Lambda$ to be given as indicated.
Then $N = \bigcup_\Lambda N_\lambda$ is a submodule, and there is an idempotent $e \in S$ with $Me \subset N$ and $N(1-e) \ll M(1-e)$. Therefore, for every $\lambda \in \Lambda$, we have $N_\lambda(1-e) \subset N(1-e) \ll M(1-e)$, and by (2), $N_\lambda \cap M(1-e) = 0$. This implies $N \cap M(1-e) = 0$ and $M = N \oplus M(1-e)$.

(4) By (3), the set of direct summands L of M with $a \notin L$ is inductive, and hence has a maximal element M_1, by Zorn's Lemma. Assume $M = M_1 \oplus M_2$ for a suitable $M_2 \subset M$.

If there is a proper non-superfluous submodule in M_2, then a non-trivial decomposition $M_2 = U \oplus V$ exists and $M = M_1 \oplus U \oplus V$. By the maximality of M_1, we conclude $a \in M_1 \oplus U$ and $a \in M_1 \oplus V$. But this means $a \in M_1$, contradicting the choice of M_1. Hence all proper submodules in M_2 are superfluous, i.e. M_2 is hollow.

(5) From $M = N \oplus L$, we get $N + H = N \oplus ((N + H) \cap L)$, and hence

$$(N + H) \cap L \simeq (N + H)/N \simeq H/(H \cap N)$$

is a hollow module. If $(N + H) \cap L$ is not superfluous in M, it has to contain a direct summand of M and hence it is a direct summand. Then $N + H$ is also a direct summand in M. Because of $N \cap H \neq H$, we have $N \cap H \ll M$ and $N \cap H$ is a direct summand in $N + H$ and M, i.e. $N \cap H = 0$.

Now assume $(N + H) \cap L \ll M$. If $M = H \oplus H'$, we obtain

$$M = (N + H) + H' = N + [(N + H) \cap L] + H' = N + H'.$$

Then $N \cap H'$ is a direct summand of M and $N = N' \oplus (N \cap H')$ for some $N' \subset N$. From this we derive $M = (N' \oplus (N \cap H')) + H' = N' \oplus H'$ and $N' \simeq H$.

Let $p : M \to N$ be an epimorphism of R-modules. We say a decomposition of $N = \bigoplus_\Lambda N_\lambda$ can *be lifted* to M (under p) if there is a decomposition $M = \bigoplus_\Lambda M_\lambda$ such that $(M_\lambda)p = N_\lambda$ for every $\lambda \in \Lambda$.

We will be mainly interested in lifting decompositions of $M/Rad(M)$ under the canonical map $M \to M/Rad(M)$ as in the following situation:

41.17 Supplemented π-projective modules. Decompositions. *Assume M is a non-zero supplemented π-projective R-module. Then:*

(1) There is a decomposition $M = \bigoplus_\Lambda H_\lambda$ with hollow modules H_λ, and, for every direct summand N of M, there exists a subset $\Lambda' \subset \Lambda$ with

$$M = (\bigoplus\nolimits_{\Lambda'} H_\lambda) \oplus N.$$

(2) If $M = \sum_\Lambda N_\lambda$ is an irredundant sum with indecomposable N_λ, then $M = \bigoplus_\Lambda N_\lambda$.

(3) If $Rad(M) \ll M$, then $M = \bigoplus_\Lambda L_\lambda$ with local modules L_λ.

(4) There is a decomposition $M = \bigoplus_\Lambda L_\lambda \oplus K$ with local modules L_λ, $Rad(\bigoplus_\Lambda L_\lambda) \ll \bigoplus_\Lambda L_\lambda$ and $K = Rad(K)$.

(5) Every direct decomposition of $M/Rad(M)$ can be lifted to M under $M \to M/Rad(M)$.

Proof: (1) Denote by $\mathcal{H}$ the set of all hollow submodules in M and consider

$$\{\mathcal{H}' \subset \mathcal{H} \,|\, \sum_{H\in\mathcal{H}'} H \text{ is a direct sum and a direct summand in } M\}.$$

This set is non-empty and inductive with respect to inclusion by 41.16,(3), and, by Zorn's Lemma, has a maximal element $\{H_\lambda\}_\Lambda$. By construction $H = \bigoplus_\Lambda H_\lambda$ is a direct summand, i.e. $M = H \oplus K$ for some $K \subset M$. Assume $K \neq 0$.

Since K is also supplemented and π-projective, it possesses a hollow direct summand $H_1 \neq 0$ (see 41.16). Then the direct summand $H \oplus H_1$ of M is properly larger than H. This contradicts the maximality of H, hence $K = 0$ and we conclude $M = H$.

For the direct summand N of M, we consider subsets $I \subset \Lambda$ with the properties $N \cap (\bigoplus_I H_i) = 0$ and $N \oplus (\bigoplus_I H_i)$ are direct summands of M. From 41.16,(3), we derive that these subsets are inductively ordered and hence have a maximal element $\Lambda' \subset \Lambda$.

Assume $L := N \oplus (\bigoplus_{\Lambda'} H_\lambda) \neq M$. By the proof of 41.16,(4), we can find a decomposition $M = K \oplus H$ with $L \subset K$ and H hollow. Let $p : M \to H$ denote the related projection. If $(H_\mu)p = H$ holds for some $\mu \in \Lambda$, then $M = K + H_\mu$. Since $K \cap H_\mu \neq H_\mu$, $K \cap H_\mu \ll M$. Considering 41.15,(c), this means $K \cap H_\mu = 0$, i.e. $M = K \oplus H_\mu$.

Hence $L \oplus H_\mu$ is a direct summand of M. Since $\mu \notin \Lambda'$, this is a contradiction to the maximality of Λ'. Consequently we have $(H_\lambda)p \neq H$ for every $\lambda \in \Lambda$. This implies, for every finite partial sum $T = H_1 \oplus H_2 \oplus \cdots \oplus H_n$ of $\bigoplus_\Lambda H_\lambda$, the relation $(T)p = (H_1)p + \cdots + (H_n)p \ll H$. Then, from 41.16,(2), we get $T \cap H = 0$ and finally $(\bigoplus_\Lambda H_\lambda) \cap H = 0$, i.e. $H = 0$. This means $M = L$.

(2) Since the sum $\sum_\Lambda N_\lambda$ is irredundant, none of the N_λ can be superfluous in M. Therefore every (indecomposable) N_λ contains a direct summand, and hence has to be a direct summand of M and is hollow (see 41.12, 41.16). We show that, for every finite subset $E \subset \Lambda$, the sum $\sum_E N_\lambda$ is direct:

Let $F \subset E$ be maximal with respect to the properties

$$\sum_F N_\lambda \text{ is a direct sum and a direct summand in } M.$$

For $\lambda_0 \in E \setminus F$, 41.16 implies $(\bigoplus_F N_\lambda) + N_{\lambda_o} = (\bigoplus_F N_\lambda) \oplus K$ with $K \ll M$. But this yields $M = \sum_{\lambda\neq\lambda_0} N_\lambda$, contradicting the irredundance of the given sum.

(3) We have already seen in 41.6 that supplemented modules with superfluous radical are irredundant sums of local modules. Therefore (3) follows from (2).

(4) Let M' be a supplement of $Rad(M)$ in M and K a supplement of M' in M, with $K \subset Rad(M)$. Then we have $Rad(M') \ll M'$, $K = Rad(K)$ and $M = M' \oplus K$ (see also 41.7,(3)).

(5) By (4), we may assume $Rad(M) \ll M$. Since $M/Rad(M)$ is semisimple, every decomposition $M/Rad(M) = \bigoplus_\Lambda K_\lambda$ can be refined to a decomposition with simple summands. Hence we may assume the K_λ to be simple modules. Then there are indecomposable modules $F_\lambda \subset M$ with $(F_\lambda + Rad(M))/Rad(M) \simeq K_\lambda$ (see 41.12), and $M = \sum_\Lambda F_\lambda$ is an irredundant sum. By (2), this sum is direct.

Let us now introduce a further projectivity property, which is of interest in connection with supplemented modules:

An R-module M is called *direct projective* if, for every direct summand X of M, every epimorphism $M \to X$ splits.

It is clear that self-projective modules are direct projective. Also modules M, with the property that for every $f \in End(M)$ the kernel $Ke\, f$ is a direct summand, are direct projective. Hollow modules are π-projective but need not be direct projective.

41.18 Properties of direct projective modules.
Assume M to be a direct projective R-module. Then:

(1) If U, V are direct summands of M, then every epimorphism $U \to V$ splits.

(2) Every direct summand of M is direct projective.

(3) If U, V are direct summands with $U + V = M$, then $U \cap V$ is a direct summand in U (and M) and $M = U \oplus V'$ for some $V' \subset V$.

(4) If every submodule of M lies above a direct summand, then M is π-projective.

Proof: (1) Let $h : U \to V$ be an epimorphism and $p : M \to U$ a projection. Then ph is epic and hence splits. Thus h also splits.

(2) follows directly from (1).

(3) If $M = V \oplus X$, then $X \simeq M/V \simeq U/U \cap V$, and, by (1), the exact sequence $0 \to U \cap V \to U \to U/U \cap V \to 0$ splits. Hence $U \cap V$ is a direct summand in U, M and also in V. Therefore there is a $V' \subset V$ with $V = (U \cap V) \oplus V'$ and $M = U + ((U \cap V) \oplus V') = U \oplus V'$.

(4) follows from 41.15 because of (3).

41.19 Endomorphism ring of direct projective modules.
Assume M to be a direct projective R-module and $S = End(M)$. Then:

(1) $K(S) := \{f \in S \mid Im\ f \ll M\} \subset Jac(S)$.

(2) The following assertions are equivalent:

(a) For every $f \in S$, the image $Im\ f$ lies above a direct summand of M;
(b) ${}_SS$ is f-supplemented and $K(S) = Jac(S)$.

(3) If M is hollow, then S is a local ring.

Proof: (1) If $Im\ f \ll M$, then $M = Mf + M(1-f) = M(1-f)$. Hence $1-f$ is surjective and splits, i.e. there exists $g \in S$ with $g(1-f) = 1$ and f is left quasi-regular. Since $K(S)$ is a left ideal, this yields $K(S) \subset Jac(S)$ (see 21.11).

(2) $(a) \Rightarrow (b)$ By 41.13, it is sufficient to show that every cyclic left ideal Sf, $f \in S$, lies above a direct summand: By (a), there is an idempotent $e \in S$ with $Me \subset Mf$ and

$$Mf \cap M(1-e) = Mf(1-e) \ll M.$$

Since $fe : M \to Me$ is epic, there exists $g \in S$ with $gfe = e$. Setting $h = egf \in S$ we get $h^2 = h$, $e = he$, $Sh \subset Sf$, and

$$Sf(1-h) \subset Sf(1-e) \ll S,$$

since, by (1), $f(1-e) \in Jac(S)$. If $Im\ f$ is not superfluous in M, then, in the construction above, e and h are non-zero and

$$Im\ f(1-h) = Im\ (f - f(eg)f) \ll M.$$

Therefore the factor ring $S/K(S)$ is (von Neumann) regular and hence $K(S) = Jac(S)$.

$(b) \Rightarrow (a)$ Let ${}_SS$ be finitely supplemented. By 41.15, for $f \in S$, there exists an idempotent $e \in S$ with $Se \subset Sf$ and $Sf(1-e) \subset Jac(S) = K(S)$. Hence $M = Me + M(1-e)$, $Me \subset Mf$ and $Mf(1-e) \ll M$.

(3) If $f \in S$ is surjective, then there exists $g \in S$ with $gf = id_M$. Consequently, $Im\ g$ can not be superfluous. Therefore, g is surjective and f is injective, i.e. f is an isomorphism.

If f is not surjective, then we have $Im\ f \ll M$, and $1-f$ is surjective and hence an isomorphism. Thus S is a local ring.

It is interesting to look at notions dual to some of the preceding ones. Instead of supplements we will examine complements (see 17.5). By Zorn's

Lemma, we obtain that in every module there are *ample* complements, i.e. for every two submodules U, V of M with $U \cap V = 0$ there exists a complement V' of U with $V \subset V'$.

An R-module is called *π-injective* (or *quasi-continuous*) if, for every two submodules U, V of M with $U \cap V = 0$, there exists $f \in End(M)$ with

$$U \subset Ke\,(f) \quad \text{and} \quad V \subset Ke\,(1-f).$$

This is true if and only if the following monomorphism splits

$$M \to M/U \oplus M/V, \quad m \mapsto (m+U, m+V).$$

Dualizing the proofs of 41.14 we can show:

41.20 Properties of π-injective modules.
Assume M to be a π-injective R-module. Then:

(1) Every direct summand of M is π-injective.

(2) If $M = U \oplus V$, then V is U-injective (and U is V-injective).

(3) If $M = U \oplus V$ and $U \simeq V$, then M is self-injective.

(4) If U, V are direct summands of M and $U \cap V = 0$, then $U \oplus V$ is also a direct summand of M.

We say a submodule $U \subset M$ *lies under a direct summand* if there is an idempotent $e \in End(M)$ with $U \trianglelefteq Me$ (U is essential in Me). With this we obtain dual to 41.15:

41.21 Characterization of π-injective modules.
For an R-module M the following statements are equivalent:
(a) M is π-injective;
(b) the sum of two mutual complements is M;
(c) (i) every submodule of M lies under a direct summand, and
(ii) if U, V are direct summands of M with $U \cap V = 0$, then $U \oplus V$ is also a direct summand of M;
(d) for submodules U, V of M with $U \cap V = 0$ there is an idempotent $e \in End(M)$ with $U \subset Ke\,(e)$ and $V \subset Ke\,(1-e)$;
(e) if $\widehat{M}$ is the M-injective hull of M, then, for every idempotent $e \in End(\widehat{M})$, we have $Me \subset M$.

Proof: The equivalence of (a), (b), (c) and (d) follows dually to 41.15.

$(b) \Rightarrow (e)$ For any idempotent $e \in End(\widehat{M})$, the submodules $M \cap \widehat{M}e$ and $M \cap \widehat{M}(1-e)$ are mutual complements: Consider a submodule $X \subset M$ with

$$M \cap \widehat{M}e \subset X \text{ and } \quad X \cap \widehat{M}(1-e) = 0.$$

For $x = xe + x(1-e) \in X$, assume $x(1-e) \neq 0$. Then $M \cap Rxe \neq 0$, and hence there exists $r \in R$ with $0 \neq rx(1-e) \in M \cap Rx(1-e)$. Therefore

$$rx - rxe = rx(1-e) \in X \cap \widehat{M}(1-e) = 0,$$

a contradiction. From this we get $x(1-e) = 0$ and $X = M \cap \widehat{M}e$. Now, by (b), we find an idempotent $f \in End(M)$ with

$$M \cap \widehat{M}e = Mf, \quad M \cap \widehat{M}(1-e) = M(1-f)$$

and hence $Mf(1-e) = 0$ and $M(1-f)e = 0$, i.e. $me = mf$ for all $m \in M$ and $Me = Mf \subset M$.

$(e) \Rightarrow (b)$ If U and V are mutual complements, then $U + V \trianglelefteq M$ and $\widehat{M} = \widehat{U} \oplus \widehat{V}$. If $e : \widehat{M} \to \widehat{U}$ is the corresponding projection, then

$$U \subset Me \subset M \text{ and } Me \cap V \subset \widehat{M}e \cap \widehat{M}(1-e) = 0.$$

Therefore $U = Me$ and we have $M = U \oplus V$.

An R-module M is called *direct injective* if, for every direct summand X of M, every monomorphism $X \to M$ splits.

Examples include self-injective modules and modules whose images of endomorphisms are direct summands (see 39.11). Dual to 41.19 we obtain:

41.22 Endomorphism ring of direct injective modules.
For a direct injective module M with $S = End(M)$ we have:

(1) $W(S) := \{f \in S \mid Ke\, f \trianglelefteq M\} \subset Jac(S)$.

(2) The following assertions are equivalent:

(a) for every $f \in S$, the kernel $Ke\, f$ lies under a direct summand;
(b) S_S is f-supplemented and $W(S) = Jac(S)$.

(3) If every submodule of M is essential, then S is a local ring.

Proof: (1) $Ke\, f \cap Ke\,(1-f) = 0$ always holds. If $Ke\, f$ is essential, $1-f$ has to be monic and hence splits, i.e. there exists $h \in S$ with $(1-f)h = 1$. Therefore, $W(S)$ is a quasi-regular right ideal and $W(S) \subset Jac(S)$ (see 21.11).

Similarly, the remaining assertions follow dually to 41.19.

Characterizing the f-supplemented rings in 42.11, we will see that in 41.22 ${}_SS$ and S_S are f-supplemented, and that (2)(b) generalizes 22.1.

41.23 Exercises.

(1) Let M be an R-module with M-injective hull $\widehat{M}$. Prove:

(i) The following assertions are equivalent:

(a) M is π-injective;

(b) if $\widehat{M} = L_1 \oplus L_2$, then $M = (L_1 \cap M) \oplus (L_2 \cap M)$;

(c) if $\widehat{M} = \bigoplus_\Lambda L_\lambda$, then $M = \bigoplus_\Lambda (L_\lambda \cap M)$, Λ index set.

(ii) M is uniform (every submodule essential) if and only if M is π-injective and indecomposable.

(2) An R-module M is said to be an extending module *if every submodule of M lies under a direct summand. By 41.21, π-injective modules have this property. Prove (compare Okado):*

(i) Let M be a locally noetherian extending module and $\{N_\lambda\}_\Lambda$ an independent family of submodules of M. Assume, for every finite subset $A \subset \Lambda$, $\bigoplus_A N_\lambda$ is a direct summand in M.

Then $\bigoplus_\Lambda N_\lambda$ is also a direct summand in M.

(ii) A locally noetherian extending module is a direct sum of uniform modules.

(iii) A module M is locally noetherian if and only if every extending module in $\sigma[M]$ is a direct sum of indecomposable (uniform) modules.

(3) Prove that for an R-module M the following are equivalent:

(a) $M^{(\mathbb{N})}$ is π-injective;

(b) $M^{(\mathbb{N})}$ is M-injective;

(c) $M^{(\mathbb{N})}$ is self-injective.

(4) For $n \geq 2$, let S be the (n,n)-matrix ring over a ring R. Prove that the following assertions are equivalent:

(a) ${}_SS$ is π-injective;

(b) ${}_SS$ is S-injective.

(5) Prove that for an R-module M the following are equivalent:

(a) M is direct injective and every submodule of M lies under a direct summand;

(b) M is π-injective and direct injective;

(c) M is π-injective and every monomorphism $f \in End(M)$ with $Im\, f \trianglelefteq M$ is an isomorphism.

Modules with theses properties are called continuous.

(6) Prove that, for every prime number p, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is hollow but not local.

(7) Recall that every $\mathbb{Z}$-module M is a direct sum of an injective (divisible) and a reduced $\mathbb{Z}$-module (without non-zero injective submodules) (see 39.17, (6)).

Prove that a $\mathbb{Z}$-module M is supplemented if and only if it is a torsion module and, for every prime number p, the divisible part of the p-component $p(M)$ of M (see 15.10) is artinian and the reduced part of $p(M)$ is bounded. (Zöschinger [3])

Literature: HARADA; Ahsan [2], Birkenmeier [2], Fleury, Goel-Jain, Golan [2], Harada [5,7], Hauger, Hausen-Johnson [1], Inoue [1], Jain-Müller, Jain-Saleh, Jain-Singh,G., Jeremy, Li-Zelmanowitz, Mohamed-Müller [1,2], Mohamed-Müller-Singh, Mohamed-Singh, Müller-Rizvi [2], Oshiro [2,3], Okado, Rangaswamy [2], Satyanarayana, Singh-Jain, Takeuchi [1,3], Tiwary-Chaubey, Varadarajan, Zöschinger [1,2,3,4,6].

42 Semiperfect modules and rings

1.Supplements and projective covers. 2.Projective covers and lifting of decompositions. 3.Semiperfect modules. 4.Direct sums and semiperfect modules. 5.Projective semiperfect modules. 6.Semiperfect rings. 7.Nil ideals and lifting of idempotents. 8.f-semiperfect modules. 9.Direct sums and f-semiperfect modules. 10.Projective f-semiperfect modules. 11.f-semiperfect rings. 12.(f-) semiperfect endomorphism rings. 13.Exercises.

In 19.4 projective covers of modules in $\sigma[M]$ were introduced. The following definition is based on this notion:

Let M be an R-module. We call a module N in $\sigma[M]$ *semiperfect in* $\sigma[M]$ if every factor module of N has a projective cover in $\sigma[M]$.

N is called *f-semiperfect in* $\sigma[M]$ if, for every finitely generated submodule $K \subset N$, the factor module N/K has a projective cover in $\sigma[M]$.

Obviously, a simple module is (f-) semiperfect if and only if it has a projective cover. The connection to the notions of the preceding paragraph is given by:

42.1 Supplements and projective covers.

If the R-module N is projective in $\sigma[M]$, then, for a submodule $U \subset N$, the following statements are equivalent:

(a) There exists a direct summand $V \subset N$ with $U + V = N$ and $U \cap V \ll V$;

(b) N/U has a projective cover in $\sigma[M]$.

Proof: $(a) \Rightarrow (b)$ V is projective in $\sigma[M]$, and for the epimorphism $p : V \to N \to N/U$, we have $Ke\, p = U \cap V \ll V$.

$(b) \Rightarrow (a)$ If $\pi : P \to N/U$ is a projective cover, then we can complete the diagram with canonical epimorphism p,

$$\begin{array}{ccccc} & & N & & \\ & & \downarrow p & & \\ P & \xrightarrow{\pi} & N/U & \longrightarrow & 0 \end{array}$$

commutatively by an $f : N \to P$. f is surjective, hence it splits. Therefore there is a $g : P \to N$ with $gf = id_P$ and hence $\pi = gf\pi = gp$. From this we derive $U + (P)g = N$, $(P)g$ is a projective cover of N/U and consequently $U \cap (P)g \ll (P)g$ (see also 19.5).

For further proofs, the following observation will be helpful:

42.2 Projective covers and lifting of decompositions.

Let M be an R-module and $\pi : P \to N$ a projective cover of N in $\sigma[M]$. Assume $N = \bigoplus_\Lambda N_\lambda$ and every N_λ has a projective cover $\pi_\lambda : P_\lambda \to N_\lambda$ in $\sigma[M]$. Then there is an isomorphism $g : \bigoplus_\Lambda P_\lambda \to P$ with $(P_\lambda)g\pi = N_\lambda$ for all $\lambda \in \Lambda$.

Proof: Let $\bigoplus_\Lambda \pi_\lambda : \bigoplus_\Lambda P_\lambda \to \bigoplus_\Lambda N_\lambda = N$ be the canonical epimorphism. The diagram

$$\begin{array}{ccccc} & & \bigoplus_\Lambda P_\lambda & & \\ & & \downarrow & & \\ P & \xrightarrow{\pi} & N & \longrightarrow & 0 \end{array}$$

can be commutatively completed by some $g : \bigoplus_\Lambda P_\lambda \to P$. Since $Ke\, \pi \ll P$ the map g is an epimorphism, therefore it splits and $Ke\, g$ is direct a summand of $\bigoplus_\Lambda P_\lambda$. Now we have

$$Ke\, g \subset Ke\, (\bigoplus_\Lambda \pi_\lambda) = \bigoplus_\Lambda Ke\, \pi_\lambda \subset Rad(\bigoplus_\Lambda P_\lambda).$$

However, by 22.3, the radical of a projective module in $\sigma[M]$ cannot contain a non-zero direct summand. Hence g is an isomorphism.

42.3 Semiperfect modules.

Let M be an R-module.

(1) A projective module in $\sigma[M]$ is semiperfect in $\sigma[M]$ if and only if it is (amply) supplemented.

(2) If N is a semiperfect module in $\sigma[M]$, then:

(i) Every factor module of N is semiperfect;

(ii) if $\pi : P \to N$ is an epimorphism in $\sigma[M]$ with $Ke\, \pi \ll P$, then P is also semiperfect;

(iii) $Rad(N) \ll N$ and N is amply supplemented, hence the factor module $N/Rad(N)$ is semisimple.

(3) A module in $\sigma[M]$ is semiperfect if and only if it has a semiperfect (supplemented) projective cover in $\sigma[M]$.

Proof: (1) follows from 42.1 by observing that in a (π-) projective module every supplement is a direct summand (see 41.16).

(2)(i) follows directly from the definition.

(ii) For $U \subset P$ we have the following diagram with canonical maps

$$\begin{array}{ccc} P & \xrightarrow{\pi} & N \\ \downarrow & & \downarrow \\ P/U & \xrightarrow{g} & N/(U)\pi \end{array} ,$$

where g is epic with superfluous kernel. If $\pi' : Q \to N/(U)\pi$ is a projective cover, then there exists $h : Q \to P/U$ with $hg = \pi'$ and $Ke\, h \ll Q$. Hence h is a projective cover of P/U.

(iii) Let $\pi : P \to N$ be a projective cover of N. Then by (1) and (ii), P is amply supplemented, and every supplement is a direct summand. By 41.12, every non-superfluous submodule contains a non-zero direct summand. We saw in 22.3 that the radical of a projective module in $\sigma[M]$ contains no non-trivial direct summand. Hence $Rad(P) \ll P$.

Since $P/Rad(P)$ is semisimple (P is supplemented), we have $Rad\, N = (Rad\, P)\pi$ (see 23.3), and therefore also $Rad\, N \ll N$. Being the image of the amply supplemented module P, the module N is also amply supplemented.

(3) is a consequence of (1) and (2).

42.4 Direct sums and semiperfect modules.
Let M be an R-module.

(1) For a projective module P in $\sigma[M]$ the following are equivalent:

(a) P is semiperfect in $\sigma[M]$;

(b) $Rad(P) \ll P$ and $P = \bigoplus_\Lambda L_\lambda$, with the L_λ's projective covers of simple modules.

(2) A direct sum $\bigoplus_\Lambda P_\lambda$ of projective modules P_λ is semiperfect in $\sigma[M]$ if and only if every P_λ is semiperfect in $\sigma[M]$ and $Rad(\bigoplus_\Lambda P_\lambda) \ll \bigoplus_\Lambda P_\lambda$.

(3) A direct sum $N = \bigoplus_\Lambda N_\lambda$ in $\sigma[M]$ is semiperfect if and only if every N_λ is semiperfect, N has a projective cover, and $Rad(N) \ll N$.

(4) Assume N is a semisimple module with projective cover P in $\sigma[M]$. Then P is semiperfect if and only if every simple summand of N has a projective cover in $\sigma[M]$.

Proof: (1) $(a) \Rightarrow (b)$ If P is semiperfect, then $Rad(P) \ll P$ by 42.3. By the decomposition properties of π-projective supplemented modules (see 41.17), we see that $P = \bigoplus_\Lambda L_\lambda$ with L_λ local. Being local and projective modules, the L_λ are projective covers of simple modules in $\sigma[M]$ (see 19.7).

$(b) \Rightarrow (a)$ Assume $P = \bigoplus_\Lambda L_\lambda$ with local projective modules L_λ and $Rad(P) \ll P$. For a submodule $U \subset P$, we set $U' = U + Rad(P)$. Since $P/Rad(P)$ is semisimple, this also holds for P/U'. For a suitable subset $\Lambda' \subset \Lambda$, we obtain an epimorphism

$$\pi : \bigoplus\nolimits_{\Lambda'} L_\lambda \to P/U' \text{ with } Ke\,\pi = Rad(\bigoplus\nolimits_{\Lambda'} L_\lambda) \ll \bigoplus\nolimits_{\Lambda'} L_\lambda,$$

hence a projective cover of P/U'. Then, by 42.1, there is a supplement of U' in P and this is also a supplement of U (since $Rad(P) \ll P$).

(2) is a direct consequence of (1).

(3) The necessity of the given conditions is obvious.

If on the other hand, $P \to N$ is a projective cover of N and $P_\lambda \to N_\lambda$ are projective covers of the N_λ, then, by 42.2, $P \simeq \bigoplus_\Lambda P_\lambda$. If $Rad(N) \ll N$, the map $P \to N \to N/Rad(N)$ has a superfluous kernel (see 19.3). Since $Rad(P)$ is contained in the kernel of this map, this means $Rad(P) \ll P$. Hence, by (2), P is semiperfect in $\sigma[M]$.

(4) can be derived from 42.2 and (1).

42.5 Projective semiperfect modules.

Assume the R-module M to be projective in $\sigma[M]$. *Then the following statements are equivalent:*

(a) M is semiperfect in $\sigma[M]$*;*

(b) M is (amply) supplemented;

(c) every finitely M-generated module is semiperfect in $\sigma[M]$*;*

(d) every finitely M-generated module has a projective cover in $\sigma[M]$*;*

(e) every finitely M-generated module is (amply) supplemented;

(f) (α) $M/Rad(M)$ *is semisimple and* $Rad(M) \ll M$, *and*
(β) *decompositions of* $M/Rad(M)$ *can be lifted under* $M \to M/Rad(M)$*;*

(g) every proper submodule is contained in a maximal submodule of M, and
(α) every simple factor module of M has a projective cover in $\sigma[M]$, *or*
(β) every maximal and every cyclic submodule has a supplement in M;

(h) M is a direct sum of local modules and $Rad(M) \ll M$.

Proof: $(a) \Leftrightarrow (b)$ is shown in 42.3.

The equivalence of (a), (c), (d) and (e) follows from the facts that all M^k, $k \in I\!N$, are projective and supplemented, and that factor modules of (amply) supplemented modules are again (amply) supplemented (see 41.1, 41.2).

$(a) \Rightarrow (f)$ (α) is shown in 42.3. (β) was obtained in 41.17 for π-projective supplemented modules. It can also be derived from 42.2.

$(f) \Rightarrow (a)$ Let $U \subset M$ and $U' = U + Rad(M)$. Then M/U' is a direct summand of $M/Rad(M)$ and so, by (β), there exists a decomposition $M = M_1 \oplus M_2$ with $M_1/Rad(M_1) \simeq M/U'$. Since $Rad(M_1) \ll M_1$, the module M_1 is a projective cover of M/U'. The kernel of $M/U \to M/U'$ is superfluous and hence M_1 also provides a projective cover of M/U (by 19.3,(1)).

$(a) \Rightarrow (g)$ We have seen earlier (e.g. 41.1) that in a supplemented module every submodule is contained in a maximal submodule. (α) and (β) are clear.

$(g) \Rightarrow (h)$ If every proper submodule is contained in a maximal one, then $Rad(M) \ll M$ (see 21.6). Assume (β). Let U be a maximal submodule of M. Then a supplement V of U is cyclic and has itself a supplement. But mutual supplements in (π-) projective modules are direct summands. By 42.1, M/U has a projective cover, i.e. (α) holds.

From the properties in (g) we conclude (by 41.5) that M is a sum of local modules, and hence $M/Rad(M)$ is semisimple (see proof of 41.5). Every simple summand of $M/Rad(M)$ is a simple factor module of M and therefore has a projective cover in $\sigma[M]$. Hence, by 42.4, M is a direct sum of local modules.

$(h) \Rightarrow (a)$ is an assertion of 42.4.

We call an idempotent $e \in R$ *local* if $eRe \simeq End_R(Re)$ is a local ring. For $M = R$, the assertions in 42.5 can be formulated in the following way:

42.6 Semiperfect rings. Characterizations.

For a ring R the following statements are equivalent:

(a) ${}_RR$ *is semiperfect;*

(b) ${}_RR$ *is supplemented;*

(c) every finitely generated R-module is semiperfect in R-MOD;

(d) every finitely generated R-module has a projective cover in R-MOD;

(e) every finitely generated R-module is (amply) supplemented;

(f) $R/Jac(R)$ *is left semisimple and idempotents in* $R/Jac(R)$ *can be lifted to R;*

(g) every simple R-module has a projective cover in R-MOD;

(h) every maximal left ideal has a supplement in R;

(i) ${}_RR$ *is a (direct) sum of local (projective covers of simple) modules;*

(j) $R = Re_1 \oplus \cdots \oplus Re_k$ *for local orthogonal idempotents* e_i*;*

(k) R_R *is semiperfect.*

If R satisfies these conditions, then R is called a *semiperfect ring.* The assertions $(b) - (j)$ hold similarly for right modules.

Proof: Most of the equivalences result from 42.5.

For $(b) \Leftrightarrow (h) \Leftrightarrow (i)$ we refer to 41.6.

$(i) \Leftrightarrow (j)$ can be derived from the definition of local idempotents in view of 19.7.

The characterization in (f) is left-right-symmetric, and hence $(f) \Leftrightarrow (k)$ follows in a similar way to $(a) \Leftrightarrow (f)$.

The lifting of idempotents from $R/Jac(R)$ to R is always possible if $Jac(R)$ is a nil ideal. This is a consequence of

42.7 Nil ideals and lifting of idempotents.
If J is a nil ideal in the ring R, then idempotents in R/J can be lifted under $R \to R/J$.

Proof: Let $g \in R$ with $g^2 + J = g + J$. We look for an idempotent $e \in R$ with $e + J = g + J$. Since $g^2 - g \in J$, there exists $k \in I\!N$ with $(g^2 - g)^k = 0$. This yields $0 = g^k(1-g)^k = g^k - g^{k+1}p$, where $p = p(g)$ is an element of the ring $Z\!\!Z[g]$ and therefore commutes with g. Setting $e = g^k p^k$ we get

$$e = g^k p^k = (g^{k+1}p)p^k = g^{k+1}p^{k+1} = \cdots = g^{2k}p^{2k} = e^2 \text{ and}$$

$$g + J = g^k + J = g^{k+1}p + J = (g^{k+1} + J)(p + J) = (g + J)(p + J) = gp + J,$$

hence $g + J = (g + J)^k = (gp + J)^k = e + J$.

Analogously to semiperfect modules and rings, f-semiperfect modules and rings can be described. Fundamental for this are the

42.8 Properties of f-semiperfect modules.
Let M be an R-module.

(1) For a projective module P in $\sigma[M]$ the following are equivalent:
(a) P is f-semiperfect;
(b) every finitely generated submodule lies above a direct summand of P.

If P is finitely generated this is also equivalent to:
(c) P is (amply) f-supplemented.

(2) For an f-semiperfect module N in $\sigma[M]$ we have:
(i) For superfluous and for finitely generated submodules $L \subset N$, the factor module N/L is f-semiperfect in $\sigma[M]$;
(ii) if $\pi : P \to N$ is an epimorphism with $Ke\,\pi \ll P$, then P is also f-semiperfect in $\sigma[M]$;
(iii) N is finitely supplemented, and if $Rad(N) \ll N$, every finitely generated submodule of $N/Rad(N)$ is a direct summand.

(3) A module in $\sigma[M]$ is f-semiperfect if and only if it has a projective cover which is f-semiperfect in $\sigma[M]$.

Proof: (1) $(a) \Leftrightarrow (b)$ follows directly from 42.1 and 41.14,(3).

$(b) \Leftrightarrow (c)$ If P is finitely generated, then supplements in P are finitely generated (see 41.1). Moreover, mutual supplements are direct summands in P (see 41.14).

(2) (i) Let L, K be submodules of N, K finitely generated.

We have $(N/L)/[(L+K)/L] \simeq N/(L+K)$. If L is finitely generated, then this module has a projective cover. If $L \ll N$, then the kernel of $N/K \to N/(L+K)$ is superfluous in N/K, and the projective cover of N/K yields a projective cover of $N/(L+K)$ (see 19.3).

(ii) can be seen from the proof of (2)(ii) in 42.3.

(iii) By (ii), the projective cover of N is f-semiperfect and, by (1), finitely supplemented. Therefore N, and if $Rad(N) \ll N$ also $N/Rad(N)$, are finitely supplemented.

(3) is a consequence of (2).

42.9 Direct sums and f-semiperfect modules.
Let M be an R-module.

(1) Direct summands of a projective, f-semiperfect module in $\sigma[M]$ are f-semiperfect in $\sigma[M]$.

(2) A direct sum of finitely generated, projective modules in $\sigma[M]$ is f-semiperfect if and only if every summand is f-semiperfect in $\sigma[M]$.

(3) A direct sum of finitely generated, f-semiperfect modules in $\sigma[M]$ is f-semiperfect in $\sigma[M]$ if and only if it has a projective cover in $\sigma[M]$.

(4) If P is projective and f-semiperfect in $\sigma[M]$ with $Rad(P) \ll P$, then P is a direct sum of cyclic (f-semiperfect) modules.

Proof: (1) Let N be a direct summand of a projective, f-semiperfect module P. Then every finitely generated submodule of N lies above a direct summand X of P. X is also a direct summand of N, hence N is f-semiperfect.

(2) Let $\{P_\lambda\}_\Lambda$ be a family of finitely generated, projective and f-semiperfect modules in $\sigma[M]$ and set $P = \bigoplus_\Lambda P_\lambda$. A finitely generated submodule $U \subset P$ is contained in a finite partial sum P'. By 41.3, P' is finitely supplemented since all P_λ are finitely supplemented, and therefore P' is f-semiperfect (by 42.8). Hence P'/U has a projective cover Q in $\sigma[M]$. If $P = P' \oplus P''$, then $Q \oplus P''$ yields a projective cover of $P/U \simeq P'/U \oplus P''$.

(3) Let $\{N_\lambda\}_\Lambda$ be a family of finitely generated, f-semiperfect modules in $\sigma[M]$. If $N = \bigoplus_\Lambda N_\lambda$ is f-semiperfect, it has a projective cover.

On the other hand, let $\pi : P \to N$ be a projective cover. If $\pi_\lambda : P_\lambda \to N_\lambda$ are projective covers of the N_λ, then, by 42.2, $P \simeq \bigoplus_\Lambda P_\lambda$, and, by (2), P – and hence also N – is f-semiperfect (see 42.8).

(4) If P is projective in $\sigma[M]$, then, by Kaplansky's Theorem 8.10, P is a direct sum of countably generated modules. Hence it is enough to prove the assertion for countably generated modules. This is done by induction.

Consider $P = \sum_{i\in I\!N} Rm_i$, $m_i \in P$. Then, by 42.8,(1), there is a decomposition $P = P_1 \oplus Q_1$ such that $P_1 \subset Rm_1$, $Rm_1 \subset P_1 + K_1$ with $K_1 = Rm_1 \cap Q_1 \ll M$. As a direct summand of Rm_1, the module P_1 is cyclic. Assume, for $k \in I\!N$, we have found cyclic modules $P_i \subset P$ with

$$P = (\sum_{i\le k} P_i) \oplus Q_k \text{ and } \sum_{i\le k} Rm_i \subset (\bigoplus_{i\le k} P_i) + K_k, \quad K_k \ll M.$$

Now Q_k is f-semiperfect, i.e. there is a decomposition $Q_k = P_{k+1} \oplus Q_{k+1}$, with $P_{k+1} \subset Rm_{k+1}$, $Rm_{k+1} \subset P_{k+1} + K'_{k+1}$ and $K'_{k+1} = Rm_{k+1} \cap Q_{k+1} \ll P$.

Hence we have $P = (\bigoplus_{i\le k+1} P_i) \oplus Q_{k+1}$ and

$$\sum_{i\le k+1} Rm_i \subset (\bigoplus_{i\le k+1} P_i) + K_{k+1} \text{ with } K_{k+1} = K'_{k+1} + K_k \ll M.$$

Since $\sum_{i\in I\!N} K_i \subset Rad(P) \ll P$, we finally get

$$P = \sum_{i\in I\!N} Rm_i = (\bigoplus_{i\in I\!N} P_i) + \sum_{i\in I\!N} K_i = \bigoplus_{i\in I\!N} P_i.$$

Similarly to 42.5 we have here:

42.10 Projective f-semiperfect modules.
For a finitely generated, self-projective R-module M, the following statements are equivalent:
(a) M is f-semiperfect in $\sigma[M]$;
(b) M is finitely supplemented;
(c) all finitely presented, M-generated modules in $\sigma[M]$
  (i) are f-semiperfect in $\sigma[M]$, or
  (ii) are (amply) f-supplemented, or
  (iii) have projective covers in $\sigma[M]$;
(d) $M/Rad(M)$ is regular in $\sigma[M/Rad(M)]$, and every decomposition of $M/Rad(M)$ can be lifted under $M \to M/Rad(M)$.

Proof: $(a) \Leftrightarrow (b)$ was shown in 42.8.

$(a) \Leftrightarrow (c)$ The finitely presented, M-generated modules are of the form M^k/L with finitely generated $L \subset M^k$, $k \in I\!N$. Hence the assertions follow from 42.8 and 41.3.

$(a) \Rightarrow (d)$ $M/Rad(M)$ is self-projective and f-semiperfect, and so every finitely generated submodule in it is a direct summand. Hence $M/Rad(M)$ is regular in $\sigma[M]$ (see 37.4). The summands of $M/Rad(M)$ have projective covers and, by 42.2, decompositions of $M/Rad(M)$ can be lifted.

$(d) \Rightarrow (a)$ can be seen with the same argument as $(f) \Rightarrow (a)$ in 42.5.

In particular, for $M = R$ we obtain:

42.11 f-semiperfect rings. Characterizations.
For a ring R, the following properties are equivalent:
(a) ${}_RR$ is f-semiperfect in R-MOD;
(b) ${}_RR$ is finitely supplemented;
(c) all finitely presented modules in R-MOD
(i) are f-semiperfect, or
(ii) are (amply) finitely supplemented, or
(iii) have projective covers;
(d) $R/Jac(R)$ is regular and idempotents in $R/Jac(R)$ can be lifted to R;
(e) every cyclic left ideal has a supplement in ${}_RR$;
(f) every cyclic left ideal lies above a direct summand of ${}_RR$;
(g) for every $a \in R$ there is an idempotent $e \in Ra$ with $a(1-e) \in Jac(R)$;
(h) R_R is f-semiperfect in MOD-R.

The corresponding assertions $(b) - (g)$ for right modules are also equivalent to the above.

If R satisfies these properties, then R is called an *f-semiperfect ring.*

Proof: The equivalences of (a) to (d) result from 42.10.

$(a) \Rightarrow (e)$ is clear.

$(e) \Rightarrow (f)$ Every supplement in R is cyclic, and mutual supplements are direct summands.

$(f) \Rightarrow (b)$ follows from 41.13 and 42.8.

$(f) \Leftrightarrow (g)$ follows from the properties of $Jac(R)$ (see 21.11).

Since (d) is left-right-symmetric, the equivalence $(d) \Leftrightarrow (h)$ is obtained similarly to $(a) \Leftrightarrow (d)$.

As we have seen in 22.1, the endomorphism rings of self-injective modules are examples for f-semiperfect rings.

Our knowledge about endomorphism rings of direct projective modules (see 41.19) enables us to prove:

42.12 (f-) semiperfect endomorphism rings.
Assume the R-module M is projective in $\sigma[M]$, and put $S = End_R(M)$.

(1) S is semiperfect if and only if M is finitely generated and semiperfect in $\sigma[M]$.

(2) If M is semiperfect, or M is finitely generated and f-semiperfect in $\sigma[M]$, then S is f-semiperfect.

(3) If S is f-semiperfect, then $Rad(M) \ll M$ and M is a direct sum of cyclic modules.

Proof: (1) If S is semiperfect, then, by 42.6, $S = Se_1 \oplus \cdots \oplus Se_k$ with local idempotents $e_i \in S$. So we have a decomposition $M = Me_1 \oplus \cdots \oplus Me_k$, where the $End(Me_i) \simeq e_iSe_i$ (see 8.7) are local rings. We know from 19.7 that the Me_i's are local modules. Hence M is finitely generated and, by 42.5, semiperfect in $\sigma[M]$.

If M is semiperfect, then, for every $f \in S$, the image $Im\, f$ lies above a direct summand and, by 41.19, S is f-semiperfect.

If, moreover, M is finitely generated, then $M/Rad(M)$ is semisimple (by 42.3) and finitely generated, and $S/Jac(S) \simeq End(M/Rad(M))$ is a left semisimple ring (see 22.2, 20.6). Hence, by 42.6, S is semiperfect.

(2) Under the given conditions, for every $f \in S$, the image of f lies above a direct summand and S is f-semiperfect by 41.19 and 42.11.

(3) Let S be f-semiperfect. Assume $Rad(M)$ is not superfluous in M and $Rad(M) + K = M$ with $K \neq M$. Then $Rad(M) \to M \to M/K$ is epic and there exists $f : M \to Rad(M)$ with $Im\, f + K = M$. Hence $Im\, f$ is not superfluous in M and, by 41.19, it lies above a non-zero direct summand of M. However, by 22.3, the radical of a projective module in $\sigma[M]$ cannot contain a non-zero direct summand. Hence $Rad(M) \ll M$.

In view of Kaplansky's Theorem 8.10, we may assume M is countably generated, i.e. $M = \sum_{i \in \mathbb{N}} Rm_i$. Since the canonical map $f : \bigoplus_{\mathbb{N}} Rm_i \to M$ splits, there exists $g : M \to \bigoplus_{\mathbb{N}} Rm_i$ with $gf = id_M$. Forming, with the canonical projections π_i, the morphisms $g_i = g\pi_i \in End(M)$, we obtain $M = \sum_{\mathbb{N}} (M)g_i$. By 41.19, $Im\, g_1$ lies above a direct summand. Hence

$$M = P_1 \oplus Q_1 \text{ with } P_1 \subset Im\, g_1 \text{ and } Im\, g_1 \cap Q_1 \ll M.$$

As a direct summand of Rm_1, the module P_1 is cyclic.

Since also for every $h \in End(Q_1)$, the image $Im\, h$ lies above a direct summand we can confirm the assertion similarly to the proof of (4) in 42.9 by induction.

42.13 Exercises.

(1) Let R be a semiperfect ring. Prove that for a projective R-module P, the following statements are equivalent:
(a) Every epimorphism $P \to P$ is an isomorphism (P is a Hopf module);
(b) P is finitely generated.

(2) Let R be a semiperfect ring. Prove: Every R-projective module P with $Rad(P) \ll P$ is projective in R-MOD.

(3) Let R be a semiperfect ring. Prove that, for an R-module M, the following statements are equivalent:
(a) M is R-projective;
(b) $Hom_R(M,-)$ is exact with respect to exact sequences
$0 \to K \to R \to R/K \to 0$ *in R-MOD with* $K \ll R$.
(compare 19.10,(8))

(4) Consider the following subring of the rational numbers:

$R = \{\frac{m}{n} \mid m,n \in \mathbb{Z},\ (m,n) = 1,$ *2 and 3 are not divisors of n*$\}$.

Prove that $R/Jac(R)$ is semisimple but R is not semiperfect.

(5) Let R be a ring and J an ideal of R contained in $Jac(R)$. Show that the following are equivalent:
(a) Idempotents in R/J can be lifted under $R \to R/J$;
(b) every direct summand in ${}_RR/J$ has a projective cover;
(c) any finite set of orthogonal idempotents in R/J can be lifted to orthogonal idempotents under $R \to R/J$.

(6) Show that for a commutative ring R the following are equivalent:
(a) R is semiperfect;
(b) R is a finite direct product of local rings.

Literature: ANDERSON-FULLER, KASCH, RENAULT;
Ahsan [1], Azumaya [1], Fieldhouse [2], Golan [1,2], Hausen-Johnson [2], Hill [1], Hiremath [8], Jansen, Jøndrup-Simson, Ketkar-Vanaja [1,2], Koh, Nicholson [1,2,4], Oshiro [2], Rangaswamy-Vanaja [3], Rowen [2,3], Snider, Szeto [2], Varadarajan, Ware, Wisbauer [4,5].

43 Perfect modules and rings

1.Projective covers and perfect modules. 2.Perfect modules. 3.Sequences of homomorphisms. 4.Endomorphism rings and finiteness conditions. 5.t-nilpotent ideals. 6.Ascending chain condition for M-cyclic modules. 7.Supplements under $Hom(M,-)$. 8.Finitely generated, perfect modules. 9.Left perfect rings. 10.Right perfect endomorphism rings. 11.Perfect modules over commutative rings. 12.Exercises.

We saw in the preceding paragraph that the direct sum of semiperfect modules need not be semiperfect. In this section we want to study the question when, for a module N, every sum $N^{(\Lambda)}$ is semiperfect.

Let M be an R-module and N in $\sigma[M]$. We call N *perfect in* $\sigma[M]$ if, for every index set Λ, the sum $N^{(\Lambda)}$ is semiperfect in $\sigma[M]$.

To begin with let us point out that we can restrict our investigations to projective modules in $\sigma[M]$:

43.1 Projective covers and perfect modules.
Let M be an R-module.

(1) For N in $\sigma[M]$, the following statements are equivalent:
(a) N is perfect in $\sigma[M]$;
(b) N has a projective cover P and P is perfect in $\sigma[M]$.

(2) For M, the following statements are equivalent:
(a) M is projective and perfect in $\sigma[M]$;
(b) for every set Λ, $M^{(\Lambda)}$ is supplemented and π-projective.

Proof: (1) $(a) \Rightarrow (b)$ Let N be perfect and P a projective cover of N in $\sigma[M]$. Then P is semiperfect and, by 42.2, $P^{(\Lambda)}$ is a projective cover of $N^{(\Lambda)}$. Since $Rad\,(N^{(\Lambda)}) \ll N^{(\Lambda)}$ (see 42.3), the kernel of

$$P^{(\Lambda)} \to N^{(\Lambda)} \to N^{(\Lambda)}/Rad\,(N^{(\Lambda)})$$

is also superfluous in $P^{(\Lambda)}$ (see 19.3). Hence, by 42.4, $P^{(\Lambda)}$ is semiperfect for every set Λ and consequently P is perfect in $\sigma[M]$.

$(b) \Rightarrow (a)$ If P is perfect, then $P^{(\Lambda)}$ is semiperfect in $\sigma[M]$. Since $P^{(\Lambda)} \to N^{(\Lambda)}$ is epic, $N^{(\Lambda)}$ is also semiperfect in $\sigma[M]$.

(2) $(a) \Rightarrow (b)$ is clear.

$(b) \Rightarrow (a)$ Since $M^{(\Lambda)} \oplus M^{(\Lambda)}$ is π-projective, $M^{(\Lambda)}$ is self-projective by 41.14, and M is $M^{(\Lambda)}$-projective, i.e. M is projective in $\sigma[M]$. Therefore $M^{(\Lambda)}$ is projective and supplemented, i.e. semiperfect in $\sigma[M]$ (see 42.3).

43.2 Perfect modules. Characterizations and properties. *Let M be an R-module and P a projective module in $\sigma[M]$.*

(1) The following statements are equivalent:

(a) P is perfect in $\sigma[M]$;

(b) P is semiperfect in $\sigma[M]$ and $Rad\,P^{(\Lambda)} \ll P^{(\Lambda)}$ for every set Λ;

(c) every P-generated module has a projective cover in $\sigma[M]$;

(d) every P-generated module is (amply) supplemented.

(2) If P is perfect in $\sigma[M]$, then:

(i) Every P-generated, flat module in $\sigma[M]$ is projective in $\sigma[M]$;

(ii) $P/Rad(P)$ is semisimple, and every P-generated module has a superfluous radical;

(iii) $End_R(P^{(\Lambda)})$ is f-semiperfect for every set Λ.

Proof: (1) $(a) \Leftrightarrow (b)$ follows from 42.4,(3). The other implications follow from 42.5 since every P-generated module is a factor module of $P^{(\Lambda)}$ for suitable Λ.

(2)(i) P is a direct sum of local modules (see 42.4) which obviously are finitely presented in $\sigma[M]$. Therefore every P-generated module L is generated by finitely presented modules in $\sigma[M]$. By (1)(c), L has a projective cover. If L is flat in $\sigma[M]$, then we see from 36.4 that L has to be projective in $\sigma[M]$.

(ii) Since $Rad\,P^{(\Lambda)} \ll P^{(\Lambda)}$ and $P^{(\Lambda)}$ is a good module (see 23.3, 23.4), every factor module of $P^{(\Lambda)}$ has a superfluous radical.

(iii) follows from 42.12.

We will see later on (in 51.4) that any of the properties given in 43.2,(2) characterizes P as a perfect module. Moreover, we will find out that some of the properties in 43.2 need not be demanded for all sets Λ but only for $\Lambda = I\!N$. For this the following technical lemma is helpful:

43.3 Sequences of homomorphisms.

Let $\{N_i\}_{I\!N}$ be a family of R-modules, $\{f_i : N_i \to N_{i+1}\}_{I\!N}$ a family of homomorphisms and $N = \bigoplus_{I\!N} N_i$.

With the canonical inclusions $\varepsilon_i : N_i \to N$ define

$g_i : N_i \to N$ *by* $g_i = \varepsilon_i - f_i\varepsilon_{i+1}$, *and*

$g : N \to N$ *with* $\varepsilon_i g = g_i,\ i \in I\!N$.

(1) $\{Im\ g_i\}_{I\!N}$ is an independent family of submodules and, for every $k \in I\!N$, the partial sum $\bigoplus_{i\leq k} Im\ g_i$ is a direct summand in N.

(2) If $\sum_{I\!N} Im\ f_i\varepsilon_{i+1} \ll N$, then, for every $m \in N_1$, there exists $r \in I\!N$ with $(m)f_1 \cdots f_r = 0$.

(3) If $Im\ g$ is a direct summand in N, then, for any finitely many $m_1, \ldots, m_t \in N_1$, there exist $r \in I\!N$ and $h_{r+1,r} \in Hom(N_{r+1}, N_r)$ with

$$(m_i)f_1 \cdots f_{r-1} = (m_i)f_1 \cdots f_r h_{r+1,r} \quad \textit{for} \quad i = 1, \ldots, t.$$

If N_1 is finitely generated, then $f_1 \cdots f_{r-1} = f_1 \cdots f_r h_{r+1,r}$ for some $r \in I\!N$.

Proof: (1) For $k = 2$, we have $Im\ g_1 \cap Im\ g_2 = 0$ and

$$N = Im\ g_1 \oplus Im\ g_2 \oplus (\bigoplus\nolimits_{i \geq 3} N_i).$$

In a similar way the assertion can be confirmed for every $k \in I\!N$.

(2) We have $N = Im\ g + \sum_{i \in I\!N} Im\ f_i \varepsilon_{i+1} = Im\ g$. Hence, for $m \in N_1$, there is a representation $m\varepsilon_1 = \sum_{i \leq r} m_i(\varepsilon_i - f_i \varepsilon_{i+1})$ for some $m_i \in N_i$. Comparing the components we derive $m\varepsilon_1 = m_1\varepsilon_1$, $m_{i+1}\varepsilon_{i+1} = m_i f_i \varepsilon_{i+1}$, hence $m_{i+1} = m_i f_i$ for $i \leq r$, and consequently

$$0 = (m_r)f_r = (m_{r-1})f_{r-1}f_r = \cdots = (m)f_1 \cdots f_r.$$

(3) g is monic, hence there exists $g^{-1} : Im\ g \to N$. If $Im\ g$ is a direct summand, then there is a projection $e : N \to Im\ g$ with $ge = g$. Then, for $h = eg^{-1} \in End(N)$, we have $gh = geg^{-1} = id_N$. Thus, with the canonical projections $\pi_j : N \to N_j$, we have, for $x \in N_i$, the relations

$$x\varepsilon_i = x\varepsilon_i gh = x(\varepsilon_i - f_i\varepsilon_{i+1})h = \sum\nolimits_j x(\varepsilon_i - f_i \varepsilon_{i+1})h\pi_j\varepsilon_j.$$

Setting $h_{i,j} = \varepsilon_i h \pi_j$ we obtain, by comparing the components,

$$-xf_ih_{i+1,i} = x - xh_{i,i} \text{ und } xh_{i,j} - xf_ih_{i+1,j} = 0 \text{ for } j \neq i.$$

Using these relations step by step, beginning with $x = (m)f_1 \cdots f_{r-1} \in N_r$ for $m \in N_1$, we get

$$\begin{array}{rl} -mf_1 \ldots f_{r-1}f_rh_{r+1,r} & = mf_1 \ldots f_{r-1} - mf_1 \ldots f_{r-1}h_{r,r} \\ & = mf_1 \ldots f_{r-1} - mf_1 \ldots f_{r-2}h_{r-1,r} \\ & \vdots \\ & = mf_1 \cdots f_{r-1} - mf_1h_{1,r}. \end{array}$$

Choosing r large enough to obtain $mf_1h_{1,r} = 0$ for all $m \in \{m_1, \ldots, m_t\}$, we have $mf_1 \cdots f_{r-1} = mf_1 \cdots f_r(-h_{r+1,r})$ for all these m.

Let us apply this knowledge to endomorphism rings of finitely generated modules.

We call a subset X of a ring R *right t-nilpotent* if, for every sequence $x_1, x_2, \dots$ of elements in X, there is a $k \in \mathbb{N}$ with $x_1 x_2 \cdots x_k = 0$.

Similarly *left t-nilpotent* is defined.

Recall that a subset J of a ring S is said to *act t-nilpotently* on a right S-module M_S if, for every sequence $s_1, s_2, \dots$ of elements in J and $m \in M$, $m s_1 s_2 \cdots s_i = 0$ for some $i \in \mathbb{N}$ (see 31.8). J is right t-nilpotent if it acts t-nilpotently on S_S.

43.4 Endomorphism rings and finiteness conditions.
Let M be an R-module and $S = End_R(M)$.

(1) If M is projective in $\sigma[M]$ and $Rad(M^{(\mathbb{N})}) \ll M^{(\mathbb{N})}$, then $Jac(S)$ acts locally t-nilpotently on M_S.

If, in addition, ${}_RM$ is finitely generated, $Jac(S)$ is right t-nilpotent.

(2) Assume M is finitely generated and satisfies one of the conditions
(i) M is self-projective and perfect in $\sigma[M]$,
(ii) M is self-projective, and M-generated flat modules in $\sigma[M]$ are projective in $\sigma[M]$,
(iii) $M^{(\mathbb{N})}$ is (pure) injective,
(iv) $End_R(M^{(\mathbb{N})})$ is regular.

Then S satisfies the descending chain condition for cyclic right ideals. Hence $S/Jac(S)$ is left semisimple and $Jac(S)$ is right t-nilpotent.

Proof: (1) If M is projective and $s_i \in Jac(S)$, then $Im\ s_i \ll M$ (see 22.2) and, by 43.3, the set $\{s_i : M \to M\}_{\mathbb{N}}$ acts locally t-nilpotently on M_S.

(2) Every descending chain of cyclic right ideals in S can be written as $f_1 S \supset f_1 f_2 S \supset f_1 f_2 f_3 S \supset \cdots$ with a sequence $\{f_i\}_{\mathbb{N}}$ in S. By 43.3, such a sequence becomes stationary if – with analogous notation – $Im\ g$ is a direct summand in $M^{(\mathbb{N})}$.

$(i) \Rightarrow (ii)$ This we already know from 43.2.

Assume (ii). Since $Im\ g$ is a direct limit of direct summands $\bigoplus_{i \le n} Im\ g_i$ of $M^{(\mathbb{N})}$, we obtain $M^{(\mathbb{N})}/Im\ g$ as a direct limit of projective modules. Hence $M^{(\mathbb{N})}/Im\ g$ is flat and therefore projective, i.e. $Im\ g$ is a direct summand in $M^{(\mathbb{N})}$.

Assume (iii). $Im\ g$ is a direct limit of direct summands and hence is a pure submodule of $M^{(\mathbb{N})}$ (see 33.8). Since $M^{(\mathbb{N})} \simeq Im\ g$ is pure injective, it is a direct summand.

If $End_R(M^{(\mathbb{N})})$ is regular, then $Im\ g$ is a direct summand by 37.7.

The last assertions result from 31.8 (changing sides).

The importance of t-nilpotent ideals lies in the fact that they allow us to extend the Nakayama Lemma 21.13 for quasi-regular left ideals and finitely generated modules to arbitrary modules:

43.5 t-nilpotent ideals and superfluous submodules.

For a left ideal I in the ring R, the following statements are equivalent:

(a) I is right t-nilpotent;

(b) $IM \neq M$ for every non-zero left R-module M;

(c) $IM \ll M$ for every non-zero left R-module M;

(d) $IR^{(\mathbb{N})} \ll R^{(\mathbb{N})}$.

Proof: $(a) \Rightarrow (b)$ Assume, for a non-zero R-module M, that $IM = M$. Then there exists $a_1 \in I$ with $a_1 M = a_1 IM \neq 0$. Now there exists $a_2 \in I$ with $a_1 a_2 M = a_1 a_2 IM \neq 0$. In this way we find a sequence $\{a_i\}_{\mathbb{N}} \in I$ with $a_1 \cdots a_n \neq 0$ for every $n \in \mathbb{N}$.

$(b) \Rightarrow (c)$ Consider a submodule $N \subset M$ with $IM + N = M$. Then

$$I(M/N) = (IM + N)/N = M/N,$$

and, by (b), this implies $M/N = 0$ and $M = N$.

$(c) \Rightarrow (d)$ is clear.

$(d) \Rightarrow (a)$ For a sequence $\{s_i\}_{\mathbb{N}}$ of elements in I we get

$$\bigoplus_{\mathbb{N}} Rs_i \subset I^{(\mathbb{N})} \subset I(R^{(\mathbb{N})}) \ll R^{(\mathbb{N})}.$$

By 43.3, the sequence is right t-nilpotent.

There is another finiteness condition characterizing perfect modules:

Let M be an R-module. We say the *ascending chain condition (acc) for M-cyclic modules* holds if in every module any ascending chain of M-cyclic submodules becomes stationary after finitely many steps.

Obviously, this property carries over to factor modules of M. If M has finite length, then M satisfies this condition since the length of any proper M-cyclic module is smaller than the length of M.

43.6 Ascending chain condition for M-cyclic modules.

Let M be a left R-module and $S = End_R(M)$.

(1) The following statements are equivalent:

(a) $Rad(M) = 0$ and acc for M-cyclic modules holds;

(b) M is semisimple and finitely generated.

(2) Assume acc for M-cyclic modules holds. Then $M/Rad(M)$ is finitely generated and semisimple and $K(S) = \{f \in S \mid Im\, f \ll M\}$ acts locally t-nilpotently on M.

(3) Assume M to be finitely generated, $M/Rad(M)$ to be semisimple and $Rad(M^{(\mathbb{N})}) \ll M^{(\mathbb{N})}$. Then acc for M-cyclic modules holds.

Proof: (1) $(a) \Rightarrow (b)$ We know that the intersection of the maximal submodules of M is zero. If there are finitely many maximal submodules $M_i \subset M$ with $M_1 \cap \cdots \cap M_k = 0$, then M is isomorphic to a submodule of the semisimple module $M/M_1 \oplus \cdots \oplus M/M_k$ and hence it is finitely generated and semisimple.

Assume that there is an infinite set $\{M_i\}_{\mathbb{N}}$ of maximal submodules $M_i \subset M$ such that

$$M_1 \supset M_1 \cap M_2 \supset M_1 \cap M_2 \cap M_3 \supset \dots$$

form a properly descending chain. Then for $N_k = M_1 \cap \cdots \cap M_k$ we have

$$M/N_k \simeq M/M_1 \oplus \cdots \oplus M/M_k$$

(see 9.12), and $M/N_1 \subset M/N_2 \subset M/N_3 \subset \cdots$ is a properly ascending chain of M-cyclic modules, contradicting (a).

$(b) \Rightarrow (a)$ M has finite length.

(2) The first assertion follows from (1).

Let $\{s_i\}_{\mathbb{N}}$ be a sequence of elements in $K(S)$.
With $M_i = M$ and $s_{ij} = s_i \cdots s_{j-1}$ for $i < j$, $i, j \in \mathbb{N}$, $(M_i, s_{ij})_{\mathbb{N}}$ forms a direct system of modules. If $(u_i, \varinjlim M_i)_{\mathbb{N}}$ is the direct limit of it, then $u_i = s_i u_{i+1}$ holds and hence $Mu_i \subset Mu_{i+1}$ for all $i \in \mathbb{N}$.

Since $\varinjlim M_i = \bigcup_{\mathbb{N}} Mu_i$ (see 24.3), and because of the ascending chain condition, there exists $k \in \mathbb{N}$ with $\varinjlim M_i = M_k u_k$. From the commutative diagram

$$\begin{array}{ccc} & & M_{k+1} = M \\ & {}^{s_k} \nearrow & \downarrow u_{k+1} \\ M_k & \xrightarrow{u_k} & \varinjlim M_i \end{array}$$

we see that $M = Ms_k + Ke\ u_{k+1} = Ke\ u_{k+1}$ holds (notice $Ms_k \ll M$), and therefore $\varinjlim M_i = Mu_{k+1} = 0$. Then $(ms_1 \cdots s_k)u_{k+1} = 0$ for any $m \in M_1$, and, by the properties of direct limits (see 24.3), we find some $r \in \mathbb{N}$ with $ms_1 \cdots s_k s_{k+1,r+1} = ms_1 \cdots s_r = 0$.

(3) Let $\{M_i\}_{\mathbb{N}}$ be an ascending sequence of M-cyclic submodules of a module N'. Then $\bigcup_{\mathbb{N}} M_i = N$ is a factor module of $M^{(\mathbb{N})}$, and from $Rad\, M^{(\mathbb{N})} \ll M^{(\mathbb{N})}$ we also get $Rad(N) \ll N$ (M is good).

The modules $M_i + Rad(N)/Rad(N)$ form an ascending chain of $M/Rad(M)$-cyclic submodules of $N/Rad(N)$. By (1), it has to become stationary after finitely many steps.

Hence $N/Rad(N) = M_k + Rad(N)/Rad(N)$ holds for a suitable $k \in \mathbb{N}$, and therefore $N = M_k + Rad(N) = M_k$.

For the description of perfect modules we still need another lemma:

43.7 Supplements under $Hom(M, -)$.
Assume M to be a finitely generated, self-projective R-module and $S = End_R(M)$. Then an M-generated module N is supplemented if and only if $Hom_R(M, N)$ is supplemented as a left S-module.

Proof: Let N be supplemented and $I \subset Hom_R(M, N)$ an S-submodule. By 18.4, $I = Hom_R(M, MI)$. There is a supplement V of MI in N and we have (also 18.4)

$$Hom_R(M, N) = Hom_R(M, MI + V) = I + Hom_R(M, V).$$

$Hom_R(M, V)$ is a supplement of I in $Hom_R(M, N)$:
If, for $Y \subset Hom_R(M, V)$, we have $I + Y = Hom_R(M, N)$, then this yields

$$N = M Hom_R(M, N) = MI + MY \text{ with } MY \subset V.$$

Minimality of V implies $MY = V$ and $Y = Hom_R(M, MY) = Hom_R(M, V)$.
Now let ${}_S Hom_R(M, N)$ be supplemented and $U \subset N$. For a supplement X of $Hom_R(M, U)$ in $Hom_R(M, N)$ we have

$$N = M Hom_R(M, N) = MX + M Hom_R(M, U) = MX + U.$$

MX is a supplement of U in N: Assume for $V \subset MX$ we have $N = U + V$. Then we conclude

$$Hom_R(M, N) = Hom_R(M, U + V) = Hom_R(M, U) + Hom_R(M, V),$$

where $Hom_R(M, V) \subset Hom_R(M, MX) = X$. Now minimality of X implies $Hom_R(M, V) = X$, and hence $MX = M Hom_R(M, V) \subset V$.

The preceding reflections yield:

43.8 Finitely generated, perfect modules. Characterizations.
Let M be a finitely generated, self-projective R-module with endomorphism ring $S = End_R(M)$. The following statements are equivalent:
(a) M is perfect in $\sigma[M]$;
(b) every (indecomposable) M-generated flat module in $\sigma[M]$ is projective in $\sigma[M]$;

(c) $M^{(\mathbb{N})}$ *is semiperfect in* $\sigma[M]$*;*
(d) $M/Rad(M)$ *is semisimple and* $Rad\, M^{(\mathbb{N})} \ll M^{(\mathbb{N})}$*;*
(e) the ascending chain condition for M*-cyclic modules holds;*
(f) $S/Jac(S)$ *is left semisimple and* $Jac(S)$ *is right t-nilpotent;*
(g) S *satisfies the descending chain condition for cyclic right ideals;*
(h) ${}_SS$ *is perfect in S-MOD;*
(i) $End_R(M^{(\mathbb{N})})$ *is f-semiperfect.*

Proof: $(a) \Rightarrow (c) \Rightarrow (d)$ is clear by definition and 42.5.

$(d) \Rightarrow (e)$ has been shown in 43.6.

$(a) \Rightarrow (b) \Rightarrow (g)$ results from 43.2 and 43.4. By 36.4, we are able to restrict the condition (b) to indecomposable modules.

$(g) \Rightarrow (f)$ is a result of 31.9 (notice change of sides).

$(e) \Rightarrow (f)$ By 43.6, $M/Rad(M)$ is semisimple and $K(S) = Jac(S)$ (see 22.2) is right t-nilpotent.

$(f) \Rightarrow (h)$ Since $Jac(S)$ is a nil ideal, idempotents of $S/Jac(S)$ can be lifted (see 42.7). Hence, by 42.6, S is semiperfect. The right t-nilpotence of $Jac(S)$ implies, by 43.5, for every index set Λ,

$$Rad(S^{(\Lambda)}) = Jac(S)S^{(\Lambda)} \ll S^{(\Lambda)}.$$

Therefore S is perfect in S-MOD, by 43.2.

$(a) \Leftrightarrow (h)$ $M^{(\Lambda)}$ is supplemented (and hence semiperfect in $\sigma[M]$) if and only if $Hom(M, M^{(\Lambda)}) \simeq S^{(\Lambda)}$ is supplemented as a left S-module. This has been shown in 43.7.

$(c) \Rightarrow (i)$ is included in 42.12.

$(i) \Rightarrow (f)$ If $End(M^{(\mathbb{N})})$ is f-semiperfect, by 42.12, $Rad\, M^{(\mathbb{N})}$ is superfluous in $M^{(\mathbb{N})}$ and according to 43.4, $Jac(S)$ is right t-nilpotent. From the properties of endomorphism rings of projective modules in 22.2, we derive, for $\overline{M} = M/Rad(M)$,

$$End(\overline{M}^{(\mathbb{N})}) \simeq End(M^{(\mathbb{N})}/RadM^{(\mathbb{N})}) \simeq End(M^{(\mathbb{N})})/Jac(End(M^{(\mathbb{N})})).$$

By (i) and 42.11, this is a regular ring. Hence $End(\overline{M}) \simeq S/Jac(S)$ is a regular ring satisfying the descending chain condition for cyclic right ideals by 43.2,(2), i.e. it is left semisimple by 31.9.

Some of the characterizations of finitely generated perfect modules can be shown for arbitrary perfect modules (see 51.4).

For $M = R$, 43.8 yields most of the implications in

43.9 Left perfect rings. Characterizations.

For a ring R the following assertions are equivalent:

(a) ${}_RR$ *is perfect in R-MOD;*

(b) *every (indecomposable) flat R-left module is projective;*

(c) *every left R-module (or only $R^{(\mathbb{N})}$) is semiperfect;*

(d) *every left R-module has a projective cover;*

(e) *every left R-module is (amply) supplemented;*

(f) $R/Jac(R)$ *is left semisimple and* $Rad\, R^{(\mathbb{N})} \ll {}_RR^{(\mathbb{N})}$;

(g) *the ascending chain condition for cyclic R-left modules holds;*

(h) $End_R(R^{(\mathbb{N})})$ *is f-semiperfect;*

(i) $R/Jac(R)$ *is left semisimple and* $Jac(R)$ *is right t-nilpotent;*

(j) *R satisfies the descending chain condition for cyclic right ideals;*

(k) *R contains no infinite set of orthogonal idempotents and every non-zero right R-module has non-zero socle.*

A ring with these properties is called *left perfect*.

Proof: The equivalence of (a), (b), (c), (f), (g), (h), (i) and (j) follows directly from 43.8. The equivalence of (a), (d) and (e) can be derived from 43.2.

$(j) \Rightarrow (k)$ was shown in 31.9.

$(k) \Rightarrow (i)$ Assume that there is a sequence $\{a_i\}_{\mathbb{N}}$ with $a_i \in Jac(R)$, such that $a_1 \cdots a_k \neq 0$ holds for all $k \in \mathbb{N}$. By Zorn's Lemma, there exists a right ideal $K \subset R$ maximal with respect to $a_1 \cdots a_k \notin K$ for all $k \in \mathbb{N}$.

Since $Soc(R/K) \neq 0$, there exists a right ideal $L \subset R$ with $K \subset L$ and L/K simple. By the choice of K, this means $a_1 \cdots a_r \in L$ for some $r \in \mathbb{N}$ and hence also $a_1 \cdots a_r a_{r+1} \in L \setminus K$. Since L/K is simple, we can find a $t \in R$ with

$$a_1 \cdots a_r + K = a_1 \cdots a_r a_{r+1} t + K, \text{ i.e. } a_1 \cdots a_r (1 - a_{r+1} t) \in K.$$

Since $a_{r+1} \in Jac(R)$, the element $1 - a_{r+1}t$ is invertible, hence $a_1 \cdots a_r \in K$, a contradiction to the choice of K. From this we conclude that $Jac(R)$ is right t-nilpotent.

Now, idempotents of $\overline{R} = R/Jac(R)$ can be lifted, and hence there is no infinite set of orthogonal idempotents in $\overline{R}$, i.e. $\overline{R} = f_1\overline{R} \oplus \cdots \oplus f_r\overline{R}$ with idempotents $f_i \in \overline{R}$ and indecomposable $f_i\overline{R}$. Since $Soc(f_i\overline{R}) \neq 0$, there is a simple right ideal $E_i \subset f_i\overline{R}$. Since $E_i^2 \neq 0$ (because of $Jac(\overline{R}) = 0$), by 2.7, E_i is a direct summand of $\overline{R}$ and therefore $f_i\overline{R} = E_i$. Hence $\overline{R}$ is right semisimple.

Notice that *left* perfect rings need not be *right* perfect. Left artinian rings R are left and right perfect since in this case $Jac(R)$ is nilpotent and hence left and right t-nilpotent.

In 43.8 we saw that the endomorphism ring of a finitely generated, perfect module is left perfect. In the following situation the endomorphism ring is right perfect:

43.10 Right perfect endomorphism rings.

For a finitely generated, semi-projective R-module M with endomorphism ring $S = End_R(M)$, the following statements are equivalent:
(a) M satisfies dcc for M-cyclic submodules;
(b) S satisfies dcc for cyclic (finitely generated) left ideals;
(c) S is right perfect.

If M is self-projective, the above is also equivalent to:
(d) M satisfies dcc for finitely M-generated submodules.

Proof: $(a) \Leftrightarrow (b)$ For every cyclic left ideal $Sf \subset S$, $f \in S$, we have $Sf = Hom(M, Mf)$ (see before 31.10). For $g \in S$, we have $Sf \supset Sg$ if and only if $Hom(M, Mf) \supset Hom(M, Mg)$.

$(b) \Leftrightarrow (c)$ follows from 43.9 (notice change of sides).

$(b) \Leftrightarrow (d)$ If M is self-projective, then $I = Hom(M, MI)$ for every finitely generated left ideal $I \subset S$ (by 18.4).

Of course, for commutative rings, *left perfect* and *right perfect* are equivalent. This is also true for endomorphism rings of finitely generated, self-projective modules over commutative rings:

43.11 Perfect modules over commutative rings.

Let R be a commutative ring, M a finitely generated, self-projective R-module and $S = End_R(M)$. Then the following statements are equivalent:
(a) M is perfect in $\sigma[M]$;
(b) acc for M-cyclic modules holds;
(c) M satisfies dcc for M-cyclic (finitely M-generated) submodules;
(d) M satisfies dcc for cyclic (finitely generated) submodules;
(e) S is left perfect;
(f) S is right perfect;
(g) $\overline{R} = R/An(M)$ is a perfect ring.

Proof: $(a) \Leftrightarrow (b) \Leftrightarrow (e)$ is known from 43.8.

$(c) \Leftrightarrow (f)$ was shown in 43.10.

Observing that, under the given assumptions, M is a projective generator in $\sigma[M] = \overline{R}$-MOD (see 18.11), the equivalences $(a) \Leftrightarrow (g)$ and $(c) \Leftrightarrow (d) \Leftrightarrow (g)$ are easily confirmed.

43.12 Exercises

(1) Prove that for a ring R the following statements are equivalent:
(a) R is left perfect;
(b) the direct limit of projective modules in R-MOD is self-projective;
(c) every flat module in R-MOD is self-projective.

(2) Let R be a ring and $e_1, \ldots, e_n$ *orthogonal idempotents in R with* $e_1 + \cdots + e_n = 1$. *Prove that the following properties are equivalent:*
(a) R is left perfect;
(b) the rings $e_i Re_i$ *are left perfect for* $i = 1, \ldots, n$;
(c) $R/Jac(R)$ *is left artinian, and, for every R-right module L,* $Soc(L) \trianglelefteq L$.

(3) Let R be a left perfect ring. Prove that in R-MOD every R-projective module is projective (see 42.13,(2)).

(4) Let P be a self-projective generator in R-MOD. Show: If $End(P)$ *is left perfect, then P is finitely generated and R-projective (see 19.10,(7)).*

Literature: ANDERSON-FULLER, KASCH;
Ashan [1], Anh [2], Azumaya [1], Brodskii [1], Colby-Rutter, Cunningham-Rutter, Dlab, Faticoni, Fuller-Hill, Golan [1], Hauger, Hausen-Johnson [2], Hiremath, Izawa [2], Jonah, Osofsky, Rangaswamy [4], Rangaswamy-Vanaja [2], Rant, Renault [2], de Robert, Tuganbaev [3], Whitehead, Wisbauer [4,5].

Chapter 9

Relations between functors

First of all we want to introduce morphisms of functors between arbitrary categories and then investigate applications in particular for module categories. Thereby we shall learn about adjoint functors, equivalences and dualities between categories. For definitions and basic properties of functors we refer to § 11.

44 Functorial morphisms

1.Definition. 2.Functor category. 3.Functorial morphisms to Mor-functors. 4.Isomorphism of Mor-functors. 5.Representation functor. 6.Morphisms of functors on module categories.

44.1 Functorial morphisms. Definition.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F\colon \mathcal{C} \to \mathcal{D}$, $F'\colon \mathcal{C} \to \mathcal{D}$ two covariant functors. By a *functorial morphism* $\eta\colon F \to F'$ we mean a class of morphisms $\eta_A\colon F(A) \to F'(A)$ of $\mathcal{D}$, $A \in \mathcal{C}$, such that for every morphism $f\colon A \to B$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(A) & \stackrel{F(f)}{\to} & F(B) \\ \downarrow \eta_A & & \downarrow \eta_B \\ F'(A) & \stackrel{F'(f)}{\to} & F'(B) \end{array}$$

is commutative.

If all η_A, $A \in \mathcal{C}$, are isomorphisms in $\mathcal{D}$, $\eta\colon F \to F'$ is called a *functorial isomorphism*. Then F is said to be *isomorphic to* F' and we write $F \simeq F'$.

If F, F' are two *contravariant* functors of $\mathcal{C}$ to $\mathcal{D}$, for a functorial morphism η: $F \to F'$ we only have to convert the horizontal arrows in the above diagram.

Instead of 'functorial morphism' we also say *natural transformation.* Functorial isomorphisms are also called *natural isomorphisms* or *natural equivalences* of functors.

If η: $F \to F'$ and ψ: $F' \to F''$ are functorial morphisms, we obtain by composition $\eta_A\psi_A$: $F(A) \to F''(A)$ in $\mathcal{D}$, a functorial morphism $\eta\psi$: $F \to F''$. Moreover, the identities in $\mathcal{D}$, $id_{F(A)}$: $F(A) \to F(A)$, $A \in \mathcal{C}$, yield a functorial isomorphism id: $F \to F$. It is easily verified that the functorial morphisms satisfy the conditions for the composition of morphisms of a category. Hereby the covariant functors $\mathcal{C} \to \mathcal{D}$ are regarded as objects.

However, the totality of functorial morphisms between two functors in general need not be a set. In the following case this is true:

We call a category $\mathcal{C}$ *small* if the class of objects of $\mathcal{C}$ form a set.

44.2 Functor category.

Let $\mathcal{C}$ be a small, and $\mathcal{D}$ an arbitrary category. Taking as

objects:	*the class of the covariant functors of $\mathcal{C}$ to $\mathcal{D}$,*
morphisms:	*the functorial morphisms,*
composition:	*the composition of functorial morphisms,*

we get a category, the category of covariant functors from $\mathcal{C}$ to $\mathcal{D}$, *for short: the* functor category *from $\mathcal{C}$ to $\mathcal{D}$.*

Proof: After the explanations above, it only remains to prove that the functorial morphisms between two functors F, F': $\mathcal{C} \to \mathcal{D}$ always form a set. In fact, these sets may be regarded as subsets of the cartesian product $\prod_{A\in\mathcal{C}} Mor_{\mathcal{D}}(F(A), F'(A))$.

We already know examples of functorial isomorphisms. So each of the following functors is isomorphic to the identity on R-MOD:

$$Hom_R(R, -) : R\text{-}MOD \to R\text{-}MOD, \text{ with the isomorphisms}$$
$$\eta_M : Hom_R(R, M) \to M,\ \alpha \mapsto (1)\alpha, \text{ (see 11.11).}$$
$$R \otimes_R - : R\text{-}MOD \to R\text{-}MOD, \text{ with the isomorphisms}$$
$$\psi_M : R \otimes_R M \to M,\ r \otimes m \to rm \text{ (see 12.6).}$$

In 11.5, for every object A in a category $\mathcal{C}$ we have defined the covariant functor $Mor_{\mathcal{C}}(A, -)$: $\mathcal{C} \to ENS$. Considering the functorial morphisms of $Mor(A, -)$ to an arbitrary covariant functor F: $\mathcal{C} \to ENS$, we obtain a remarkable assertion known as the *Yoneda Lemma* :

44.3 Functorial morphisms to *Mor*-functors.

Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to ENS$ a covariant functor. For $A \in \mathcal{C}$ denote by $[Mor_{\mathcal{C}}(A,-), F]$ the class of functorial morphisms between these functors. Then the Yoneda map

$$Y : [Mor_{\mathcal{C}}(A,-), F] \to F(A),\ \eta \mapsto (id_A)\eta_A,$$

is bijective.

Proof: Let η: $Mor_{\mathcal{C}}(A,-) \to F$ be a functorial morphism. Then η_A: $Mor_{\mathcal{C}}(A,A) \to F(A)$ is a map and, for every $f : A \to B$ in $\mathcal{C}$, the following diagram is commutative:

$$\begin{array}{ccc} Mor_{\mathcal{C}}(A,A) & \xrightarrow{\eta_A} & F(A) \\ \downarrow {\scriptstyle Mor(A,f)} & & \downarrow {\scriptstyle F(f)} \\ Mor_{\mathcal{C}}(A,B) & \xrightarrow{\eta_B} & F(B) \end{array} .$$

In particular, for $id_A \in Mor_{\mathcal{C}}(A,A)$ we have $(id_A)\eta_A F(f) = (f)\eta_B$. Therefore all η_B are already completely determined by $(id_A)\eta_A$, i.e. Y is injective.

For an arbitrary element $\alpha \in F(A)$ and $B \in \mathcal{C}$, we define a map

$$\bar{\eta}_B : Mor_{\mathcal{C}}(A,B) \to F(B),\ f \mapsto (\alpha)F(f).$$

We have to show that, for every $g : B \to C$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} Mor_{\mathcal{C}}(A,B) & \xrightarrow{\bar{\eta}_B} & F(B) \\ \downarrow {\scriptstyle Mor(A,g)} & & \downarrow {\scriptstyle F(g)} \\ Mor_{\mathcal{C}}(A,C) & \xrightarrow{\bar{\eta}_C} & F(C) \end{array}$$

is commutative. Now we have for $f \in Mor_{\mathcal{C}}(A,B)$,

$$(f)\bar{\eta}_B F(g) = (\alpha)F(f)F(g) = (\alpha)F(fg) \text{ and}$$
$$(f)Mor(A,g)\bar{\eta}_C = (fg)\bar{\eta}_C = (\alpha)F(fg).$$

Hence the $\bar{\eta}_C$'s define a functorial morphism $\bar{\eta} : Mor(A,-) \to F$ with $Y(\bar{\eta}) = (id_A)\bar{\eta}_A = \alpha$, i.e. Y is surjective.

For objects A, A' and a morphism α: $A' \to A$ in $\mathcal{C}$, the mappings

$$Mor(\alpha, B) : Mor_{\mathcal{C}}(A,B) \to Mor_{\mathcal{C}}(A',B),\ B \in \mathcal{C},$$

yield a functorial morphism

$$Mor(\alpha,-) : Mor_{\mathcal{C}}(A,-) \to Mor_{\mathcal{C}}(A',-).$$

In the proof of 44.3 we see, with $F = Mor(A', -)$, that $Mor(\alpha, -)$ just corresponds to the functorial morphism $\bar{\eta}$ and we obtain:

44.4 Isomorphism of *Mor*-functors.
Let A, A' and A'' be objects in a category $\mathcal{C}$.

(1) The map $\alpha \mapsto Mor(\alpha, -)$ of $Mor(A', A)$ into the set of functorial morphisms $[Mor(A, -),\ Mor(A', -)]$ is bijective.

(2) For $\alpha \in Mor(A', A)$, $\beta \in Mor(A'', A')$ we have

$$Mor(\beta\alpha, -) = Mor(\alpha, -)Mor(\beta, -) : Mor(A, -) \to Mor(A'', -).$$

(3) Every functorial isomorphism between $Mor(A, -)$ and $Mor(A', -)$ is induced by an isomorphism $A' \to A$ in $\mathcal{C}$.

Proof: (1) follows - as suggested above - from 44.3.

(2) is easily verified by referring to the corresponding definitions.

(3) Obviously, for an isomorphism $\alpha \in Mor(A', A)$, the morphism $Mor(\alpha, -)$: $Mor(A, -) \to Mor(A', -)$ is also an isomorphism.

On the other hand, let $Mor(A, -)$ and $Mor(A', -)$ be isomorphic. By (1), there are morphisms $\alpha : A' \to A$ and $\beta : A \to A'$, such that
$Mor(\alpha, -) : Mor(A, -) \to Mor(A', -)$ and
$Mor(\beta, -) : Mor(A', -) \to Mor(A, -)$
are functorial isomorphisms which are inverse to each other. In particular, $Mor(\beta\alpha, A) = id_{Mor(A,A)}$ and hence $\beta\alpha = id_A$.

Similarly we get $\alpha\beta = id_{A'}$.

44.5 Representation functor.
If $\mathcal{C}$ is a small category, then the assignments

$$\begin{array}{ccl} A & \sim\sim> & Mor_{\mathcal{C}}(A, -), \\ A' \xrightarrow{f} A & \sim\sim> & Mor_{\mathcal{C}}(f, -) : Mor(A, -) \to Mor(A', -) \end{array}$$

for $A,\ A' \in Obj(\mathcal{C})$ and $f \in Mor(\mathcal{C})$ define a fully faithful, contravariant functor of $\mathcal{C}$ into the functor category of $\mathcal{C}$ to ENS.

It is called the representation functor *(of $\mathcal{C}$).*

Proof: Since $\mathcal{C}$ is a small category, the covariant functors $\mathcal{C} \to ENS$ (see 44.2) form a category. The functor properties of the given assignment are easy to verify. By 44.4,(2), this functor is contravariant.

The property 'fully faithful' just means an isomorphism between $Mor_{\mathcal{C}}(A, B)$ and the functorial morphisms $[Mor(A, -),\ Mor(B, -)]$. This is given by the Yoneda Lemma 44.3 and 44.4.

If $\mathcal{C}$ is a full subcategory of a module category, hence an additive category, the functors $Hom_R(A,-)\colon \mathcal{C} \to AB$ are additive (see 11.7). In the preceding constructions the covariant functors $\mathcal{C} \to ENS$ are in fact additive covariant functors $\mathcal{C} \to AB$.

The sum of two functorial morphisms η, ψ between additive functors F, $F'\colon \mathcal{C} \to AB$ is defined by

$$(\eta+\psi)_A = \eta_A + \psi_A \text{ for } A \in Obj(\mathcal{C}),$$

where $\eta_A + \psi_A$ is the sum of two homomorphisms of $\mathbb{Z}$-modules.

With this operation, for every additive covariant functor $F\colon \mathcal{C} \to AB$, the set of functorial morphisms $[Hom_R(A,-), F]$ form an abelian group. Let us sum up for this case some of our results:

44.6 Morphisms of functors on module categories.
Let $\mathcal{C}$ be a full subcategory of R-MOD.

(1) For an additive covariant functor $F : \mathcal{C} \to AB$ and $N \in \mathcal{C}$, the Yoneda map

$$Y : [Hom_R(N,-), F] \to F(N),\ \ \eta \mapsto (id_N)\eta_N,$$

is a group isomorphism.

(2) For an additive contravariant functor $G : \mathcal{C} \to AB$ and $N \in \mathcal{C}$, the Yoneda map

$$[Hom_R(-,N), G] \to G(N),\ \ \eta \mapsto (id_N)\eta_N,$$

is a group isomorphism.

(3) If for $N, N' \in \mathcal{C}$, the functors $Hom_R(N,-)$, $Hom_R(N',-) : \mathcal{C} \to AB$ are isomorphic, then N and N' are isomorphic R-modules.

(4) If for $N, N' \in \mathcal{C}$, the functors $Hom_R(-,N)$, $Hom_R(-,N') : \mathcal{C} \to AB$ are isomorphic, then N and N' are isomorphic R-modules.

Proof: (1) By the general Yoneda Lemma 44.3, the map Y is bijective. Moreover, for two functorial morphisms $\eta, \psi : Hom_R(N,-) \to F$ we have

$$\begin{aligned} Y(\eta+\psi) &= (id_N)(\eta+\psi)_N = (id_N)(\eta_N+\psi_N) \\ &= (id_N)\eta_N + (id_N)\psi_N = Y(\eta) + Y(\psi). \end{aligned}$$

(2) This also follows from 44.3, since the contravariant functor $Hom_R(-,N) : \mathcal{C} \to AB$, $N \in \mathcal{C}$, can be regarded as a covariant functor on the dual category $\mathcal{C}^o$ (see 7.3, (5)).

(3) and (4) can be obtained from 44.4.

Literature: HILTON-STAMMBACH, JACOBSON, STENSTRÖM; Sklyarenko [2].

45 Adjoint pairs of functors

1.The bifunctor $Mor_{\mathcal{C}}(-,-)$. 2.Adjoint pairs of functors. 3.Representable functors and limits. 4.Adjoint and representable functors. 5.Functorial morphisms to the identity. 6.Adjoint covariant functors on module categories. 7.Functors R-$MOD \to \mathcal{D}$ preserving limits. 8.The pair of functors $U \otimes_S -$, $Hom_R(U,-)$. 9.Adjoint contravariant functors. 10.The pair of functors $Hom_R(-,U)$, $Hom_S(-,U)$. 11.The inclusion $\sigma[M] \to R$-MOD.

For two categories $\mathcal{C}$, $\mathcal{D}$ we form the *product category* $\mathcal{C} \times \mathcal{D}$ with
objects: the ordered pairs (C, D) with C from $\mathcal{C}$, D from $\mathcal{D}$,
morphisms: $Mor((C,D),(C',D')) := Mor_{\mathcal{C}}(C,C') \times Mor_{\mathcal{D}}(D,D')$,
composition by components: $(f,g)(f',g') := (ff', gg')$,
if $ff' \in \mathcal{C}$ and $gg' \in \mathcal{D}$ are defined.

It is easy to verify that $id_{(C,D)} = (id_C, id_D)$ and the conditions for a category are satisfied.

Functors of $\mathcal{C} \times \mathcal{D}$ into a category $\mathcal{E}$ are called *bifunctors*. For a bifunctor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ there is, for every object $D \in \mathcal{D}$, a *partial functor*

$$F(-,D) : \mathcal{C} \to \mathcal{E} \quad \text{with} \quad \begin{cases} C & \mapsto \quad F(C,D) \text{ for } C \in Obj(\mathcal{C}) \\ f & \mapsto \quad F(f, id_D) \text{ for } f \in Mor(\mathcal{C}). \end{cases}$$

Similarly, for every C in $\mathcal{C}$ we obtain a functor $F(C,-) : \mathcal{D} \to \mathcal{E}$.

F is said to be *covariant (contravariant)* if all partial functors $F(-,D)$ and $F(C,-)$ are covariant (contravariant).

For additive categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, the bifunctor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is called *biadditive* if all partial functors are additive.

Denote by $\mathcal{C} \to \mathcal{C}^o$ the transition from $\mathcal{C}$ to the dual category $\mathcal{C}^o$ (see 11.1). As an important example for the above notions we have:

45.1 The bifunctor $Mor_{\mathcal{C}}(-,-)$.
(1) For any category $\mathcal{C}$, a covariant bifunctor

$$Mor_{\mathcal{C}}(-,-) : \ \mathcal{C}^o \times \mathcal{C} \to ENS$$

is defined by assigning to objects $A, B \in \mathcal{C}$,

$$\mathcal{C}^o \times \mathcal{C} \ni (A^o, B) \sim\!\sim\!> Mor_{\mathcal{C}}(A,B) \in ENS\,,$$

and to morphisms $f : A_2 \to A_1$, $g : B_1 \to B_2$ in $\mathcal{C}$,

$$(f^o, g) : (A_1^o, B_1) \to (A_2^o, B_2) \rightsquigarrow$$
$$Mor(f^o, g) : Mor_{\mathcal{C}}(A_1, B_1) \to Mor_{\mathcal{C}}(A_2, B_2),\ h \mapsto fhg.$$

(2) For A, B in $\mathcal{C}$ the partial functors

$$Mor_{\mathcal{C}}(A^o, -) : \mathcal{C} \to ENS \text{ and } Mor_{\mathcal{C}^o}(-, B) : \mathcal{C}^o \to ENS,$$

are the usual covariant Mor-functors (see 11.5).

(3) If $\mathcal{C}$ is a full subcategory of R-MOD, the homomorphisms determine a biadditive covariant bifunctor

$$Hom_R(-, -) : \mathcal{C}^o \times \mathcal{C} \to AB.$$

These assertions are easy to verify. The bifunctor $Mor(-, -)$, of course, can also be interpreted as a contravariant bifunctor $\mathcal{C} \times \mathcal{C}^o \to ENS$.

Let F: $\mathcal{C} \to \mathcal{D}$ and G: $\mathcal{D} \to \mathcal{C}$ be two covariant functors between the categories $\mathcal{C}$, $\mathcal{D}$. Then F can also be understood as a covariant functor $\mathcal{C}^o \to \mathcal{D}^o$, and by composing the functors

$$\begin{array}{ll} F \times id : & \mathcal{C}^o \times \mathcal{D} \to \mathcal{D}^o \times \mathcal{D} \text{ with } Mor_{\mathcal{D}}(-, -) : \mathcal{D}^o \times \mathcal{D} \to ENS, \quad \text{and} \\ id \times G : & \mathcal{C}^o \times \mathcal{D} \to \mathcal{C}^o \times \mathcal{C} \text{ with } Mor_{\mathcal{C}}(-, -) : \mathcal{C}^o \times \mathcal{C} \to ENS, \end{array}$$

we obtain two functors $Mor_{\mathcal{D}}(F(-), -)$ and $Mor_{\mathcal{C}}(-, G(-))$ from $\mathcal{C}^o \times \mathcal{D}$ to ENS.

45.2 Adjoint pairs of functors. Definitions.

(1) A pair (F, G) of covariant functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ is called *adjoint* if there is a functorial isomorphism

$$Mor_{\mathcal{D}}(F(-), -) \simeq Mor_{\mathcal{C}}(-, G(-)).$$

We say F is *left adjoint* to G and G is *right adjoint* to F.

So F is right adjoint to G if the bifunctors

$$Mor_{\mathcal{C}}(G(-), -) \text{ and } Mor_{\mathcal{D}}(-, F(-)) :\ \mathcal{D}^o \times \mathcal{C} \to ENS$$

are isomorphic.

(2) A pair (F', G') of contravariant functors $F' : \mathcal{C} \to \mathcal{D}$, $G' : \mathcal{D} \to \mathcal{C}$ is called *right adjoint* if the pair of covariant functors $F^* : \mathcal{C} \to \mathcal{D} \to \mathcal{D}^o$ and $G^* : \mathcal{D}^o \to \mathcal{D} \to \mathcal{C}$ is adjoint, i.e. if the bifunctors

$$Mor_{\mathcal{D}^o}(F^*(-), -) \text{ and } Mor_{\mathcal{C}}(-, G^*(-)) :\ \mathcal{C}^o \times \mathcal{D}^o \to ENS$$

are isomorphic.

(F', G') is called *left adjoint* if (G^*, F^*) is adjoint, i.e. if the following bifunctors are isomorphic:

$$Mor_{\mathcal{C}}(G^*(-), -) \text{ and } Mor_{\mathcal{D}^o}(-, F^*(-)) : \ \mathcal{D} \times \mathcal{C} \to ENS.$$

Hence an adjoint pair (F, G) of covariant functors is characterized by a family of bijective mappings

$$\eta_{A,D} : Mor_{\mathcal{D}}(F(A), D) \to Mor_{\mathcal{C}}(A, G(D)), A \in \mathcal{C}, \ D \in \mathcal{D},$$

such that, for morphisms $f : A' \to A$ in $\mathcal{C}$ and $h : D \to D'$ in $\mathcal{D}$, the following diagrams are commutative:

$$\begin{array}{ccc}
Mor_{\mathcal{D}}(F(A), D) & \xrightarrow{\eta_{A,D}} & Mor_{\mathcal{C}}(A, G(D)) \\
\downarrow {\scriptstyle Mor(F(f), D)} & & \downarrow {\scriptstyle Mor(f, G(D))} \\
Mor_{\mathcal{D}}(F(A'), D) & \xrightarrow{\eta_{A',D}} & Mor_{\mathcal{C}}(A', G(D))
\end{array} \quad ,$$

$$\begin{array}{ccc}
Mor_{\mathcal{D}}(F(A), D) & \xrightarrow{\eta_{A,D}} & Mor_{\mathcal{C}}(A, G(D)) \\
\downarrow {\scriptstyle Mor(F(A), h)} & & \downarrow {\scriptstyle Mor(A, G(h))} \\
Mor_{\mathcal{D}}(F(A), D') & \xrightarrow{\eta_{A,D'}} & Mor_{\mathcal{C}}(A, G(D'))
\end{array} \quad .$$

For short we say that the $\eta_{A,D}$ are *natural in each variable*.

The pair (G, F) is adjoint if there are bijective maps

$$\psi_{D,A} : Mor_{\mathcal{C}}(G(D), A) \to Mor_{\mathcal{D}}(D, F(A)), D \in \mathcal{D}, \ A \in \mathcal{C},$$

which are natural in each variable.

Thus a pair (F', G') of contravariant functors is *right adjoint* if there are bijective maps, natural in each variable,

$$\eta_{A,D} : Mor_{\mathcal{D}}(D, F'(A)) \to Mor_{\mathcal{C}}(A, G'(D)), \ A \in \mathcal{C}, \ D \in \mathcal{D}.$$

It is *left adjoint* if bijections are given, natural in each variable,

$$\psi_{D,A} : Mor_{\mathcal{C}}(G'(D), A) \to Mor_{\mathcal{D}}(F'(A), D), \ A \in \mathcal{C}, \ D \in \mathcal{D}.$$

A *covariant* functor T: $\mathcal{C} \to ENS$ is called *representable* if there is a functorial isomorphism $Mor_{\mathcal{C}}(A, -) \to T$ for some object A in $\mathcal{C}$.

Contravariant representable functors are defined by functorial isomorphisms to $Mor_{\mathcal{C}}(-, A)$.

The behavior of representable functors towards limits corresponds to that of Mor-functors. For an inverse system $(C_i, f_{ji})_\Delta$, resp. a direct system $(E_i, h_{ij})_\Lambda$, and an object A in $\mathcal{C}$ we have isomorphisms

$$Mor_{\mathcal{C}}(A, \varprojlim C_i) \simeq \varprojlim Mor_{\mathcal{C}}(A, C_i), \;\; Mor_{\mathcal{C}}(\varinjlim E_i, A) \simeq \varprojlim Mor_{\mathcal{C}}(E_i, A)$$

in ENS if the given limits exist in $\mathcal{C}$ (see 29.5). These relations also hold if the index sets are not directed, i.e. products and coproducts in $\mathcal{C}$ are included as special cases.

A covariant functor is said to *preserve* limits if it turns limits into limits.

45.3 Representable functors and limits.

(1) (i) A covariant representable functor $T\colon \mathcal{C} \to ENS$ preserves inverse limits (in case they exist in $\mathcal{C}$).

(ii) A contravariant representable functor $T\colon \mathcal{C} \to ENS$ converts direct limits from $\mathcal{C}$ (in case they exist) into inverse limits in ENS.

(2) A covariant functor $G\colon \mathcal{D} \to \mathcal{C}$ preserves inverse limits if and only if, for every object C in $\mathcal{C}$, the functor $Mor_{\mathcal{C}}(C, G(-))\colon \mathcal{D} \to ENS$ preserves inverse limits.

(3) A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves direct limits if and only if, for every $D \in \mathcal{D}$, the functor $Mor_{\mathcal{D}}(F(-), D)\colon \mathcal{C} \to ENS$ converts direct limits into inverse limits.

In all assertions products, resp. coproducts, are included as special cases.

Proof: (1)(i) Let $\eta : Mor_{\mathcal{C}}(A, -) \to T$ be a functorial isomorphism and $(C_i, f_{ji})_\Delta$ an inverse system of objects in $\mathcal{C}$ with canonical morphisms $f_j\colon \varprojlim C_i \to C_j$. Then $\{Mor_{\mathcal{C}}(A, C_i)\}_\Delta$ and $\{T(C_i)\}_\Delta$ in a canonical way form inverse systems and we obtain the commutative diagram in ENS

$$\begin{array}{ccccc}
\varprojlim Mor_{\mathcal{C}}(A, C_i) & \simeq & Mor_{\mathcal{C}}(A, \varprojlim C_i) & \xrightarrow{\simeq} & T(\varprojlim C_i) \\
 & & \downarrow {\scriptstyle Mor_{\mathcal{C}}(A, f_j)} & & \downarrow {\scriptstyle T(f_j)} \\
 & & Mor_{\mathcal{C}}(A, C_j) & \xrightarrow{\simeq} & T(C_j)
\end{array} .$$

From this we conclude that $(T(\varprojlim C_i), T(f_j))_\Delta$ form an inverse limit of $(T(C_i), T(f_{ji}))_\Delta$, i.e. $T(\varprojlim C_i) \simeq \varprojlim T(C_i)$.

(ii) can be shown in a similar fashion to (i) by using the properties of $Mor_{\mathcal{C}}(-, A)$.

(2) Let $(D_i, g_{ji})_\Delta$ be an inverse system in $\mathcal{D}$ with canonical morphisms $g_j\colon \varprojlim D_i \to D_j$, and $\{h_j : A \to G(D_j)\}_\Delta$ an inverse family of morphisms

in $\mathcal{C}$. If $Mor_{\mathcal{C}}(C, G(-))$ preserves inverse limits, then the diagram

$$\begin{array}{ccc} Mor_{\mathcal{C}}(C, G(\varprojlim D_i)) & \xrightarrow{Mor(C,G(g_j))} & Mor_{\mathcal{C}}(C, G(D_j)) \\ & & \nearrow{\scriptstyle Mor(C,h_j)} \\ & Mor_{\mathcal{C}}(C, A) & \end{array}$$

can be completed commutatively by a map

$$\delta_C : Mor(C, A) \to Mor(C, G(\varprojlim D_i)).$$

Setting $C = A$, we obtain a morphism $h := (id_A)\delta_A : A \to G(\varprojlim D_i)$. From the diagram we get $h_j = hG(g_j)$. Thus $(G(\varprojlim D_i), G(g_j))_\Delta$ is an inverse limit of $(G(D_i), G(g_{ji}))_\Delta$, i.e. $G(\varprojlim D_i) \simeq \varprojlim G(D_i)$.

The other implication follows from (1),(i).

(3) This can be proved like (2) by using the conversion of limits by $Mor_{\mathcal{D}}(-, D)$.

45.4 Adjoint and representable functors. Properties. *Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor between the categories $\mathcal{C}, \mathcal{D}$.*

(1) For F, the following properties are equivalent:

(a) F has a right adjoint functor G: $\mathcal{D} \to \mathcal{C}$;

(b) for every object D in $\mathcal{D}$, the functor $Mor_{\mathcal{D}}(F(-), D)$: $\mathcal{C}^o \to ENS$ is representable.

(2) If both functors $G, G' : \mathcal{D} \to \mathcal{C}$ are right adjoint to F, then G and G' are isomorphic.

(3) If F has a right adjoint functor $G :\ \mathcal{D} \to \mathcal{C}$, then
F preserves direct limits (including coproducts) in $\mathcal{C}$,
G preserves inverse limits (including products) in $\mathcal{D}$.

Proof: (1) (a) $\Rightarrow$ (b) follows from the definition of adjoint pairs.

(b) $\Rightarrow$ (a) For $D \in \mathcal{D}$ choose $G(D) \in \mathcal{C}$ with a functorial isomorphism

$$\eta^D : Mor_{\mathcal{C}}(-, G(D)) \longrightarrow Mor_{\mathcal{D}}(F(-), D)\,.$$

By 44.4, $G(D)$ is uniquely determined up to isomorphisms.

For a morphism f: $D \to D'$ in $\mathcal{D}$, there is a unique morphism $G(f)$: $G(D) \to G(D')$ (see 44.4) making the following diagram with functorial morphisms commutative

$$\begin{array}{ccc} Mor_{\mathcal{C}}(-, G(D)) & \xrightarrow{\eta^D} & Mor_{\mathcal{D}}(F(-), D) \\ \downarrow {\scriptstyle Mor_{\mathcal{C}}(-, G(f))} & & \downarrow {\scriptstyle Mor_{\mathcal{D}}(F(-), f)} \\ Mor_{\mathcal{C}}(-, G(D')) & \xrightarrow{\eta^{D'}} & Mor_{\mathcal{D}}(F(-), D') \end{array} \quad .$$

Now it is easy to check that the functor $G : \mathcal{D} \to \mathcal{C}$, defined this way, is right adjoint to F.

(2) The assertion follows from (1) since the given construction is unique up to isomorphism.

(3) Since the functors $Mor_{\mathcal{C}}(C, G(-))$ are representable they preserve inverse limits. Thus the assertion for G follows from 45.3,(2).

Similarly, the assertion concerning F is obtained from 45.3,(3).

A further characterization of adjoint functors is included in

45.5 Functorial morphisms to the identity.

(1) Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be covariant functors. Then (F,G) is an adjoint pair if and only if there are functorial morphisms

$$\eta : id_{\mathcal{C}} \to GF \text{ and } \psi : FG \to id_{\mathcal{D}},$$

for which the composed morphisms

$$F(C) \xrightarrow{F(\eta_C)} FGF(C) \xrightarrow{\psi_{F(C)}} F(C),\ C \in \mathcal{C},$$

$$G(D) \xrightarrow{\eta_{G(D)}} GFG(D) \xrightarrow{G(\psi_D)} G(D),\ D \in \mathcal{D},$$

yield the identity on $F(C)$ resp. $G(D)$.

(2) Let $F'\colon \mathcal{C} \to \mathcal{D}$ and $G'\colon \mathcal{D} \to \mathcal{C}$ be contravariant functors. Then (F', G') is a right adjoint pair if and only if there are functorial morphisms

$$\eta : id_{\mathcal{C}} \to G'F' \text{ and } \psi : id_{\mathcal{D}} \to F'G',$$

for which the composed morphisms

$$F'(C) \xrightarrow{\psi_{F'(c)}} F'G'F'(C) \xrightarrow{F'(\eta_C)} F'(C),\ C \in \mathcal{C},$$

$$G'(D) \xrightarrow{\eta_{G'(D)}} G'F'G'(D) \xrightarrow{G'(\psi_D)} G'(D),\ D \in \mathcal{D},$$

yield the identity on $F'(C)$ resp. $G'(D)$.

(3) Left adjoint pairs (F', G') are characterized in an analogous way by functorial morphisms

$$G'F' \to id_{\mathcal{C}} \text{ and } \quad F'G' \to id_{\mathcal{D}}.$$

Proof: (1) $\Rightarrow$ Let (F,G) be an adjoint pair with bijections

$$\eta_{C,D} : Mor_{\mathcal{D}}(F(C),D) \longrightarrow Mor_{\mathcal{C}}(C,G(D)).$$

In particular, for $D = F(C)$ we obtain morphisms in $\mathcal{C}$

$$\eta_C = (id_{F(C)})\eta_{C,F(C)} : C \longrightarrow GF(C).$$

These determine a functorial morphism $\eta : id_{\mathcal{C}} \to GF$: For a morphism $f : C \to C'$ in $\mathcal{C}$ we have commutative partial diagrams

$$\begin{array}{ccccc}
Mor_{\mathcal{D}}(F(C),F(C)) & \to & Mor_{\mathcal{D}}(F(C),F(C')) & \leftarrow & Mor_{\mathcal{D}}(F(C'),F(C')) \\
\downarrow \eta_{C,F(C)} & & \downarrow \eta_{C,F(C')} & & \downarrow \eta_{C',F(C')} \\
Mor_{\mathcal{C}}(C,GF(C)) & \to & Mor_{\mathcal{C}}(C,GF(C')) & \leftarrow & Mor_{\mathcal{C}}(C',GF(C'))
\end{array} .$$

Considering the images of $id_{F(C)}$ resp. $id_{F(C')}$ we derive the desired condition $\eta_C GF(f) = f\eta_{C'}$.

If $\psi_{C,D}$ is the map inverse to $\eta_{C,D}$, then we get a morphisms in $\mathcal{D}$,

$$\psi_D = (id_{G(D)})\psi_{G(D),D} : FG(D) \to D,\ D \in \mathcal{D},$$

which determine a functorial morphism ψ: $FG \to id_{\mathcal{D}}$.

From the commutative diagram, with $g : C \to G(D)$ in $\mathcal{C}$,

$$\begin{array}{ccc}
Mor_{\mathcal{C}}(G(D),G(D)) & \xrightarrow{\psi_{G(D),D}} & Mor_{\mathcal{D}}(FG(D),D) \\
\downarrow Mor_{\mathcal{C}}(g,G(D)) & & \downarrow Mor_{\mathcal{D}}(F(g),D) \\
Mor_{\mathcal{C}}(C,G(D)) & \xrightarrow{\psi_{C,D}} & Mor_{\mathcal{D}}(F(C),D)
\end{array} ,$$

we obtain considering the image of $id_{G(D)}$, $(g)\psi_{C,D} = F(g)\psi_D$. Thus the triangle in the following diagram (with $D = F(C)$) is commutative:

$$\begin{array}{ccccc}
Mor_{\mathcal{D}}(F(C),F(C)) & \xrightarrow{\eta_{C,F(C)}} & Mor_{\mathcal{C}}(C,GF(C)) & \xrightarrow{\psi_{C,F(C)}} & Mor_{\mathcal{D}}(F(C),F(C)) \\
& & F_{C,GF(C)} \searrow & & \nearrow Mor(F(C),\psi_{F(C)}) \\
& & & Mor_{\mathcal{D}}(F(C),FGF(C)) &
\end{array} .$$

From this we see

$$id_{F(C)} = (id_{F(C)})\eta_{C,F(C)}\psi_{C,F(C)} = (\eta_C)\psi_{C,F(C)} = F(\eta_C)\psi_{F(C)} .$$

Therefore $F(C) \xrightarrow{F(\eta_C)} FGF(C) \xrightarrow{\psi_{F(C)}} F(C)$ yields the identity on $F(C)$.

The given relation for $G(D)$ can be obtained in an analogous way.

$\Leftarrow$ If functorial morphisms are given with the properties indicated, the desired functorial morphisms are obtained by

$$Mor(F(C), D) \overset{G_{F(C),D}}{\longrightarrow} Mor(GF(C), G(D)) \overset{Mor(\eta_C, G(D))}{\longrightarrow} Mor(C, G(D))$$
$$Mor(C, G(D)) \overset{F_{C,G(D)}}{\longrightarrow} Mor(F(C), FG(D)) \overset{Mor(F(C), \psi_{G(D)})}{\longrightarrow} Mor(F(C), D).$$

These are isomorphisms which are inverse to each other: Because of the Yoneda Lemma, it is sufficient to check whether the composition leaves $id_{F(C)}$ resp. $id_{G(D)}$ unchanged. This follows from the given conditions.

(2) is obtained by applying (1) to the covariant functors

$$F^* : \mathcal{C} \to \mathcal{D} \to \mathcal{D}^o \text{ and } G^* : \mathcal{D}^o \to \mathcal{D} \to \mathcal{C}\ .$$

45.6 Adjoint covariant functors on module categories.

For rings R, S, let $\mathcal{C} \subset R\text{-}MOD$ and $\mathcal{D} \subset S\text{-}MOD$ be full subcategories which are closed under finite products.

Assume that the covariant functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ form an adjoint pair with functorial isomorphism

$$\eta : Hom_S(F(-), -) \to Hom_R(-, G(-)).$$

Then (1) F and G are additive functors and for A in $\mathcal{C}$, D in $\mathcal{D}$,

$$\eta_{A,D} :\ Hom_S(F(A), D) \longrightarrow Hom_R(A, G(D))$$

is a group isomorphism.

(2) F preserves direct limits (hence direct sums and cokernels),
G preserves inverse limits (hence direct products and kernels).

(3) If $\mathcal{C}$ and $\mathcal{D}$ are closed under forming sub- and factor modules (exact categories), then F is right exact and G is left exact and
(i) if F is exact, then G preserves injective objects;
(ii) if G is exact, then F preserves projective objects.

(4) For $\mathcal{C} = R\text{-}MOD$, or $\mathcal{C} =$ the category of finitely presented modules in $R\text{-}MOD$, $F(R)$ is an (S,R)-bimodule and

$$G \simeq Hom_S(F(R), -) \text{ and } F \simeq F(R) \otimes_R -.$$

Proof: (1) By 45.4, the functors F and G preserve, in particular, finite products and hence they are additive by 11.9. (2) follows from 45.4, (3).

(3) Under the given assumptions, F (resp. G) is right exact (left exact) if and only if it preserves cokernels (kernels) (see 11.8).

(i) We have a functorial isomorphism

$$Hom(F(-), D) \simeq Hom(-, G(D)).$$

If D is injective and F an exact functor, then these functors are exact and hence $G(D)$ is injective.

(ii) can be shown in a similar way to (i).

(4) Since $R \simeq Hom_R(R,R)$, the object $F(R)$ becomes an (S,R)-bimodule via the ring homomorphism $F_{R,R}$: $Hom_R(R,R) \longrightarrow Hom_S(F(R),F(R))$. Hence for L in $\mathcal{D}$, $Hom_S(F(R),L)$ is a left R-module (with $(x)r\psi := (xr)\psi)$ and we have a $\mathbb{Z}$-isomorphism, which is in fact R-linear,

$$Hom_S(F(R), L) \xrightarrow{\eta_{R,L}} Hom_R(R, G(L)) \simeq G(L),$$

yielding the desired functorial isomorphism $G \simeq Hom_S(F(R), -)$.

The characterization of F follows from more general observations in:

45.7 Functors *R-MOD* $\to \mathcal{D}$ preserving limits.

For rings R, S, let $\mathcal{E}$ be the category of finitely presented R-modules and $\mathcal{D} \subset S$-MOD, $\mathcal{D}' \subset$ MOD-S full subcategories.

(1) If $F : R$-MOD $\to \mathcal{D}$ is a covariant functor preserving direct limits, then $F(R)$ is an (S,R)-bimodule and $F \simeq F(R) \otimes_R -$.

(2) If $F : \mathcal{E} \to \mathcal{D}$ is a covariant, additive and right exact functor, then $F(R)$ is an (S,R)-bimodule and $F \simeq F(R) \otimes_R -$.

(3) If $F' : R$-MOD $\to \mathcal{D}'$ is a contravariant functor converting direct limits into inverse limits, then $F'(R)$ is an (R,S)-bimodule and
$F' \simeq Hom_R(-, F'(R))$.

(4) If $F' : \mathcal{E} \to \mathcal{D}'$ is a contravariant, additive functor converting kernels into cokernels, then $F'(R)$ is an (R,S)-bimodule and
$F' \simeq Hom_R(-, F'(R))$.

Proof: (1) The (S,R)-bimodule structure of $F(R)$ follows from the ring homomorphism $F_{R,R} : Hom_R(R,R) \to Hom_S(F(R),F(R))$. For an R-module K, we obtain, from the isomorphism $K \simeq Hom_R(R,K)$, a $\mathbb{Z}$-bilinear map

$$F(R) \times K \longrightarrow F(R) \times Hom_S(F(R), F(K)) \xrightarrow{\mu} F(K)\ ,$$

where μ denotes the map $(a,f) \mapsto (a)f$. This map is in fact R-balanced and therefore leads to an S-homomorphism

$$\eta_K : {}_SF(R) \otimes_R K \to {}_SF(K),$$

which determines a functorial morphism $\eta : F(R) \otimes_R - \to F$.

For $K = R$, the map $\eta_R : F(R) \otimes_R R \to F(R)$ is an isomorphism. Since F and $F(R)\otimes_R-$ commute with direct sums, η_K is an isomorphism for every free R-module. For an arbitrary R-module K, we have an exact sequence $R^{(\Lambda)} \to R^{(\Omega)} \to K \to 0$. Since F and $F(R) \otimes_R -$ preserve cokernels, we can form the following commutative exact diagram

$$\begin{array}{ccccccc}
F(R) \otimes R^{(\Lambda)} & \longrightarrow & F(R) \otimes R^{(\Omega)} & \longrightarrow & F(R) \otimes K & \longrightarrow 0 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \eta_K & \\
F(R^{(\Lambda)}) & \longrightarrow & F(R^{(\Omega)}) & \longrightarrow & F(K) & \longrightarrow 0 & .
\end{array}$$

Hence η_K is an isomorphism for any K, and η is a functorial isomorphism.

(2) can be seen from the proof of (1), observing that additive functors preserve finite products and that for a finitely presented K the sets Λ and Ω can be chosen finite.

(3) The (R, S)-bimodule structure of $F'(R)$ results from the ring homomorphism

$$F'_{R,R} : Hom_R(R, R)^{op} \to Hom_S(F'(R), F'(R))$$

(we write S-homomorphisms on the left). For $N \in R\text{-}MOD$ we have

$$N \simeq Hom_R(R, N) \xrightarrow{F'_{R,N}} Hom_S(F'(N), F'(R))$$

which yields, for every $n \in N$, an S-homomorphism $F'_{R,N}(n)$. Consider

$$\psi_N : F'(N) \to Hom_R(N, F'(R)),\ x \mapsto [n \mapsto F'_{R,N}(n)(x)].$$

It is easy to see that this defines S-homomorphisms describing a functorial morphism ψ: $F' \to Hom_R(-, F'(R))$. Since ψ_R is an isomorphism, this is also true for ψ_K, if K is a free R-module. Similarly to (1), we conclude from this that ψ_N is an isomorphism for every $N \in R\text{-}MOD$, i.e. ψ is a functorial isomorphism.

(4) follows from the proof of (3) (see the proof of (2)).

We have seen in 45.6 that, in certain situations, adjoint functors can be represented by Hom- and tensor functors. We already know from 12.2 that these functors are adjoint to each other. In fact we have:

45.8 The pair of functors $U \otimes_S -$, $Hom_R(U, -)$.

(1) For an (R, S)-bimodule ${}_RU_S$, the pair of functors

$$U \otimes_S - :\ S\text{-}MOD \to \sigma[{}_RU] \text{ and } Hom_R(U, -) : \sigma[{}_RU] \to S\text{-}MOD$$

is adjoint via the isomorphisms (for $L \in S\text{-}MOD$, $N \in \sigma[_RU]$)

$$\psi_{L,N} : Hom_R(U \otimes_S L, N) \to Hom_S(L, Hom_R(U,N)),\ \delta \mapsto [l \mapsto (- \otimes l)\delta].$$

(2) Associated with this are the functorial morphisms

$$\nu_L : L \to Hom_R(U, U \otimes_S L), \quad l \mapsto [u \mapsto u \otimes l],$$
$$\mu_N : U \otimes_S Hom_R(U,N) \to N, \quad u \otimes f \mapsto (u)f,$$

where $Im\ \mu_N = Tr(U,N)$.

(3) We have $(id_U \otimes \nu_L)\mu_{U\otimes L} = id_{U\otimes L}$ in R-MOD and $\nu_{Hom(U,N)} \cdot Hom_R(U, \mu_{Hom(U,N)}) = id_{Hom(U,N)}$ in S-MOD.

Proof: (1) For $L \in S\text{-}MOD$ there is an exact sequence $S^{(\Lambda)} \to L \to 0$. Then also $U \otimes_S S^{(\Lambda)} \to U \otimes_S L \to 0$ is exact and $U \otimes_S L$ is generated by $_RU$, hence contained in $\sigma[_RU]$.

The mappings $\psi_{L,N}$ we already know from 12.2 as isomorphisms of abelian groups. It is easy to see that they are functorial in L and N.

(2) The mappings ν_L and μ_N yield the functorial morphisms to the identity considered in 45.5.

(3) These relations follow from 45.5 but can also be verified directly.

From 45.6 and 45.7 we see that, for a functor $F : R\text{-}MOD \to AB$, there is a right adjoint functor if and only if $F \simeq F(R) \otimes_R -$.

Dual to the assertions in 45.6 we obtain:

45.9 Adjoint contravariant functors.

For rings R, S, let $\mathcal{C} \subset R\text{-}MOD$ and $\mathcal{D} \subset MOD\text{-}S$ be full subcategories closed under finite products. Assume $F' : \mathcal{C} \to \mathcal{D}$ and $G' : \mathcal{D} \to \mathcal{C}$ to be contravariant functors.

(1) If the pair (F', G') is right or left adjoint, then F' and G' are additive functors and the related morphisms

$$Hom_S(L, F'(N)) \to Hom_R(N, G'(L)) \ \textit{resp.}\ ,$$
$$Hom_R(G'(L), N) \to Hom_S(F'(N), L),$$

with $L \in MOD\text{-}S$, $N \in R\text{-}MOD$, are group isomorphisms.

(2) If the pair (F', G') is right adjoint, then:

(i) F' and G' convert direct limits into inverse limits (if they exist).

(ii) If $\mathcal{C}$, $\mathcal{D}$ are closed under sub- and factor modules, and if F' (resp. G') is exact, then G' (resp. F') converts projectives into injectives.

(iii) If $\mathcal{C}$ is closed under factor modules and R is in $\mathcal{C}$, or if $\mathcal{C}$ is the category of finitely presented R-modules, then $F'(R)$ is an (R,S)-bimodule,

$$F' \simeq Hom_R(-, F'(R)) \text{ and } G' \simeq Hom_S(-, F'(R)).$$

(3) If the pair (F', G') is left adjoint, then:

(i) F' and G' convert inverse limits into direct limits (if they exist).

(ii) If $\mathcal{C}$, $\mathcal{D}$ are closed under sub- and factor modules, and if F' (resp. G') is exact, then G' (resp. F') converts injectives into projectives.

Proof: Most of these assertions follow by applying 45.6 to the covariant functors $\mathcal{C} \to \mathcal{D} \to \mathcal{D}^o$ and $\mathcal{D}^o \to \mathcal{D} \to \mathcal{C}$.

(2)(*iii*) If $\mathcal{C}$ consists of all finitely presented R-modules, then the isomorphism $F' \simeq Hom_S(-, F'(R))$ follows from 45.7 (notice (2)(*i*)).

If $\mathcal{C}$ is closed under factor modules and $R \in \mathcal{C}$, then every object in $\mathcal{C}$ is a direct limit (formed in R-MOD) of finitely presented modules (in $\mathcal{C}$). Since F' converts this into an inverse limit we also get $F' \simeq Hom_R(-, F'(R))$.

Two functors adjoint to F' are always isomorphic by 45.4. Therefore the isomorphism $G' \simeq Hom_S(-, F'(R))$ follows from the next statement:

45.10 The pair of functors $Hom_R(-,U), Hom_S(-,U)$.

(1) For an (R,S)-bimodule ${}_RU_S$, the functors

$$Hom_R(-,U): \ R\text{-}MOD \to MOD\text{-}S, \quad Hom_S(-,U): \ MOD\text{-}S \to R\text{-}MOD$$

form a right adjoint pair by the isomorphisms ($L \in MOD$-S, $N \in R$-MOD):

$$\begin{array}{rrcl} \Phi_{L,N}: & Hom_S(L, Hom_R(N,U)) & \to & Hom_R(N, Hom_S(L,U)), \\ & f & \mapsto & [n \to (n)[f(-)]]. \end{array}$$

(2) Associated with this are the (evaluation) homomorphisms

$$\begin{array}{rll} \Phi_N: & N \to Hom_S(Hom_R(N,U),U), & n \mapsto [\beta \mapsto (n)\beta], \\ \Phi_L: & L \to Hom_R(Hom_S(L,U),U), & l \mapsto [\alpha \mapsto \alpha(l)], \end{array}$$

where $Ke\,\Phi_L = \{l \in L \mid \alpha(l) = 0 \text{ for all } \alpha \in Hom(L,U)\} = Re(L,U)$ and $Ke\,\Phi_N = Re(N,U)$.

Therefore Φ_L is injective if and only if L is cogenerated by U_S, and Φ_N is injective if and only if N is cogenerated by ${}_RU$.

(3) If we denote $Hom_R(-,U)$ and $Hom_S(-,U)$ by $(\ )^*$, then for Φ_L: $L \to L^{**}$, Φ_N: $N \to N^{**}$ in (2) we have:

$$\Phi_{L^*}(\Phi_L)^* = id_{L^*} \text{ in } R\text{-}MOD \text{ and } (\Phi_N)^*\Phi_{N^*} = id_{N^*} \text{ in } MOD\text{-}S.$$

Thus Φ_{L^*} and Φ_{N^*} are always monic.

Proof: (1) $Hom_R(N, Hom_S(-,U)) : MOD\text{-}S \to \mathbb{Z}\text{-}MOD$ is a contravariant functor converting direct limits into inverse limits. Now, by 45.7,(3),

$$\begin{aligned} Hom_R(N, Hom_S(-,U)) &\simeq Hom_S(-, Hom_R(N, Hom_S(S,U))) \\ &\simeq Hom_S(-, Hom_R(N,U)). \end{aligned}$$

This implies that $\Phi_{L,N}$ is an isomorphism and is functorial in L.

Similarly it can be seen that it is also functorial in N.

(2), (3) By evaluating it can be seen that Φ_N and Φ_L are just the functorial morphisms to the identity considered in 45.5,(2), and we obtain the relations given in 45.5. They also can be verified (more easily) directly from the given definitions. (Notice the different way of writing the composition in *R-MOD* and *MOD-S*.) By 14.5, the reject $Re(N,U)$ is zero if and only if N is cogenerated by U.

Also we already know the next example of adjoint functors:

45.11 The inclusion $\sigma[M] \to R\text{-}MOD$.

Let M be an R-module and L a generator in $\sigma[M]$. Then the inclusion functor

$$I : \sigma[M] \to R\text{-}MOD$$

is left adjoint to the trace functor $Tr(L,-) : R\text{-}MOD \to \sigma[M]$.

For all $N \in \sigma[M]$ and $K \in R\text{-}MOD$, we have

$$Hom_R(I(N), K) \simeq Hom_R(N, Tr(L,K)).$$

Since I is obviously exact, we conclude from 45.5:

(i) I preserves direct sums;

(ii) $Tr(L,-)$ is left exact and preserves products and injective objects.

Here (i) is just the known fact that (direct sums and) direct limits in $\sigma[M]$ are also direct limits in *R-MOD*.

From (ii) we see that, for a family $\{N_\lambda\}_\Lambda$ of modules from $\sigma[M]$, the trace $Tr(L, \prod_\Lambda N_\lambda)$ represents the product of the N_λ in $\sigma[M]$ (see 15.1), and that, for an injective object K in *R-MOD*, the trace $Tr(L,K)$ is injective in $\sigma[M]$ (see 16.8).

By the way, for any $M \in R\text{-}MOD$ and

$$Gen(M) := \{N \in R\text{-}MOD \mid N \text{ is M-generated }\},$$

the inclusion $I : Gen(M) \to R\text{-}MOD$ is left adjoint to the trace functor $Tr(M,-) : R\text{-}MOD \to Gen(M)$.

Literature: FAITH [1], HILTON-STAMMBACH, STENSTRÖM; Garcia-Gomez [3], Kashu, Lambek [2], Nishida [2], Zimmermann-Huisgen.

46 Equivalences of categories

1.Characterization and properties of equivalences. 2.Equivalence between $\sigma[M]$ and S-MOD. 3.Properties of equivalences $\sigma[M] \rightarrow$ S-MOD. 4.Morita equivalent rings. 5.Ideal structure of equivalent rings. 6.Matrix rings and equivalences. 7.Equivalences determined by $Hom(M,-)$. 8.Properties of M-faithful and M-torsion modules. 9.M-presented modules with projective M. 10.M-faithful and M-presented modules. 11.Equivalences for self-projective modules.

Two categories $\mathcal{C}$, $\mathcal{D}$ are called *equivalent* if there are covariant functors F: $\mathcal{C} \rightarrow \mathcal{D}$ and G: $\mathcal{D} \rightarrow \mathcal{C}$ with functorial isomorphisms $GF \simeq id_{\mathcal{C}}$ and $FG \simeq id_{\mathcal{D}}$.

In this case the functors F and G are called *equivalences*. We say that G is the *(equivalence) inverse* of F.

First of all we want to find out which of the properties formulated in 11.2 a functor has to possess to be an equivalence:

46.1 Characterization and properties of equivalences.

Let F: $\mathcal{C} \rightarrow \mathcal{D}$ be a covariant functor between categories $\mathcal{C}$, $\mathcal{D}$.

(1) F is an equivalence if and only if F is faithful, full and representative.

(2) If F is an equivalence with inverse G: $\mathcal{D} \rightarrow \mathcal{C}$, then:

(i) (F,G) and (G,F) are pairs of adjoint functors;

(ii) F (and G) preserve direct and inverse limits (if they exist).

Proof: (1) $\Rightarrow$ Let F be an equivalence with inverse G: $\mathcal{D} \rightarrow \mathcal{C}$ and η: $id_{\mathcal{C}} \rightarrow GF$, ψ: $id_{\mathcal{D}} \rightarrow FG$ be the related functorial isomorphisms. For every morphism f: $A \rightarrow B$ in $\mathcal{C}$, we have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \eta_A & & \downarrow \eta_B \\ GF(A) & \xrightarrow{GF(f)} & GF(B) \end{array}$$

with isomorphisms η_A, η_B. If f_1: $A \rightarrow B$, f_2: $A \rightarrow B$ are different morphisms in $\mathcal{C}$, then we have $GF(f_1) \neq GF(f_2)$ and hence $F(f_1) \neq F(f_2)$ has to hold. Therefore $F_{A,B}$: $Mor_{\mathcal{C}}(A,B) \rightarrow Mor_{\mathcal{D}}(F(A),F(B))$ is injective, i.e. F is faithful. For reasons of symmetry G has to be faithful, too.

For $g \in Mor_{\mathcal{D}}(F(A),F(B))$ we obtain with $h = \eta_A G(g) \eta_B^{-1}$ the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow \eta_A & & \downarrow \eta_B \\ GF(A) & \xrightarrow{G(g)} & GF(B) \end{array}$$

from which we derive $GF(h) = G(g)$. Since G is faithful, this means $F(h) = g$. Therefore $F_{A,B}$ is surjective and F is full.

Finally, F is representative since for every D in $\mathcal{D}$ there is an isomorphism ψ_D: $D \to FG(D)$.

$\Leftarrow$ Assume the functor F to be faithful, full and representative. Then for every object D in $\mathcal{D}$ there is an object $G(D)$ in $\mathcal{C}$ with an isomorphism $\gamma_D : F(G(D)) \to D$. A morphism $g \in Mor_{\mathcal{D}}(D, H)$ leads to a morphism

$$\bar{g} := \gamma_D g \gamma_H^{-1} : F(G(D)) \longrightarrow F(G(H)).$$

Since F is full and faithful, there is a unique morphism $G(g)$: $G(D) \to G(H)$ with $F(G(g)) = \bar{g}$.

It is easy to check that this determines a functor G: $\mathcal{D} \to \mathcal{C}$ and a functorial isomorphism $\gamma : FG \to id_{\mathcal{D}}$. From this we derive an isomorphism $\gamma_{F(A)}$: $FGF(A) \to F(A)$ for A in $\mathcal{C}$. Considering again the fact that F is faithful and full we obtain a unique isomorphism (notice 11.3)
δ_A: $GF(A) \to A$ with $F(\delta_A) = \gamma_{F(A)}$.

It remains to verify that this yields a functorial isomorphism
δ: $GF \to id_{\mathcal{C}}$: For $f \in Mor_{\mathcal{C}}(A, B)$ we form the diagrams

$$\begin{array}{ccc} GF(A) & \xrightarrow{GF(f)} & GF(B) \\ \downarrow \delta_A & & \downarrow \delta_B \\ A & \xrightarrow{f} & B \end{array} \quad and \quad \begin{array}{ccc} FGF(A) & \xrightarrow{FGF(f)} & FGF(B) \\ \downarrow F(\delta_A) & & \downarrow F(\delta_B) \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array} .$$

Because of $F(\delta_A) = \gamma_{F(A)}$, the right diagram is commutative. Since F is faithful, the left diagram has to be commutative, too.

(2) (i) Let η: $id_{\mathcal{C}} \to GF$ be a functorial isomorphism. Since G is full and faithful by (1), we have, for objects C in $\mathcal{C}$ and D in $\mathcal{D}$, isomorphisms

$$Mor_{\mathcal{D}}(F(C), D) \xrightarrow{G_{F(C),D}} Mor_{\mathcal{C}}(GF(C), G(D)) \xrightarrow{Mor(\eta, G(D))} Mor_{\mathcal{C}}(C, G(D)),$$

which make (F, G) an adjoint pair.

In a similar way we see that (G, F) is also an adjoint pair.

(ii) Since F is a left and right adjoint functor, this follows from 45.4.

Equivalences of module categories are of main interest for us. From the preceding considerations we derive without difficulty:

46.2 Equivalence between $\sigma[M]$ and S-MOD.
For an R-module M and a ring S, the following are equivalent:
(a) $\sigma[M]$ is equivalent to S-MOD;
(b) there is a finitely generated, projective generator P in $\sigma[M]$ with $End_R(P) \simeq S$;
(c) there is a finitely generated generator P in $\sigma[M]$ with $End_R(P) \simeq S$ and P_S is faithfully flat;
(d) there is a module P in $\sigma[M]$ with $End_R(P) \simeq S$ such that $Hom_R(P,-) : \sigma[M] \to S\text{-}MOD$ is an equivalence with inverse $P \otimes_S -$.

Proof: $(a) \Rightarrow (b)$ Let $F : \sigma[M] \to S\text{-}MOD$ be an equivalence with inverse $G : S\text{-}MOD \to \sigma[M]$. By 45.6, there is an (R,S)-bimodule P ($\simeq {}_RG(S)_S$) with $G \simeq P \otimes_S -$ and $F \simeq Hom_R(P,-)$. Therefore $Hom_R(P,-)$ is (full and) faithful by 46.1, and P is a generator (see 13.6).

Since $(P \otimes_S -, Hom_R(P,-))$ form a pair of adjoint functors, $Hom_R(P,-)$ is exact, by 45.6, and preserves direct limits. Hence P is projective in $\sigma[M]$ (see 18.3) and finitely generated (see 25.2). Also, we have isomorphisms

$$End_R(P) \simeq Hom_R(G(S), G(S)) \simeq Hom_S(FG(S), FG(S)) \simeq S.$$

$(b) \Leftrightarrow (c)$ was shown earlier in 18.5.

$(b) \Rightarrow (d)$ We show

$$P \otimes_S Hom_R(P,-) \simeq id_{\sigma[M]} \text{ and } Hom_R(P, P \otimes_S -) \simeq id_{MOD-S}.$$

Under the assumptions in (b), these functors are obviously right exact and commute with direct sums.

For $N \in \sigma[M]$, we form an exact sequence $P^{(\Lambda)} \to P^{(\Omega)} \to N \to 0$ and obtain from it the following commutative exact diagram

$$\begin{array}{ccccccc}
P \otimes Hom(P, P^{(\Lambda)}) & \to & P \otimes Hom(P, P^{(\Omega)}) & \to & P \otimes Hom(P, N) & \to & 0 \\
\downarrow \mu_\Lambda & & \downarrow \mu_\Omega & & \downarrow \mu_N & & \\
P^{(\Lambda)} & \to & P^{(\Omega)} & \to & N & \to & 0,
\end{array}$$

where the μ denote the evaluation homomorphisms. Since μ_Λ and μ_Ω are isomorphisms, this is also true for μ_N.

Similarly we obtain for $L \in S\text{-}MOD$ (from $S^{(\Lambda)} \to S^{(\Omega)} \to L \to 0$) that $Hom_R(P, P \otimes_S L) \simeq L$.

$(d) \Rightarrow (a)$ is clear.

A finitely generated, projective generator in $\sigma[M]$ is called a *progenerator* (in $\sigma[M]$).

46.3 Properties of equivalences $\sigma[M] \to S$-MOD.

Let M be a left R-module, S a ring and $F\colon \sigma[M] \to S$-MOD an equivalence. Then:

(1) F preserves and reflects

(i) injective and projective objects,
(ii) direct and inverse limits,
(iii) generators and cogenerators,
(iv) finitely generated and finitely presented objects,
(v) finitely cogenerated and finitely copresented objects,
(vi) essential monomorphisms and superfluous epimorphisms.

(2) If $F \simeq Hom_R(P,-)$, for a progenerater $P \in \sigma[M]$, then for $N \in \sigma[M]$, the map $Hom_R(P,-)$:

$$\{\ R\text{-submodules of } N\ \} \longrightarrow \{\ S\text{-submodules of } Hom_R(P,N)\}$$

is bijective and order preserving (lattice isomorphism).

(3) F preserves and reflects

(i) artinian and noetherian modules,
(ii) simple modules,
(iii) (the length of) composition series of modules,
(iv) indecomposable modules.

Proof: (1) Let G be an equivalence inverse of F.

(i)-(v) Since (F,G) and (G,F) are adjoint pairs we derive from 45.4 that F and G preserve the given properties. E.g., for C in $\sigma[M]$ assume the object $F(C)$ to be injective. Then $GF(C) \simeq C$ is also injective and hence F reflects injective objects.

The other assertions are seen in an analogous way.

(vi) Let $f\colon\ L \to M$ be an essential monomorphism in $\sigma[M]$. If $F(g)\colon\ F(N) \to F(K)$ is a morphism in S-MOD and $F(f)F(g) = F(fg)$ is monic, then, by (ii), fg is also monic. Since f is essential, g is also monic and (by (e)) $F(g)$ is monic. Hence $F(f)$ is an essential monomorphism.

The further assertion can be seen in a similar way.

(2) Since P is finitely generated and projective, for every S-submodule $I \subset Hom_R(P,N)$ we have $I = Hom_R(P,PI)$ (by 18.4). Thus the assignment is surjective. Since P is a generator, for every submodule $K \subset N$, $K = Tr(P,K) = P\,Hom_R(P,K)$ and the map is injective.

(3) All given properties are characterized by the lattice of submodules.

Obviously, from 46.3 we derive that for a finitely generated, self-projective self-generator ${}_RM$ (progenerator in $\sigma[M]$) the following holds:

$End_R(M)$ is left artinian, noetherian, semiperfect, perfect, resp. injective, if and only if ${}_RM$ has the corresponding property.

Some of these relations we have observed earlier.

Two rings R and S are said to be *Morita equivalent* or just *equivalent* if *R-MOD* and *S-MOD* are equivalent categories. Applying 46.2, we obtain the following description of this situation, which shows that equivalence is a left-right-symmetric notion.

46.4 Morita equivalent rings. Characterizations.

(1) For two rings R and S the following assertions are equivalent:

(a) There is an equivalence $F\colon$ R-MOD $\to$ S-MOD;

(b) there is a progenerator P in R-MOD with $S \simeq End_R(P)$;

(c) there is a generator P in R-MOD with $S \simeq End_R(P)$, such that P_S is a generator in MOD-S;

(d) there is a progenerator P^* in S-MOD with $R \simeq End_S(P^*)$;

(e) there is a progenerator P_S in MOD-S with $R \simeq End(P_S)$;

(f) there is an equivalence $F'\colon$ MOD-R $\to$ MOD-S.

(2) Assume G and G' to be equivalence inverses to F resp. F' in (1). Putting $P := G(S)$, we can choose $P^* = Hom_R(P,R)$ to get functorial isomorphisms

$$\begin{aligned} &F \simeq Hom_R(P,-) \simeq P^* \otimes_R -\,, \quad G \simeq P \otimes_S -\,,\\ &F' \simeq - \otimes_R P \simeq Hom_R(P^*,-)\,, \quad G' \simeq Hom_S(P_S,-)\,. \end{aligned}$$

In particular, we have bimodule isomorphisms

$$P^* \otimes_R P \to S,\ \varphi \otimes p \mapsto (-)\varphi p, \quad P \otimes_S P^* \to R,\ p \otimes \varphi \mapsto \varphi(p).$$

Proof: (1) $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (d)$ and $(e) \Leftrightarrow (f)$ follow from 46.2.

$(b) \Rightarrow (c)$ Being a generator in *R-MOD*, the module P is finitely generated and projective over its endomorphism ring S and $R \simeq End(P_S)$ (see 18.8). Hence P_S is also finitely generated and projective over its endomorphism ring R, implying that P_S is a generator in *MOD-S* (also 18.8).

$(c) \Rightarrow (b)$ Since P_S is a generator in *MOD-S* and $R \simeq End_S(P)$, it follows again by 18.8 that ${}_RP$ is finitely generated and projective.

$(b) \Leftrightarrow (e)$ follows from the proof of $(b) \Leftrightarrow (c)$.

(2) The equivalence $Hom_R(P,-)$ turns the projective generator R into the projective generator ${}_SP^* = Hom_R(P,R)$ in *S-MOD* (see 46.3). If P is a

progenerator in *R-MOD*, then P is a direct summand of ${}_RR^k$ and ${}_RR$ is a direct summand of P^l for suitable $k, l \in \mathbb{N}$. Hence P^* is a direct summand of R^k_R and R_R is a direct summand of P^{*l}, i.e. P^* is a progenerator in *MOD-R*.

The isomorphisms $F \simeq Hom_R(P, -)$ and $G \simeq P \otimes_S -$ (for $P = G(S)$) are known from 45.6. Under the given assumptions we get, from 45.7,(1),

$$Hom_R(P, -) \simeq Hom_R(P, R) \otimes_R - \ ,$$
$$Hom_R(P^*, -) \simeq - \otimes_R Hom_R(P^*, R) \simeq - \otimes {}_RP \quad .$$

The representation of G' follows from the uniqueness of adjoint functors (see 45.4). Now the isomorphisms can easily be verified.

If R and S are Morita equivalent rings, then it is clear that particular properties of *R-MOD* (resp. *MOD-R*) we can find again in *S-MOD* (resp. *MOD-S*). Many module theoretic properties of a ring (e.g. perfect) carry over to rings equivalent to R. It is remarkable that in addition equivalent rings also have the same ideal structure:

46.5 Ideal structure of equivalent rings.
If R and S are Morita equivalent rings, then:

(1) There is an order preserving, bijective map between the sets of two-sided ideals in R and two-sided ideals of S (lattice isomorphism).

(2) The center of R is isomorphic to the center of S.

Proof: (1) Let P be a progenerator in *R-MOD* with $S = End_R(P)$ and $P^* = Hom_R(P, R)$. For two-sided ideals $I \subset R$ and $B \subset S$ we obtain from the exact sequences $0 \to I \to R$ and $0 \to B \to S$ the exact sequences

$$0 \to P^* \otimes_R I \otimes_R P \to P^* \otimes_R R \otimes_R P \simeq S \quad ,$$
$$0 \to P \otimes_S B \otimes_S P^* \to P \otimes_S S \otimes_S P^* \simeq R \quad .$$

With their help we can regard $P^* \otimes_R I \otimes_R P$ and $P \otimes_S B \otimes_S P^*$ as ideals in S resp. R. From the isomorphisms $P^* \otimes_R P \simeq S$ and $P \otimes_S P^* \simeq R$ given in 46.4 we conclude that this assignment is bijective.

(2) We may assume $R = End(P_S)$ and regard center(R) as a subring of center(S). On the other hand, we can consider center(S) as a subring of the center of $End(P_S) = R$. Thus center(R) = center(S).

We call a property of a ring R a *Morita invariant* if is carried over to every ring equivalent to R.

For example, it follows from 46.4 that left semisimple, regular, left artinian and left noetherian are examples of such properties. From 46.5 we see that the simplicity of a ring and the ascending (descending) chain condition of two-sided ideals are also Morita invariants. As special cases of the preceding considerations we obtain the following assertions, part of which we already knew before.

46.6 Matrix rings and equivalences.
Let R be a ring, and, for $n \in I\!N$, let $R^{(n,n)}$ denote the ring of $n \times n$ matrices over R. Then:

(1) The rings R and $R^{(n,n)}$ are Morita equivalent.

(2) The map $Hom_R(R^n, -)\colon I \to Hom(R^n, I) \subset R^{(n,n)}$ is a bijective, order preserving map from left ideals in R to left ideals of $R^{(n,n)}$.

(3) Ideals in R are in one-to-one correspondence with ideals in $R^{(n,n)}$ under $I \to I^{(n,n)}$.

(4) The center of $R^{(n,n)}$ is isomorphic to the center of R.

(5) Two rings R and S are equivalent if and only if there exist $n \in I\!N$ and an idempotent $e \in R^{(n,n)}$ with

$$S \simeq eR^{(n,n)}e \quad \text{and} \quad R^{(n,n)}eR^{(n,n)} = R^{(n,n)}.$$

Proof: Since R^n is a progenerator in R-MOD and $End(R^n) \simeq R^{(n,n)}$, (1) and (2) follow from 46.3 and 46.4. (3) and (4) result from 46.5.

(5) If P is a progenerator in R-MOD with $End_R(P) = S$, then P is a direct summand of some R^n, $n \in I\!N$. Thus $P \simeq R^n e$ for some idempotent $e \in R^{(n,n)}$ and $S \simeq End(R^n e) \simeq eR^{(n,n)}e$. Since $R^n e$ is a generator, we have

$$R^n = Tr(R^n e, R^n) = R^n e Hom(R^n, R^n) = R^n e R^{(n,n)} \text{ and}$$
$$R^{(n,n)} = Hom_R(R^n, R^n e R^{(n,n)}) = Hom(R^n, R^n) e R^{(n,n)} = R^{(n,n)} e R^{(n,n)}.$$

On the other hand, assume n and e to be as in (5). Then $S \simeq End(R^n e)$ and from $R^{(n,n)}eR^{(n,n)} = R^{(n,n)}$ we have, by applying it to R^n,

$$(R^n e)R^{(n,n)} = R^n R^{(n,n)} e R^{(n,n)} = R^n R^{(n,n)} = R^n.$$

Hence $R^n e$ generates the module R^n, i.e. $R^n e$ is a projective generator in R-MOD.

At the beginning of this paragraph (see 46.2) we have studied the problem of how to describe equivalences of given categories. On the other hand, we can also ask whether a given functor determines an equivalence between suitable categories. For arbitrary modules we can show:

46.7 Equivalences determined by $Hom_R(M,-)$.
Let M be an R-module and $S = End_R(M)$.

(1) The functor $Hom_R(M,-)$ determines an equivalence between the full subcategory of $\sigma[M]$, consisting of direct summands of finite direct sums of copies of M, and the full subcategory of finitely generated, projective modules in S-MOD.

(2) If M is finitely generated, then $Hom_R(M,-)$ determines an equivalence between the direct summands of (arbitrary) direct sums of copies of M and all projective modules in S-MOD.

In both cases the inverse functor is given by $M \otimes_S -$.

Proof: (1) The functor $M \otimes_S Hom_R(M,-)$ commutes with finite sums and $M \otimes_S Hom_R(M,M) \simeq M$. Hence, for every direct summand of M^n, we also have an isomorphism $M \otimes_S Hom(M,K) \simeq K$. Similarly we see that, for every finitely generated projective S-module L, an isomorphism $L \simeq Hom_R(M, M \otimes_S L)$ is given. Thus $Hom_R(M,-)$ and $M \otimes_S -$ are equivalences between the given categories.

(2) If M is finitely generated, then $Hom_R(M,-)$ commutes with arbitrary sums and the proof above again yields the assertion.

In general the subcategories of $\sigma[M]$ occuring in 46.7 do not have any special properties. However, if M is finitely generated and self-projective, then we can find two subcategories of $\sigma[M]$ which are equivalent to the full category *S-MOD*. For this we need another definition:

Let M be a finitely generated, self-projective R-module. An R-module N with $Hom_R(M,N) = 0$ is called an *M-torsion module.*

Denote by $t_M(N) = \sum\{K \subset N \mid Hom_R(M,K) = 0\}$, the sum of all M-torsion submodules of N.

N is called *M-faithful* (or *M-torsion free*) if $t_M(N) = 0$.

For every N, the factor module $N/Tr(M,N)$ is an M-torsion module. Thus every module in $\sigma[M]$ is M-faithful if and only if M is a generator in $\sigma[M]$.

46.8 Properties of M-faithful and M-torsion modules.

Let M be a finitely generated, self-projective left R-module. With the above notation we put $\overline{N} := N/t_M(N)$, for $N \in R$-MOD. Then:

(1) $t_M(N)$ is an M-torsion module and $t_M(U) = U \cap t_M(N)$ for all $U \subset N$.

(2) For $f \in Hom_R(N, N')$, we have $(t_M(N))f \subset t_M(N')$.

(3) $Hom_R(M, N) \simeq Hom_R(M, \overline{N}) \simeq Hom_R(\overline{M}, \overline{N})$, $End_R(M) \simeq End_R(\overline{M})$ and $\overline{N}$ is M-faithful.

(4) For any family of R-modules $\{N_\lambda\}_\Lambda$, we have

$$t_M(\bigoplus_\Lambda N_\lambda) = \bigoplus_\Lambda t_M(N_\lambda).$$

(5) If M generates N, then $t_M(N) \subset Rad(N)$.

(6) For the map $\mu\colon M \otimes_S Hom_R(M, N) \to N$, $(m, f) \mapsto (m)f$, we obtain $Hom_R(M, Ke\mu) = 0$.

Proof: (1) From 18.4,(3) we derive

$$Hom(M, t_M(N)) = \sum\{Hom(M, K) \mid Hom(M, K) = 0\} = 0.$$

(2) From the exact sequence $t_M(N) \to (t_M(N))f \to 0$ we obtain the exact sequence $0 = Hom(M, t_M(N)) \to Hom(M, (t_M(N))f) \to 0$.

(3) Because of (1), the first part follows from the exact sequence $0 \to t_M(N) \to N \to \overline{N} \to 0$. Assume $U \subset N$ and $t_M(N) \subset U$. Then $Hom(M, U/t_M(N)) = 0$ implies $Hom(M, U) = 0$ and $U \subset t_M(N)$.

(4) Of course, $\bigoplus_\Lambda t_M(N_\lambda) \subset t_M(\bigoplus_\Lambda N_\lambda)$. Assume $Hom(M, U) = 0$ for $U \subset \bigoplus_\Lambda N_\lambda$. Then for $\pi_\mu : \bigoplus_\Lambda N_\lambda \to N_\mu$ we also have $Hom(M, U\pi_\mu) = 0$ and $U\pi_\mu \subset t_M(N_\mu)$ implying $U \subset \bigoplus_\Lambda t_M(N_\lambda)$.

(5) follows from (2), since simple factor modules of N are M-faithful.

(6) Since $Hom_R(M, N) \simeq Hom_R(M, Tr(M, N))$ we may assume N to be M-generated. From the exact sequence $0 \to X \to M^{(\Lambda)} \to N \to 0$ we derive the exact commutative diagram

$$\begin{array}{ccccccc}
 & M \otimes Hom(M, X) & \to & M \otimes Hom(M, M^{(\Lambda)}) & \to & M \otimes Hom(M, N) & \to 0 \\
 & \downarrow \mu_X & & \downarrow \simeq & & \downarrow \mu_N & \\
0 \to & X & \to & M^{(\Lambda)} & \to & N & \to 0 \\
 & \downarrow & & & & & \\
 & X/Tr(M, X) & & & & & .
\end{array}$$

From the Kernel Cokernel Lemma, we deduce $Ke\mu_N \simeq X/Tr(M, X)$ and $Hom(M, Ke\mu_N) \simeq Hom_R(M, X/Tr(M, X)) = 0$.

Let us call an R-module N *M-presented* if there is an exact sequence $M^{(\Lambda)} \to M^{(\Omega)} \to N \to 0$.

N is called *finitely M-presented* if in this sequence the sets Λ and Ω can be chosen finite.

46.9 M-presented modules with projective M.
Assume the R-module M to be projective in $\sigma[M]$.

(1) If in the exact sequence $0 \to K \to L \to N \to 0$ in $\sigma[M]$ the modules K and N are M-generated, then L is also M-generated.

(2) For an M-generated R-module N the following are equivalent:

(a) N is M-presented;

(b) there is an exact sequence $0 \to K \to L \to N \to 0$ with K M-generated and L projective in $\sigma[M]$;

(c) in any exact sequence $0 \to K' \to L' \to N \to 0$ with L' M-generated, K' is also M-generated.

Proof: (1) From the exact sequence $M^{(\Lambda)} \to N \to 0$ we form the exact pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & M^{(\Lambda)} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & . \end{array}$$

We see that $P \simeq K \oplus M^{(\Lambda)}$. Thus P is M-generated and hence L is also M-generated.

(2) $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$ are clear.

$(b) \Rightarrow (c)$ Since L is projective we are able to use the two exact sequences to form the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow f & & \| & & \\ 0 & \longrightarrow & K' & \longrightarrow & L' & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & Coke\, g & \simeq & Coke\, f & & & & \end{array}$$

Herein $Im\ g$ and $K'/Im\ g \simeq Coke\, g$ are M-generated modules. Hence, by (1), K' is also M-generated.

Notice the following relationship between these notions.

46.10 M-faithful and M-presented modules.
Let the R-module M be projective in $\sigma[M]$. Then the following statements are equivalent:

(a) Every M-generated, M-faithful module is M-presented;

(b) every M-generated module is M-faithful;
(c) every M-presented module is M-faithful;
(d) M is a generator in $\sigma[M]$.

Proof: $(a) \Rightarrow (b)$ Let N be an M-generated R-module and p: $M^{(\Omega)} \to N$ an epimorphism. Then $\overline{N} = N/t_M(N)$ is M-generated and M-faithful, thus, by (a), M-presented. We form the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & t_M(N) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & Ke\,p & \longrightarrow & M^{(\Omega)} & \xrightarrow{p} & N & \longrightarrow & 0 \\
 & & \downarrow & & || & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M^{(\Omega)} & \longrightarrow & \overline{N} & \longrightarrow & 0 \ .
\end{array}$$

By 46.9, K is M-generated. Then $t_M(N) \simeq K/Ke\,p$ is also M-generated, hence zero and $N = \overline{N}$ is M-faithful.

$(b) \Rightarrow (c)$ and $(d) \Rightarrow (a)$ are obvious.

$(c) \Rightarrow (d)$ For every submodule $U \subset M^k$, $k \in I\!N$, $U/Tr(M,U)$ is an M-torsion submodule of the M-presented module $M^k/Tr(M,U)$. By (c), we have $U/Tr(M,U) = 0$, i.e. $U = Tr(M,U)$ and M is a generator in $\sigma[M]$ (see 15.5).

46.11 Equivalences for self-projective modules.

Let M be a self-projective R-module and $S = End_R(M)$. Consider the full subcategories of $\sigma[M]$:

$FMP[M]$ with all finitely M-presented modules as objects,

$MP[M]$ with all M-presented modules as objects,

$MT[M]$ with all M-faithful, M-generated modules as objects.

Let $FP[S]$ denote the category of all finitely presented S-modules.

(1) The functor $Hom_R(M,-) : FMP[M] \to FP[S]$ is an equivalence with inverse $M \otimes_S -$.

(2) If M is finitely generated, then:

(i) $Hom_R(M,-) : MP[M] \to S$-MOD is an equivalence with inverse $M \otimes_S -$.

(ii) $Hom_R(M,-) : MT[M] \to S$-MOD is an equivalence with inverse

$$\overline{M \otimes_S -} : {}_SL \mapsto (M \otimes_S L)/t_M(M \otimes_S L).$$

Proof: (1) The functor $Hom_R(M, M \otimes_S -)$ is right exact and commutes with finite sums. By 45.6 we have

$$Hom_R(M, M \otimes_S -) \simeq Hom_R(M, M \otimes_S S) \otimes_S - \simeq S \otimes_S -.$$

Similarly we derive, from the isomorphism $M \otimes_S Hom_R(M, M) \simeq M$, that for every finitely M-presented module N, we have$M \otimes_S Hom(M, N) \simeq N$. Hence the two functors are equivalences which are inverse to each other.

(2) (i) If M is finitely generated, then $Hom_R(M, -)$ commutes with arbitrary sums and the proof of (1) works again.

(ii) For every M-generated, M-faithful module N and
μ: $M \otimes_S Hom_R(M, N) \to N$, we get $Ke\, \mu = t_M(M \otimes_S Hom_R(M, N))$ (see 46.8). Therefore we have isomorphisms

$\overline{M \otimes_S Hom(M, N)} \simeq N$ and $\overline{M \otimes_S Hom(M, -)} \simeq id_{MT(M)}$.

Moreover, for every S-module L (see (1) and 46.8),

$$Hom_R(M, \overline{M \otimes_S L}) \simeq Hom_R(M, M \otimes_S L) \simeq L.$$

Literature: Abrams, Anh-Márki, Azumaya [2], Bolla, Brodskii [4,5], Camillo [1], Fuller [2], Garcia-Gomez [4], Gregorio, Kato [2], Lambek [2], Liu, Nishida [2], Onodera [4], Sato, Yao, Zimmermann-Huisgen.

47 Dualities between categories

1.Dualities. 2.Properties of the category $\sigma_f[M]$. 3.Dualities between module categories. 4.U-reflexive modules. 5.Morita duality. 6.U-dense modules. 7.Relations between $_RU$ and U_S. 8.Cogenerators and linearly compact modules. 9.Modules and rings with $AB5^*$. 10.Injective modules with essential socle. 11.Dualities determined by $Hom(-,U)$. 12.Characterization of Morita dualities. 13.Duality between finitely generated modules. 14.Characterization of linearly compact rings. 15.Characterization of Morita rings. 16.Exercise.

Two categories $\mathcal{C}, \mathcal{D}$ are called *dual* (to each other) if there are contravariant functors $F' : \mathcal{C} \to \mathcal{D}$ and $G' : \mathcal{D} \to \mathcal{C}$ with functorial isomorphisms

$$G'F' \simeq id_{\mathcal{C}} \text{ and } \quad F'G' \simeq id_{\mathcal{D}}.$$

Then the functors F' and G' are called *dualities*. G' is called a *(duality) inverse* of F'.

Obviously the functor $(-)^o$: $\mathcal{C} \to \mathcal{C}^o$ is a duality between $\mathcal{C}$ and $\mathcal{C}^o$ in this sense since for the compositions we have $(-)^{oo} = id_{\mathcal{C}}$ and $(-)^{oo} = id_{\mathcal{C}^o}$.

Applying this functor it is possible to assign to a duality F': $\mathcal{C} \to \mathcal{D}$, equivalences $\mathcal{C} \to \mathcal{D} \to \mathcal{D}^o$ and $\mathcal{C}^o \to \mathcal{C} \to \mathcal{D}$ which in turn determine the duality F'. From the categorial point of view we therefore may replace dualities between $\mathcal{C}$ and $\mathcal{D}$ by equivalences between $\mathcal{C}$ and $\mathcal{D}^o$. However, to investigate dualities between module categories this will not always be helpful since, e.g., $(R\text{-}MOD)^o$ is not a module category. In fact, we will see that there cannot exist any duality between full module categories (remark after 47.3). Therefore we also work out the basic properties of dualities.

47.1 Characterization and properties of dualities.
Let $F' : \mathcal{C} \to \mathcal{D}$ be a contravariant functor between categories $\mathcal{C}$, $\mathcal{D}$.

(1) F' is a duality if and only if F' is faithful, full and representative.

(2) If F' is a duality with inverse $G' : \mathcal{D} \to \mathcal{C}$, then:

(i) The pair (F', G') is right and left adjoint;

(ii) F' and G' convert direct into inverse limits and inverse into direct limits if they exist.

Proof: (1) The proof of 46.1 works similarly for contravariant F'. We also could apply 46.1 to the covariant functor $\mathcal{C} \to \mathcal{D} \to \mathcal{D}^o$.

(2) (i) If η: $id_{\mathcal{C}} \to G'F'$ is a functorial isomorphism, then we have, for

$C \in \mathcal{C}$, $D \in \mathcal{D}$, the isomorphisms

$$Mor_{\mathcal{D}}(D, F'(C)) \xrightarrow{G'_{D,F'(C)}} Mor_{\mathcal{C}}(G'F'(C), G'(D)) \xrightarrow{[\eta_C, G'(D)]} Mor_{\mathcal{C}}(C, G'(D)).$$

Thus (F', G') is a right adjoint pair (see 45.2).

Similarly we see that (F', G') is left adjoint, too.

(ii) follows from 45.4,(3) taking into account that $\mathcal{C} \to \mathcal{D} \to \mathcal{D}^o$ and $\mathcal{D}^o \to \mathcal{D} \to \mathcal{C}$ are right and left adjoint covariant functors.

Studying dualities, categories of the following type play an important role: For an R-module M let $\sigma_f[M]$ denote the full subcategory of $\sigma[M]$ whose objects are submodules of *finitely* M-generated modules.

$\sigma_f[R]$ just consists of submodules of finitely generated R-modules.

47.2 Properties of the category $\sigma_f[M]$.
Let M be an R-module. Then:

(1) $\sigma_f[M]$ is closed under finite products, sub- and factor modules.

(2) $\sigma_f[M]$ contains all finitely generated modules in $\sigma[M]$.

(3) For N in $\sigma_f[M]$ we have:

(i) N is injective in $\sigma_f[M]$ if and only if it is M-injective (= injective in $\sigma[M]$);

(ii) if N is a cogenerator in $\sigma_f[M]$, then N cogenerates all simple modules in $\sigma[M]$;

(iii) N is an injective cogenerator in $\sigma_f[M]$ if and only if it is an injective cogenerator in $\sigma[M]$;

(iv) N is projective in $\sigma_f[M]$ if and only if it is M-projective;

(v) N is a generator in $\sigma_f[M]$ if and only if it is a generator in $\sigma[M]$.

Proof: (1) can be obtained in the same way as corresponding assertions for $\sigma[M]$ (see 15.1), (2) is clear.

(3) (i) follows from 16.3. (ii) is a consequence of (2). (iii) follows from 16.5 (by (i) and (ii)). (iv) is obtained from 18.2. (v) follows from 15.1,(3).

47.3 Dualities between module categories (Morita duality).

Let R, S be rings, and $\mathcal{C} \subset R$-MOD, $\mathcal{D} \subset$ MOD-S full subcategories which are closed under finite products, sub- and factor modules.

Assume $F' \colon \mathcal{C} \to \mathcal{D}$ is a duality with inverse $G' \colon \mathcal{D} \to \mathcal{C}$. Then:

(1) All modules in $\mathcal{C}$ and $\mathcal{D}$ are linearly compact R- resp. S-modules.

(2) In case $S \in \mathcal{D}$, there is an R-module $U \in \mathcal{C}$ with $S \simeq End(U)$ and the properties:

(i) $F' \simeq Hom_R(-, {}_RU)$ *and* $G' \simeq Hom_S(-, U_S)$*;*
(ii) ${}_RU$ *is an injective cogenerator in* $\sigma[U]$ *(and* $\mathcal{C}$*);*
(iii) U_S *is an injective cogenerator in MOD-S.*

(3) For a left R-module U with $End({}_RU) = S$*, satisfying the properties (ii) and (iii) above, the functor*

$$Hom_R(-, U) : \sigma_f[{}_RU] \to \sigma_f[S_S]$$

determines a duality with inverse $Hom_S(-, U_S)$*.*

Proof: (1) F' and G' are exact functors since they convert limits (see 47.1) and $\mathcal{C}$, $\mathcal{D}$ are exact categories. Let $N \in \mathcal{C}$ and $\{K_\lambda\}_\Lambda$ be an inverse family of submodules of N. Then $\{F'(K_\lambda)\}_\Lambda$ and $\{F'(N/K_\lambda)\}_\Lambda$ form direct systems of modules in $\mathcal{D}$ in a canonical way and we have the exact sequence

$$0 \longrightarrow \varinjlim F'(N/K_\lambda) \longrightarrow F'(N) \longrightarrow \varinjlim F'(K_\lambda) \longrightarrow 0\,,$$

yielding the commutative diagram with exact upper row

$$\begin{array}{ccccccccc}
0 & \to & \varprojlim G'F'(K_\lambda) & \to & G'F'(N) & \to & \varprojlim G'F'(N/K_\lambda) & \to & 0 \\
 & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\
0 & \to & \varprojlim K_\lambda & \to & N & \to & \varprojlim N/K_\lambda & \to & 0
\end{array} \;,$$

where the limits are taken in *R-MOD*. Then the lower row also has to be exact and hence N is linearly compact (see 29.7).

(2) For $U = G'(S)$ we obtain from 45.9 the isomorphisms

$$F' \simeq Hom_R(-, {}_RU), \quad G' \simeq Hom_S(-, U_S).$$

Since both functors are exact and faithful, ${}_RU$ and U_S have to be injective and cogenerators for $\mathcal{C}$, resp. $\mathcal{D}$ (see 14.6). It is easy to see that they are also injective cogenerators for $\sigma[U]$, resp. *MOD-S* (notice $U \in \mathcal{C}$, $S_S \in \mathcal{D}$).

From this (i) - (iii) follows.

(3) This will be shown in 47.4,(3).

In case there is a duality $F' : \mathcal{C} \to \mathcal{D}$, with the conditions given in 47.3, all modules in $\mathcal{C}$ (and $\mathcal{D}$) are linearly compact. Since infinite direct sums of non-zero modules are *not* linearly compact (by 29.8), $\mathcal{C} = \sigma[M]$ or $\mathcal{C} =$*R-MOD* cannot occur in 47.3. In particular, the category (*R-MOD*)$^{\rm o}$ dual to *R-MOD* is not a (full) module category.

For further descriptions of dualities let us have a closer look at the contravariant Hom-functors:

Let ${}_RU$ be an R-module with $S = End({}_RU)$. We denote both functors $Hom_R(-,{}_RU)$ and $Hom_S(-,U_S)$ by $(-)^*$ and by

$$\Phi_N : N \to N^{**},\ n \mapsto [\beta \mapsto (n)\beta],$$

(for N in *R-MOD* resp. in *MOD-S*) the (functorial) evaluation morphism. From 45.10 we know that Φ_N is monic if and only if N is cogenerated by ${}_RU$ (resp. U_S). Moreover, Φ_{N^*} is always monic and splits.

N is called *U-reflexive* if Φ_N is an isomorphism.

47.4 Properties of U-reflexive modules.
Let U be a left R-module with $S = End({}_RU)$.

(1) (i) A finite direct sum of modules is U-reflexive if and only if every summand is U-reflexive.

(ii) If the left R-module N is U-reflexive, then N^* is also U-reflexive.

(2) Let N be U-reflexive and $K \subset N$. Then:

(i) If ${}_RU$ is N-injective, then K is U-reflexive if and only if N/K is cogenerated by U.

(ii) If U_S is $Hom_R(N,U)$-injective, then $\Phi_{N/K}$: $N/K \to (N/K)^{**}$ is epic.

(3) If ${}_RU$ is an injective cogenerator in $\sigma[{}_RU]$ and U_S an injective cogenerator in MOD-S, then all modules in $\sigma_f[{}_RU]$ and $\sigma_f[S_S]$ are U-reflexive.

Proof: (1)(i) is clear since $(-)^{**}$ preserves finite direct sums.

(ii) From 45.10, we know the relation $(\Phi_N)^*\Phi_{N^*} = id_{N^*}$ in *MOD-S*. If N is reflexive, i.e. Φ_N is an isomorphism, then $(\Phi_N)^*$ is an isomorphism and hence Φ_{N^*} is also an isomorphism.

(2)(i) If ${}_RU$ is N-injective, we have the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{i} & N & \xrightarrow{p} & N/K & \longrightarrow & 0 \\
 & & \downarrow \Phi_K & & \downarrow\simeq & & \downarrow \Phi_{N/K} & & \\
0 & \longrightarrow & K^{**} & \xrightarrow{i^{**}} & N^{**} & \xrightarrow{p^{**}} & (N/K)^{**} & & .
\end{array}$$

By the Kernel Cokernel Lemma, Φ_K is epic if and only if $\Phi_{N/K}$ is monic.

(ii) If U_S is N^*_S-injective, in the above diagram p^{**} is epic and hence $\Phi_{N/K}$ is also epic.

(3) Since ${}_RU$ and S_S are U-reflexive, the assertion follows from (1) and (2). As $id_{\sigma_f[U]} \simeq (-)^{**}$ and $id_{\sigma_f[S]} \simeq (-)^{**}$, the categories $\sigma_f[{}_RU]$ and $\sigma_f[S_S]$ are dual to each other. This also shows 47.3, (3).

47.5 Properties of Morita dualities.

Let U be a left R-module and $S = End(_RU)$.
Assume $Hom_R(-,U)$: $\sigma_f[_RU] \to \sigma_f[S_S]$ to be a duality. Then:

(1) $_RU$ and U_S are injective cogenerators in $\sigma_f[U]$, resp. in $\sigma_f[S_S]$.

(2) For every N in $\sigma_f[_RU]$ the annihilator mappings

$$\begin{aligned} Hom_R(N/-,U): &\quad \{R\text{-submodules of } N\} \\ &\quad \to \{S\text{-submodules of } Hom_R(N,U)\}, \\ Hom_R(U/-,U): &\quad \{(R,S)\text{-submodules of } U\} \to \{\text{ideals in } S\} \end{aligned}$$

are bijective and order reversing.

(3) $_RU$ is finitely cogenerated (in $\sigma[_RU]$) and the following classes correspond under $(-)^*$:
(i) finitely cogenerated modules of one category to finitely generated modules of the other category;
(ii) finitely cogenerated, injective modules in $\sigma[_RU]$ to finitely generated, projective modules in MOD-S.

(4) If there is a finitely generated injective cogenerator in MOD-S, then there is a finitely cogenerated projective generator in $\sigma[_RU]$.

(5) $R \in \sigma[_RU]$ holds if and only if $_RU$ is faithful and U_S is finitely generated. In this case we have:
(i) $R \simeq End(U_S)$ and $_RR$ and U_S are finitely cogenerated and linearly compact,
(ii) the lattices of the (two-sided) ideals of R and S are isomorphic,
(iii) the centers of R and S are isomorphic.

(6) $_RU$ is finitely generated if and only if $S \in \sigma[U_S]$.

Proof: (1) has already been stated in 47.3.

(2) Denote $An(-) = Hom_R(N/-,U)$, and for S-submodules L of $Hom_R(N,U)$, we set $Ke\ (L) = \bigcap\{Ke\ f \mid f \in L\}$. From 28.1 we know that $Ke\ An(K) = K$ for all $K \subset N$ (U is a cogenerator) and $AnKe\ (L') = L'$ for all finitely generated submodules $L' \subset Hom_R(N,U)_S$.

For an arbitrary $L \subset Hom_R(N,U)$, let $\{L_\lambda\}_\Lambda$ be the family of finitely generated submodules of L. The $\{Ke\ L_\lambda\}_\Lambda$ form an inverse system of submodules with $Ke\ L = \bigcap_\Lambda Ke L_\lambda$ and $N/Ke\ L \simeq \varprojlim N/Ke\ L_\lambda$, since N is linearly compact by 47.3. With canonical identifications we obtain

$$\begin{aligned} AnKe\ (L) = Hom_R(N/Ke\ L, U) &= \varinjlim Hom(N/Ke\ L_\lambda, U) \\ &= \varinjlim AnKe\ (L_\lambda) = \varinjlim L_\lambda = L\ . \end{aligned}$$

Hence $An(-)$ and $Ke\ (-)$ are inverse to each other. Obviously, they are order reversing.

(3) Let $\{K_\lambda\}_\Lambda$ be a family of submodules of $_RU$ with $\bigcap_\Lambda K_\lambda = 0$. By 14.7, we have to show that $\bigcap_E K_\lambda = 0$ holds for a finite subset $E \subset \Lambda$. Since S_S is finitely generated we conclude from (1):

$$S = An(\bigcap\nolimits_\Lambda K_\lambda) = \sum\nolimits_\Lambda An(K_\lambda) = \sum\nolimits_E An(K_\lambda), \text{ with finite } E \subset \Lambda,$$

and hence $0 = Ke\, S = Ke\, (\sum_E An(K_\lambda)) = \bigcap_E K_\lambda$.

Therefore all finitely cogenerated modules in $\sigma[U]$ are submodules of some finite sum U^k, $k \in I\!N$, and the given relations are easy to verify.

(4) If Q is an injective cogenerator in $\sigma_f[S_S]$, then $Hom_S(Q, U)$ is a projective generator in $\sigma_f[U]$ and hence also in $\sigma[U]$ (see 47.2).

(5) If $_RU$ is faithful and U_S finitely generated, then, by the proof of 15.4, $R \subset U^k$ for some $k \in I\!N$. On the other hand, since $_RU$ is injective, this relation yields the exact sequence $S^k \to U_S \to 0$.

(i) R is U-reflexive by 47.4,(3), i.e. $R \simeq End(U_S)$. By (3), $_RR$ and U_S are finitely cogenerated and, by 47.3, they are linearly compact.

(ii) Symmetrically to (2) we obtain now an order reversing bijection between the ideals of R and the (R, S)-submodules of U. Together with (2) this yields the given relation between the ideals of R and S.

(iii) can be seen as the corresponding assertion in 46.5.

(6) If $_RU$ is finitely generated, then $S \in \sigma[U_S]$ and hence $S \subset U^l$, $l \in I\!N$ (see 15.4, 15.3). On the other hand, since U_S is injective, this relation yields the exact sequence $(R^{**})^l \to U \to 0$, i.e. U is finitely generated as an R^{**}-module. By 15.7, R is dense in $R^{**} \simeq End(U_S)$ and hence $\sigma[_RU] = \sigma[_{R^{**}}U]$ (see 15.8). Therefore $_RU$ is finitely generated as an R-module.

Asking when Φ_N is epic, the following weaker property is of interest:

The map $\Phi_N\colon N \to N^{**}$ is called *dense* if, for any $h \in N^{**}$ and finitely many $f_1, \ldots, f_k \in N^* = Hom_R(N, U)$, there exists $n \in N$ with

$$h(f_i) = [(n)\Phi_N](f_i) = (n)f_i \quad \text{for} \quad i = 1, \ldots, k\,.$$

Then we say that the module N is *U-dense*.

In particular, for $N = R$ this corresponds to the property of $R \to End(U_S)$ described in the Density Theorem 15.7.

Clearly every U-reflexive R-module is U-dense. Note that for an R-module N with N^* finitely generated as S-module, Φ_N is U-dense if and only if it is epic.

47.6 Properties of U-dense modules.
Let U be a left R-module and $S = End(_RU)$.

(1) A finite direct sum of modules is U-dense if and only if every direct summand is U-dense.

(2) If N is U-dense and $N \subset U^k$ for some $k \in I\!N$, then N is U-reflexive.

(3) The following assertions are equivalent:

(a) Every (finitely generated, cyclic) R-module is U-dense;

(b) every (finitely generated, cyclic) R-submodule of U^k, $k \in I\!N$, is U-reflexive.

(4) If U is a cogenerator in $\sigma[_RU]$, then every R-module is U-dense.

Proof: (1) Necessity is clear. Let X, Y be U-dense R-modules and

$$\Phi_{X\oplus Y} : X \oplus Y \to X^{**} \oplus Y^{**}.$$

For elements $(h, k) \in X^{**} \oplus Y^{**}$ and $(f_1, g_1), \ldots, (f_n, g_n)$ in $X^* \oplus Y^*$, there exist $x \in X$, $y \in Y$ with $(x)f_i = h(f_i)$ and $(y)g_i = k(g_i)$ for $1 \le i \le n$. Then $(x, y)(f_i, g_i) = (h, k)(f_i, g_i)$ for $1 \le i \le n$.

(2) For $N \subset U^k$, Φ_N is monic by 45.10 and there are finitely many

$$f_1, \ldots, f_k \in N^* = Hom_R(N, U) \quad \text{with } \bigcap_{i\le k} Ke f_i = 0.$$

Since Φ_N is dense, for every $h \in X^{**}$ we find an $x \in N$ with $(x)f_i = h(f_i)$ for all $i \le k$. Moreover, for every $g \in N^*$ there exists $y \in N$ with $(y)g = h(g)$ and $(y)f_i = h(f_i)$ for all $i \le k$. But this means $x - y \in \bigcap_{i\le k} Ke\ f_i = 0$, thus $(x)\Phi_N(g) = h(g)$ for all $g \in N^*$ and hence $(x)\Phi_N = h$, i.e. Φ_N is epic.

(3) $(a) \Rightarrow (b)$ follows directly from (2).

$(b) \Rightarrow (a)$ Let N be a (finitely generated, cyclic) R-module, $h \in N^{**}$, $f_1, \ldots, f_k \in N^*$ and $f : N \to U^k$, $n \mapsto ((n)f_i)_{i\le k}$.

Then $(N)f$ is a (finitely generated, cyclic) submodule of U^k and hence U-reflexive. We have the commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{f} & Nf & \subset & U^k \\ \downarrow \Phi_N & & \downarrow \Phi_{Nf} & & \\ N^{**} & \xrightarrow{f^{**}} & (Nf)^{**} & & \end{array} .$$

Therefore $f\Phi_{Nf}$ is epic, i.e. there exists $x \in N$ with

$$(x)f\Phi_{Nf} = (h)f^{**} = h(f^*) = (x)\Phi_N f^{**} .$$

Let π_i' denote the restrictions of the projections π_i: $U^k \to U$ to Nf. Then for all $i \le k$ we have

$$hf^*(\pi_i') = h(f\pi_i') = h(f_i) = [(x)\Phi_N](f\pi_i) = (x)f\pi_i = (x)f_i .$$

Hence Φ_N is dense.

(4) follows from 47.7.

The following interrelation between cogenerator properties of ${}_RU$ and injectivity properties of U_S is of great importance:

47.7 Relation between ${}_RU$ and U_S.
Let U be a left R-module and $S = End({}_RU)$.

(1) For $X \in R$-MOD the following assertions are equivalent:
(a) For any $k \in I\!N$ and $f : X \to {}_RU^k$, Coke f is U-cogenerated;
(b) (i) X is U-dense and
(ii) $Hom_S(-, U_S)$ is exact relative to all exact sequences $0 \to L \xrightarrow{\varepsilon} X^*$ in MOD-S with L finitely generated.

(2) U_S is FP-injective (absolutely pure) if and only if ${}_RU$ cogenerates the cokernels of the morphisms ${}_RU^n \to {}_RU^k$, $n, k \in I\!N$.

Proof: (1) $(a) \Rightarrow (b)(ii)$ Let $f_1, \ldots, f_k$ be a generating set of $L \subset X^*$ and $\varepsilon : L \to X^*$ the canonical inclusion. With the mappings

$$f : X \to U^k,\ x \mapsto ((x)f_i)_{i \le k},\ \text{and}\quad \rho : L^* \to U^k,\ h \mapsto (h(f_i))_{i \le k},$$

the following diagram is commutative:

$$\begin{array}{ccc} X^{**} & \xrightarrow{\varepsilon^*} & L^* \\ \uparrow \Phi_X & & \downarrow \rho \\ X & \xrightarrow{f} & U^k \end{array} .$$

Hence $Im\ f \subset Im\ \rho$ and we show $Im\ f = Im\ \rho$:

Let us assume that, for some $h \in L^*$, we have $(h)\rho \notin Im\ f$. Since U cogenerates $U^k/Im\ f$, there is a morphism $g = \sum g_i \colon U^k \to U$ with $fg = 0$ and $(h)\rho g \neq 0$. However, we have the relations

$$(h)\rho g = \sum\nolimits_{i \le k} h(f_i)g_i = h(\sum\nolimits_{i \le k} f_i g_i) = h(fg) = 0\ .$$

From this contradiction we derive $Im\ f = Im\ \rho$. Since ρ is injective, for every $h \in L^*$, there exists $x \in X$ with $h = (x)\Phi_X\varepsilon^*$. Hence $\Phi_X\varepsilon^*$ and ε^* are surjective and (ii) is shown.

(i) For $\widetilde{h} \in X^{**}$ and $f_1, \ldots, f_k \in X^*$ set $L = \sum_{i \le k} f_i S$. Then $h = (\widetilde{h})\varepsilon^* \in L^*$ and, by the observations above, there exists $x \in X$ with

$$\widetilde{h}(v) = h(v) = [(x)\Phi_X](v) = (x)v \quad \text{for all}\ \ v \in L\ ,$$

in particular, $h(f_i) = (x)f_i$ for $i \leq k$, i.e. Φ_X is dense.

$(b) \Rightarrow (a)$ In R-MOD let $f : X \to U^k$ with $k \in I\!N$ be given. With the inclusion $\varepsilon : Xf \to U^k$ we obtain the sequence

$$(U^k)^* \xrightarrow{\varepsilon^*} Im\, \varepsilon^* \xrightarrow{\delta} (Xf)^* \xrightarrow{f^*} X^*, \;\; \delta \text{ inclusion.}$$

From this we derive the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Phi_X"] & Xf \arrow[d, "\Phi_{Xf}"] & \\
X^{**} \arrow[r, "f^{**}"] & (Xf)^{**} \arrow[r, "\delta^*"] & (Im\varepsilon^*)^*
\end{tikzcd} .$$

Since $(U^k)^* \simeq S^k$, the image $Im\, \varepsilon^*$ is finitely generated as an S-module and hence $f^{**}\delta^*$ is epic by (ii). Therefore, for any $h \in (Im\varepsilon^*)^*$, there exists

$$\widetilde{h} \in X^{**} \quad \text{with} \quad h = (\widetilde{h})f^{**}\delta^* = \widetilde{h}f^*\delta,$$

and for $v \in Im\, \varepsilon^*$ we get $h(v) = \widetilde{h}f^*\delta(v) = \widetilde{h}(fv)$.

For any generating set $v_1, \ldots, v_k$ of $Im\, \varepsilon^*$, we have $fv_i \in X^*$. Since Φ_X is dense, there exists $x \in X^*$ with $[(x)\Phi_X](fv_i) = \widetilde{h}(fv_i) = h(v_i)$ and hence

$$[(x)\Phi_X f^{**}\delta^*](v_i) = [(x)\Phi_X](fv_i) = h(v_i) \text{ for all } i \leq k.$$

Therefore $\Phi_X f^{**}\delta^*$ is surjective, and we see from the diagram above that $\Phi_{Xf}\delta^*$ has to be surjective.

Let us construct the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Xf \arrow[r, "\varepsilon"] \arrow[d, "\Phi_{Xf}"'] & U^k \arrow[r] \arrow[dd, "\Phi_{U^k}"] & Cokef \arrow[r] \arrow[dd, "\Phi_{Cokef}"] & 0 \\
 & (Xf)^{**} \arrow[d, "\delta^*"'] \arrow[dr, "\varepsilon^{**}"] & & & \\
0 \arrow[r] & (Im\varepsilon^*)^* \arrow[r] & (U^k)^{**} \arrow[r] & (Cokef)^{**} &
\end{tikzcd} .$$

Therein Φ_{U^k} is an isomorphism and we have seen above that $\Phi_{Xf}\delta^*$ is epic. By the Kernel Cokernel Lemma, Φ_{Cokef} is monic, i.e. $Cokef$ is cogenerated by U (see 45.10).

(2) This is obtained replacing X in (1) by the modules U^n, $n \in I\!N$.

47.8 Cogenerators and linearly compact modules.
Let the left R-module U be a cogenerator in $\sigma_f[_RU]$ and $S = End(_RU)$.

(1) For every linearly compact R-module X we have:
(i) Φ_X: $X \to X^{**}$ is epic and X^* is U-reflexive.
(ii) U_S is $Hom_R(X,U)$-injective.

(2) If every factor module of a left R-module N is cogenerated by U (e.g. $N \in \sigma_f[U]$), then the following assertions are equivalent:
(a) N is linearly compact;
(b) N is U-reflexive and U_S is $Hom_R(N,U)$-injective.

(3) $_RU$ is linearly compact if and only if U_S is S-injective.

Proof: (1) (i) Let $\{L_\lambda\}_\Lambda$ be the family of finitely generated S-submodules of $Hom_R(X,U) = X^*$. Then the $K_\lambda := Ke\ L_\lambda$ form an inverse family of submodules of X and the $Hom_R(K_\lambda, U)$ a direct family of submodules of X^*. Since by construction $L_\lambda \subset Hom_R(X/K_\lambda, U) = (X/K_\lambda)^* \subset X^*$, we have

$$X^* = \varinjlim L_\lambda = \varinjlim (X/K_\lambda)^* \text{ and } X^{**} \simeq \varprojlim (X/K_\lambda)^{**}.$$

Also, by construction, $X/K_\lambda \subset U^k$ for some $k \in I\!N$. Since all R-modules are U-dense, X/K_λ is U-reflexive (see 47.6). By the linear compactness of X, the first row of the following commutative diagram is exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigcap K_\lambda & \longrightarrow & X & \longrightarrow & \varprojlim X/K_\lambda & \longrightarrow & 0 \\ & & & & \downarrow \Phi_X & & \downarrow\simeq & & \\ & & & & X^{**} & \stackrel{\simeq}{\longrightarrow} & \varprojlim (X/K_\lambda)^{**} & & . \end{array}$$

From this we see that Φ_X is epic. Hence $(\Phi_X)^*$ is monic and from the relation $(\Phi_X)^*\Phi_{X^*} = id_{X^*}$ in *MOD-S* (see 45.10) it follows that Φ_{X^*} has to be an isomorphism, i.e. X^* is U-reflexive.

(ii) Let $0 \to F \to X^*$ be an exact sequence in *MOD-S* and $\{F_\lambda\}_\Lambda$ the family of finitely generated submodules of F. Applying $Hom_R(-, U_S)$, we obtain, by 47.7, exact sequences $X^{**} \to F_\lambda^* \to 0$. By (i), X^{**} is linearly compact (see 29.8,(2)) and hence the sequence $X^{**} \to \varprojlim F_\lambda^* \to 0$ is exact.

Because $\varprojlim F_\lambda^* = (\varinjlim F_\lambda)^* \simeq F^*$, the sequence $X^{**} \to F^* \to 0$ is also exact, i.e. U_S is X^*-injective.

(2) $(a) \Rightarrow (b)$ follows from (1) since Φ_N is monic.

$(b) \Rightarrow (a)$ Let $\{K_\lambda\}_\Lambda$ be an inverse family of submodules of N with N/K_λ finitely cogenerated. Then the N/K_λ are submodules of U^k, $k \in I\!N$, and, by 47.6, they are U-reflexive.

$\{(N/K_\lambda)^*\}_\Lambda$ is a direct system of submodules of N^* and the sequence $0 \to \varinjlim (N/K)^* \to N^*$ is exact. Since U_S is N^*-injective, the lower row of the following commutative diagram is exact

$$\begin{array}{ccccc} N & \longrightarrow & \varprojlim N/K_\lambda & \longrightarrow & 0 \\ \downarrow \simeq & & \downarrow \simeq & & \\ N^{**} & \longrightarrow & \varprojlim (N/K_\lambda)^{**} & \longrightarrow & 0 \end{array} .$$

Hence the upper row is also exact and, by 29.7, N is linearly compact.

(3) is a special case of (2) (with $N = U$).

Now consider a generalization of linearly compact modules:

We say an R-module N satisfies the property AB 5* if, for every submodule $K \subset N$ and every inverse family $\{N_\lambda\}_\Lambda$ of submodules of N,

$$K + \bigcap\nolimits_\Lambda N_\lambda = \bigcap\nolimits_\Lambda (K + N_\lambda).$$

The dual relation for a direct family $\{Y_\lambda\}_\Lambda$ of submodules of N,

$$K \cap (\sum\nolimits_\Lambda Y_\lambda) = \sum\nolimits_\Lambda (K \cap Y_\lambda),$$

holds in every R-module N. A. Grothendieck denoted the corresponding property in general categories by AB 5, and the upper property by AB 5*. We have seen in 29.8 that AB 5* is satisfied in linearly compact modules.

47.9 Properties of modules and rings with AB5*.

(1) Assume the R-module N satisfies AB 5. Then:*

(i) Submodules and factor modules of N satisfy AB 5;*

(ii) N is amply supplemented;

(iii) if $N \simeq E^{(\Lambda)}$ for an R-module E, then Λ is finite;

(iv) if there are only finitely many non-isomorphic simple modules in $\sigma[N]$, then every factor module of N^k, $k \in \mathbb{N}$, has a finitely generated socle.

(2) If ${}_RR$ satisfies AB 5, then:*

(i) R is semiperfect;

(ii) every finitely generated left R-module has a finitely generated socle;

(iii) if R is right perfect, then ${}_RR$ is artinian.

Proof: (1) (i) is easy to verify.

(ii) follows from the proof of 41.10,(1).

(iii) Assume $N \simeq E^{(\Lambda)}$ for an infinite set Λ. By (i), we may assume $\Lambda = \mathbb{N}$, i.e. $N = \bigoplus_{\mathbb{N}} E_j$, $E_j = E$.

With the canonical projections $\pi_i : \bigoplus_{I\!N} E_j \to E_i$ we form submodules

$$N_k = \bigoplus\nolimits_{j\geq k} E_j \quad \text{and} \quad K = \{a \in N \mid \sum\nolimits_{I\!N}(a)\pi_i = 0\}.$$

Then $K + N_k = N$ for every $k \in I\!N$ and $\bigcap_{I\!N} N_k = 0$, i.e. we get the contradiction

$$N = \bigcap\nolimits_{I\!N}(K + N_k) = K + \bigcap\nolimits_{I\!N} N_k = K.$$

(iv) First of all, $Soc(N)$ is finitely generated: The homogeneous components of $Soc(N)$ (see 20.5) are finitely generated by (iii), and, by assumption, there are only finitely many of them.

Then, by (i), every factor module of N has a finitely generated socle. Now it can be shown by induction on $k \in I\!N$ that every factor module of N^k also has finitely generated socle.

(2) (i) ${}_RR$ is supplemented and hence a semiperfect module by 42.6.

(ii) By (i), there are only finitely many non-isomorphic simple R-modules (see 42.6) and hence the assertion follows from (1)(iv).

(iii) Every factor module of ${}_RR$ has finitely generated (see (ii)) and essential socle (see 43.9), hence is finitely cogenerated and ${}_RR$ is artinian.

47.10 Injective modules with essential socle.
Let U be a self-injective left R-module and $S = End({}_RU)$.

(1) Assume ${}_RU$ is finitely cogenerated. Then every simple S-module is cogenerated by U_S.

(2) Assume $Soc_RU \trianglelefteq {}_RU$ and let X be an R-module with AB5* and with only finitely many non-isomorphic simple modules in $\sigma[X]$. Then:
(i) U_S cogenerates all factor modules of X^*.
(ii) If ${}_RU$ is X-injective, then X^* is linearly compact.

(3) Assume $Soc_RU \trianglelefteq {}_RU$ and X is a linearly compact R-module. Then:
(i) U_S is a cogenerator for $\sigma_f[X^*_S]$.
(ii) If ${}_RU$ is X-injective, then X^* is linearly compact.

(4) Assume ${}_RU$ is a self-cogenerator. Then $Soc\,U_S \trianglelefteq U_S$ and, if the simple S-modules are cogenerated by U_S, then $Soc_RU \trianglelefteq {}_RU$.

Proof: (1) Under the given assumptions we know, from 22.1, that $Jac(S) \simeq Hom_R(U/Soc\,U, U)$ and $S/Jac(S) \simeq Hom_R(Soc\,U, U)$ is a semisimple S-module which contains a copy of every simple S-module. $Soc\,U$ is generated by R and hence $Hom_R(Soc\,U, U)$ is cogenerated by U_S.

(2)(i), (3)(i) Let $Soc\,_R U \trianglelefteq_R U$, $X \in R\text{-}MOD$ and $L \subset X^*$ be an S-submodule. We have the commutative diagram with exact rows

$$(D)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & X^* & \longrightarrow & X^*/L & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & (X/Ke\,L)^* & \longrightarrow & X^* & \longrightarrow & (Ke\,L)^* & & . \end{array}$$

If $L = (X/Ke\,L)^* = AnKe\ (L)$, then X^*/L is a submodule of $(Ke\,L)^* = Hom_R(Ke\,L, U)$ and hence is U_S-cogenerated. We show $L = AnKe\ (L)$.

Let $f \in AnKe(L)$ and $\{L_\lambda\}_\Lambda$ be the family of finitely generated submodules of L. By 28.1, $L_\lambda = AnKe\ (L_\lambda)$ always holds. If $Ke\,L_{\lambda'} \subset Ke\ f$ for some $\lambda' \in \Lambda$, then $f \in AnKe(L_{\lambda'}) = L_{\lambda'} \subset L$.

Assume $Ke\,L_\lambda \not\subset Ke\ f$ for all $\lambda \in \Lambda$. Then $\{Ke\ f + Ke\,L_\lambda/Ke\,f\}_\Lambda$ is an inverse system of non-zero submodules of $X/Ke\,f \subset U$. Since $Soc\,_R U \trianglelefteq_R U$ the module $X/Ke\,f$ also has an essential socle.

If X is linearly compact, then so is also $X/Ke\,f$ and consequently $Soc(X/Ke\,f)$ is finitely generated (see 29.8).

In case X satisfies AB 5* and there are only finitely many non-isomorphic simple modules in $\sigma[X]$, by 47.9, $Soc(X/Ke\,f)$ is finitely generated. Hence $X/Ke\,f$ is finitely cogenerated (see 21.3). By 29.10, there is a non-zero submodule $N/Ke\,f$ ($N \subset X$) which is contained in every module of the system considered above, thus $N \subset Ke\ f + Ke\ L_\lambda$ and

$$Ke\,f \neq N \subset \bigcap\nolimits_\Lambda (Ke\,f + Ke\ L_\lambda) = Ke\ f + \bigcap\nolimits_\Lambda Ke\ L_\lambda \ \text{(by AB 5}^*\text{)}.$$

Since $\bigcap_\Lambda Ke\ L_\lambda = Ke\ (\sum_\Lambda L_\lambda) = Ke\ (L) \subset Ke\ f$, this means $N/Ke\,f = 0$, a contradiction. Hence $f \in L$ and $L = AnKe\ (L)$.

If X is linearly compact then so is X^k, $k \in I\!N$, and, from the above considerations, we deduce that U_S cogenerates all factor modules of $(X^*)^k$, and hence all modules in $\sigma_f[X^*]$.

(2)(ii), (3)(ii) Let $_RU$ be X-injective, X as given in (2), resp. (3), and $\{L_\lambda\}_\Lambda$ be an inverse family of submodules in X^*. By the proof of (i), $L_\lambda = AnKe\ (L_\lambda)$, thus $L_\lambda = (X/Ke\,L_\lambda)^*$ and from the diagram (D) with L replaced by L_λ, we conclude $X^*/L_\lambda \simeq (Ke\,L_\lambda)^*$. The $Ke\,L_\lambda$ form a direct system of submodules of X and we have the exact sequence

$$0 \longrightarrow \varinjlim Ke\ L_\lambda \longrightarrow X \longrightarrow \varinjlim X/Ke\,L_\lambda \longrightarrow 0\,,$$

which yields with $(-)^*$ the commutative diagram with exact lower row

$$\begin{array}{ccccccccc} 0 & \to & \varprojlim L_\lambda & \to & X^* & \to & \varprojlim X^*/L_\lambda & \to & 0 \\ & & \downarrow\simeq & & \| & & \downarrow\simeq & & \\ 0 & \to & \varprojlim (X/KeL_\lambda)^* & \to & X^* & \to & \varprojlim (KeL_\lambda)^* & \to & 0 \quad . \end{array}$$

But then the first row is also exact, i.e. X^* is linearly compact.

(4) By assumption, ${}_RU$ is an injective cogenerator in $\sigma[{}_RU]$ (see 16.5). Let L be a cyclic S-submodule of $U_S = Hom_R(R,U)$, i.e. $L = fS$ with $f \in Hom_R(R,U)$. By 28.1, $L = AnKe\,(f) = Hom_R(R/Ke\,f, U)$.

Since $R/Ke\,f \in \sigma[{}_RU]$, any simple factor module E of $R/Ke\,f$ also belongs to $\sigma[{}_RU]$ and $E^* \neq 0$ is a simple S-submodule of $(R/Ke\,f)^* = L$. This means $Soc(U_S) \trianglelefteq U_S$.

Now let U_S be a cogenerator for the simple S-modules. For any non-zero R-submodule $K \subset U$, the module K^* is a factor module of $U^* \simeq S$ and there is a simple factor module F of K^*. By 47.4,(2), K is U-reflexive and hence $F^* \neq 0$ is a submodule of $K^{**} \simeq K$. Since U_S is injective with respect to exact sequences $0 \to F \to U_S$ (see 47.7), F^* is a simple module in $\sigma_f[{}_{R^{**}}U] = \sigma_f[{}_RU]$ (see 15.8). Therefore $Soc_R U \trianglelefteq {}_RU$.

Let us turn now to the question of which dualities are determined by an R-module U. For this another definition is helpful:

We call an R-module K *finitely U-copresented* if there is an exact sequence $0 \to K \to U^k \to U^l$ with $k, l \in I\!N$.

47.11 Dualities determined by $Hom_R(-,U)$.
Let U be a left R-module with $S = End({}_RU)$. Then

$Hom_R(-,{}_RU)$ determines a duality between the full subcategories of the U-reflexive modules in R-MOD and the U-reflexive modules in MOD-S, with inverse $Hom_S(-,U_S)$.

(1) ${}_RU$ and S_S are U-reflexive, and direct summands of U^k, $k \in I\!N$, are turned into finitely generated, projective modules in MOD-S (and vice versa).

(2) If ${}_RU$ is self-injective, then finitely U-copresented R-modules are turned into finitely presented S-modules (and vice versa).

(3) If ${}_RU$ is linearly compact and a cogenerator in $\sigma_f[{}_RU]$, then all modules in $\sigma_f[{}_RU]$ are U-reflexive.

(4) If ${}_RU$ is linearly compact and self-injective with $Soc_R U \trianglelefteq_R U$, then all modules in $\sigma_f[S_S]$ are U-reflexive.

Proof: By 47.4, with N, N^* is also U-reflexive. Hence $Hom_R(-,U)$ is a functor between the given categories. In each case $(-)^{**}$ is isomorphic to the identity.

(1) By 47.3, all given modules are U-reflexive and obviously correspond to each other.

(2) It is easy to verify that, for self-injective ${}_RU$, kernels of morphisms between U-reflexive R-modules in $\sigma[{}_RU]$ and cokernels of morphisms between U-reflexive S-modules are again U-reflexive.

(3), (4) follow from 47.8 resp. 47.10, since for a linearly compact ${}_RU$ every module in $\sigma_f[{}_RU]$ is linearly compact.

With the above knowledge of the interrelation between ${}_RU$ and U_S we are now able to show:

47.12 Characterization of Morita dualities.

For an R-module U with $S = End({}_RU)$, the following are equivalent:

(a) $Hom_R(-, U)$: $\sigma_f[{}_RU] \to \sigma_f[S_S]$ is a duality;

(b) ${}_RU$ is an injective cogenerator in $\sigma[{}_RU]$, and U_S is an injective cogenerator in MOD-S;

(c) ${}_RU$ is linearly compact, finitely cogenerated and an injective cogenerator in $\sigma[{}_RU]$;

(d) U_S is an injective cogenerator in MOD-S, $Soc(U_S) \trianglelefteq U_S$, S_S is linearly compact, and R is U-dense;

(e) all factor modules of ${}_RU$ and S_S are U-reflexive.

Proof: $(a) \Leftrightarrow (b)$ is already known from 47.3.

$(b) \Rightarrow (c)$ By 47.3, ${}_RU$ has to be linearly compact in $\sigma[{}_RU]$. In 47.5 we have seen that ${}_RU$ is finitely cogenerated.

$(c) \Rightarrow (d)$ From 47.8, we derive that U_S is S-injective. It was shown in 47.10 that U_S is an injective cogenerator in $\sigma_f[S_S]$, S_S is linearly compact, and $Soc(U_S) \trianglelefteq U_S$. By 47.6 (or the Density Theorem), R is U-dense.

$(d) \Rightarrow (b)$ Set $B = End(U_S) \simeq R^{**}$ and consider the bimodule ${}_BU_S$. By 47.8, ${}_BU \simeq Hom_S(S, U)$ is self-injective (notice change of sides) and, by 47.10, a cogenerator for $\sigma_f[{}_BU]$. Now R is U-dense, hence $\sigma[{}_RU] = \sigma[{}_BU]$ (see 15.8) and therefore ${}_RU$ is an injective cogenerator in $\sigma[{}_RU]$.

$(b) \Rightarrow (e)$ has been proved in 47.4.

$(e) \Rightarrow (b)$ First we show that ${}_RU$ is self-injective, i.e. for every monomorphism $f : K \to U$ the map $f^* : U^* \to K^*$ is epic. From the inclusion $\delta : Im\, f^* \to K^*$ we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & K & \xrightarrow{f} & U & \to & U/K & \to & 0 \\
 & & {\scriptstyle \Phi_K}\downarrow & & & & & & \\
 & & K^{**} & & \downarrow{\scriptstyle \Phi_U} & & \downarrow{\scriptstyle \Phi_{U/K}} & & \\
 & & {\scriptstyle \delta^*}\downarrow & & & & & & \\
0 & \to & (Im\, f^*)^* & \to & U^{**} & \to & (U/K)^{**} & & .
\end{array}$$

Since, by assumption, Φ_U and $\Phi_{U/K}$ are isomorphisms, $\Phi_K\delta^*$ also has to be an isomorphism. $Im\ f^*$ as a factor module of $U^* \simeq S$ is again U-reflexive and hence, by 47.4, $(Imf^*)^* \simeq K$ is U-reflexive. Therefore Φ_K and δ^* are isomorphisms. Then δ^{**} is also an isomorphism and from the commutative diagram

$$\begin{array}{ccc} Im\ f^* & \xrightarrow{\delta} & K^* \\ \downarrow \Phi_{Im\ f^*} & & \downarrow \Phi_{K^*} \\ (Im\ f^*)^{**} & \xrightarrow{\delta^{**}} & K^{***} \end{array}$$

we derive that δ is an isomorphism and so f^* is epic.

Since all factor modules of U are U-reflexive and hence U-cogenerated, ${}_RU$ is a cogenerator in $\sigma[U]$.

A similar proof shows that U_S is an injective cogenerator in *MOD-S*.

47.13 Duality between finitely generated modules.

Let R, S be rings and $M \in R$-MOD. Denote by $\mathcal{C}$ the subcategory of finitely generated modules in $\sigma[M]$ and mod-S the category of finitely generated S-modules.

(1) Assume F': $\mathcal{C} \to$ mod-S is a duality.

Then there is a module $U \in \mathcal{C}$ with $S \simeq End({}_RU)$ and

(i) ${}_RU$ is an injective cogenerator in $\sigma[U]$ and U_S is an injective cogenerator in MOD-S;

(ii) ${}_RU$ and S_S are modules of finite length;

(iii) ${}_RU$ is an injective cogenerator in $\sigma[M]$, $\mathcal{C} = \sigma_f[U]$ and $\sigma[U] = \sigma[M]$.

(2) If ${}_RU$ is an injective cogenerator of finite length in $\sigma[M]$ with $\sigma[U] = \sigma[M]$ and $S = End({}_RU)$, then $Hom_R(-, U)$: $\mathcal{C} \to$ mod-S is a duality.

Proof: (1)(i) Let $G' : mod\text{-}S \to \mathcal{C}$ be the functor inverse to F'. For the (finitely generated) R-module $U = G'(S) \in \mathcal{C}$, there are functorial isomorphisms $G' \simeq Hom_S(-, U_S)$ and $F' \simeq Hom_R(-, {}_RU)$ (by 45.9). In addition we have $End({}_RU) = F'(U) \simeq F'G'(S) \simeq S$.

Hence, for all $N \in \mathcal{C}$, $N \simeq G'F'(N) \simeq N^{**}$. Therefore, by 45.10, N is cogenerated by ${}_RU$, i.e. ${}_RU$ is a cogenerator in $\mathcal{C}$ and also in $\sigma_f[U]$.

For every submodule $K \subset U$, the module U/K is finitely generated and $F'(U/K) \simeq (U/K)^*$ is also finitely generated. Hence, by 47.7, U_S is injective relative to $0 \to (U/K)^* \to U^*$, and from the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & U & \longrightarrow & U/K & \longrightarrow & 0 \\ & & & & \downarrow \Phi_U & & \downarrow \Phi_{U/K} & & \\ & & & & U^{**} & \longrightarrow & (U/K)^{**} & \longrightarrow & 0 \end{array}$$

we conclude that $\Phi_{U/K}$ is an isomorphism, i.e. every factor module of U is U-reflexive.

Analogously we obtain that every factor module of S_S is also U-reflexive. Hence, by 47.12, ${}_RU$ and U_S are injective cogenerators in $\sigma[U]$ resp. *MOD-S*, i.e. we have a Morita duality between $\sigma_f[U]$ and $\sigma_f[S_S]$.

(*ii*) For any submodule $K \subset_R U$, $U/K \simeq (U/K)^{**}$ and so U/K is dual to the finitely generated S-module $(U/K)^*$ and hence finitely cogenerated by 47.5. From 31.1 we know that in this case ${}_RU$ is artinian. Because of the order reversing bijection between submodules of ${}_RU$ and right ideals of S (see 47.5), we obtain the ascending chain condition for right ideals of S, i.e. S_S is noetherian. Symmetrically it follows that ${}_RU$ is also noetherian and S_S is artinian. Hence both modules have finite length.

(*iii*) For every R-module X the map $\Phi_X : X \to X^{**}$ is dense by 47.7. If X is in $\sigma[M]$ and finitely generated, then $X^* \simeq F'(X)$ is a finitely generated S-module and hence Φ_X is an isomorphism. Then $X \simeq X^{**}$ is a submodule of a finite sum ${}_RU^k$ and therefore has finite length. This means $\mathcal{C} = \sigma_f[U]$. In particular, every finitely generated submodule $N \subset M$ is in $\sigma_f[U]$ and U is N-injective. Hence, by 16.3, U is also M-injective.

Since simple modules in $\sigma[M]$ belong to $\sigma_f[U]$, the module ${}_RU$ is an injective cogenerator in $\sigma[M]$. The finitely generated submodules N_λ of M, $\lambda \in \Lambda$, belong to $\sigma_f[U] \subset \sigma[U]$ and $M = \varinjlim N_\lambda \in \sigma[U]$, i.e. $\sigma[U] = \sigma[M]$.

(2) A module of finite length is linearly compact and has essential socle. Hence, by 47.12, $Hom_R(-, U) : \sigma_f[U] \to \sigma_f[S_S]$ determines a duality. Since finitely generated modules in $\sigma[U] = \sigma[M]$ have finite length, we conclude $\sigma_f[U] = \mathcal{C}$. With ${}_RU$, S_S also has finite length implying $\sigma_f[S_S] = mod\text{-}S$.

The preceding results now permit a

47.14 Characterization of linearly compact rings.

For a ring R the following assertions are equivalent:

(a) ${}_RR$ is linearly compact;

(b) $R \simeq End(U_S)$ for a ring S and an S-module U_S which is finitely cogenerated, self-injective, and
(i) U_S is linearly compact, or
(ii) U_S satisfies $AB5^*$ and is a self-cogenerator, or
(iii) U_S satisfies $AB5^*$ and there are only finitely many non-isomorphic simple modules in $\sigma[U_S]$.

Proof: $(a) \Rightarrow (b)$ If R is linearly compact, then there are only finitely many non-isomorphic simple R-modules (R is semiperfect). If we choose

${}_RU$ as the minimal cogenerator in *R-MOD*, then ${}_RU$ is finitely cogenerated. For $S = End({}_RU)$, by 47.10, $R^* \simeq U_S$ is a linearly compact S-module with essential socle and is a cogenerator for $\sigma_f[U_S]$. We learn from 47.8 that U_S is self-injective, and that R is U-reflexive, hence $R \simeq End(U_S)$. Obviously, (i), (ii) and (iii) now hold.

$(b) \Rightarrow (a)$ Assume U_S to have the corresponding properties, then, by 47.10, $End(U_S)$ is left linearly compact (notice $(b)(ii) \Rightarrow (b)(iii)$).

A ring R is called a *left Morita ring* if there is an injective cogenerator ${}_RU$ in *R-MOD* such that (for $S = End({}_RU)$) U_S is an injective cogenerator in *MOD-S* and $R \simeq End(U_S)$.

47.15 Characterization of Morita rings.
Let R be a ring, ${}_RQ$ a minimal cogenerator in R-MOD and $S = End_R(Q)$.

(1) The following assertions are equivalent:
(a) R is a left Morita ring;
(b) ${}_RR$ and ${}_RQ$ are linearly compact;
(c) ${}_RR$ is linearly compact and ${}_RQ$ satisfies $AB\,5^*$;
(d) ${}_RR$ is linearly compact and S_S is linearly compact;
(e) ${}_RR$ is linearly compact and S_S satisfies $AB\,5^*$.

(2) The following assertions are equivalent:
(a) R is a left artinian left Morita ring;
(b) R is left artinian and ${}_RQ$ is finitely generated;
(c) ${}_RR$ satisfies $AB\,5^*$ and ${}_RQ$ has finite length;
(d) ${}_RR$ is linearly compact and S_S is artinian;
(e) there is a duality between the finitely generated modules in R-MOD and MOD-S.

Proof: (1) $(a) \Rightarrow (b)$ Let ${}_RU$ be an injective cogenerator in *R-MOD*, $S' = End({}_RU)$ such that $U_{S'}$ is an injective cogenerator in $MOD\text{-}S'$ and $R \simeq End(U_{S'})$. Then, by 47.8, all U-reflexive modules are linearly compact, in particular ${}_RR$ and ${}_RQ$ are.

$(b) \Rightarrow (c)$ and $(d) \Rightarrow (e)$ are clear by 29.8.

$(c) \Rightarrow (a)$ Since ${}_RR$ is linearly compact, there are only finitely many simple R-modules and ${}_RQ$ is finitely cogenerated. Therefore we conclude from 47.10 that Q_S is a cogenerator for $\sigma_f[Q_S]$, Q_S cogenerates all cyclic S-modules, and S_S is linearly compact (results from AB 5^* for ${}_RQ$). From 47.8 we now derive $R \simeq R^{**}$ and that all sub- and factor modules of S are

Q-reflexive. Hence, for every submodule $L \subset S_S$, the sequence

$$0 \longrightarrow L^{**} \longrightarrow S^{**} \longrightarrow (S/L)^{**} \longrightarrow 0$$

is exact. With the cogenerator ${}_RQ$, the functor $(-)^* = Hom_R(-, {}_RQ)$ reflects exact sequences, i.e. $S^* \to L^* \to 0$ is also exact. Therefore Q_S is S-injective and an injective cogenerator in *MOD-S*.

$(a) \Rightarrow (d)$ Since ${}_RR$ is linearly compact, ${}_RQ$ has an essential socle (${}_RQ$ is finitely cogenerated). By 47.8 and 47.10, Q_S is an injective cogenerator in *MOD-S*. As a Q-reflexive module, S_S is linearly compact (again by 47.8).

$(e) \Rightarrow (b)$ By 47.10, Q_S has an essential socle. Since ${}_RR$ is linearly compact, $R \simeq R^{**}$ and Q_S is self-injective (see 47.8). Now we deduce from 47.10,(2) that ${}_RQ \simeq Hom_S(S, Q_S) = S^*$ is linearly compact.

(2) $(a) \Rightarrow (b)$ Since R is artinian, $Rad({}_RQ) \ll {}_RQ$ and $Q/Rad(Q)$ is semisimple (see 31.5). By (1), Q and $Q/Rad(Q)$ are linearly compact and, by 29.8, $Q/Rad(Q)$ is finitely generated. Then, by 19.6, Q is also finitely generated.

$(b) \Rightarrow (a)$ This follows from (1) since, as a finitely generated R-module, ${}_RQ$ has finite length and hence is linearly compact.

$(b) \Rightarrow (c)$ Since ${}_RR$ is an artinian module, it satisfies AB 5^* and ${}_RQ$ has finite length.

$(c) \Rightarrow (b)$ Let $J = Jac(R)$. Since ${}_RQ$ is a module of finite length, the descending chain of submodules $JQ \supset J^2Q \supset \ldots$ becomes stationary. Hence, for some $n \in I\!N$, we have $J^nQ = J(J^nQ)$ and, by Nakayama's Lemma, $J^nQ = 0$. Since Q is a faithful R-module, the ideal J has to be nilpotent. Because of AB 5^* in R, the factor ring R/J is left semisimple and R is right (and left) perfect and left artinian by 47.9.

$(b) \Rightarrow (d)$ Since ${}_RQ$ is self-injective and has finite length, S_S is artinian (see 31.11,(3) and 31.12).

$(d) \Rightarrow (b)$ By (1), R is a left Morita ring. Therefore we have, by 47.5, a bijection between the R-submodules of ${}_RQ$ and the right ideals of S. Hence ${}_RQ$ has finite length.

$(a) \Rightarrow (e)$ follows from 47.13.

47.16 Exercise.

Let R be a commutative artinian ring and E the injective hull of $R/Jac\,R$. Prove (see 32.9,(5)):

(i) The injective hulls of the simple R-modules are finitely generated.

(ii) $Hom_R(-, E) : R\text{-}mod \to R\text{-}mod$ *is a duality.*

Literature: ANDERSON-FULLER, FAITH [2], KASCH, SOLIAN; Anh [1], Brodskii [3,4,5], Colby-Fuller [1,2,3], Couchot [6], Damiano [2], Dikranjan-Orsatti, Garcia-Gomez [2], Gregorio, Kerner, Kitamura, Kraemer, Lambek-Rattray, Lemonnier, Macdonald, Masaike [1], Menini-Orsatti [1,2], Miller-Turnidge [1], Müller [1,2,3], Onodera [2], Orsatti-Roselli, Ringel-Tachikawa, Roux [1,2,4], Sandomierski, Schulz [1], Sklyarenko [2], Upham, Vamos [1], Yamagata [1], Zelmanowitz [5], Zelmanowitz-Jansen [1,2].

48 Quasi-Frobenius modules and rings

1.Weak cogenerators. 2.Quasi-Frobenius modules. 3.QF modules with coherence properties. 4.Duality between finitely presented modules. 5.Weakly injective and flat modules. 6.Self-projective, coherent QF modules. 7.QF rings. 8.Coherent QF (IF) rings. 9.Progenerators as QF modules. 10.Projective cogenerators. 11.M as projective cogenerator in $\sigma[M]$. 12.$_RR$ as cogenerator. 13.$_RR$ as linearly compact cogenerator. 14.Self-projective noetherian QF modules I,II. 15.Noetherian QF rings. 16.Cogenerator with commutative endomorphism ring. 17.Commutative PF rings. 18.Exercises.

In this section we will mainly be occupied with cogenerator properties of modules and rings. These considerations are closely related to the statements concerning dualities.

We call an R-module M a *weak cogenerator (in $\sigma[M]$)* if, for every finitely generated submodule $K \subset M^{(\mathbb{N})}$, the factor module $M^{(\mathbb{N})}/K$ is M-cogenerated.

Obviously, this is equivalent to the property that M cogenerates all modules M^n/K with finitely generated $K \subset M^n$, $n \in \mathbb{N}$.

If M is a cogenerator in $\sigma_f[M]$, then M is a weak cogenerator.

48.1 Characterization of weak cogenerators.

For an R-module M with $S = End(_RM)$, the following are equivalent:

(a) $_RM$ is a weak cogenerator (in $\sigma[M]$);

(b) M_S is weakly M_S-injective, and

(i) every finitely generated R-module is M-dense, or

(ii) every finitely generated submodule of $M^{(\mathbb{N})}$ is M-reflexive, or

(iii) R is M-dense.

If in this case M is finitely generated, then M_S is weakly S-injective (FP-injective).

Proof: $(a) \Rightarrow (b)(i)$ can be derived directly from 47.7.

$(b)(i) \Leftrightarrow (ii)$ was shown in 47.6. $(i) \Rightarrow (iii)$ is clear.

$(b)(iii) \Rightarrow (a)$ If R is M-dense, then, by 47.6, this also holds for every R^k, $k \in \mathbb{N}$. Hence the assertion follows from 47.7. If M is finitely generated, then the cokernel of $f\colon M^k \to M^l$ is cogenerated by M (see 47.7).

An R-module M is called a *Quasi-Frobenius module* or a *QF module* if M is weakly M-injective and a weak cogenerator in $\sigma[M]$.

48.2 Characterization of Quasi-Frobenius modules.

For an R-module M with $S = End(_RM)$, the following are equivalent:

(a) $_RM$ *is a QF module;*

(b) (i) $_RM$ *is weakly M-injective and*
(ii) *every finitely generated submodule of* $_RM^{(\mathbb{N})}$ *is M-reflexive;*

(c) (i) $_RM$ *is weakly* $_RM$*-injective and*
(ii) M_S *is weakly* M_S*-injective and*
(iii) R *is M-dense;*

(d) M_S *is a QF module, and R is M-dense;*

(e) $_RM$ *and* M_S *are weak cogenerators in* $\sigma[_RM]$*, resp.* $\sigma[M_S]$*;*

(f) (i) $_RM$ *is weakly M-injective and*
(ii) $Hom_R(-, {_RM})$ *determines a duality between the finitely generated submodules of* $_RM^{(\mathbb{N})}$ *and* $M_S^{(\mathbb{N})}$.

Proof: $(a) \Leftrightarrow (b)$ For a finitely generated submodule $K \subset M^n$, $n \in \mathbb{N}$, we have the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & M^n & \longrightarrow & M^n/K & \longrightarrow & 0 \\ & & \downarrow \Phi_K & & \downarrow \simeq & & \downarrow \Phi_{M^n/K} & & \\ 0 & \longrightarrow & K^{**} & \longrightarrow & (M^n)^{**} & \longrightarrow & (M^n/K)^{**} & & . \end{array}$$

Φ_K is always monic. By the Kernel Cokernel Lemma, Φ_K is epic if and only if $\Phi_{M^n/K}$ is monic, i.e. M^n/K is cogenerated by M (see 45.10).

$(a) \Leftrightarrow (c)$ follows from the characterization of weak cogenerators in 48.1.

$(c) \Leftrightarrow (d)$ This results from $(a) \Leftrightarrow (c)$ shown already and the fact that density of R in $B = End(M_S)$ implies $\sigma[_RM] = \sigma[_BM]$ (see 15.8).

$(d) \Rightarrow (e)$ is clear by (a)$\Leftrightarrow$(d).

$(e) \Rightarrow (c)$ This can be seen from 48.1 (by density of R in $End(M_S)$).

$(a) \Rightarrow (f)$ We saw in (b) that the finitely generated submodules of $_RM^{(\mathbb{N})}$ and $M_S^{(\mathbb{N})}$ are M-reflexive. Observing that R is M-dense we verify easily that the given modules are mapped into each other by $Hom_R(-, M)$. Thus, by 47.11, we have a duality.

$(f) \Rightarrow (b)$ If the given functors determine a duality, then, for every finitely generated submodule $K \subset M^n$, $n \in \mathbb{N}$, there is an isomorphism $K \simeq K^{**}$ and K^* is a finitely generated submodule of M_S^n.

By 28.1, for every finitely generated submodule $L \subset M_S^k$, $k \in \mathbb{N}$, we have $L = AnKe(L)$ and hence M_S^k/L is cogenerated by M_S. Therefore M_S is a weak cogenerator in $\sigma[M_S]$ and, by 48.1, all finitely generated submodules of $M_S^{(\mathbb{N})}$ are M-reflexive. Hence K^* is M-reflexive and, by 47.4, K^{**} and $K \simeq K^{**}$ are also M-reflexive.

48.3 QF modules with coherence properties.
Let M be an R-module and $S = End(_RM)$.

(1) If the R-module X is cogenerated by M and $Hom_R(X, M)$ is a finitely generated S-module, then $X \subset M^n$ for some $n \in I\!N$.

(2) If $_RM$ is finitely generated, then the following are equivalent:
(a) $_RM$ is a QF module and M_S is locally coherent in MOD-S;
(b) $_RM$ is weakly M-injective and every factor module $M^{(I\!N)}/K$, with finitely generated $K \subset M^{(I\!N)}$, is isomorphic to a submodule of $_RM^{(I\!N)}$.

Proof: (1) For a generating set $f_1, \dots, f_n$ of $Hom_R(X, M)_S$ we have

$$0 = Re(X, M) = \bigcap \{Kef \mid f \in Hom_R(X, M)\} = \bigcap\nolimits_{i \le n} Kef_i \ .$$

Thus $\prod f_i : X \to M^n$ is monic.

(2) Since $_RM$ is finitely generated, we know that $S \in \sigma[M_S]$.

$(a) \Rightarrow (b)$ Let $0 \to K \to M^n \to N \to 0$ be exact, $n \in I\!N$, and K finitely generated. With $(\ )^* = Hom(-, {}_RM)$ we obtain the exact sequence

$$0 \longrightarrow N^* \longrightarrow S^n \longrightarrow K^* \longrightarrow 0 \ .$$

Hence K^* is finitely generated and – as a submodule of the locally coherent module $M_S^{(I\!N)}$ – even finitely presented. Therefore N^* is finitely generated and, by (1), $N \subset_R M^m$ for some $m \in I\!N$.

$(b) \Rightarrow (a)$ $_RM$ is obviously a weak cogenerator in $\sigma[M]$ and hence a QF module. By 28.1, every finitely generated submodule $L \subset M_S$ is of the form $Hom(\overline{R}, M)$ with $\overline{R} = R/Ke\ L \subset M^k$ for some $k \in I\!N$. Hence we have an exact sequence

$$0 \longrightarrow Hom_R(M^k/\overline{R}, M) \longrightarrow S^k \longrightarrow L \longrightarrow 0 \ .$$

Since $M^k/\overline{R}$ is a finitely generated R-submodule of some M^r, $r \in I\!N$, the S-module $Hom_R(M^k/\overline{R}, M)$ is finitely generated and L is finitely presented in *MOD-S*.

48.4 Duality between finitely presented modules.
Let M be an R-module and $S = End(_RM)$.

(1) The following assertions are equivalent:
(a) (i) $_RM$ is weakly M-injective, and
(ii) $Hom_R(-, M)$ determines a duality between the finitely presented modules in $\sigma[_RM]$ and the finitely presented modules in MOD-S;
(b) (i) $_RM$ is a QF module and coherent in $\sigma[M]$, and
(ii) M_S is locally coherent in MOD-S.

(2) If ${}_RM$ is faithful and ${}_RM$ and M_S are finitely generated, then the following statements are equivalent:

(a) $Hom_R(-,M)$ determines a duality between the finitely presented modules in R-MOD and MOD-S;

(b) ${}_RM$ is a QF module and ${}_RM$ and M_S are coherent in R-MOD, resp. MOD-S;

(c) (i) every factor module ${}_RM^{(\mathbb{N})}/K$, with K finitely generated, is isomorphic to a submodule of ${}_RM^{(\mathbb{N})}$;
(ii) every factor module $M_S^{(\mathbb{N})}/L$, with L_S finitely generated, is isomorphic to a submodule of $M_S^{(\mathbb{N})}$.

Proof: (1) $(a) \Rightarrow (b)$ If $Hom_R(-,M)$ is a duality, then the adjoint functor $Hom_S(-,M_S)$ is inverse to it.

By (ii), $Hom_S(S,M_S) \simeq {}_RM$ is finitely presented in $\sigma[M]$, and all finitely presented modules $P \in \sigma[M]$ are cogenerated by M (since $P \simeq P^{**}$). Then, for any finitely generated submodule $K \subset M^r$, $r \in \mathbb{N}$, the modules M^r/K and $(M^r/K)^*$ are also finitely presented in $\sigma[M]$, resp. in *MOD-S*.

From 48.3 we learn that $M^r/K \subset {}_RM^{(\mathbb{N})}$ and M_S is locally coherent in *MOD-S*. From the exact sequence

$$0 \longrightarrow (M^r/K)^* \longrightarrow S^r \longrightarrow K^* \longrightarrow 0$$

we deduce that K^* is finitely presented in *MOD-S* and hence $K \simeq K^{**}$ is finitely presented in $\sigma[M]$. Therefore ${}_RM$ is coherent in $\sigma[M]$.

$(b) \Rightarrow (a)$ A finitely presented module $P \in \sigma[M]$ is a submodule of a factor module M^n/K, with $n \in \mathbb{N}$ and K finitely generated (see 25.1). If (b) holds, then (by 48.3) the finitely presented modules in $\sigma[M]$ are exactly the finitely generated submodules of ${}_RM^{(\mathbb{N})}$. Since $S \in \sigma[M_S]$, the finitely presented modules in $\sigma[M_S]$ are just the finitely generated submodules of $M_S^{(\mathbb{N})}$. Hence the assertion follows from 48.2.

(2) $(a) \Rightarrow (b)$ First we see (as in (1)) that the modules $Hom_R(R,M) \simeq M_S$ and $Hom_S(S,M_S) \simeq {}_RM$ are finitely presented. Moreover, for finitely generated submodules $K \subset {}_RM^{(\mathbb{N})}$, $L \subset M_S^{(\mathbb{N})}$, we know by assumption

$${}_RM^{(\mathbb{N})}/K \subset_R M^{(\mathbb{N})} \text{ and } \quad M_S^{(\mathbb{N})}/L \subset M_S^{(\mathbb{N})}.$$

Since R is M-dense and $R \subset M^{\mathbb{N}}$ (see 15.7, 15.4), we get $R \simeq End(M_S)$. Hence, by 48.3, ${}_RM$ is a QF module with the given properties.

$(b) \Rightarrow (a)$ Since $\sigma[{}_RM] = R\text{-}MOD$ and $\sigma[M_S] = MOD\text{-}S$, the finitely presented submodules of $M^{(\mathbb{N})}/K$, K finitely generated, are in *R-MOD*, resp. *MOD-S*. Hence, by 48.3, they are submodules of ${}_RM^{(\mathbb{N})}$, resp. $M_S^{(\mathbb{N})}$, and the assertion follows from 48.2.

$(b) \Rightarrow (c)$ Since M_S is finitely generated, we have $R \subset M^{\mathbb{N}}$, R is M-dense and, by 47.6, M-reflexive. Now the assertions follow from 48.3.

$(c) \Rightarrow (b)$ From (i) and (ii) we see that ${}_RM$ and M_S are weak cogenerators in $\sigma[{}_RM]$, resp. $\sigma[M_S]$. Hence, by 48.2, ${}_RM$ is a QF module. The remaining assertions again follow from 48.3.

To prepare for a further investigation of QF modules with finiteness condition we show (by 'submodule of' we shall mean 'isomorphic to a submodule of'):

48.5 Weakly injective and flat modules.
Let M be an R-module.

(1) If every M-injective module is flat in $\sigma[M]$, then every finitely presented module in $\sigma[M]$ is a submodule of $M^{(\mathbb{N})}$.

(2) If M is projective in $\sigma[M]$ and finitely presented modules in $\sigma[M]$ are submodules of $M^{(\mathbb{N})}$, then weakly M-injective modules are flat in $\sigma[M]$.

(3) If M is finitely generated and self-projective, then the following statements are equivalent:
(a) Every finitely presented module in $\sigma[M]$ is a submodule of $M^{(\mathbb{N})}$;
(b) every factor module $M^{(\mathbb{N})}/K$, with finitely generated $K \subset M^{(\mathbb{N})}$, is a submodule of $M^{(\mathbb{N})}$;
(c) every (weakly) M-injective module is flat in $\sigma[M]$.

Proof: (1) Let P be a finitely presented module in $\sigma[M]$ and $P \to \widehat{P}$ its M-injective hull. Since $\widehat{P}$ is M-generated (see 16.3), we have the following diagram with exact row

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ M^{(\Lambda)} & \longrightarrow & \widehat{P} & \longrightarrow & 0 \,. \end{array}$$

By assumption $\widehat{P}$ is flat. Hence the row is pure and there is a monomorphism $P \to M^{(\Lambda)}$ which completes the diagram commutatively (see § 36). Since ${}_RP$ is finitely generated, it is a submodule of a finite partial sum M^k.

(2) Let P be finitely presented in $\sigma[M]$ and $N \in \sigma[M]$ weakly M-injective. Then P is a submodule of $M^{(\mathbb{N})}$ and we have to show that P is projective with respect to every exact sequence $X \to N \to 0$ in $\sigma[M]$. For

$f\colon P \to N$ we have the exact diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & P & \longrightarrow & M^{(\mathbb{N})} \\ & & \downarrow f & & \\ X & \longrightarrow & N & \longrightarrow & 0 \end{array} .$$

Since N is weakly M-injective, there exists $g : M^{(\mathbb{N})} \to N$ and also, by the projectivity of M, $h : M^{(\mathbb{N})} \to X$ which complete the diagram commutatively. Hence every exact sequence $X \to N \to 0$ is pure, i.e. N is flat in $\sigma[M]$ (see § 36).

(3) Any finitely presented $P \in \sigma[M]$ is a submodule of M^n/K (see 25.1). Since M is finitely presented in $\sigma[M]$, these modules are also finitely presented (see 25.1) and the assertion follows from (1) and (2).

48.6 Self-projective coherent QF modules.

Assume M is a finitely generated, self-projective left R-module and $S = End(_RM)$. Then the following statements are equivalent:

(a) (i) $_RM$ is a QF module and is coherent in $\sigma[M]$,
(ii) M_S is locally coherent in MOD-S;

(b) (i) $_RM$ is weakly M-injective and is coherent in $\sigma[M]$,
(ii) every weakly M-injective module is flat in $\sigma[M]$;

(c) (i) $_RM$ is coherent in $\sigma[M]$,
(ii) every M-generated flat module in $\sigma[M]$ is weakly M-injective,
(iii) every factor module $M^{(\mathbb{N})}/K$, with finitely generated $K \subset M^{(\mathbb{N})}$, is a submodule of $M^{(\mathbb{N})}$;

(d) for M-generated R-modules, weakly M-injective is equivalent to flat in $\sigma[M]$.

Proof: $(a) \Leftrightarrow (b)$ If (a) holds, then, by 48.3, every finitely presented module in $\sigma[M]$ is a submodule of $M^{(\mathbb{N})}$, and, by 48.5, weakly M-injective modules are flat in $\sigma[M]$.

The reverse statement follows from the same propositions.

$(a) \Rightarrow (c)$ If the R-module F is M-generated and flat in $\sigma[M]$, then there is a pure epimorphism $M^{(\Lambda)} \to F$. Since $M^{(\Lambda)}$ is weakly M-injective and M is coherent in $\sigma[M]$, we know from 35.5 that F is also weakly M-injective. $(a) \Rightarrow (c)(iii)$ was shown in 48.3.

$(c) \Rightarrow (a)$ Since M is projective in $\sigma[M]$, it is flat and hence weakly M-injective. Therefore the assertion follows again from 48.3.

$(a) \Rightarrow (d)$ is clear by the equivalences already proved.

$(d) \Rightarrow (a)$ Since direct limits of flat modules are flat in $\sigma[M]$, by (d), direct limits of weakly M-injective (= absolutely pure) modules in $\sigma[M]$ are weakly M-injective. Hence, by 35.6, ${}_RM$ is coherent in $\sigma[M]$. By 48.5, every finitely presented module in $\sigma[M]$ is a submodule of $M^{(\mathbb{N})}$. This implies, by 48.3, that ${}_RM$ is a QF module and M_S is locally coherent.

We call a ring R a *Quasi-Frobenius ring* or a *QF ring* if ${}_RR$ is a QF module. This definition is in accordance with our usual way of defining module properties for rings. However, in the literature a *QF ring* often means a *noetherian* QF ring in our sense (see 48.15).

48.7 Characterization of QF rings.

For a ring R the following properties are equivalent:

(a) ${}_RR$ is a QF module;

(b) ${}_RR$ is FP-injective and cogenerates all finitely presented modules in R-MOD;

(c) ${}_RR$ is FP-injective, and the finitely generated submodules of ${}_RR^{(\mathbb{N})}$ are R-reflexive;

(d) ${}_RR$ and R_R cogenerate the finitely presented modules in R-MOD, resp. MOD-R;

(e) ${}_RR$ and R_R are FP-injective (in R-MOD resp. MOD-R);

(f) R_R is a QF module.

Proof: The equivalence of these assertions follows from 48.2.

Well-known examples of QF rings (in our sense) are (von Neumann) regular rings (see 37.6). These rings are, in addition, both-sided coherent and hence also satisfy the conditions formulated in the next proposition:

48.8 Characterization of coherent QF rings (IF rings).

For a ring R the following properties are equivalent:

(a) R is a QF ring, ${}_RR$ and R_R are coherent;

(b) (i) ${}_RR$ is FP-injective and coherent,
(ii) finitely presented modules in R-MOD are submodules of ${}_RR^{(\mathbb{N})}$;

(c) (i) ${}_RR$ is FP-injective and coherent,
(ii) every FP-injective module is flat in R-MOD;

(d) (i) ${}_RR$ is coherent,
(ii) every flat module is FP-injective in R-MOD,
(iii) finitely presented modules in R-MOD are submodules of ${}_RR^{(\mathbb{N})}$;

(e) $Hom_R(-,R)$ determines a duality between the finitely presented modules in R-MOD and MOD-R;

(f) in R-MOD, FP-injective is equivalent to flat;

(g) in R-MOD and MOD-R, the FP-injective modules are flat;

(h) in R-MOD and MOD-R, the finitely presented modules are submodules of $R^{(\mathbb{N})}$.

Proof: $(a) \Leftrightarrow (b)$ is a special case of 48.3. The equivalence of (a), (c), (d) and (f) can be derived from 48.6.

$(a) \Leftrightarrow (e)$ was shown in 48.4.

$(a) \Rightarrow (g)$ follows as $(a) \Rightarrow (c)$ by reasons of symmetry.

$(g) \Leftrightarrow (h)$ is a result of 48.5.

$(h) \Rightarrow (a)$ From the given properties we see at once that ${}_RR$ and R_R are weak cogenerators in *R-MOD*, resp. *MOD-R*. Hence, by 48.2, R is a QF ring and, by 48.3, ${}_RR$ and R_R are coherent.

Coherent QF rings are precisely the rings whose injective left and right modules are flat (see 48.5,(1)). Hence they are also called *IF rings* (Colby).

48.9 Progenerators as QF modules.

Assume M is a finitely generated, self-projective left R-module with $S = End({}_RM)$. Then the following assertions are equivalent:

(a) (i) ${}_RM$ is a QF module, is coherent and is a generator in $\sigma[M]$,
(ii) M_S is locally coherent in MOD-S;

(b) ${}_RM$ is a self-generator and S is an IF ring;

(c) ${}_RM$ is a QF module, S is an IF ring and M_S is locally coherent in MOD-S;

(d) (i) every (weakly) M-injective module is flat in $\sigma[{}_RM]$,
(ii) every (weakly) S-injective module is flat in MOD-S;

(e) (i) every finitely presented module in $\sigma[{}_RM]$ is a submodule of $M^{(\mathbb{N})}$,
(ii) every finitely presented module in MOD-S is a submodule of $S^{(\mathbb{N})}$.

Proof: If M is a self-generator, then, under the given conditions, it is a generator in $\sigma[M]$ (see 18.5) and so $Hom_R(M,-) : \sigma[M] \to S\text{-}MOD$ is an equivalence (see 46.2). Hence it is obvious that ${}_RM$ is a coherent QF module if and only if this holds for ${}_SS = Hom_R(M,M)$.

$(a) \Rightarrow (b)$ By the remark above, it remains to prove that S_S is coherent: Since ${}_RM$ is finitely generated $S_S \subset M_S^k$, $k \in \mathbb{N}$, and the desired property follows from $(a)(ii)$.

$(b) \Rightarrow (a)$ Here the local coherence of M_S remains to be shown: Every finitely generated submodule $L \subset M_S$ is of the form (see 28.1)

$$L = Hom(\overline{R}, M) \text{ with } \overline{R} = R/Ke\, L \subset M^n,\ n \in \mathbb{N}.$$

Since M is a generator in $\sigma[M]$, we have exact sequences $M^k \to \overline{R} \to 0$ and $0 \to Hom_R(\overline{R}, M) \to Hom(M^k, M)$, i.e. $L \subset S_S^k$ and hence L is coherent.

$(a) \Rightarrow (c)$ is clear by $(a) \Leftrightarrow (b)$.

$(c) \Rightarrow (b)$ M_S is weakly S-injective and hence, by 48.8, flat in *MOD-S*. Then, by 15.9, ${}_RM$ generates the kernels of the morphisms $f\colon M^k \to M^n$, $k, n \in I\!N$. For a finitely generated $K \subset_R M$, we have the exact sequence

$$0 \longrightarrow (M/K)^* \longrightarrow S \longrightarrow K^* \longrightarrow 0 .$$

K^* is also a submodule of $M_S^{(I\!N)}$ and therefore finitely presented. Then $(M/K)^*$ is finitely generated and, by 48.3, $M/K \subset M^n$, $n \in I\!N$. Hence K is the kernel of a morphism $M \to M/K \subset M^n$ and hence is generated by M, i.e. M is a self-generator.

$(a) \Rightarrow (d)$ (i) follows from 48.6. (ii) is a property of IF rings (see 48.8).

$(d) \Leftrightarrow (e)$ follows from 48.5.

$(e) \Rightarrow (a)$ By $(e)(i)$, ${}_RM$ is a weak cogenerator in $\sigma[M]$ (see 25.1) and M_S is weakly S-injective (see 48.1). From $(d) \Leftrightarrow (e)$ we learn that M_S is flat, i.e. ${}_RM$ generates the kernels of the morphisms $f\colon M^k \to M^n$, $k, n \in I\!N$. Since, for every finitely generated submodule $K \subset M$, by (i), $M/K \subset M^{(I\!N)}$ holds, K is the kernel of such a morphism and hence is generated by M, i.e. M is a self-generator. Now the equivalence of $\sigma[M]$ and *S-MOD* tells us that every (weakly) S-injective module in *S-MOD* is flat. Together with $(d)(ii)$ this yields that S is an IF ring (see 48.8).

The existence of a projective cogenerator in $\sigma[M]$ has important consequences for the structure of this category :

48.10 Projective cogenerators. Properties.

Let M be an R-module and assume there exists a projective cogenerator Q in $\sigma[M]$. If $Soc(Q)$ is finitely generated, then:

(1) There is an injective, projective generator in $\sigma[M]$;

(2) every projective module in $\sigma[M]$ is weakly M-injective;

(3) every cogenerator is a generator in $\sigma[M]$;

(4) every generator is a cogenerator in $\sigma[M]$;

(5) M is a generator and a cogenerator in $\sigma[M]$.

Proof: (1) As a cogenerator, Q contains an M-injective hull of every simple module in $\sigma[M]$ and $Soc(Q)$ contains a copy of every simple module in $\sigma[M]$. Since $Soc(Q)$ is finitely generated, there are only finitely many non-isomorphic simple modules $E_1, \ldots, E_k$ in $\sigma[M]$.

The M-injective hulls $\widehat{E}_i \subset Q$ are (directly) indecomposable and - being direct summands of Q - projective in $\sigma[M]$. $End(\widehat{E}_i)$ is a local ring (see 19.9). Hence, by 19.7, $\widehat{E}_i$ is a projective cover of a simple module in $\sigma[M]$.

The factor module $\widehat{E}_i/Rad\,\widehat{E}_i$ is simple. For $E_i \not\simeq E_j$, also $\widehat{E}_i \not\simeq \widehat{E}_j$ and - by the uniqueness of projective covers - $\widehat{E}_i/Rad\,\widehat{E}_i \not\simeq \widehat{E}_j/Rad\,\widehat{E}_j$. Hence $\{\widehat{E}_i/Rad\,\widehat{E}_i\}_{i\leq k}$ forms a representing set of simple modules in $\sigma[M]$.

$G := \bigoplus_{i\leq k}\widehat{E}_i$ is an injective cogenerator. Moreover, G is projective and generates every simple module in $\sigma[M]$. Hence it is a generator in $\sigma[M]$.

(2) Every projective module in $\sigma[M]$ is a direct summand of a weakly M-injective module $G^{(\Lambda)}$, for suitable Λ.

(3) Every cogenerator contains a direct summand isomorphic to G, hence it generates G and therefore every module in $\sigma[M]$.

(4) If P is a generator in $\sigma[M]$, then G is isomorphic to a direct summand of $P^{(\Lambda)}$, for suitable Λ.

(5) M generates every injective module in $\sigma[M]$ (see 16.3) and in particular G. So M is a generator and, by (4), also a cogenerator in $\sigma[M]$.

48.11 M as a projective cogenerator in $\sigma[M]$.

For a finitely generated, self-projective R-module M, the following are equivalent:

(a) M is a cogenerator in $\sigma[M]$, and there are only finitely many non-isomorphic simple modules in $\sigma[M]$;

(b) every cogenerator is a generator in $\sigma[M]$;

(c) M is finitely cogenerated, M-injective and a (self-)generator;

(d) M is semiperfect in $\sigma[M]$, M-injective, a self-generator and $Soc\,M \trianglelefteq M$;

(e) every module which cogenerates M is a generator in $\sigma[M]$.

Proof: $(a) \Rightarrow (b)$ If $E_1, \dots, E_k$ are the simple modules in $\sigma[M]$, then the sum of the M-injective hulls $\widehat{E}_i$ is a cogenerator and a direct summand of M. Hence it is projective in $\sigma[M]$ and the assertion follows from 48.10.

$(b) \Rightarrow (c)$ Let $\{E_\lambda\}_\Lambda$ be a (not necessarily finite) representing set of simple modules in $\sigma[M]$. Then $\bigoplus_\Lambda \widehat{E}_\lambda$ is a cogenerator and (by (b)) a generator in $\sigma[M]$. Therefore $_RM$ is a direct summand of a finite direct sum of copies of the $\widehat{E}_\lambda$, $\lambda \in \Lambda$, and hence it is M-injective and finitely cogenerated. Since all $\widehat{E}_\lambda$ are M-generated this also holds for $\bigoplus_\Lambda \widehat{E}_\lambda$, and hence M is a generator in $\sigma[M]$.

$(c) \Rightarrow (d)$ Being finitely cogenerated, M has a finite essential socle, i.e. $Soc\,M = \bigoplus_{i\leq n} E_i$ with simple E_i (see 21.3). Then $M \simeq \bigoplus_{i\leq n}\widehat{E}_i$ where the

$\widehat{E}_i$ are projective modules with local $End(\widehat{E}_i)$ and hence are semiperfect in $\sigma[M]$. Therefore M is also semiperfect in $\sigma[M]$ (see 42.5).

$(d) \Rightarrow (a)$ $M/Rad\,M$ is finitely generated and semisimple (see 42.3) containing a copy of every simple module in $\sigma[M]$.

By assumption, $M = \bigoplus_{i\leq n}\widehat{E}_i$ where the $\widehat{E}_i$ are M-injective hulls of simple modules E_i in $\sigma[M]$. By 19.7, the $\widehat{E}_i$ are also projective covers of simple modules.

Since $M/Rad\,M \simeq \bigoplus \widehat{E}_i/Rad\,\widehat{E}_i$, every simple module in $\sigma[M]$ is isomorphic to some $\widehat{E}_i/Rad\,\widehat{E}_i$. Now $\widehat{E}_i/Rad\,\widehat{E}_i \not\simeq \widehat{E}_j/Rad\,\widehat{E}_j$ implies $E_i \not\simeq E_j$ (uniqueness of projective covers and injective hulls). Since there are only finitely many distinct simple modules in $\sigma[M]$, any of them is isomorphic to some $E_i \subset M$. Hence M is a cogenerator in $\sigma[M]$.

$(c) \Rightarrow (e)$ Let Q be a module in $\sigma[M]$ which cogenerates M. Then M is a direct summand of a finite sum Q^k, $k \in I\!N$, and hence is Q-generated. Since M is a generator in $\sigma[M]$, Q is also a generator in $\sigma[M]$.

$(e) \Rightarrow (b)$ Every cogenerator in $\sigma[M]$ also cogenerates M.

48.12 $_RR$ as a cogenerator. Characterizations.

For a ring R the following are equivalent:

(a) $_RR$ is a cogenerator, and there are only finitely many non-isomorphic simple modules in R-MOD;

(b) every cogenerator is a generator in R-MOD;

(c) $_RR$ is injective and finitely cogenerated;

(d) $_RR$ is injective, semiperfect and $Soc_R R \trianglelefteq {}_RR$;

(e) every faithful module is a generator in R-MOD;

(f) $_RR$ is a cogenerator in R-MOD, and R_R cogenerates all simple modules in MOD-R.

These rings are called *left Pseudo-Frobenius rings* or *left PF rings.*

Proof: The equivalences of (a) to (e) are derived immediately from 48.11, by observing that faithful R-modules are those which cogenerate R.

$(c) \Rightarrow (f)$ We obtain from 47.10,(1) that R_R cogenerates the simple right R-modules. By $(a) \Leftrightarrow (c)$, $_RR$ is a cogenerator.

$(f) \Rightarrow (e)$ Let $_RN$ be a faithful R-module. We know from 13.5,(3) that $Tr(N,R) = N\,Hom(N,R)$ is an ideal in R.

If $Tr(N,R) \neq R$, then $Tr(N,R)$ is contained in a maximal right ideal K. Since R/K – by assumption – is cogenerated by R_R, there exists $a \in R$ with $a \neq 0$ and $aK = 0$, and hence $aN\,Hom(N,R) = 0$. This means

$$aN \subset \bigcap \{Kef \mid f \in Hom(N,R)\} = 0$$

and therefore $a = 0$ since ${}_RN$ is faithful. Hence $Tr(N, R) = R$, and N is a generator in R-MOD.

The properties of R considered in 48.12 obviously are one-sided. An example of a left PF ring which is not a right PF ring is given in Dischinger-Müller [2].

B. Osofsky proved (in *J. Algebra* 1966) that left PF rings can also be characterized by the property ${}_RR$ *is an injective cogenerator in R-MOD.*

A left-right-symmetric situation can be found in

48.13 ${}_RR$ as linearly compact cogenerator.

For a ring R the following properties are equivalent:

(a) ${}_RR$ *is a linearly compact cogenerator in R-MOD;*

(b) ${}_RR$ *is injective, linearly compact and* $Soc_R R \trianglelefteq_R R$*;*

(c) ${}_RR$ *and* R_R *are cogenerators in R-MOD, resp. MOD-R;*

(d) ${}_RR$ *is a cogenerator and* R_R *is injective;*

(e) ${}_RR$ *and* R_R *are injective, and* ${}_RR$ *(and* R_R*) is finitely cogenerated;*

(f) all finitely generated (cyclic) modules in R-MOD and MOD-R are R-reflexive;

(g) all finitely cogenerated modules in R-MOD and MOD-R are R-reflexive.

The rings described here are special Morita rings (see 47.15).

Proof: $(a) \Leftrightarrow (b)$ Being linearly compact, the module ${}_RR$ has finitely generated socle. Therefore the assertion follows from 48.12.

$(a) \Leftrightarrow (c)$ Observing $(a) \Leftrightarrow (b)$ and 48.12 this follows from the characterization of Morita dualities (47.12).

$(a) \Leftrightarrow (d)$ is a special case of 47.8.

$(d) \Leftrightarrow (e)$ Noting 47.8, this can be deduced from 48.12.

$(a) \Leftrightarrow (f)$ This follows from 47.12.

$(a) \Rightarrow (g)$ If (a) holds, all finitely cogenerated modules in R-MOD are submodules of $R^{\mathbb{N}}$ and linearly compact, hence R-reflexive (see 47.8).

The corresponding statement holds in MOD-R (since (a)$\Leftrightarrow$(c)).

$(g) \Rightarrow (c)$ Injective hulls of simple modules are finitely cogenerated. Being R-reflexive modules, they are cogenerated by R (see 45.10). Hence R is a cogenerator in R-MOD resp. MOD-R.

If M is an injective cogenerator in $\sigma[M]$, then M is a QF module (in the sense of 48.2). On the other hand, locally noetherian QF modules M are just injective cogenerators in $\sigma[M]$. If, in addition, M is self-projective and finitely generated, we obtain remarkable equivalences:

48.14 Self-projective, noetherian QF modules, I.

For a finitely generated, self-projective R-module M with $S = End(_RM)$, the following are equivalent:

(a) $_RM$ *is noetherian and a QF module;*

(b) $_RM$ *is noetherian and a cogenerator in* $\sigma[M]$;

(c) $_RM$ *is noetherian, M-injective and a self-generator;*

(d) $_RM$ *is a cogenerator in* $\sigma[M]$ *and* S_S *is artinian;*

(e) $_RM$ *is artinian and cogenerator in* $\sigma[M]$;

(f) $_RM$ *has finite length and injective hulls of simple modules in* $\sigma[M]$ *are projective in* $\sigma[M]$;

(g) *every injective module in* $\sigma[M]$ *is projective in* $\sigma[M]$;

(h) $_RM$ *is a self-generator, and projectives are injective in* $\sigma[M]$;

(i) $_RM^{(\mathbb{N})}$ *is an injective cogenerator in* $\sigma[M]$;

(j) $_RM$ *is perfect and (weakly) M-injective modules are flat in* $\sigma[M]$.

Proof: $(a) \Leftrightarrow (b)$ As noted above, a noetherian QF module is an M-injective cogenerator in $\sigma[M]$ and the assertion follows from 48.10.

$(a) \Rightarrow (c)$ follows from 48.11.

$(c) \Rightarrow (d)$ Because of the equivalence $Hom(M,-) : \sigma[M] \to S\text{-}MOD$, the module $_SS$ is noetherian and S-injective. Since $M^{(\mathbb{N})}$ is M-injective by 31.12, S is semiprimary and, by 31.4, $_SS$ is artinian. Hence $_RM$ is artinian, thus finitely cogenerated and, by 48.11, $_RM$ is a cogenerator in $\sigma[M]$. Now, by 47.13, $Hom_R(-,M)$ determines a duality between the finitely generated modules in $\sigma[M]$ and MOD-S, and S_S has finite length (is artinian).

$(d) \Rightarrow (b)$ Since $_RM$ is a cogenerator, $KeAn(K) = K$ holds for every submodule $K \subset {_RM}$ (see 28.1). The descending chain condition for submodules of the type $An(K) \subset S_S$ yields the ascending chain condition for submodules $K \subset_R M$, i.e. $_RM$ is noetherian.

$(d) \Rightarrow (e)$ S_S being an artinian module, it is also noetherian and the assertion is obtained as in $(d) \Rightarrow (b)$.

$(e) \Rightarrow (b)$ By 48.10, $_RM$ is a generator in $\sigma[M]$. Therefore we have an equivalence $Hom_R(M,-) : \sigma[M] \to S\text{-}MOD$ and $_SS$ is artinian, hence noetherian and $_RM$ is also noetherian.

$(a) \Rightarrow (f)$ By $(a) \Leftrightarrow (e)$, $_RM$ has finite length. The injective hulls of the simple modules are direct summands of $_RM$ and hence M-projective.

$(f) \Rightarrow (g)$ Since M has finite length, by 32.5, injective modules in $\sigma[M]$ are direct sums of injective hulls of simple modules which, by (f), are projective in $\sigma[M]$. Hence injectives are projective in $\sigma[M]$.

$(g) \Rightarrow (b)$ If (g) holds, every injective module (being M-generated) is a direct summand of some $M^{(\Lambda)}$. Hence every module in $\sigma[M]$ is a submodule of a sum $M^{(\Lambda)}$ and M is a cogenerator in $\sigma[M]$. Also, by Kaplansky's Theorem 8.10, every injective module in $\sigma[M]$ is a direct sum of countably generated modules and, by 27.5, M is noetherian.

$(c) \Rightarrow (h)$ Every projective module in $\sigma[M]$ is a direct summand of an injective module $M^{(\Lambda)}$, for suitable Λ (see 27.3).

$(h) \Rightarrow (i)$ Since, in particular, $M^{(\mathbb{N})}$ is M-injective, S is a semiprimary ring (see 28.4, 31.12). Hence in M, the descending chain condition for M-cyclic submodules holds (see 43.10). Since M is a self-generator this means $Soc\, M \trianglelefteq M$ and, by 48.11, M is a cogenerator in $\sigma[M]$. Therefore $M^{(\mathbb{N})}$ is an injective cogenerator in $\sigma[M]$.

$(i) \Rightarrow (a)$ follows from 28.4, $(j) \Rightarrow (g)$ from 43.8.

$(d) \Leftrightarrow (j)$ Since S is a (left) perfect ring, ${}_RM$ is a perfect module in $\sigma[M]$. Then M-generated flat modules are projective in $\sigma[M]$ (see 43.8).

In the characterizations of QF modules considered in 48.14 we always assumed the module M to be self-projective. It is interesting to observe that projectivity can also be deduced from other properties:

48.14 Self-projective, noetherian *QF* modules, II.

For an R-module M and $S = End_R(M)$, the following are equivalent:

(a) ${}_RM$ is a noetherian, injective generator in $\sigma[M]$;

(b) ${}_RM$ is an artinian, projective cogenerator in $\sigma[M]$;

(c) ${}_RM$ is a noetherian, projective cogenerator in $\sigma[M]$;

(d) ${}_RM$ is an injective generator in $\sigma[M]$ and ${}_SS$ is artinian.

Proof: $(a) \Rightarrow (b)$ We know from 31.12 that S is semi-primary and hence, by 51.12, there is a finitely generated projective generator M' in $\sigma[M]$. Since M' is M-generated, it is a direct summand of a finite direct sum of copies of M. Hence M' is also injective in $\sigma[M]$. So by 48.14.I., M' is an artinian cogenerator in $\sigma[M'] = \sigma[M]$ and it follows that M is also artinian and cogenerator in $\sigma[M]$. This implies that M is a direct summand of a finite direct sum of copies of M' and hence is projective in $\sigma[M]$.

$(b) \Rightarrow (c)$ If $\sigma[M]$ has a projective cogenerator with finitely generated socle, then every cogenerator in $\sigma[M]$ is a generator by 48.10. In particular, M is an artinian generator in $\sigma[M]$ and hence is noetherian (see 32.8).

$(b) \Rightarrow (a)$ follows from the proof of $(b) \Rightarrow (c)$ (using 48.10,(2)).

$(a) \Rightarrow (d)$ is clear from the above implications since $Hom_R(M, -)$ defines an equivalence between $\sigma[M]$ and *S-MOD*.

$(d) \Rightarrow (a)$ As a left artinian ring, S is left noetherian. It is obvious that, for a generator M, *acc* on left ideals in S implies *acc* on submodules of M.

Applied to $M = R$ the preceding propositions yield characterizations of noetherian QF rings in our terminology (often just called 'QF rings'):

48.15 Noetherian QF rings. Characterizations.

For a ring R the following properties are equivalent:

(a) ${}_RR$ is noetherian and a QF module;

(b) ${}_RR$ is noetherian and a cogenerator in R-MOD;

(c) ${}_RR$ is noetherian and injective;

(d) ${}_RR$ is a cogenerator in R-MOD and R_R is noetherian;

(e) ${}_RR$ is a cogenerator in R-MOD and R_R is artinian;

(f) ${}_RR$ is artinian and a cogenerator in R-MOD;

(g) ${}_RR$ is artinian, and injective hulls of simple modules are projective;

(h) every injective module is projective in R-MOD;

(i) every projective module is injective in R-MOD;

(j) $R^{(\mathbb{N})}$ is an injective cogenerator in R-MOD;

(k) ${}_RR$ is perfect and every FP-injective module is flat;

(l) R_R is noetherian and a QF module.

Proof: $(d) \Rightarrow (b)$ If ${}_RR$ is a cogenerator, R_R is FP-injective by 47.7, hence injective in *MOD-R*, since R_R is noetherian. By 47.8, ${}_RR$ is linearly compact and, by 47.5, there is an order reversing bijection between left ideals and right ideals of R. Hence ${}_RR$ is artinian and therefore noetherian.

All other implications follow from 48.14.

For cogenerators with commutative endomorphism rings the results in 48.11 can be refined. This applies in particular to any unital ring R considered as (R, R)-bimodule, since in this case the endomorphism ring is isomorphic to the center. For commutative rings we will obtain an improvement of 48.12.

Consider a module $M = M_1 \oplus M_2$ and the idempotent e in $S = End(M)$ defined by the projection $M \to M_1$. For every $0 \neq t \in Hom(M_1, M_2)$, considered as element in S, we have $0 \neq t = et \neq te = 0$. Hence, if S is commutative, we have $Hom(M_1, M_2) = 0$. This simple observation is crucial for our next proof.

Recall that an R-submodule of M is said to be *fully invariant* if it is also an $End(M)$-submodule. As an example for the following result one may take the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ or any submodule of it:

48.16 Cogenerator with commutative endomorphism ring.

Let M be an R-module and assume $S = End_R(M)$ to be commutative. Choose $\{E_\lambda\}_\Lambda$ as a minimal representing set of simple modules in $\sigma[M]$ and denote by $\widehat{E}_\lambda$ the injective hull of E_λ in $\sigma[M]$. Then the following statements are equivalent:

(a) M is a cogenerator in $\sigma[M]$;

(b) M is self-injective and self-cogenerator;

(c) $M \simeq \bigoplus_\Lambda \widehat{E}_\lambda$;

(d) M is a direct sum of indecomposable modules N which are cogenerators in $\sigma[N]$.

Under these conditions we have:

(1) Every R-submodule of M is fully invariant and hence self-injective and self-cogenerator.

(2) For every $\lambda \in \Lambda$, the category $\sigma[\widehat{E}_\lambda]$ contains only one simple module (up to isomorphism).

(3) If the $\widehat{E}_\lambda$'s are finitely generated R-modules, then M generates all simple modules in $\sigma[M]$.

(4) If M is projective in $\sigma[M]$, then M is a generator in $\sigma[M]$.

(5) If M is finitely generated, then M is finitely cogenerated.

Proof: By 17.12, a module is a cogenerator in $\sigma[M]$ if and only if it contains a copy of an injective hull for every simple module in $\sigma[M]$. Of course, $(b) \Rightarrow (a)$ and $(c) \Rightarrow (a)$ are trivial.

$(a) \Rightarrow (c)$ Set $K = \bigoplus_\Lambda \widehat{E}_\lambda$. Since M is a cogenerator in $\sigma[M]$, we have $K \subset M$. Assume $K \neq M$. M cogenerates M/K, and hence there is a $0 \neq t \in S$ with $Kt = 0$. Since M is also cogenerated by K,

$$Re(M,K) = \bigcap \{Ke\, f \mid f \in Hom(M,K)\} = 0.$$

Considering $Hom(M,K)$ as subset of S, we obtain, by the commutativity of S, $tHom(M,K) = Hom(M,K)t = 0$.

This means $Mt \subset Re(M,K) = 0$ and $t = 0$, a contradiction.

$(c) \Rightarrow (b)$ We have to prove that M is self-injective. By 16.2, it is enough to show that M is $\widehat{E}_\lambda$-injective for every $\lambda \in \Lambda$.

Let $U \subset \widehat{E}_\lambda$ be a submodule and $f \in Hom(U,M)$. Since $Hom(\widehat{E}_\lambda, \widehat{E}_\mu) = 0$ for all $\lambda \neq \mu \in \Lambda$, we find $Uf \subset \widehat{E}_\lambda$, i.e. we have the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & U & \longrightarrow & \widehat{E}_\lambda \\ & & \downarrow f & & \\ & & \widehat{E}_\lambda & \subset & M \end{array} ,$$

which can be completed in the desired way by injectivity of $\widehat{E}_\lambda$.

$(c) \Rightarrow (d)$ $\widehat{E}_\lambda$ is cogenerator in $\sigma[\widehat{E}_\lambda]$ (property (2)).

$(d) \Rightarrow (a)$ Assume $M = \bigoplus_A N_\alpha$, with N_α indecomposable and cogenerator in $\sigma[N_\alpha]$. Since $End(N_\alpha)$ is commutative, we know from $(a) \Rightarrow (b)$ above that N_α is self-injective and has simple socle. With the proof $(c) \Rightarrow (b)$ we see that M is M-injective.

Since any simple module in $\sigma[M]$ is isomorphic to a simple module in one of the $\sigma[N_\alpha]$'s, it has to be isomorphic to the socle of one of the N_α's and hence is cogenerated by M. Thus M is a cogenerator in $\sigma[M]$.

Now let us prove the properties indicated:

(1) We may assume M to be a faithful R-module. By the Density Theorem, we know that R is dense in $B = End(M_S)$ and that the R-submodules of M are exactly the B-submodules (see 15.7, 15.8). Since the commutative ring S can be considered as subring of B, we conclude that R-submodules of M are also S-submodules. By 17.11, fully invariant submodules of self-injective modules are again self-injective. It is easily checked that fully invariant submodules of self-cogenerators are again self-cogenerators.

(2) The socle of $\widehat{E}_\lambda$ is simple and, by (1), every simple module in $\sigma[\widehat{E}_\lambda]$ has to be isomorphic to it.

(3) If $\widehat{E}_\lambda$ is finitely generated, it has a maximal submodule $V_\lambda \subset \widehat{E}_\lambda$, and $\widehat{E}_\lambda/V_\lambda$ has to be isomorphic to E_λ. Hence every simple module E_λ is isomorphic to a factor module of M.

(4) If M is projective in $\sigma[M]$, then the $\widehat{E}_\lambda$ are also projective in $\sigma[M]$. Since they have local endomorphism rings, they are (M-)projective covers of simple modules in $\sigma[M]$ and hence cyclic (see 19.7). Now, by (3), M generates all simple modules in $\sigma[M]$ and hence is a generator in $\sigma[M]$.

(5) This is easily seen from (c).

As a special case of the preceding theorem we state (compare 48.12):

48.17 Commutative PF rings.

For a commutative ring R with unit, the following are equivalent:

(a) R is a cogenerator in R-MOD;

(b) R is an injective cogenerator in R-MOD;

(c) R is injective and finitely cogenerated.

48.18 Exercises.

(1) Prove that for a ring R the following assertions are equivalent:

(a) R is an IF ring (see 48.8);

(b) ${}_RR$ and R_R are coherent, and, for finitely generated left ideals I and right ideals J, the 'double annihilator conditions' hold (notation as in 2.2):

$$I = An^l An^r(I), \quad J = An^r An^l(J) \ .$$

(2) Prove that for a ring R the following assertions are equivalent:
(a) Every finitely presented module in R-MOD is a submodule of a free module;
(b) every injective module is flat in R-MOD;
(c) the injective hulls of the finitely presented modules are flat in R-MOD.
Rings with these properties are called left IF rings.

(3) Let R be a left IF ring (see (2)) and $T \subset R$ a subring with the properties: ${}_TR$ is flat and T_T is a direct summand in R_T.
Prove that T is also a left IF ring.

(4) An R-module is called small *if it is a superfluous submodule in some R-module. Prove that for the ring R the following are equivalent:*
(a) R is a noetherian QF ring (see 48.15);
(b) every module in R-MOD is a direct sum of a projective and a superflous module (see Rayar [1]);
(c) R_R is perfect, ${}_RR$ is coherent and FP-injective;
(d) for every free R-left module F, the endomorphism ring $End(F)$ is left (FP-) injective (see Menal [2]).

(5) In 36.8, exercise (11), semi-flat *R-modules are defined. Prove that for a ring R the following assertions are equivalent (Hauptfleisch-Döman):*
(a) Every injective module in R-MOD is (semi-) flat;
(b) every module in R-MOD is semi-flat;
(c) every finitely presented module in R-MOD is semi-flat;
(d) every module in R-MOD is a submodule of a flat module.

*(6) An R-module M is called an R-*Mittag-Leffler module *if, for every index set Λ, the canonical map $R^\Lambda \otimes_R M \to M^\Lambda$ (see 12.9) is monic. Prove that in R-MOD the following assertions are equivalent:*
(a) Every finitely generated module is a submodule of a finitely presented module;
(b) every injective module is an R-Mittag-Leffler module;
(c) the injective hulls of the finitely generated modules are R-Mittag-Leffler modules. (Jones)

(7) Let R be a finite dimensional algebra over the field K.
Regard $R^* = Hom_K(R, K)$ in the canonical way as a right R-module.

(α) *Prove: (i)* $R/Jac(R) \simeq SocR^*$*;*
(ii) R^* *is an injective hull of* $R/Jac(R)$ *in MOD-R;*
(iii) R^* *is an injective cogenerator in MOD-R.*

(β) *R is called a* Frobenius algebra *if* $R_R \simeq R^*$ *in MOD-R. Prove:*
(i) If $Jac(R) = 0$*, then R is a Frobenius algebra ;*
(ii) for a finite group G, the group ring KG is a Frobenius algebra over K;
(iii) if R is a Frobenius algebra, then R is a noetherian QF ring.

Literature: DROZD-KIRICHENKO, FAITH [2], HARADA, KASCH, RENAULT, STENSTRÖM, TACHIKAWA;
Albu-Wisbauer, Bican, Birkenmeier [2,3], Brodskii [2], Brodskii-Grigorjan, Chatters-Hajarnavis, Cheatham-Enochs [1], Colby, Colby-Fuller [1,2,3], Colby-Rutter [1], Couchot [3,5,6,7], Damiano [1], Dischinger-Müller [2], Enochs-Jenda, Faith [1,3], Franzsen-Schultz, Garcia-Gomez [3], Gomez [2], Gomez-Martinez, Gomez-Rodriguez [1,2], Grigorjan, Harada [1,3,4,5,6], Hauger-Zimmermann, Hauptfleisch-Döman, Jain, Johns [2], Jones, Kasch-Pareigis, Kato [1], Kirichenko-Lebed, Kitamura, Kraemer, Lemonnier, Macdonald, Martin, Masaike [1,2], Matlis [2], Menal [1,2], Menini [1], Menini-Orsatti [1], Miller-Turnidge [1], Miyashita, Müller [1,3], Okninski, Onodera [1,2,6,7], Oshiro [4], Page [1,2,3], Popescu, Rayar [1], Ringel-Tachikawa, Roux [1,2,4], Rutter, Skornjakov, Tachikawa, Tsukerman, Tuganbaev [5,7,8], Wisbauer [7,14], Würfel [1], Xu Yan, Yamagata [2], Zelmanowitz [5].

Chapter 10

Functor Rings

Investigating rings and modules the study of certain functor rings turned out to be useful. In particular the category of functors from the finitely generated (or finitely presented) modules in $\sigma[M]$ to abelian groups is of considerable interest.

This category can be viewed as a category over a suitable ring T without unit but with enough idempotents. In the next paragraph we will develop the theory of these rings and their modules.

Then we will study the functors $\widehat{Hom}(V, -)$ (§ 51) which will provide a connection between $\sigma[M]$ and the T-modules in § 52. Thereby we get effective methods to study pure semisimple rings and rings of finite representation type.

49 Rings with local units

1.T-MOD for T with local units. 2.Special objects in T-MOD. 3.Canonical isomorphisms in T-MOD. 4.Pure sequences in T-MOD. 5.Flat modules in T-MOD. 6.The Jacobson radical of T. 7.Nakayama's Lemma for T. 8.t-nilpotent ideals and superfluous submodules in T-MOD. 9.Left perfect rings T. 10.Semiperfect rings T. 11.Exercises.

Let T be an associative ring (not necessarily with unit). We call T a *ring with local units* if for any finitely many $a_1, \dots, a_k \in T$ there exists an idempotent $e \in T$ with $\{a_1, \dots, a_k\} \subset eTe$. For such rings $T^2 = T$ holds.

We say that *T has enough idempotents*, if there exists a family $\{e_\alpha\}_A$ of pairwise orthogonal idempotents $e_\alpha \in T$ with $T = \bigoplus_A e_\alpha T = \bigoplus_A T e_\alpha$. In this case $\{e_\alpha\}_A$ is called a *complete family* of idempotents in T.

A ring T with enough idempotents is a ring with local units:
For $a_1, \ldots, a_k \in T$ there are finite subsets E, $F \subset A$ with $a_i \in \bigoplus_E Te_\alpha$ and $a_i \in \bigoplus_F e_\alpha T$ for $i = 1, \ldots, k$. With the idempotent $e = \sum_{\alpha \in E \cup F} e_\alpha$ we have $\{a_1, \ldots, a_k\} \subset eTe$.

If T has a unit e, then $\{e\}$ is a complete family of idempotents. On the other hand, $e_1 + \cdots + e_n$ is a unit in T if *finitely many* $e_1, \ldots, e_n$ form a complete family of idempotents in T.

We shall encounter rings with enough idempotents mainly as subrings of endomorphism rings (§ 51).

A ring T without unit is not necessarily a generator for all T-modules. Looking for relations between properties of T and T-modules it makes sense to restrict to 'suitable' T-modules, namely the submodules of ${}_TT$-generated modules:

For T we construct the Dorroh overring T^* with unit (see 1.5). Then T is a unitary left module over T^*, and the left ideals of T are exactly the T^*-submodules of T (see 6.3). We can now consider the categories $\sigma[{}_{T^*}T]$ and $\sigma[T_{T^*}]$.

Recalling results shown for categories of the type $\sigma[M]$ we can develop in this context a homological characterization for arbitrary rings T without unit, where T in general is neither projective nor a generator in $\sigma[{}_{T^*}T]$. However we will concentrate our interest on rings T with local units which, of course, have special properties. In this case we denote by *T-MOD* the category $\sigma[{}_{T^*}T]$ and by *MOD-T* the category $\sigma[T_{T^*}]$. For rings with unit these are the usual categories of unitary modules.

49.1 *T-MOD* for *T* with local units.

(1) Let T be a ring with local units. Then

(i) $\{Te \mid e^2 = e \in T\}$ is a generating set of finitely generated, projective modules in T-MOD;

(ii) T is flat and a generator in T-MOD;

(iii) for every T-module N the equality $Tr(T, N) = TN$ holds;

(iv) a T-module N is in T-MOD if and only if $TN = N$;

(v) if $N \in$ T-MOD then for finitely many $n_1, \ldots, n_k \in N$ there exists an idempotent $e \in T$ with $en_i = n_i$ for $i = 1, \ldots, k$.

(2) A ring T with enough idempotents is a projective generator in T-MOD (and MOD-T).

Proof: $(1)(i)$ For an idempotent $e \in T$ we consider the following diagram in *T-MOD* with exact row

$$\begin{array}{ccccc} & & Te & & \\ & & \downarrow f & & \\ T & \xrightarrow{g} & N & \longrightarrow & 0 \end{array} .$$

Choosing an element $c \in T$ with $(c)g = (e)f$, we get by $te \mapsto tec$ a morphism $Te \to T$ which yields a commutative diagram. Therefore Te is T-projective.

For every $a \in T$ there exists an idempotent $e \in T$ with $a = ea$, and we have an epimorphism $Te \to Ta$, $te \mapsto ta$. So the modules of the form Te generate all submodules of ${}_TT$ and therefore all simple modules in *T-MOD*. By 18.5, $\bigoplus\{Te \mid e^2 = e \in T\}$ – and also ${}_TT$ – is a generator in *T-MOD*.

(ii) For any idempotents e_1, $e_2 \in T$ there exists an idempotent $e_3 \in T$ with $Te_1 \subset Te_3$, $Te_2 \subset Te_3$. Hence ${}_TT$ is the direct limit of the projective modules $\{Te\}$ and ${}_TT$ is flat in *T-MOD* by 36.2.

(iii) The morphism $T^{(N)} \to TN$, $(t_n) \mapsto \sum t_n n$, is surjective for every T-module N. This implies $TN \subset Tr(T,N)$.

On the other hand, we have an epimorphism $h : T^{(\Lambda)} \to K$ for every T-generated submodule $K \subset N$. Using $T^2 = T$ we have $K = (TT^{(\Lambda)})h = TK$ and $Tr(T,N) \subset TN$.

(iv) Since T is a generator in *T-MOD*, the assertion follows from (iii).

(v) Because $N = TN$, we have $n = t_1n_1 + \cdots + t_rn_r$, with $t_i \in T$, $n_i \in N$, for every $n \in N$. For an idempotent $e \in T$ with $et_i = t_i$ $(i = 1, \ldots, r)$ we also have $en = n$.

(2) follows directly from (1)(i).

We call a T-module *quasi-free* if it is isomorphic to a direct sum of modules of the form Te with $e^2 = e \in T$. With this definition there are analogous results as for free modules over rings with unit:

49.2 Special objects in *T-MOD*.
Let T be a ring with local units:

(1) A T-module is in T-MOD if and only if it is an image of a quasi-free T-module.

(2) A module in T-MOD is finitely generated if and only if it is an image of a finitely generated, quasi-free T-module.

(3) A module in T-MOD is (finitely generated and) projective in T-MOD if and only if it is a direct summand of a (finitely generated) quasi-free T-module.

(4) A module N in T-MOD is finitely presented in T-MOD if and only if there exists an exact sequence $L_1 \to L_0 \to N \to 0$ with L_0, L_1 finitely generated and quasi-free.

(5) Every module in T-MOD is a direct limit of finitely presented modules in T-MOD.

(6) For a family $\{N_\lambda\}_\Lambda$ of modules in T-MOD, the product in T-MOD is

$$\prod\nolimits_\Lambda^T N_\lambda = Tr(T, \prod\nolimits_\Lambda N_\lambda) = T \cdot \prod\nolimits_\Lambda N_\lambda,$$

with $\prod_\Lambda N_\lambda$ denoting the cartesian product.

Proof: The statements (1), (2) and (3) follow from the fact that $\{Te \,|\, e^2 = e \in T\}$ is a generating set of finitely generated, projective modules in *T-MOD* (see 49.1).

(4) follows from (2) and the properties of finitely presented modules in *T-MOD* (see 25.1).

(5) Since every module in *T-MOD* is generated by finitely presented modules, we get the assertion by 25.3.

(6) The first equality is given by the description of the product in *T-MOD* ($= \sigma[_{T^*}T]$, see 15.1), the second follows from the characterization of the trace of T in 49.1.

Furthermore, in the case under consideration, we have important isomorphisms which are well-known for rings with unit:

49.3 Canonical isomorphisms in *T-MOD*.

Let T be a ring with local units, S a ring with unit, K a (T,S)-bimodule.

(1) The map $Hom_T(Te, K) \to eK$, $f \mapsto (e)f$, is an S-isomorphism for every idempotent $e \in T$.

(2) The map $\mu_T : T \otimes_T K \to TK$, $t \otimes k \mapsto tk$, is a (T,S)-isomorphism.

(3) The map $\mu_{eT} : eT \otimes_T K \to eK$, $et \otimes k \mapsto etk$, is an S-isomorphism for every idempotent $e \in T$.

(4) The functor $F \otimes_T - : T\text{-}MOD \to AB$ is exact for every projective module F in MOD-T.

(5) For $P \in T$-MOD and $L \in$ MOD-S, the map

$$\lambda_P : Hom_S(K, L) \otimes_T P \to Hom_S(Hom_T(P, K), L),\ f \otimes p \mapsto [g \mapsto f((p)g)],$$

is a $\mathbb{Z}$-isomorphism (functorial in P) whenever

(i) P is finitely generated and projective in T-MOD, or

(ii) P is finitely presented in T-MOD and L is K_S-injective.

Proof: (1) Obviously the map is an injective S-morphism.

For every $ek \in eK$, the map $f : Te \to K$, $te \mapsto tek$, is a T-morphism with $(e)f = ek$. Hence the map in (1) is also surjective.

(2) is shown in 12.6. (3) follows by (2) and the equality $eTK = eK$.

(4) With the isomorphism in (2) the functor $T \otimes_T - : T\text{-}MOD \to AB$ is exact. By 12.14, we have that the functor $F \otimes_T -$ is exact for all projective modules $F \in MOD\text{-}T$.

(5) Considering (1) and (3), we find that the given map λ_P, for $P = Te$ with $e^2 = e \in T$, is an isomorphism. Since, for every finitely presented module P in T-MOD, there exists an exact sequence $L_o \to L_1 \to P \to 0$, with L_o, L_1 finitely generated and quasi-free, we can follow the proof of 25.5,(1) for rings with unit.

Using the above relations, we can now characterize the pure sequences in T-MOD in the same way as in R-MOD. Again denote $\overline{\mathbb{Q}} = \mathbb{Q}/\mathbb{Z}$.

49.4 Pure sequences in *T-MOD*. Characterizations.
Let T be a ring with local units. For a short exact sequence in T-MOD

$$(*) \qquad 0 \to K \to L \to N \to 0\,,$$

the following statements are equivalent:

(a) The sequence $(*)$ is pure in T-MOD;

(b) the sequence $0 \to F \otimes_T K \to F \otimes_T L \to F \otimes_T N \to 0$ is exact for
  (i) every finitely presented module F in MOD-T, or
  (ii) every module F in MOD-T, or
  (iii) every right T-module F;

(c) the sequence $0 \to Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(L,\overline{\mathbb{Q}}) \to Hom_{\mathbb{Z}}(K,\overline{\mathbb{Q}}) \to 0$
  (i) remains exact under $- \otimes_T P$, P finitely presented in T-MOD, or
  (ii) splits as a sequence of right T-modules;

(d) every finite system of equations over K which is solvable in L is solvable in K;

(e) if equations $\sum_{j=1}^{k} a_{ij}X_j = m_i$, $i = 1, \ldots, n$, $a_{ij} \in T$, $m_i \in K$, $n, k \in \mathbb{N}$, have a solution in L, then they also have a solution in K;

(f) for every commutative diagram

$$\begin{array}{ccccc} & & L_o & \xrightarrow{g} & L_1 \\ & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & L \end{array}$$

  with L_o, L_1 finitely generated and quasi-free, there exists $h : L_1 \to K$ with $f = gh$;

(g) the sequence $(*)$ is a direct limit of splitting sequences.

Proof: Using the isomorphisms given in 49.3, we can follow the proof of 34.5. It only remains to show:

$(b)(ii) \Rightarrow (iii)$ Let F be an arbitrary right T-module. Then we have $FT \in MOD\text{-}T$ (see 49.1). The assertions follows from the isomorphism

$FT \otimes_T N \simeq F \otimes_T T \otimes_T N \simeq F \otimes_T N$, for any $N \in T\text{-}MOD$.

The characterization of pure sequences leads to a description of flat modules in T-MOD corresponding to the situation for rings with unit:

49.5 Flat modules in T-MOD. Characterizations.
Let T be a ring with local units.

(1) For $N \in T\text{-}MOD$ the following assertions are equivalent:

(a) N is flat in T-MOD (def. before 36.1);

(b) the functor $- \otimes_T N : MOD\text{-}T \to AB$ is exact;

(c) the functor $- \otimes_T N$ is exact on exact sequences of the form $0 \to J_T \to T_T$ (with J_T finitely generated);

(d) $J \otimes N \to JN$, $i \otimes n \mapsto in$, is monic (an isomorphism) for every (finitely generated) right ideal $J \subset T$;

(e) N is a direct limit of (finitely generated) projective (quasi-free) modules in T-MOD;

(f) $Hom_T(P,T) \otimes_T N \to Hom_T(P,N)$, $h \otimes n \mapsto [p \mapsto (p)h \cdot n]$, is epic for every finitely presented module $P \in T\text{-}MOD$;

(g) $Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}})_T$ is (weakly) T_T-injective.

(2) If $0 \to K \to L \to N \to 0$ is a pure exact sequence in T-MOD, then L is flat if and only if K and N are flat in T-MOD.

Proof: (1) The equivalence of (b), (c) and (d) is given by 12.15 and 12.16. $(a) \Leftrightarrow (e)$ has been shown in 36.2.

$(e) \Rightarrow (b)$ The functor $- \otimes_T N : MOD\text{-}T \to AB$ is exact for every projective module N in T-MOD (notice change of sides). Since tensor products commute with direct limits (see 24.11), $- \otimes_T N$ is also exact if N is a direct limit of projective modules.

$(b) \Rightarrow (f)$ With the isomorphisms given in 49.3 it is easy to see that the map considered is an isomorphism for $P = Te$ ($e^2 = e \in T$). Then this is also true for finitely generated quasi-free T-modules, and finally it can be shown for finitely presented modules in T-MOD (see 25.5,(2)).

$(f) \Rightarrow (a)$ To prove this we only have to transfer the proof of the corresponding assertion for rings with unit (see 36.5,$(f) \Rightarrow (a)$).

$(b) \Leftrightarrow (g)$ Let $0 \to J \to T_T \to V \to 0$ be an exact sequence in $MOD\text{-}T$. Using the functor $Hom_T(-, Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}}))$ and the canonical isomorphisms given in 12.12 (which are also true in the case of rings without unit) we get the commutative diagram (see proof of 34.6)

$$\begin{array}{ccccccccc}
0 \to & Hom(V, Hom(N,\overline{\mathbb{Q}})) & \to & Hom(T, Hom(N,\overline{\mathbb{Q}})) & \to & Hom(J, Hom(N,\overline{\mathbb{Q}})) & \to 0 \\
& \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & \\
0 \to & Hom(V \otimes_T N,\overline{\mathbb{Q}}) & \to & Hom(T \otimes_T N,\overline{\mathbb{Q}}) & \to & Hom(J \otimes_T N,\overline{\mathbb{Q}}) & \to 0.
\end{array}$$

If (b) holds, then the bottom row is exact and therefore the top row is also exact, i.e. $Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}})_T$ is T_T-injective.

On the other hand, if $Hom_{\mathbb{Z}}(N,\overline{\mathbb{Q}})_T$ is weakly T_T-injective (= absolutely pure in *MOD-T*, see 35.4), and if, in the above sequence, J_T is finitely generated, then the upper row in the diagram is exact. Hence the lower row is also exact implying the exactness of $0 \to J \otimes_T N \to T \otimes_T N$ ($\overline{\mathbb{Q}}$ is a cogenerator in $\mathbb{Z}$-*MOD*, see 14.6). So (c) holds and therefore (d) is also true (see above).

(2) Because of more general assertions in 36.1, we only have to show that K is flat whenever L is flat: For a right ideal $J \subset T$ we have the commutative diagram with exact row

$$\begin{array}{ccccc}
0 & \longrightarrow & J \otimes K & \longrightarrow & J \otimes L \\
 & & \downarrow \mu_J & & \downarrow \simeq \\
 & & JK & \longrightarrow & JL
\end{array} .$$

Hence μ_J is monic and K is flat by (1),(d).

In 21.16 characterizations of the Jacobson radical of rings without unit are given. The rings considered in this chapter also allow well-known descriptions of the Jacobson radical of rings with unit:

49.6 The Jacobson radical of T.

Let T be a ring with local units.

(1) The radical $Jac(T)$ can be characterized as

(a) $\bigcap\{An(E) \mid E$ *a simple module in T-MOD*$\}$;

(b) $\bigcap\{K \subset T \mid K$ *a maximal left ideal in* $T\}$ (= $Rad_T T$);

(c) the sum of all superfluous left ideals in T;

(d) the largest left quasi-regular left ideal in T;

(e) the largest quasi-regular ideal in T;

(a)* $\bigcap\{An(E) \mid E$ *a simple module in MOD-T*$\}$.

Analogously the right hand versions of (b), (c) and (d) are true.

(2) $T/Jac(T)$ is left semisimple if and only if it is right semisimple.

(3) $Jac(T) = TJac(End(_TT))$.

Proof: (1) Except of (c) all characterizations are given by 21.16, because all simple T-modules E are in T-*MOD* (notice $TE = E$), and every maximal left ideal in T is modular.

Since $Jac(T) = Rad(_TT)$ we obtain (c) by 21.5. Notice that $Jac(T)$ is not necessarily superfluous in $_TT$.

(2) Since $T/Jac(T)$ has also local units we may assume $Jac(T) = 0$. If T is left semisimple, then $T = \sum_\Lambda Te_\lambda = \sum_\Lambda e_\lambda T$ with idempotents e_λ and simple modules Te_λ.

By Schur's Lemma, $End(e_\lambda T) \simeq e_\lambda T e_\lambda \simeq End(Te_\lambda)$ is a division ring. So the projective right module $e_\lambda T$ has a semiperfect endomorphism ring and therefore is semiperfect by 42.12.

Since $e_\lambda T$ is indecomposable and $Rad(e_\lambda T) \subset Jac(T) = 0$, $e_\lambda T$ is simple and $T = \sum_\Lambda e_\lambda T$ is right semisimple.

(3) Let $S = End(_TT)$. We may assume $T \subset S$ and $Jac(T)$ is an S-submodule of $T_S \subset S$ and hence a quasi-regular right ideal in S (see (1)). This implies $Jac(T) \subset TJac(S)$.

On the other hand, for every $a \in TJac(S) \subset Jac(S)$ there exists $b \in S$ with $b+a-ba = 0$. Then, for an idempotent $e \in T$ with $ea = a$, the relation $eb + a - (eb)a = 0$ holds with $eb \in T$. So $TJac(S)$ is a quasi-regular left ideal in T and, by (1), $TJac(S) \subset Jac(T)$.

The following two propositions, which turned out to be so useful for rings with unit, are now true in almost the same form:

49.7 Nakayama's Lemma for T.

For a left ideal I in a ring T with local units, the following are equivalent:

(a) $I \subset Jac(T)$;

(b) $IN \neq N$ *for every finitely generated module* $0 \neq N \in T$-*MOD;*

(c) $IN \ll N$ *for every finitely generated module* $0 \neq N \in T$-*MOD;*

(d) $Ie \ll Te$ *for every idempotent* $e \in T$.

Proof: Since $Jac(T)$ annihilates all simple modules in T-*MOD*, the relation $Jac(T)(N/Rad(N)) = 0$ holds for every $N \in T$-*MOD*, implying $Jac(T)N \subset Rad(N)$. Hence we have $(a) \Rightarrow (b) \Rightarrow (c)$ as in 21.13.

$(c) \Rightarrow (d)$ This follows from $Ie \subset ITe \ll Te$.

$(d) \Rightarrow (a)$ If $Ie \ll Te$ is true for every idempotent $e \in T$, then we have $Ie \subset Jac(T)$ (see 49.6,(c)) and therefore $I \subset \sum\{Ie \mid e^2 = e \in T\} \subset Jac(T)$.

49.8 t-nilpotent ideals and superfluous submodules.

For a left ideal I in a ring T with local units, the following are equivalent:

(a) I is right t-nilpotent;

(b) $IN \neq N$ for every module $0 \neq N \in T$-MOD;

(c) $IN \ll N$ for every module $0 \neq N \in T$-MOD;

(d) $I\,T^{(\mathbb{N})} \ll T^{(\mathbb{N})}$.

Proof: The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are obtained by using the corresponding proofs in 43.5.

$(d) \Rightarrow (a)$ Let $\{s_i\}_{\mathbb{N}}$ be a family of elements of I. Multiplying from the right hand side we get a family of T-morphisms $\{s_i : T \to T\}_{\mathbb{N}}$ with

$$\bigoplus_{\mathbb{N}} T s_i \subset I^{(\mathbb{N})} \subset I\,T^{(\mathbb{N})} \ll T^{(\mathbb{N})}.$$

By 43.3, there exists $r \in \mathbb{N}$ with $ts_1 \cdots s_r = 0$ for every $t \in T$. Choosing t as an idempotent with $ts_1 = s_1$ (by 49.1), we have $ts_1 \cdots s_r = s_1 \cdots s_r = 0$ for some $r \in \mathbb{N}$. So I is right t-nilpotent.

The properties of t-nilpotent ideals just proved allow a description of perfect rings with enough idempotents which will be useful in § 53:

49.9 Left perfect rings T. Characterizations.

For a ring T with enough idempotents the following are equivalent:

(a) ${}_TT$ is perfect in T-MOD (T is left perfect, def. § 43);

(b) every module has a projective cover in T-MOD;

(c) every module in T-MOD is (amply) supplemented (see 41.6);

(d) every flat module is projective in T-MOD;

(e) every indecomposable flat module is projective in T-MOD;

(f) $T^{(\mathbb{N})}$ is semiperfect in T-MOD;

(g) $End(T^{(\mathbb{N})})$ is f-semiperfect;

(h) $T/Jac(T)$ is left semisimple and $Rad(T^{(\mathbb{N})}) \ll {}_TT^{(\mathbb{N})}$;

(i) $T/Jac(T)$ is left semisimple and $Jac(T)$ is right t-nilpotent;

(j) the descending chain condition for cyclic right ideals in T holds.

If these assertions hold, then every module in MOD-T has a semisimple submodule.

Proof: Since T is a projective generator in T-MOD, the equivalence of (a), (b) and (c) and the implication $(a) \Rightarrow (d)$ can be deduced from 43.2.

$(d) \Leftrightarrow (e)$ By 36.4, all flat factor modules of projective modules are projective if and only if this is true for indecomposable flat factor modules.

$(a) \Rightarrow (f)$ is evident, $(f) \Rightarrow (g)$ has been shown in 42.12.

$(g) \Rightarrow (h)$ Let $B = End({}_TT^{(\mathbb{N})})$ be f-semiperfect. For any $e^2 = e \in T$, we have $T^{(\mathbb{N})} \simeq (Te)^{(\mathbb{N})} \oplus X$, for some T-module X. For the idempotent $\gamma \in B$ belonging to the above decomposition, $End_T(Te^{(\mathbb{N})}) \simeq \gamma B\gamma$. Now $B\gamma$ is an f-semiperfect B-module (see 42.9) and therefore $\gamma B\gamma = End_B(B\gamma)$ is an f-semiperfect ring (see 42.12).

Thus the ring $End((Te)^{(\mathbb{N})})$ is f-semiperfect for the finitely generated, projective module Te in T-*MOD* and, by 43.8, Te is perfect in $\sigma[Te]$. In particular, $Te/Rad(Te)$ is semisimple. Then, for a complete family $\{e_\alpha\}_A$ of idempotents in T, $T/Jac(T) \simeq \bigoplus_\Lambda Te_\alpha/Rad(Te_\alpha)$ is semisimple.

Finally we conclude by 42.12, $Rad\,(T^{(\mathbb{N})}) \ll T^{(\mathbb{N})}$.

$(h) \Rightarrow (i)$ Because $Jac(T)T^{(\mathbb{N})} = Rad\,(T^{(\mathbb{N})}) \ll T^{(\mathbb{N})}$, the ideal $Jac(T)$ is right t-nilpotent by 49.8.

$(i) \Rightarrow (a)$ Since $Jac(T)$ is a nil ideal, idempotents in $T/Jac(T)$ can be lifted to T (see 42.7). Because $Jac(T) \ll T$ (see 49.8), T is semiperfect in T-*MOD* by 42.5. Now we have $Rad(T^{(\Lambda)}) \ll T^{(\Lambda)}$ for every index set Λ (again 49.8), and so $T^{(\Lambda)}$ is semiperfect in T-*MOD* by 42.4, i.e. ${}_TT$ is perfect in T-*MOD*.

$(d) \Rightarrow (j)$ A descending chain of cyclic right ideals in T is of the form

$$f_1T \supset f_1f_2T \supset f_1f_2f_3T \supset \cdots,$$

for some sequence $\{f_i\}_{\mathbb{N}}$, $f_i \in T \subset End({}_TT)$. Applying 43.3 to $N_i = T$, $N = T^{(\mathbb{N})}$ and $\{f_i : T \to T,\, t \mapsto tf_i\}$ we conclude, with the notation of 43.3, that $N/Im\,g$ is flat and therefore, by assumption (d), is projective. So $Im\,g$ is a direct summand and, for $t \in T$, there exists $r \in \mathbb{N}$ with

$$tf_1 \cdots f_{r-1} = tf_1 \cdots f_r h_{r+1,r} \text{ and } h_{r+1,r} \in End({}_TT).$$

If, in particular, t is an idempotent with $tf_1 = f_1$ and we choose an idempotent $e \in T$ with $f_re = f_r$, then we have

$$f_1 \cdots f_{r-1} = f_1 \cdots f_r((e)h_{r+1,r}), \quad \text{with } (e)h_{r+1,r} \in T.$$

So the above descending chain of right ideals in T terminates.

$(j) \Rightarrow (i)$ By 31.8, we see that $T/Jac(T)$ is right semisimple and, by 49.6, also left semisimple.

For a family $\{s_i\}_{\mathbb{N}}$ of elements of $Jac(T)$ we construct the descending chain of cyclic right ideals $s_1T \supset s_1s_2T \supset s_1s_2s_3T \supset \cdots$ which is finite by assumption. So we have for a suitable $r \in \mathbb{N}$,

$$s_1 \cdots s_rT = s_1 \cdots s_rs_{r+1}T \subset s_1 \cdots s_rJac(T).$$

By Nakayama's Lemma 49.7, we have $s_1 \cdots s_r = 0$, i.e. $Jac(T)$ is right t-nilpotent.

It follows from 31.8 that, because of (j), every module in MOD-T has a simple submodule.

The equivalences given in 49.9 are also true for those rings with local units which are projective as left modules.

Whereas the description of left perfect rings T is almost the same as the characterization of left perfect rings with unit, we have to be more careful considering semiperfect rings:

A ring T with local units is called *left semiperfect* if every simple module in T-MOD has a projective cover.

In this case T need not be a semiperfect module in T-MOD $(= \sigma[_TT])$ in the sense of § 42, since $Jac(T)$ is not necessarily superfluous in $_TT$. On the other hand, many properties of unital semiperfect rings are preserved, especially symmetry of sides:

49.10 Semiperfect rings *T*. Characterizations.

For a ring T with local units, the following assertions are equivalent:

(a) T is left semiperfect;

(b) every finitely generated, projective module in T-MOD is a direct sum of local modules;

(c) in T-MOD every finitely generated module has a projective cover;

(d) in T-MOD every finitely generated module is semiperfect;

(e) every finitely generated module in T-MOD is supplemented;

(f) Te is semiperfect in T-MOD for every idempotent $e \in T$;

(g) eTe is a (unital) semiperfect ring for every idempotent $e \in T$;

(h) T is right semiperfect.

Proof: If T is left semiperfect the projective covers of the simple modules form a set of generators in T-MOD. By 42.4, every finite direct sum of these modules is semiperfect in T-MOD. Therefore we obtain the equivalences of (a) to (f) from the first part of § 42.

For an idempotent $e \in T$, we have $End(Te) \simeq eTe \simeq End(eT)$. By 42.12, Te (resp. eT) is semiperfect if and only if eTe is semiperfect. So (f), (g) and (h) are equivalent.

49.11 Exercises.

(1) Let T be a ring (without unit).

A left T-module M is called s-unital *if and only if* $m \in Tm$ *holds for every* $m \in M$. *Prove:*

(i) If ${}_TM$ *is s-unital, then* ${}_TT$ *generates every module in* $\sigma[{}_TM]$.

(ii) If T is a ring with local units, then every module in T-MOD is s-unital.

(iii) If T is a left fully idempotent ring ($I^2 = I$ *for all left ideals, s. 3.15), then* ${}_TT$ *is s-unital.*

(iv) If ${}_TT$ *s-unital and noetherian, then T has a unit.*

(v) If ${}_TT$ *is artinian, then* ${}_TT$ *is also noetherian.*

(2) Let T and S be rings with local units.

A module $P \in T$*-MOD is called* locally projective *if and only if P can be written as a direct limit of finitely generated and projective direct summands. Let*

$F : T\text{-}MOD \to S\text{-}MOD$ and $\quad G : S\text{-}MOD \to T\text{-}MOD$

be additive covariant functors. Prove:

F and G are pairwise inverse equivalences if and only if there exists a bimodule ${}_TP_S$ *with* $TP = P$ *and* $PS = P$ *such that:*

(i) ${}_TP$ *and* P_S *are both locally projective generators;*

(ii) $S = SEnd({}_TP)$ *and* $T = End(P_S)T$ *with* $S \subset End({}_TP)$ *and* $T \subset End(P_S)$, *viewed canonically;*

(iii) $F \simeq S\, Hom_T(P, -)$ *and* $G \simeq P \otimes_S -$.

(Anh-Márki)

Literature: Abrams, Anh-Márki, Harada [1-4], Tominaga.

50 Global dimensions of modules and rings

1.(Pure) global dimension. 2.Comparison of exact sequences. Schanuel's Lemma. 3.Modules with global dimension ≤ 2. 4.Modules with weak global dimension ≤ 2. 5.Weak global dimension of T. 6.Inequalities between global dimensions. 7.Exercises.

The study of the global dimensions of categories of modules is important for the homological classification of modules and rings. Some aspects of this technique will be useful in studying functor rings. To prepare for this, we outline basic definitions and relations in this paragraph. Most of the results treated here hold in more general form in Homological Algebra but their proofs require some more technical effort.

Let M be an R-module and for $N \in \sigma[M]$ let

$$(*) \quad 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to N \to 0$$

be an exact sequence in $\sigma[M]$.

Assume there exists a generating set of projective modules in $\sigma[M]$. Then we define:

M has *global (projective) dimension* $\leq n$, *gl.dim* $M \leq n$, if, for every $N \in \sigma[M]$, the following is true: in every sequence $(*)$ with projective $P_0, \ldots, P_{n-1}$, the module P_n is also projective.

M has *weak global dimension* $\leq n$, *w.gl.dim* $M \leq n$, if, for every $N \in \sigma[M]$, the following holds: in every sequence $(*)$ with projective $P_0, \ldots, P_{n-1}$, the module P_n is flat in $\sigma[M]$.

Assume there exists a generating set of finitely presented modules in $\sigma[M]$. The sequence $(*)$ is called *pure* if the kernel of any of its morphisms is a pure submodule. We say M has *pure global dimension* $\leq n$, *p.gl.dim* $M \leq n$, if, for every $N \in \sigma[M]$, the following holds: in any pure sequence $(*)$ with pure projective $P_0, \ldots, P_{n-1}$, the module P_n is also pure projective in $\sigma[M]$.

It is convenient that we need only one sequence for every $N \in \sigma[M]$ to check the (pure) global dimension of M:

50.1 (Pure) global dimension.
Let M be an R-module.

(1) gl.dim $M \leq n$ if and only if for every $N \in \sigma[M]$ there exists a sequence $(*)$ with projective $P_0, \ldots, P_n$.

(2) p.gl.dim $M \leq n$ if and only if for every $N \in \sigma[M]$ there exists a pure sequence $(*)$ with pure projective $P_0, \ldots, P_n$.

The **proof** follows from our next proposition.

For rings T with local units, it is possible to show that $w.gl.dim\ {}_TT \le n$ if and only if there exists a sequence $(*)$ with *flat* $P_0, \ldots, P_n$ in T-*MOD* for every $N \in T$-*MOD*. To prove this, more knowledge of Homological Algebra would be needed.

50.2 Comparison of exact sequences. Schanuel's Lemma. *Assume M to be an R-module and $N \in \sigma[M]$.*

(1) Let $0 \to K \to P \to N \to 0$ and $0 \to L \to Q \to N \to 0$ be (pure) exact sequences in $\sigma[M]$.

If P and Q are (pure) projective in $\sigma[M]$, then $K \oplus Q \simeq L \oplus P$.

(2) Let $0 \to K \to P_n \to P_{n-1} \to \cdots \to P_0 \to N \to 0$
and $0 \to L \to Q_n \to Q_{n-1} \to \cdots \to Q_0 \to N \to 0$
be (pure) exact sequences in $\sigma[M]$.

If P_i and Q_i are (pure) projective in $\sigma[M]$, then

$$K \oplus Q_n \oplus P_{n-1} \oplus Q_{n-2} \oplus \cdots \simeq L \oplus P_n \oplus Q_{n-1} \oplus P_{n-2} \oplus \cdots .$$

Proof: (1) By forming a pullback we obtain, from the given sequences, the commutative exact diagram (see 10.3)

$$\begin{array}{ccccccccc}
 & & & & L & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & V & \to & Q & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & P & \to & N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

Since P and Q are projective, we have $V \simeq K \oplus Q \simeq L \oplus P$.

Because a pullback preserves pure epimorphisms (see 33.4), the assertion about pure exact sequences and pure projective modules also follows from the above diagram.

(2) Considering the sequences $0 \to K \to P_1 \to P_0 \to N \to 0$ and $0 \to L \to Q_1 \to Q_0 \to N \to 0$, we get, by (1), $P_1/K \oplus Q_0 \simeq Q_1/L \oplus P_0$. Now we construct the exact sequences

$$\begin{array}{ccccccccc}
0 & \to & K & \to & P_1 \oplus Q_0 & \to & (P_1/K) \oplus Q_0 & \to & 0 \quad , \\
0 & \to & L & \to & Q_1 \oplus P_0 & \to & (Q_1/L) \oplus P_0 & \to & 0 \quad ,
\end{array}$$

and, again by (1), we have $L \oplus P_1 \oplus Q_0 \simeq K \oplus Q_1 \oplus P_0$.
Then the result follows by induction.

The modules M with *global dimension zero* are exactly the semisimple modules: every module in $\sigma[M]$ is projective (see 20.3).

The modules M with *global dimension* ≤ 1 are those which are hereditary in $\sigma[M]$: every submodule of a projective module is projective in $\sigma[M]$ (see 39.8).

The modules M with *weak global dimension zero* are those which are regular in $\sigma[M]$: all modules in $\sigma[M]$ are flat (see 37.2).

The modules M with *weak global dimension* ≤ 1 are modules whose submodules are flat in $\sigma[M]$: all submodules of flat modules are flat in $\sigma[M]$ (see 39.12).

Modules with *pure global dimension zero* will be studied in § 53.
From former results we have the following for a ring R with unit:

gl.dim $_RR = 0$ *if and only if gl.dim* $R_R = 0$ *(see 20.7),*
w.gl.dim $_RR = 0$ *if and only if w.gl.dim* $R_R = 0$ *(see 37.6),*
w.gl.dim $_RR \leq 1$ *if and only if w.gl.dim* $R_R \leq 1$ *(see 39.12,(2)).*

However, *gl.dim* $_RR \leq 1$ is not necessarily equivalent to *gl.dim* $R_R \leq 1$. We will derive similar relations for rings T with local units in 50.5.

Let us now consider *global dimension* ≤ 2.

50.3 Modules with global dimension ≤ 2.

Let M be an R-module with a generating set of finitely generated, projective modules in $\sigma[M]$.

(1) The following assertions are equivalent:

(a) gl.dim $M \leq 2$;
(b) for any $f : P_1 \to P_0$ with projective modules P_1, P_0 in $\sigma[M]$, $Ke\, f$ is projective in $\sigma[M]$;
(c) for any $g : Q_0 \to Q_1$ with injective modules Q_0, Q_1 in $\sigma[M]$, $Coke\, g$ is injective in $\sigma[M]$.

(2) If M is locally coherent, then the following are equivalent:

(a) for any $f : P_1 \to P_0$ with finitely generated, projective modules P_1, P_0 in $\sigma[M]$, $Ke\, f$ is projective in $\sigma[M]$;
(b) for any $g : Q_0 \to Q_1$ with absolutely pure modules Q_0, Q_1 in $\sigma[M]$, $Coke\, g$ is absolutely pure in $\sigma[M]$.

If M is locally noetherian, then (a), (b) are equivalent to:

(c) gl.dim $M \leq 2$.

Proof: (1) $(a) \Leftrightarrow (b)$ is an easy consequence of the definitions and 50.1.

$(b) \Rightarrow (c)$ We have to show that every diagram with an exact row and P_0 (finitely generated) projective

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & P_0 \\ & & \downarrow h & & \\ & & Coke\, g & & \end{array}$$

can be extended to a commutatively by some $P_0 \to Coke\, g$ (notice 16.2).

First we extend this diagram with an epimorphism $f : P_1 \to K$, P_1 projective in $\sigma[M]$, to the exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ke\, f & \longrightarrow & P_1 & \xrightarrow{f} & K & \subset & P_0 \\ & & & & & & \downarrow & & \\ & & Q_0 & \xrightarrow{g} & Q_1 & \longrightarrow & Coke\, g & \longrightarrow & 0 \quad . \end{array}$$

Step by step we obtain commutative extensions by

$\alpha : P_1 \to Q_1$, since P_1 is projective,

$Ke\, f \to Q_0$, since $(Ke\, f)\alpha \subset Im\, g$ and $Ke\, f$ is projective by (b),

$P_1 \to Q_0$, since Q_0 is injective,

$K \to Q_1$, by the proof of 7.16,

$P_0 \to Q_1$, since Q_1 is injective.

So we arrive at the desired extension of the first diagram.

$(c) \Rightarrow (b)$ We have to show that the following diagram with exact row and Q_0 injective,

$$\begin{array}{ccccc} & & Ke\, f & & \\ & & \downarrow & & \\ Q_0 & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

can be extended to a commutative diagram by some $Ke\, f \to Q_0$.

With a monomorphism $V \to Q_1$, Q_1 injective in $\sigma[M]$, we obtain the exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ke\, f & \longrightarrow & P_1 & \xrightarrow{f} & P_0 & & \\ & & \downarrow & & & & & & \\ (Q_0 & \longrightarrow) & V & \longrightarrow & Q_1 & \longrightarrow & Q_1/V & \longrightarrow & 0 \quad . \end{array}$$

Q_1/V is M-injective by assumption and we obtain commutative extensions by morphisms $P_1 \to Q_1$, $Im\, f \to Q_1/V$, $P_0 \to Q_1/V$, $P_0 \to Q_1$, $P_1 \to V$ (Homotopy Lemma) and finally $P_1 \to Q_0$.

(2) $(a) \Leftrightarrow (b)$ Let $\{P_\lambda\}_\Lambda$ be a generating set of finitely generated, projective modules P_λ in $\sigma[M]$. Since M is locally coherent, every P_λ is coherent in $\sigma[M]$ (see 26.2). Also, 'absolutely pure' in $\sigma[M] = \sigma[\bigoplus_\Lambda P_\lambda]$ is equivalent to 'weakly $\bigoplus_\Lambda P_\lambda$-injective' by 35.4. Now the assertion follows from the proof of (1) $(b) \Leftrightarrow (c)$, taking for P_1, P_0 finite direct sums of P_λ's. Then $Ke\,f$ is always finitely generated (see 26.1).

$(b) \Leftrightarrow (c)$ If M is locally noetherian, then the definition of 'absolutely pure' in $\sigma[M]$ is equivalent to 'M-injective' (by 27.3) and the assertion follows from (1).

50.4 Modules with weak global dimension ≤ 2.
Let M be an R-module with a generating set $\{\widetilde{P}_\lambda\}_\Lambda$ of finitely generated, projective modules in $\sigma[M]$.

(1) The following assertions are equivalent:

(a) w.gl.dim $M \leq 2$;

(b) for any $f : Q_1 \to P_0$, with Q_1 flat and P_0 projective in $\sigma[M]$, $Ke\,f$ is flat in $\sigma[M]$;

(c) for any $f : P_1 \to P_0$, with P_1, P_0 finitely generated and projective in $\sigma[M]$, $Ke\,f$ is flat in $\sigma[M]$.

(2) If M is locally noetherian or $\bigoplus_\Lambda \widetilde{P}_\lambda$ is perfect in $\sigma[M]$, then gl.dim $M \leq 2$ is equivalent to w.gl.dim $M \leq 2$.

Proof: (1) $(a) \Rightarrow (b)$ Let $f : Q_1 \to P_0$ be a morphism, with Q_1 flat and P_0 projective in $\sigma[M]$. We choose an epimorphism $P_1 \xrightarrow{h} Q_1$ with projective P_1 in $\sigma[M]$, and construct the commutative exact diagram

$$\begin{array}{ccccccc}
 & & Ke\,h & = & Ke\,h & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ke\,hf & \longrightarrow & P_1 & \xrightarrow{hf} & P_0 \\
 & & \downarrow & & \downarrow h & & \| \\
0 & \longrightarrow & Ke\,f & \longrightarrow & Q_1 & \xrightarrow{f} & P_0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & .
\end{array}$$

By (a), $Ke\,hf$ is flat in $\sigma[M]$. Since Q_1 is flat in $\sigma[M]$, $Ke\,h$ is a pure submodule of P_1 and hence of $Ke\,hf$ (see 33.3). Now we conclude, by 36.1, that $Ke\,f$ is flat in $\sigma[M]$.

$(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$ For $N \in \sigma[M]$ we consider an exact sequence in $\sigma[M]$

$$0 \longrightarrow Ke\,g \longrightarrow P_1 \xrightarrow{g} P_0 \longrightarrow N \longrightarrow 0\,,$$

with P_1, P_0 direct sums of finitely generated, projective modules in $\sigma[M]$. The restriction of g to a finite partial sum of P_1 has a flat kernel because of (c). Then $Ke\,g$ is a direct limit of these flat modules and hence is also flat in $\sigma[M]$.

(2) If $\bigoplus_\Lambda \widetilde{P}_\lambda$ is perfect in $\sigma[M]$, all flat modules in $\sigma[M]$ are projective (see 43.2) and $w.gl.dim\ M = gl.dim\ M$.

Now let M be locally noetherian and assume $w.gl.dim\ M \leq 2$. Then for $f : P_1 \to P_0$ with P_1, P_0 finitely generated, projective, the kernel $Ke\,f$ is flat and finitely presented, and hence projective in $\sigma[M]$. By 50.3,(2), we conclude $gl.dim\ M \leq 2$.

50.5 Weak global dimension of T.
Let T be a ring with local units:

(1) The following assertions are equivalent:
(a) ${}_TT$ is regular in T-MOD;
(b) $w.gl.dim\ {}_TT = 0$;
(c) $w.gl.dim\ T_T = 0$;
(d) T_T is regular in MOD-T.

(2) The following properties are equivalent:
(a) every left ideal of T is flat in T-MOD;
(b) $w.gl.dim\ {}_TT \leq 1$;
(c) $w.gl.dim\ T_T \leq 1$;
(d) every right ideal of T is flat in MOD-T.

(3) The following assertions are equivalent:
(a) the kernel of morphisms between flat modules is flat in T-MOD;
(b) $w.gl.dim\ {}_TT \leq 2$;
(c) $w.gl.dim\ T_T \leq 2$;
(d) the kernel of morphisms between flat modules is flat in MOD-T.

Proof: We already know the equivalence of (a) and (b), resp. (c) and (d), in (1) and (2) from more general assertions in $\sigma[M]$ (see 37.2, 39.12).

(1) $(b) \Leftrightarrow (c)$ If every module is flat in *T-MOD*, then every short exact sequence is pure in *T-MOD*, and, for every module $K \in$ *MOD-T*, the functor $K_T \otimes - : T\text{-}MOD \to AB$ is exact (see 49.4, 49.5).

(2) $(b) \Leftrightarrow (c)$ A right ideal $K \subset T_T$ is flat in *MOD-T* if and only if, for every left ideal $L \subset {}_TT$, the canonical map $K \otimes_T L \to KL$ is monic (see 49.5 and 39.12). Flat left ideals are characterized similarly.

(3) $(a) \Rightarrow (b)$ is evident.

$(b) \Rightarrow (a)$ We obtain from 50.4 that, by hypothesis (b), for all morphisms $g : V \to P_0$, with flat V and projective P_0 in T-MOD, $Ke\, g$ is flat in T-MOD. Let $f : Q_1 \to Q_0$ be a morphism with flat modules Q_1, Q_0 in T-MOD. An epimorphism $h : P_0 \to Q_0$, with P_0 projective in T-MOD, yields, by forming a pullback, the commutative exact diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & Ke\, h & = & Ke\, h \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & Ke\, f & \longrightarrow & V & \longrightarrow & P_0 \\
 & & \| & & \downarrow & & \downarrow h \\
0 & \longrightarrow & Ke\, f & \longrightarrow & Q_1 & \xrightarrow{f} & Q_0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0 \quad .
\end{array}$$

As a pure submodule (Q_0 is flat) of the projective module P_0, $Ke\, h$ is flat in T-MOD (see 49.5,(2)). Then, by 36.1, V is flat in T-MOD as an extension of the flat modules $Ke\, h$ and Q_1. As mentioned above, therefore $Ke\, f$ is also flat in T-MOD.

$(b) \Leftrightarrow (c)$ Let $w.gl.dim\ _TT \leq 2$ and $0 \to K \to Q_1 \to Q_0$ be an exact sequence of flat modules in MOD-T. We have to show that for every (finitely generated) left ideal $L \subset\ _TT$, the canonical map $\mu_L : K \otimes L \to KL$ is monic.

In an exact sequence $0 \to U \to P \xrightarrow{h} L(\subset\ _TT)$ in T-MOD with (finitely generated) projective $P \in T$-MOD, by assumption, U is flat in T-MOD. Construct the commutative exact diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
0 & \to & K \otimes_T U & \to & Q_1 \otimes_T U & \to & Q_0 \otimes_T U \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & K \otimes_T P & \to & Q_1 \otimes_T P & \to & Q_0 \otimes_T P \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & K \otimes_T L & \xrightarrow{\alpha} & Q_1 \otimes_T L & \to & Q_0 \otimes_T L \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 \quad .
\end{array}$$

Since $_TU$ is flat, we may assume $(Q_1/K) \otimes_T U \subset Q_0 \otimes_T U$ and, by the Kernel Cokernel Lemma, we conclude that α is monic.

From the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K\otimes_T L & \longrightarrow & Q_1\otimes_T L \\ & & \downarrow \mu_L & & \downarrow\simeq \\ & & KL & \longrightarrow & Q_1 L \end{array},$$

we obtain that μ_L is also monic.

Finally, we want to point out a relationship between different global dimensions of rings with local units:

50.6 Inequalities between global dimensions.
Let T be a ring with local units. Then, if the right side is finite,

$$w.gl.dim\ _TT \le gl.dim\ _TT \le p.gl.dim\ _TT\ +\ w.gl.dim\ _TT.$$

Proof: The left inequality follows immediately from the definitions.

For the right inequality we may assume $p.gl.dim\ _TT$ and $w.gl.dim\ _TT$ are finite. For $L\in T\text{-}MOD$, we construct an exact sequence with projective P_i (*projective resolution*)

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \to L \to 0\ .$$

If $w.gl.dim\ _TT\le r$, then $K = Ke\,d_{r-1}$ is a flat module in $T\text{-}MOD$. Therefore the exact sequence $0\to Ke\,d_r \to P_r \to K \to 0$ is pure in $T\text{-}MOD$ and, by 49.5,(2), $Ke\,d_r$ is flat in $T\text{-}MOD$. By a similar argument, all other $Ke\,d_s$, for $s\ge r$, are also flat and pure submodules in P_s.

Now assume $p.gl.dim\ _TT\le k$. Then, in the pure exact sequence

$$0\to N\to P_{r+k-1}\cdots\to P_{r+1}\to P_r\to K\to 0\,,$$

$N = Ke\,d_{r+k-1}$ has to be pure projective. Being flat in $T\text{-}MOD$ by the above considerations, N is in fact projective. Then we have an exact sequence

$$0\to N\to P_{r+k-1}\to\cdots P_1\to P_0\to L\to 0$$

with projective N and P_i, implying $gl.dim\ _TT\le k+r$.

50.7 Exercises.

(1) Let M be an R-module and $K\in\sigma[M]$.

(i) Prove, for pure exact sequences in $\sigma[M]$

$$0\to K\to P\to L\to 0 \ \text{ and }\quad 0\to K\to Q\to N\to 0\ :$$

If P and Q are pure injective in $\sigma[M]$, then $Q \oplus L \simeq P \oplus N$.

(ii) Find and prove an assertion as in (i) for pure exact sequences of arbitrary length.

(2) Let T be a ring with local units. Prove that the following assertions are equivalent:
(a) gl.dim.$\,_T T \leq 2$;
(b) for every cyclic module N in T-MOD there exists an exact sequence

$$0 \to P_2 \to P_1 \to P_0 \to N \to 0$$

in T-MOD with P_0, P_1, P_2 projective.

Literature: HILTON-STAMMBACH, ROTMAN; Fedin.

51 The functor $\widehat{Hom}(V,-)$

1.The ring $\widehat{End}(V)$. 2.Properties of $\widehat{Hom}(V,-)$. 3.$\widehat{Hom}(V,-)$ with V projective. 4.Further characterizations of perfect modules. 5.$\widehat{Hom}(V,-)$ and V-generated modules. 6.The pair of functors $V\otimes_T -$, $\widehat{Hom}(V,-)$. 7.$\widehat{Hom}(V,-)$ with V a generator in $\sigma[M]$. 8.$\widehat{Hom}(V,-)$ with V a generator in R-MOD. 9. V-supported modules with V_α finitely presented. 10.$\widehat{Hom}(V,-)$ with V_α finitely presented. 11. Equivalence of $\sigma[M]$ and T-MOD. 12. Modules annihilated by $M\otimes_T -$. 13. Generators with right perfect endomorphism rings.

The functors described in this paragraph generalize the functors $Hom(K,-)$ with K finitely generated. They will enable us to give a simple relationship between a category $\sigma[M]$ and its functor ring in § 52.

Let $\{V_\alpha\}_A$ be a family of finitely generated R-modules and $V=\bigoplus_A V_\alpha$. For any $N\in R\text{-}MOD$ we define:

$$\widehat{Hom}(V,N)=\{f\in Hom(V,N)\mid (V_\alpha)f=0 \ \text{ for almost all } \ \alpha\in A\}.$$

For $N=V$, we write $\widehat{Hom}(V,V)=\widehat{End}(V)$. Note that these constructions do not depend on the decomposition of V.

51.1 The ring $\widehat{End}(V)$. Properties.
With the above notation set $T=\widehat{End}(V)$ and $S=End(V)$. Then

(1) For every $f\in T$, $Im\,f$ is finitely generated.

(2) With canonical projections π_α and injections ε_α, $\{e_\alpha=\pi_\alpha\varepsilon_\alpha\}_A$ forms a complete family of idempotents in T, i.e. T has enough idempotents.

(3) T is a right ideal in S, is projective in MOD-S and $Jac(T)=T\,Jac(S)$.

(4) V is a T-right module and $VT=V$, i.e. $V\in MOD\text{-}T$ (§ 49).

(5) $T_T\subset V_T^{(\Lambda)}$, for a suitable set Λ, and hence $\sigma[V_T]=MOD\text{-}T$.

(6) For any $N\in R\text{-}MOD$, we have

$$T\,Hom(V,N)=T\,\widehat{Hom}(V,N)=\widehat{Hom}(V,N),$$

i.e. $\widehat{Hom}(V,N)$ belongs to T-MOD, and $V\,Hom(V,N)=V\,\widehat{Hom}(V,N)$.

(7) V_T is weakly T-injective (absolutely pure in MOD-T) if and only if ${}_RV$ cogenerates the cokernels of morphisms $f\in\widehat{Hom}(V^n,V^k)$, $n,k\in\mathbb{N}$.

Proof: (1) follows directly from the definition.

(2) $e_\alpha = \pi_\alpha\varepsilon_\alpha \in T$ are pairwise orthogonal. Every $f \in T$ is non-zero only on a finite partial sum $V' = \bigoplus_E V_\alpha$, $E \subset A$. Since $\sum_{\alpha\in E} e_\alpha = id_{V'}$, we have $f = \sum_{\alpha\in E} e_\alpha f$ and $T = \bigoplus_A e_\alpha T$.

On the other hand, $Im\, f$ is contained in a finite partial sum of V and we conclude $T = \bigoplus_A T e_\alpha$.

(3) By definition, we see that T is a right ideal in S and $e_\alpha T = e_\alpha S$. So $T = \bigoplus_A e_\alpha S$, where $e_\alpha S$ are direct summands of S and hence are projective. Then

$Jac(T) = Rad(T_T) = Rad(T_S) = T Jac(S)$ (see also 49.6,(3)).

(4) Since $V_\alpha = V e_\alpha \subset VT$, for every $\alpha \in A$, $VT = V$.

(5) For every e_α, we have exact sequences $R^k \to Ve_\alpha \to 0$ and $0 \to Hom(Ve_\alpha, V) \to Hom(R, V)^k$, $k \in \mathbb{N}$. Hence $e_\alpha T = Hom(Ve_\alpha, V)$ is a submodule of V_T^k and $T = \bigoplus_A e_\alpha T \subset V^{(\Lambda)}$, for some index set Λ.

(6) If $f \in \widehat{Hom}(V, N)$ is non-zero only on the finite partial sum $\bigoplus_E V_\alpha$, then, for $e = \sum_{\alpha\in E} e_\alpha$, we have $ef = f$ and so $f \in T\widehat{Hom}(V, N)$. The rest is easy to see.

(7) The assertion is given by 47.7, recalling the fact that V_T is absolutely pure if and only if it is injective with respect to exact sequences in T-MOD of the form $0 \to L \to \bigoplus_{i\le k} e_{\alpha_i} T$ with L finitely generated (see 35.1, 35.4).

For any morphism $g : N \to N'$ in R-MOD and $f \in \widehat{Hom}(V, N)$, we have $fg \in \widehat{Hom}(V, N')$ and a functor $\widehat{Hom}(V, -) : R\text{-}MOD \to T\text{-}MOD$, with

objects: $N \rightsquigarrow \widehat{Hom}(V, N)$,

morphisms:

$g : N \to N' \rightsquigarrow \widehat{Hom}(V, g) : \widehat{Hom}(V, N) \to \widehat{Hom}(V, N')$, $f \mapsto fg$.

51.2 Properties of $\widehat{Hom}(V, -)$.

Assume $V = \bigoplus_A V_\alpha$, with finitely generated R-modules V_α, $T = \widehat{End}(V)$ and $S = End(V)$. Then

(1) The functors $\widehat{Hom}(V, -)$, $T \otimes_T Hom(V, -)$, $T \otimes_S Hom(V, -)$, $T \cdot Hom(V, -)$ and $\bigoplus_A Hom(V_\alpha, -)$ from R-MOD to T-MOD are isomorphic to each other.

(2) $\widehat{Hom}(V, -)$ is a left exact functor.

(3) $\widehat{Hom}(V, -)$ preserves direct sums and direct limits of direct systems of monomorphisms.

(4) $\widehat{Hom}(V, -)$ preserves direct products (in R-MOD).

(5) Let $M \in R$-MOD and $V \in \sigma[M]$. Then

(i) $\widehat{Hom}(V, -)$ preserves direct products in $\sigma[M]$.

(ii) $\widehat{Hom}(V,-)$ *preserves direct limits in* $\sigma[M]$ *if and only if each* V_α *is finitely presented in* $\sigma[M]$.

Proof: (1) By 49.3 we have, for $N \in R\text{-}MOD$, the isomorphisms

$$T \otimes_S Hom(V,N) \simeq T \otimes_T Hom(V,N) \simeq T Hom(V,N) = \widehat{Hom}(V,N).$$

With the injections $\varepsilon_\alpha : V_\alpha \to V$, we obtain a $\mathbb{Z}$-monomorphism

$$\widehat{Hom}(V,N) \to \prod\nolimits_A Hom(V_\alpha,N), \quad f \mapsto (\varepsilon_\alpha f)_A.$$

Since only a finite number of the $\varepsilon_\alpha f$ are non-zero, this yields a $\mathbb{Z}$-isomorphism $\widehat{Hom}(V,N) \to \bigoplus_A Hom(V_\alpha,N)$. This isomorphism allows us to define a T-module structure on $\bigoplus_A Hom(V_\alpha,N)$.

(2) follows from (1), since, e.g., $Hom(V,-)$ and $T \otimes_S -$ are left exact functors (T_S is projective in $MOD\text{-}S$ by 51.1,(3)).

(3) Since each V_α is finitely generated, the functors $Hom(V_\alpha,-)$ preserve the given limits (see 24.10). The assertion now follows from the isomorphism $\widehat{Hom}(V,-) \simeq \bigoplus_A Hom(V_\alpha,-)$ and from the fact that direct limits and direct sums commute.

(4) For a family of R-modules $\{N_\lambda\}_\Lambda$ in $R\text{-}MOD$ we have
$\widehat{Hom}(V,\prod_\Lambda N_\lambda) = T \cdot Hom(V,\prod_\Lambda N_\lambda)$
$\simeq T(\prod_\Lambda Hom(V,N_\lambda)) = T(\prod_\Lambda \widehat{Hom}(V,N_\lambda))$.
The last term is the product of the $\widehat{Hom}(V,N_\lambda)$ in $T\text{-}MOD$ (see 49.2).

(5)(i) can be proven similarly to (4), recalling the construction of the product in $\sigma[M]$ (see 15.1). It also follows from the fact that $\widehat{Hom}(V,-)$ has a right adjoint (see 51.6). (ii) This is shown in the same way as (3), using the characterization of finitely presented modules in $\sigma[M]$ (see 25.2).

Similar to the situation for $Hom(V,-)$, there exists relationships between properties of $\widehat{Hom}(V,-)$ and module properties of V (e.g. (5)(ii) above). Of course, projectivity is of special interest:

51.3 $\widehat{Hom}(V,-)$ with V projective.

Let M *be an* R*-module,* $V = \bigoplus_A V_\alpha$ *with finitely generated* $V_\alpha \in \sigma[M]$, *and* $T = \widehat{End}(V)$.

(1) The following assertions are equivalent:

(a) V *is* M*-projective;*

(b) V *is projective in* $\sigma[M]$;

(c) $\widehat{Hom}(V,-) : \sigma[M] \to AB$ *is exact.*

(2) If V is M-projective, then for any $N \in \sigma[M]$:
(i) $I = \widehat{Hom}(V, VI)$ *for every T-submodule* $I \subset \widehat{Hom}(V, N)$.
(ii) $\widehat{Hom}(V, L_1 + L_2) = \widehat{Hom}(V, L_1) + \widehat{Hom}(V, L_2)$ *for any submodules* $L_1,\ L_2 \subset N$.
(iii) $Tr(V, N)$ *is supplemented as an R-module if and only if* $\widehat{Hom}(V, N)$ *is supplemented as a T-module.*
(iv) If N is V-generated and $K \ll N$, then $\widehat{Hom}(V, K) \ll \widehat{Hom}(V, N)$.

(3) For M-projective V and $N \in R$-MOD, the following are equivalent:
(a) N has dcc for finitely V-generated submodules;
(b) $\widehat{Hom}(V, N)$ *has dcc for finitely generated (or cyclic) T-submodules.*

(4) For M-projective V and $N \in R$-MOD, the following are equivalent:
(a) N has acc for finitely V-generated submodules;
(b) $\widehat{Hom}(V, N)$ *has acc for finitely generated T-submodules.*

Proof: (1) $(a) \Leftrightarrow (b)$ Since for finitely generated V_α, M-projective is equivalent to projective in $\sigma[M]$ this is also true for V (see 18.3, 18.1).

$(b) \Leftrightarrow (c)$ is given by the isomorphism $\widehat{Hom}(V, -) \simeq \bigoplus_A Hom(V_\alpha, -)$.

(2) (i) This is obtained by using the corresponding proof in 18.4, since the image of any $g \in \widehat{Hom}(V, VI)$ is finitely generated.

(ii) The desired relation is derived from the following diagram:

$$\begin{array}{ccccc} & & V & & \\ & & \downarrow & & \\ L_1 \oplus L_2 & \longrightarrow & L_1 + L_2 & \longrightarrow & 0\,. \end{array}$$

(iii) Using (i) and (ii) we can follow the proof of the corresponding assertion for finitely generated modules in 43.7.

(iv) Assume $\widehat{Hom}(V, K) + X = \widehat{Hom}(V, N)$ for some $X \subset {}_T\widehat{Hom}(V, N)$. By (i) and (ii) we obtain

$$\widehat{Hom}(V, K + VX) = \widehat{Hom}(V, K) + \widehat{Hom}(V, VX) = \widehat{Hom}(V, N),$$

implying $K + VX = N$ and $VX = N$ since $K \ll N$. Hence $X = \widehat{Hom}(V, N)$. This means $\widehat{Hom}(V, K) \ll \widehat{Hom}(V, N)$.

(3) $(a) \Rightarrow (b)$ A descending chain $X_1 \supset X_2 \supset \cdots$ of finitely generated T-submodules in $\widehat{Hom}(V, N)$ yields a descending chain $VX_1 \supset VX_2 \supset \cdots$ of finitely V-generated R-submodules of N. This chain terminates and there exists $k \in I\!N$ with $VX_k = VX_{k+l}$, for every $l \in I\!N$. Because of (2)(i), this also means $X_k = X_{k+l}$.

$(b) \Rightarrow (a)$ A descending chain $N_1 \supset N_2 \supset \cdots$ of finitely V-generated R-submodules in N gives rise to a descending chain of finitely generated T-submodules $\widehat{Hom}(V,N_1) \supset \widehat{Hom}(V,N_2) \supset \cdots$. If this chain terminates, this is also true for the chain $N_1 \supset N_2 \supset \cdots$ since $N_i = V\widehat{Hom}(V,N_i)$.

We have seen in 31.8 that the descending chain conditions for cyclic, resp. finitely generated, submodules are equivalent (notice 51.1,(6)).

(4) is shown with the same proof as (3).

Now we use the relations given in 51.3 to extend the characterizations of finitely generated perfect modules in 43.8 to arbitrary perfect modules:

51.4 Further characterizations of perfect modules.
For an R-module M which is projective in $\sigma[M]$, the following statements are equivalent:
(a) M is perfect in $\sigma[M]$ (§ 43);
(b) $M^{(\mathbb{N})}$ is semiperfect in $\sigma[M]$;
(c) $End(M^{(\mathbb{N})})$ is f-semiperfect;
(d) $M = \bigoplus_\Lambda M_\lambda$ with finitely generated M_λ, and
(i) every M-generated flat module is projective in $\sigma[M]$, or
(ii) indecomposable M-generated flat modules are projective in $\sigma[M]$, or
(iii) $M/Rad\,M$ is semisimple, and $Rad\,(M^{(\mathbb{N})}) \ll M^{(\mathbb{N})}$, or
(iv) the ring $T = \widehat{End}(M)$ is left perfect .

Proof: $(a) \Rightarrow (b) \Rightarrow (c)$ is evident by the definition and 42.12.

$(a) \Rightarrow (d)(i)$ The (semi-) perfect module M is a direct sum of finitely generated (local) modules (see 42.5). By 43.2, every M-generated, flat module is projective in $\sigma[M]$.

$(d)(i) \Leftrightarrow (ii)$ By 36.4, the flat factor modules of $M^{(\Lambda)}$, for any set Λ, are projective if and only if this is true for the indecomposable, flat factor modules of $M^{(\Lambda)}$.

$(c) \Rightarrow (d)(iii)$ By 42.12, we have $M = \bigoplus_\Lambda M_\lambda$, with finitely generated (cyclic) M_λ, and $Rad\,(M^{(\mathbb{N})}) \ll M^{(\mathbb{N})}$.

For every M_λ, $M_\lambda^{(\mathbb{N})}$ is a direct summand of $M^{(\mathbb{N})}$. So $End(M_\lambda^{(\mathbb{N})})$ is f-semiperfect (see proof of 49.9,$(g) \Rightarrow (h)$), and, by 43.8, M_λ is perfect in $\sigma[M_\lambda]$. In particular, $M_\lambda/Rad\,M_\lambda$ is semisimple.

Therefore $M/Rad\,M = \bigoplus_\Lambda (M_\lambda/Rad\,M_\lambda)$ is also semisimple.

$(d)(iii) \Rightarrow (iv)$ Denote for $S = End(M)$. Since $Hom_R(M,-)$ and $T\otimes_S -$ are exact functors, we have the exact sequence in T-MOD (see 51.2,(1))

$$0 \to \widehat{Hom}(M, Rad\,M) \to \widehat{Hom}(M,M) \to \widehat{Hom}(M, M/Rad\,M) \to 0.$$

From $Rad\,(M^{(\mathbb{N})}) \ll M^{(\mathbb{N})}$ we conclude $Rad\,M \ll M$, and hence, by 22.2, $Jac(S) = Hom(M, Rad\,M)$. By 51.1,(3), this yields

$$Jac(T) = T Jac(S) = T \cdot Hom(M, Rad\,M) = \widehat{Hom}(M, Rad\,M).$$

If $M/Rad\,M$ is semisimple, then each $M_\lambda/Rad\,M_\lambda$ is a semisimple R-module. Hence all $\widehat{Hom}(M, M_\lambda/Rad\,M_\lambda)$ and $T/Jac(T) \simeq \widehat{Hom}(M, M/Rad\,M)$ are semisimple T-modules.

From 51.3,(2) and $Rad\,(M^{(\mathbb{N})}) \ll M^{(\mathbb{N})}$, we derive

$$Rad\,(T^{(\mathbb{N})}) = \widehat{Hom}(M, Rad\,(M^{(\mathbb{N})})) \ll \widehat{Hom}(M, M^{(\mathbb{N})}) = T^{(\mathbb{N})}.$$

Hence T is left perfect by 49.9.

$(d)(i) \Rightarrow (iv)$ We show that T has dcc for cyclic right ideals (see 49.9):

Let $s_1T \supset s_1s_2T \supset s_1s_2s_3T \supset \cdots$ be a descending chain of cyclic right ideals, $s_i \in T$.

We apply 43.3 for $N_i = M$, $N = M^{(\mathbb{N})}$ and $\{s_i : M \to M\}_{\mathbb{N}}$. With the notation of 43.3, $N/Im\,g$ is a flat module and is projective (by (i)). Hence $Im\,g$ is a direct summand in N. So, for finitely many $m_1, \ldots, m_t \in M$, there exists $r \in \mathbb{N}$ and $h \in End(M)$ with $(m_i)s_1 \cdots s_{r-1} = (m_i)s_1 \cdots s_r h$ for $i = 1, \ldots, t$.

We choose an idempotent $e \in T$ with $s_r e = s_r$. Obviously $eh \in T$. Since $s_1 \neq 0$ only on a finite partial sum of M, we have

$$s_1 \cdots s_{r-1} = s_1 \cdots s_r(eh) \in s_1 \cdots s_r T$$

for some $r \in \mathbb{N}$. So our descending chain of cyclic right ideals is finite.

$(d)(iv) \Rightarrow (a)$ If T is left perfect, then, by 49.9, every module in T-MOD is supplemented. Hence the T-module $T^{(\Lambda)} \simeq \widehat{Hom}(M, M^{(\Lambda)})$ is supplemented for any Λ, and, by 51.3, the R-module $M^{(\Lambda)}$ is also supplemented. So $M^{(\Lambda)}$ is semiperfect (see 42.3) and M is perfect in $\sigma[M]$ (§ 43).

Later on we will be interested in $\widehat{Hom}(V, -)$ for V a generator in $\sigma[M]$. The following propositions prepare for this case:

51.5 $\widehat{Hom}(V, -)$ and V-generated modules.

Let $V = \bigoplus_A V_\alpha$ with finitely generated R-modules V_α, $T = \widehat{End}(V)$ and $N \in R$-MOD.

(1) The following assertions are equivalent:

(a) N is V-generated;

(b) for every R-module X and $0 \neq f \in Hom(N, X)$ there exists $h \in \widehat{Hom}(V, N)$

with $hf \neq 0$*;*

(c) for $\Lambda = \widehat{Hom}(V,N)$ *the canonical morphism*
$p: V^{(\Lambda)} \to N$, $(v_\lambda)_\Lambda \mapsto \sum_\Lambda (v_\lambda)\lambda$, *is epic;*

(d) $N = Tr(V,N) = V\widehat{Hom}(V,N)$.

(2) If N *is* V*-generated and* $p: V^{(\Lambda)} \to N$ *is as in (1)(c), then* $\widehat{Hom}(V,p): \widehat{Hom}(V,V^{(\Lambda)}) \to \widehat{Hom}(V,N)$ *is epic.*

(3) If $\widehat{Hom}(V,N)$ *is a finitely generated* T*-module, then* $Tr(V,N)$ *is a finitely generated* R*-module.*

Proof: (1) The equivalences are obtained by using the properties of $\{V_\alpha\}_A$ as a generating set for N (see 13.3).

(2) For $f \in \widehat{Hom}(V,N) = \Lambda$, the diagram

$$\begin{array}{ccccc} & & V & & \\ & & \downarrow f & & \\ V^{(\Lambda)} & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}$$

can be extended commutatively by the canonical injection $\varepsilon_f : V \to V^{(\Lambda)}$. If $f \neq 0$ only on a finite partial sum $V_o \subset V$, then we can restrict ε_f to V_o.

(3) If $\widehat{Hom}(V,N)$ is a finitely generated T-module, then there exist finitely many idempotents $e_1,\dots,e_k \in T$ yielding an epimorphism
$\bigoplus_{i\leq k} Te_i \to \widehat{Hom}(V,N)$ (see 49.2).
Applying the functor $V\otimes_T -$ and using 49.3, we obtain an epimorphism

$$\bigoplus_{i\leq k} Ve_i \simeq V\otimes_T (\bigoplus_{i\leq k} Te_i) \to Tr(V,N),$$

where each Ve_i is a finitely generated R-module.

A left adjoint functor to the $\widehat{Hom}$-functor is obtained in the same way as to the usual *Hom*-functor:

51.6 The pair of functors $V\otimes_T -$, $\widehat{Hom}(V,-)$.
Let M *be an* R*-module,* $V = \bigoplus_A V_\alpha$ *with finitely generated* $V_\alpha \in \sigma[M]$ *and* $T = \widehat{End}(V)$, $S = End(V)$.

(1) The functor $V\otimes_T - : T\text{-}MOD \to \sigma[M]$
is left adjoint to $\widehat{Hom}(V,-): \sigma[M] \to T\text{-}MOD$
with (functorial) isomorphisms for $L \in T\text{-}MOD$, $N \in \sigma[M]$,

$$\psi_{L,N} : Hom_R(V\otimes_T L, N) \to Hom_T(L, \widehat{Hom}(V,N)),\ \ \delta \mapsto [l \mapsto (-\otimes l)\delta].$$

For $L = T$ and $N = V$ we obtain a ring isomorphism

$$\psi_{T,V} : End(_RV) \to End(_TT).$$

(2) The corresponding (functorial) morphisms are

$$\nu_L : L \to \widehat{Hom}(V, V \otimes_T L), \quad l \mapsto [v \mapsto v \otimes l],$$
$$\mu_N : V \otimes_T \widehat{Hom}(V, N) \to N, \quad v \otimes f \mapsto (v)f.$$

(i) For every projective $L \in T$-MOD, ν_L is an isomorphism.
(ii) For every V-generated $N \in \sigma[M]$, μ_N is epic.
For any direct summand N of $V^{(\Lambda)}$, Λ any set, μ_N is an isomorphism.

(3) For every $N \in \sigma[M]$, the composition of the mappings

$$\widehat{Hom}(V, N) \xrightarrow{\nu_{\widehat{Hom}(V,N)}} \widehat{Hom}(V, V \otimes_T \widehat{Hom}(V, N)) \xrightarrow{\widehat{Hom}(V,\mu_N)} \widehat{Hom}(V, N)$$

is the identity on $\widehat{Hom}(V, N)$.

(4) $\widehat{Hom}(V, -)$ is an equivalence between the full subcategory of R-MOD whose objects are direct summands of $V^{(\Lambda)}$, Λ any set, and the full subcategory of projective modules in T-MOD (with inverse $V \otimes_T -$).

Proof: (1) For $L \in T$-*MOD* we have $L = TL$, and so, by 51.2,

$$\begin{aligned} Hom_T(L, Hom_R(V, N)) &= Hom_T(L, T\,Hom_R(V, N)) \\ &= Hom_T(L, \widehat{Hom}_R(V, N)). \end{aligned}$$

Since the Hom-tensor relations 12.12 are also valid for rings without unit, we can transfer the considerations of 45.8 to the situation given here.

By the isomorphism $V \otimes_T T \simeq V$ (see 49.3 and 51,1,(4)), we obtain the ring isomorphism desired.

(2) (i) Because of $V \otimes_T T \simeq V$, ν_T is an isomorphism. Since $V \otimes_T -$ and $\widehat{Hom}(V, -)$ both preserve direct sums, ν_L is also an isomorphism for projective $L \in T$-*MOD*.

(ii) Because $Im\, \mu_N = V\widehat{Hom}(V, N) = Tr(V, N)$ (see 51.5), μ_N is epic for any V-generated N.

Again it is obvious that μ_V is an isomorphism. This isomorphism can be extended to direct summands of $V^{(\Lambda)}$.

(3) The assertion is easily verified by the definitions (see also 45.5).

(4) Because of (2), it suffices to show that the given categories are mutually the image of each other under the corresponding functors.

The morphism μ_N given in 51.6,(2) becomes an isomorphism, whenever V is a generator in $\sigma[M]$. In this case we obtain a strong relationship between the modules in $\sigma[M]$ and T-MOD:

51.7 $\widehat{Hom}(V,-)$ with V a generator in $\sigma[M]$.
Let M be an R-module, $V = \bigoplus_A V_\alpha$ with finitely generated $V_\alpha \in \sigma[M]$ and $T = \widehat{E}nd(V)$. If V is a generator in $\sigma[M]$, then:

(1) V_T is flat in MOD-T (see 49.5).

(2) V_T is faithfully flat (with respect to T-MOD) if and only if ${}_RV$ is M-projective.

(3) For every $N \in \sigma[M]$, $\mu_N : V \otimes_T \widehat{Hom}(V,N) \to N$ and

$$\nu_{\widehat{Hom}(V,N)} : \widehat{Hom}_R(V,N) \to \widehat{Hom}_R(V, V \otimes_T \widehat{Hom}_R(V,N))$$

are isomorphisms.

(4) If $0 \to L \to P_1 \to P_0$ is exact, with P_1, P_0 projective in T-MOD, then $\nu_L : L \to \widehat{Hom}_R(V, V \otimes_T L)$ is an isomorphism.

(5) $N \in \sigma[M]$ is indecomposable if and only if $\widehat{Hom}(V,N)$ is indecomposable.

(6) $N \in \sigma[M]$ is Q-injective, for $Q \in \sigma[M]$, if and only if $\widehat{Hom}(V,N)$ is $\widehat{Hom}(V,Q)$-injective.

(7) $\widehat{Hom}(V,-)$ preserves essential extensions.

(8) Consider an idempotent $e \in T$. Then:

(i) If Te is coherent in T-MOD, then Ve is coherent in $\sigma[M]$.

(ii) If Te is noetherian, then Ve is a noetherian R-module.

(iii) If Te is finitely cogenerated (as a T-module), then Ve is a finitely cogenerated R-module.

(9) If M is locally of finite length, then T is semiperfect.

Proof: (1) We have to show that for every left ideal $I \subset T$ the canonical map $V \otimes_T I \to VI$ is an isomorphism (see 49.5).

As a generator in $\sigma[M]$, V is flat over its endomorphism ring $S = End(V)$ (see 15.9). Since T_S is a projective S-module (see 51.1), we obtain the isomorphisms

$$V \otimes_T I \simeq V \otimes_T T \otimes_S SI \simeq V \otimes_S SI \simeq VSI = VI.$$

(2) If ${}_RV$ is M-projective, then, by 51.3, $VI \neq V$ for every finitely generated proper left ideal $I \subset T$, and therefore $V \otimes_T K \neq 0$ for all non-zero

modules $K \in T\text{-}MOD$. This is shown with the corresponding proof in 12.17, which remains valid for rings with enough idempotents.

On the other hand, let V_T be faithfully flat. By a simple variation of the proof in 18.5 we can show that $\widehat{Hom}(V,-)$ is exact.

(3) Let $N \in \sigma[M]$. If $\Lambda = \widehat{Hom}(V,N)$ and $p : V^{(\Lambda)} \to N$ is the canonical epimorphism, then $\widehat{Hom}(V,p)$ is epic (see 51.5,(2)) and we obtain the commutative exact diagram

$$\begin{array}{cccccccc} & V \otimes \widehat{Hom}(V, Kep) & \to & V \otimes \widehat{Hom}(V, V^{(\Lambda)}) & \to & V \otimes \widehat{Hom}(V,N) & \to 0 \\ & \downarrow \mu_{Kep} & & \downarrow \mu_{V^{(\Lambda)}} & & \downarrow \mu_N & \\ 0 \to & Kep & \to & V^{(\Lambda)} & \xrightarrow{p} & N & \to 0. \end{array}$$

Since V is a generator, μ_{Kep} and μ_N are epic, and $\mu_{V^{(\Lambda)}}$ is an isomorphism by 51.6. Then μ_N is also monic by the Kernel Cokernel Lemma. With a similar proof, (1) can also be proved directly.

Because μ_N is an isomorphism, $\widehat{Hom}(V,\mu_N)$ is also an isomorphism. Since the composition of $\nu_{\widehat{Hom}(V,N)}$ and $\widehat{Hom}(V,\mu_N)$ yields the identity on $\widehat{Hom}(V,N)$ (see 51.6), $\nu_{\widehat{Hom}(V,N)}$ has to be an isomorphism.

(4) Since $V \otimes_T -$ is exact, we have the commutative exact diagram

$$\begin{array}{ccccccc} 0 & \to & L & \to & P_1 & \to & P_0 \\ & & \downarrow \nu_L & & \downarrow \nu_{P_1} & & \downarrow \nu_{P_0} \\ 0 & \to & \widehat{Hom}(V, V \otimes_T L) & \to & \widehat{Hom}(V, V \otimes_T P_1) & \to & \widehat{Hom}(V, V \otimes_T P_0)\ . \end{array}$$

Since ν_{P_0} and ν_{P_1} are isomorphisms by 51.6, this is also true for ν_L.

(5) If $N = N_1 \oplus N_2$, then $\widehat{Hom}(V,N) = \widehat{Hom}(V,N_1) \oplus \widehat{Hom}(V,N_2)$. Since V is a generator, we have $\widehat{Hom}(V,N_i) \neq 0$ for $N_i \neq 0$ $(i = 1,2)$.

If $\widehat{Hom}(V,N) \simeq L_1 \oplus L_2$ with $L_1,\ L_2 \neq 0$, then

$$N \simeq V \otimes_T \widehat{Hom}(V,N) \simeq (V \otimes_T L_1) \oplus (V \otimes_T L_2).$$

Since $V \otimes_T L_i \to VL_i \neq 0$ is epic, we conclude $V \otimes_T L_i \neq 0$ (for $i = 1,2$).

(6) By 51.6, we have a functorial isomorphism

$$Hom_R(V \otimes_T -, N) \longrightarrow Hom_T(-, \widehat{Hom}(V,N)).$$

Let $0 \to X \to \widehat{Hom}(V,Q)$ be an exact sequence in $T\text{-}MOD$. Construct the exact sequence $0 \to V \otimes_T X \to V \otimes_T \widehat{Hom}(V,Q) \simeq Q$ and the commutative diagram

$$\begin{array}{ccccc} Hom_T(\widehat{Hom}(V,Q), \widehat{Hom}(V,N)) & \longrightarrow & Hom_T(X, \widehat{Hom}(V,N)) & \longrightarrow & 0 \\ \downarrow \simeq & & \downarrow \simeq & & \\ Hom_R(Q,N) & \longrightarrow & Hom_R(V \otimes_T X, N) & \longrightarrow & 0 \end{array}\ .$$

If N is Q-injective, then the bottom row is exact and therefore the top row is also exact and $\widehat{Hom}(V,N)$ is $\widehat{Hom}(V,Q)$-injective.

Observing the fact that every submodule of Q is of the form $V \otimes_T X$ for a suitable $X \in T\text{-}MOD$, the reverse conclusion is also derived from the above diagram.

(7) Let $K \subset N$ be an essential submodule, $N \in \sigma[M]$ and $0 \neq f \in \widehat{Hom}(V,N)$. Because of $Im\, f \cap K \neq 0$, we have also $0 \neq (K)f^{-1} \subset V$. Now $(K)f^{-1}$ is generated by V and hence there exists $g \in \widehat{Hom}(V,(K)f^{-1})$ with $0 \neq gf \in \widehat{Hom}(V,K)$ (see 51.5). Thus $\widehat{Hom}(V,K) \cap Tf \neq 0$ and $\widehat{Hom}(V,K)$ is essential in $\widehat{Hom}(V,N)$.

(8)(i) Let Te be coherent in $T\text{-}MOD$. We have to show that, for every finite sum $\bigoplus Ve_i$, with $e_i^2 = e_i \in T$, the module K in an exact sequence

$$0 \longrightarrow K \longrightarrow \bigoplus Ve_i \longrightarrow Ve$$

is finitely generated (see § 26):

Applying $\widehat{Hom}(V,-)$ we obtain the exact sequence

$$0 \longrightarrow \widehat{Hom}(V,K) \longrightarrow \bigoplus \widehat{Hom}(V,V)e_i \longrightarrow \widehat{Hom}(V,V)e.$$

Then $\widehat{Hom}(V,K)$ is finitely generated since $\widehat{Hom}(V,V)e = Te$ is coherent in $T\text{-}MOD$. By 51.5, K is also finitely generated.

(ii) Let K be a submodule of Ve. Then $\widehat{Hom}(V,K) \subset \widehat{Hom}(V,Ve) = Te$ is a finitely generated T-module and consequently $K = V\widehat{Hom}(V,K)$ is a finitely generated R-module by 51.5,(3).

(iii) Let $\{N_\lambda\}_\Lambda$ be a family of modules in $\sigma[M]$ and $0 \to Ve \to \prod_\Lambda^M N_\lambda$ exact. Then $0 \to \widehat{Hom}(V,Ve) \to \prod_\Lambda^T \widehat{Hom}(V,N_\lambda)$ is also exact (see 51.2). Since $\widehat{Hom}(V,V)e = Te$ is finitely cogenerated, we can find a finite subset $E \subset \Lambda$ such that the sequence $0 \to Te \to \prod_E \widehat{Hom}(V,N_\lambda)$ is exact. Applying $V \otimes_T -$ we obtain the exact sequence

$$0 \to Ve \to \prod\nolimits_E V \otimes_T \widehat{Hom}(V,N_\lambda) \simeq \prod\nolimits_E N_\lambda\ .$$

So Ve is finitely cogenerated.

(9) For every primitive idempotent $e \in T$, Ve is an indecomposable module of finite length. By 32.4, $End(Ve) \simeq End(Te) \simeq eTe$ is a local ring. Therefore for every idempotent $f \in T$, Tf is a direct sum of local modules and hence is semiperfect, i.e. T is semiperfect (see 49.10).

If V is a generator not only in $\sigma[M]$ but in the full category R-MOD, then V is in fact finitely generated and projective over its endomorphism ring and we can improve some of the above results:

51.8 $\widehat{Hom}(V,-)$ with V a generator in R-MOD.

Let $V = \bigoplus_A V_\alpha$ with finitely generated R-modules V_α, $T = \widehat{End}(V)$, $S = End(V)$ and $\{e_\alpha\}_A$ the canonical complete family of idempotents in T (see 51.1). Assume V is a generator in R-MOD. Then:

(1) V_T is finitely generated projective in MOD-T and MOD-T $= \sigma[V_T]$.

(2) $End(V_T) \simeq End(V_S) \simeq R$.

(3) Assume $T/Jac\,T$ is left semisimple and let K be a simple R-module. If there exists $\beta \in A$ such that $V_\beta \to K$ is a projective cover for K, then $\widehat{Hom}(V,K)$ has a simple, essential socle.

(4) Again assume $T/Jac\,T$ is left semisimple. If the family $\{V_\alpha\}_A$ contains projective covers for all simple R-modules, then for any idempotent $e \in T$ the following assertions are equivalent:

(a) Te is finitely cogenerated in T-MOD;

(b) Ve is finitely cogenerated in R-MOD.

Proof: (1) Since V is a generator in R-MOD, there are finitely many $\alpha_1, \ldots, \alpha_k \in A$, such that R is a direct summand of $\bigoplus_{i\leq k} Ve_{\alpha_i}$. Then $V_T \simeq Hom_R(R,V)$ is a direct summand in $Hom(\bigoplus_{i\leq k} Ve_{\alpha_i}, V) = \bigoplus_{i\leq k} e_{\alpha_i}T$, and V_T is finitely generated and T-projective (see 49.2).

The relation $\sigma[V_T] = MOD$-T has been shown in 51.1.

(2) Let $f \in End(V_T)$, $u \in V$ and $e^2 = e \in T$ with $ue = u$ and $f(u)e = f(u)$. For $s \in S$, we have $es \in T$ and $f(us) = f(ues) = f(u)es = f(u)s$.

Hence $f \in End(V_S)$ and $End(V_S) = End(V_T)$.

For generators V in R-MOD we have, by 18.8, $R \simeq End(V_S)$.

(3) Let K be a simple R-module with injective hull $\widehat{K}$ and projective cover $p : V_\beta \to K$.

Since $\widehat{Hom}(V,K) \subset \widehat{Hom}(V,\widehat{K})$, and the latter module is an injective T-module with local endomorphism ring, every submodule of $\widehat{Hom}(V,K)$ is essential (see 19.9, 51.7).

As a projective T-module with local endomorphism ring, $\widehat{Hom}(V,V_\beta)$ is a local module. If we show that $Jac\,T$ is in the kernel of the map

$$\widehat{Hom}(V,V) \xrightarrow{\widehat{Hom}(V,e_\beta)} \widehat{Hom}(V,V_\beta) \xrightarrow{\widehat{Hom}(V,p)} \widehat{Hom}(V,K)\ ,$$

then the image of this map is a (semi) simple submodule of $\widehat{Hom}(V,K)$, and the proof is complete.

Let us assume $te_\beta p \neq 0$ for some $t \in Jac\,T$. Then $te_\beta p$, and $te_\beta : V \to V_\beta$, have to be epic. Since V_β is projective by assumption, this map splits, i.e. there exists $h \in T$ with $hte_\beta = e_\beta \in Jac\,T$. This is a contradiction to the fact that $Jac\,T$ contains no idempotents.

(4) $(a) \Rightarrow (b)$ follows from 51.7,(8).

$(b) \Rightarrow (a)$ If Ve is a finitely cogenerated R-module, then $Soc\,Ve$ is finitely generated and essential in Ve. If P is a projective cover of $Soc\,Ve$, then, by (3), we conclude that $\widehat{Hom}(V, Soc\,Ve)$ has a finitely generated, essential socle. We know from 51.7,(7), that this socle is also essential in $\widehat{Hom}(V, Ve) = Te$, i.e. Te is finitely cogenerated.

Before investigating $\widehat{Hom}(V,-)$ for finitely presented V_α, we state the following definition:

Let $\{V_\alpha\}_A$ be a family of R-modules and $V = \bigoplus_A V_\alpha$. We say that an R-module N is *V-supported*, if N is a direct limit of a direct system of modules $\{N_\lambda\}_\Lambda$, where the N_λ's are direct sums of V_α's. Of course, every V-supported module is also V-generated.

51.9 V-supported modules with V_α finitely presented.
Let M be an R-module and $V = \bigoplus_A V_\alpha$ with V_α finitely presented in $\sigma[M]$.

(1) For an R-module N the following assertions are equivalent:

(a) N is V-supported;

(b) $N \simeq \varinjlim N_\lambda$, where the N_λ's are direct summands of direct sums of V_α's;

(c) there exists a pure epimorphism $\bigoplus_B V_\beta \to N$, $V_\beta \in \{V_\alpha\}_A$;

(d) N is generated by finitely presented modules in $\sigma[M]$ and any morphism $P \to N$, with P finitely presented, can be factorized by $P \to \bigoplus_{i\leq k} V_{\alpha_i} \to N$.

(2) Let $0 \to N' \to N \to N'' \to 0$ be a pure exact sequence in $\sigma[M]$. Then N is V-supported if and only if N' and N'' are V-supported.

(3) A V-supported module is pure projective in $\sigma[M]$ if and only if it is a direct summand of a direct sum of V_α's.

Proof: (1) $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (c)$ If $N = \varinjlim N_\lambda$, the canonical map $\bigoplus_\Lambda N_\lambda \to N$ is a pure epimorphism (see 33.9). Assume every N_λ to be a direct summand of some $\overline{V}_\lambda = V_{\alpha_1} \oplus \cdots \oplus V_{\alpha_k}$. Then $\bigoplus_\Lambda N_\lambda$ is a direct summand of $\bigoplus_\Lambda \overline{V}_\lambda \simeq \bigoplus_B V_\beta$ with a suitable index set B and $V_\beta \in \{V_\alpha\}_A$. From this we obtain a pure epimorphism $\bigoplus_B V_\beta \to N$ (see 33.2,(1)).

$(c) \Rightarrow (a)$ follows from 34.2,(2).

$(c) \Rightarrow (d)$ Let P be finitely presented and $f : P \to N$. Then we can find some $g : P \to \bigoplus_B V_\beta$ to extend the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow f & & \\ \bigoplus_B V_\beta & \longrightarrow & N & \longrightarrow & 0, \end{array}$$

commutatively. Now $Im\, g$, as a finitely generated submodule, is contained in a finite partial sum.

$(d) \Rightarrow (c)$ Since N is generated by finitely presented modules, there exists a pure epimorphism $h : \bigoplus_\Lambda P_\lambda \to N$ with finitely presented P_λ (see 33.5). Denote by $\varepsilon_\lambda : P_\lambda \to \bigoplus_\Lambda P_\lambda$ the canonical injections. Then, for every $\varepsilon_\lambda h : P_\lambda \to N$, there exists a factorization $P_\lambda \xrightarrow{f_\lambda} \overline{V}_\lambda \xrightarrow{g_\lambda} N$ with $\overline{V}_\lambda = V_{\alpha_1} \oplus \cdots \oplus V_{\alpha_k}$. We obtain the commutative diagram

$$\begin{array}{ccccc} \bigoplus_\Lambda P_\lambda & & \xrightarrow{h} & & N \\ & f \searrow & & \nearrow g & \\ & & \bigoplus_\Lambda \overline{V}_\lambda & & . \end{array}$$

Since $h = fg$ is a pure epimorphism, g is also pure by 33.2, and $\bigoplus_\Lambda \overline{V}_\lambda \simeq \bigoplus_B V_\beta$ for some set B and $V_\beta \in \{V_\alpha\}_A$.

(2) Let $0 \to N' \to N \to N'' \to 0$ be pure exact in $\sigma[M]$. If N is V-supported, then there exists a pure epimorphism $h : \bigoplus_B V \to N$. Then the composition $\bigoplus_B V \to N \to N''$ is also a pure epimorphism and therefore N'' is V-supported. With a pullback we get the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \bigoplus_B V_\beta & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \downarrow h' & & \downarrow h & & \| & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & , \end{array}$$

where h' is a pure epimorphism and the first row is pure (see 33.4). Hence it suffices to show that K is V-supported.

For a finitely generated submodule $K_o \subset K$ we construct the commutative exact diagram, with a finite subset $E \subset B$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_o & \longrightarrow & \bigoplus_E V_\beta & \longrightarrow & (\bigoplus_E V_\beta)/K_o & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & \bigoplus_B V_\beta & \longrightarrow & N'' & \longrightarrow & 0 \ . \end{array}$$

Since $(\bigoplus_E V_\beta)/K_o$ is finitely presented, it can be completed in a commutative way by morphisms

$\bigoplus_E V_\beta/K_o \to \bigoplus_B V_\beta$ and $\bigoplus_E V_\beta \to K$ (Homotopy Lemma).

From this we see that every morphism $P \to K$, with P finitely generated, can be factorized through a finite sum of V_α's. So K is generated by $\{V_\alpha\}_A$ and is V-supported by (1).

Now assume both N' and N'' to be V-supported and consider a pure epimorphism $\bigoplus_B V_\beta \to N''$. Forming a pullback, we obtain the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N' & \longrightarrow & Q & \longrightarrow & \bigoplus_B V_\beta & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & ,
\end{array}$$

where the vertical morphisms and the first row are pure. Since $\bigoplus_B V_\beta$ is pure projective, we have $Q \simeq N' \oplus (\bigoplus_B V_\beta)$, and Q is V-supported. Then N is also V-supported.

(3) Since each V_α is pure projective, this follows from (1)(c).

51.10 $\widehat{Hom}(V,-)$ with V_α finitely presented.

Let M be an R-module, $V = \bigoplus_A V_\alpha$ with V_α finitely presented in $\sigma[M]$ and $T = \widehat{End}(V)$.

(1) A module ${}_TL$ is flat in T-MOD if and only if $L \simeq \widehat{Hom}(V,N)$ for some V-supported R-module N ($\simeq V \otimes_T L$).

(2) A module ${}_TL$ is projective in T-MOD if and only if $L \simeq \widehat{Hom}(V,P)$ for some V-supported, pure projective R-module P ($\simeq V \otimes_T L$).

(3) $\widehat{Hom}(V,-)$ transforms pure sequences of V-supported modules into pure sequences of T-MOD.

(4) For a V-supported module N, the following are equivalent:
(a) N is finitely presented in $\sigma[M]$;
(b) $\widehat{Hom}(V,N)$ is finitely presented in T-MOD.

(5) The functor $\widehat{Hom}(V,-)$ induces equivalences, with inverse $V \otimes_T -$, between the categories of
(i) the V-supported, pure projective modules in $\sigma[M]$ and the projective modules in T-MOD,
(ii) the V-supported R-modules and the flat modules in T-MOD.

Proof: (1) If L is flat in T-MOD, then $L \simeq \varinjlim F_\lambda$ with F_λ finitely generated and quasi-free (see 49.5). By 51.6, $\nu_{F_\lambda} : F_\lambda \to \widehat{Hom}(V, V \otimes_T F_\lambda)$ is an isomorphism, and so we have

$$L \simeq \varinjlim F_\lambda \simeq \varinjlim \widehat{Hom}(V, V \otimes_T F_\lambda) \simeq \widehat{Hom}(V, V \otimes_T L),$$

because $\widehat{Hom}(V, -)$ and $V \otimes_T -$ both preserve direct limits (see 51.2).

If $F_\lambda \simeq Te_{\alpha_1} \oplus \cdots \oplus Te_{\alpha_k}$, with canonical idempotents $e_{\alpha_i} \in T$ (see 51.1), then $V \otimes_T F_\lambda \simeq Ve_{\alpha_1} \oplus \cdots \oplus Ve_{\alpha_k}$ and $V \otimes_T F_\lambda$ is a finite direct sum of V_α's. Hence $V \otimes_T L \simeq \varinjlim V \otimes_T F_\lambda$ and $V \otimes_T L$ is V-supported.

On the other hand, let N be a V-supported module, i.e. $N = \varinjlim N_\lambda$, with each N_λ a finite direct sum of V_α's. Then $\widehat{Hom}(V, N_\lambda)$ is a quasi-free T-module and $\widehat{Hom}(V, N) \simeq \varinjlim \widehat{Hom}(V, N_\lambda)$ is flat in T-MOD.

(2) If $_TL$ is projective, then $_TL$ is a direct summand of a direct sum of quasi-free T-modules and $V \otimes_T L$ is a direct summand of a direct sum of V_α's. Hence it is pure projective (see 51.9,(3)).

On the other hand, any module of this form is obviously turned into a projective module under $\widehat{Hom}(V, -)$ (see 51.6,(4)).

(3) Since each V_α is finitely presented, a pure sequence

$0 \to N' \to N \to N'' \to 0$ in $\sigma[M]$ becomes an exact sequence

$0 \to \widehat{Hom}(V, N') \to \widehat{Hom}(V, N) \to \widehat{Hom}(V, N'') \to 0$ in T-MOD.

If N'' is V-supported, then, by (1), $\widehat{Hom}(V, N'')$ is flat in T-MOD and the above sequence is pure.

(4) Let N be a V-supported module.

$(a) \Rightarrow (b)$ If N is finitely presented, then N is a direct summand of a finite sum of V_α's and $\widehat{Hom}(V, N)$ is a direct summand of a finitely generated, quasi-free T-module.

$(b) \Rightarrow (a)$ If $\widehat{Hom}(V, N)$ is finitely presented, then it is projective by (1), and hence a direct summand of a finitely generated quasi-free T-module. Then N is a direct summand of a finite direct sum of V_α's and so is finitely presented.

(5) The V-supported, pure projective modules are exactly the direct summands of $V^{(\Lambda)}$, for suitable Λ, and the first equivalence was already pointed out in 51.6,(4). Because of (1), the two last mentioned categories correspond under $\widehat{Hom}(V, -)$.

As we have seen in 46.2, $\sigma[M]$ is equivalent to S-MOD, S a ring with unit, if and only if there is a finitely generated, projective generator in $\sigma[M]$.

Now similarly we may ask, when $\sigma[M]$ is equivalent to T-MOD for a ring T with local units or with enough idempotents:

51.11 Equivalence of $\boldsymbol{\sigma[M]}$ to $\boldsymbol{T}$-$\boldsymbol{MOD}$.

(1) For an R-module M the following assertions are equivalent:

(a) There exists a ring T with enough idempotents, such that $\sigma[M]$ is equivalent to T-MOD;

(b) there exists a ring T with local units, such that $\sigma[M]$ is equivalent to T-MOD;

(c) there exists a generating set $\{P_\alpha\}_A$ of finitely generated, projective P_α in $\sigma[M]$;

(d) there exists a generator $P = \bigoplus_A P_\alpha$, with finitely generated P_α in $\sigma[M]$, such that P_T is faithfully flat over $T = \widehat{End}(P)$.

(2) Let T be a ring with local units and $F : \sigma[M] \to T$-MOD an equivalence with inverse $G : T$-$MOD \to \sigma[M]$.

Then $G(T)$ is an (R,T)-bimodule with $G(T)T = G(T)$ and

$$G \simeq G(T) \otimes_T -, \quad F \simeq T Hom_R(G(T), -).$$

Proof: (1) $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (c)$ We have shown in 49.1 that, for a ring T with local units, $\{Te \,|\, e^2 = e \in T\}$ is a generating set of finitely generated, projective modules in T-MOD. In an equivalent category there has to be a generating set with the same properties.

$(c) \Leftrightarrow (d)$ Let $\{P_\alpha\}_A$ be a generating set of finitely generated modules in $\sigma[M]$, $P = \bigoplus_A P_\alpha$ and $T = \widehat{End}(P)$. Then, by 51.7, P_T is faithfully flat, if and only if ${}_RP$ is M-projective.

$(d) \Rightarrow (a)$ Let $P = \bigoplus_A P_\alpha$ be a projective generator, with finitely generated $P_\alpha \in \sigma[M]$, and $T = \widehat{End}(P)$. By 51.7, for every $N \in \sigma[M]$, the map $\mu_N : P \otimes_T \widehat{Hom}(P,N) \to N$ is an isomorphism. For $L \in T$-MOD, we have an exact sequence $F_1 \to F_0 \to L \to 0$ with projective (quasi-free) F_1, F_0 in T-MOD (see 49.2,(1)) and we construct the commutative exact diagram

$$\begin{array}{ccccccc}
F_1 & \longrightarrow & F_0 & \longrightarrow & L & \longrightarrow 0 \\
\downarrow \nu_{F_1} & & \downarrow \nu_{F_0} & & \downarrow \nu_L & \\
\widehat{Hom}(P, P \otimes_T F_1) & \longrightarrow & \widehat{Hom}(P, P \otimes_T F_0) & \longrightarrow & \widehat{Hom}(P, P \otimes_T L) & \longrightarrow 0
\end{array} .$$

Since ν_{F_1} and ν_{F_0} are isomorphisms, this is also true for ν_L. Thus $\widehat{Hom}(P,-) : \sigma[M] \to T$-$MOD$ is an equivalence with inverse $P \otimes_T -$.

(2) The (R,T)-bimodule structure of $G(T)$ follows from the relation $T \subset Hom_T(T,T) \simeq Hom_R(G(T), G(T))$.

Because $G(T) = G(\varinjlim Te) = \varinjlim G(Te)$ (compare proof of 49.1,(1.*ii*)), every $q \in G(T)$ is contained in a suitable $G(Te) = G(T)e$, $e^2 = e \in T$. So we have $q = qe \in G(T)T$ and therefore $G(T)T = G(T)$.

Now, by 49.3, $G(T) \otimes_T T \simeq G(T)$ and, with the proof of 45.7,(1), we can show that $G \simeq G(T) \otimes_T -$.

By the functorial isomorphisms in 12.12, for $L \in T\text{-}MOD$, $N \in \sigma[M]$,

$$\begin{aligned} Hom_R(G(T) \otimes_T L, N) &\simeq Hom_T(L, Hom_R(G(T), N)) \\ &= Hom_T(L, T\,Hom_R(G(T), N)), \end{aligned}$$

the functor $T\,Hom_R(G(T), -) : \sigma[M] \to T\text{-}MOD$ is adjoint to G, implying $F \simeq T\,Hom_R(G(T), -)$.

In 18.12 we encountered a category $\sigma[M]$ without any projective objects. We will see in the next theorem that the existence of projective objects can be derived from certain finiteness conditions. For this end we need:

51.12 Modules annihilated by $M \otimes_T -$.

Assume $M = \bigoplus_\Lambda M_\lambda$, with all M_λ finitely generated, is a generator in $\sigma[M]$ and $T = \widehat{End}_R(M)$ is a right perfect ring. Then:

(1) The class

$$\mathcal{T} = \{X \in T\text{-}MOD \mid M \otimes_T X = 0\}$$

is closed under direct sums, submodules, factor modules and extensions.

(2) There exist an injective module Q and a projective module P in T-MOD, with the properties

$$\begin{aligned} \mathcal{T} &= \{X \in T\text{-}MOD \mid Hom_T(X, Q) = 0\} \\ &= \{X \in T\text{-}MOD \mid Hom_T(P, X) = 0\} \quad . \end{aligned}$$

(3) P is a direct sum of cyclic local T-modules, $M \otimes_T P$ is a direct sum of finitely generated R-modules, and $\widehat{End}_R(M \otimes_T P) \simeq \widehat{End}_T(P)$.

Proof: (1) Since M_T is flat by 51.7, the properties of $\mathcal{T}$ are easily verified.

(2) Let $\mathcal{E}$ denote a representative set of all simple modules in T-MOD not contained in $\mathcal{T}$, and denote by Q the injective hull of the direct sum of all objects in $\mathcal{E}$.

Assume $0 \neq f \in Hom_T(X,Q)$ for some $X \in \mathcal{T}$. Since Q has essential socle, we may find a submodule $X' \subset X$ with $(X')f = E$ and $E \in \mathcal{E}$. Since $(X')f \in \mathcal{T}$, this is a contradiction and hence

$\mathcal{T} \subset \{X \in T\text{-}MOD \mid Hom_T(X,Q) = 0\}$.

Now assume $Hom_T(X,Q) = 0$ for a non-zero T-module X. X has non-zero socle (see 49.9) and, by definition of Q, $Soc(X)$ belongs to $\mathcal{T}$.

Consider the ascending Loewy series $\{L_\alpha(X)\}_{\alpha\geq 0}$ of X with $L_0(X) = 0$, $L_1(X) = Soc(X)$ and so on (see 32.6).

If $L_1(X) \neq X$, then $Hom(X/L_1(X), Q) = 0$ and, as above, we see that $Soc(X/L_1(X)) = L_2(X)/L_1(X)$ belongs to $\mathcal{T}$. Since $\mathcal{T}$ is closed under extensions, we conclude $L_2(X) \in \mathcal{T}$. By transfinite induction we obtain that $L_\alpha(X) \in \mathcal{T}$ for all ordinals α. Since ${}_TT$ has *dcc* on finitely generated left ideals, we know (by 32.6) that $X = L_\gamma(X)$ for some ordinal γ. Hence $X \in \mathcal{T}$ and the first equality is established.

By 49.10, T is also left semi-perfect and hence every module in $\mathcal{E}$ has a projective cover in T-*MOD*. We denote the direct sum of all these projective covers by P.

Assume $0 \neq f \in Hom_T(P,X)$ for some $X \in T$-*MOD*. Then we have $(P')f \neq 0$ for one of the local summands P' of P. For a maximal submodule $K \subset (P')f$, we have an epimorphism

$$P' \to (P')f \to (P')f/K.$$

Since a local module has only one simple factor module (see 19.7), $(P')f/K$ cannot be in $\mathcal{T}$ (by construction of P) and hence is isomorphic to a submodule of Q. Thus we have a non-zero morphism $X \supset (P')f \to Q$. Since Q is injective this can be extended to a non-zero morphism $X \to Q$.

Finally, consider $0 \neq g \in Hom_T(X,Q)$ for $X \in T$-*MOD*. Then, for some simple submodule $E \subset Q$, we find a submodule $X' \subset X$ with $(X')g = E$ and we have the diagram with exact line

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ X' & \longrightarrow & E & \longrightarrow & 0 \end{array}\ .$$

By projectivity of P, this can be extended commutatively by a (non-zero) morphism $P \to X' \subset X$, establishing the second equality.

(3) The properties of P are clear by construction. Obviously, for every idempotent $e \in T$, $M \otimes_T Te \simeq Me$ is a finitely generated R-module. Hence, for any finitely generated, projective T-module ${}_TP'$, $M_\lambda \otimes_T P'$ is a finitely

generated R-module. Since the tensor product commutes with direct sums, we see that $M \otimes_T P$ is a direct sum of finitely generated R-modules.

By 51.6,(2) we have an isomorphism $\nu_P : P \to \widehat{Hom}_R(M, M \otimes_T P)$. Together with the canonical isomorphism in 51.6,(1), this yields

$$\widehat{Hom}_R(M \otimes_T P, M \otimes_T P) \simeq \widehat{Hom}_T(P, \widehat{Hom}_R(M, M \otimes_T P)) \simeq \widehat{End}_T(P)\,.$$

With this preparation we now obtain:

51.13 Generators with right perfect endomorphism rings.
Assume $M = \bigoplus_\Lambda M_\lambda$, with all M_λ finitely generated, is a generator in $\sigma[M]$ and $T = \widehat{End}_R(M)$ is a right perfect ring.

Then there exists a projective left T-module P which is a direct sum of local modules such that $M \otimes_T P$ is a projective generator in $\sigma[M]$ and

$$\widehat{Hom}_R(M \otimes_T P, -) : \sigma[M] \longrightarrow \widehat{End}_T(P)\text{-}MOD$$

is an equivalence of categories.

Proof: Take the projective T-module P as defined in 51.12 and set $S = \widehat{End}_T(P) \simeq \widehat{End}_R(M \otimes_T P)$. Let us first show that $M \otimes_T P$ is a generator in $\sigma[M]$.

For $K \in \sigma[M]$, consider the exact sequence with evaluation map μ

$$P \otimes_S \widehat{Hom}_T(P, \widehat{Hom}_R(M, K)) \xrightarrow{\mu} \widehat{Hom}_R(M, K) \longrightarrow Coke\,\mu \longrightarrow 0.$$

Since the image of μ is in fact the trace of P in $\widehat{Hom}_R(M, K)$, we see that $\widehat{Hom}_T(P, Coke\,\mu) = 0$ and hence $M \otimes_T Coke\,\mu = 0$ by 51.12. Therefore, by tensoring with M_T, we obtain the commutative exact diagram

$$\begin{array}{ccccc} M \otimes_T P \otimes_S \widehat{Hom}_T(P, \widehat{Hom}_R(M, K)) & \longrightarrow & M \otimes_T \widehat{Hom}_R(M, K) & \longrightarrow & 0 \\ \downarrow\simeq & & \downarrow\simeq & & \\ M \otimes_T P \otimes_S \widehat{Hom}_R(M \otimes_T P, K) & \longrightarrow & K & \longrightarrow & 0 \end{array},$$

where the left isomorphism is given by 51.6,(1) and the right isomorphism by 51.7,(3). From this we see that the trace of $M \otimes_T P$ in K is equal to K and hence $M \otimes_T P$ is a generator in $\sigma[M]$.

To prove that $M \otimes_T P$ is a self-projective R-module we have to show that $M \otimes_T P_S$ is a faithfully flat S-module (see 51.7,(2)). We know from 51.7,(1) that it is a flat module. Assume $(M \otimes_T P) \otimes_S S/I = 0$ for some left ideal $I \subset S$. Then $Hom_T(P, P \otimes_S S/I) = 0$ by 51.12, implying $P \otimes_S S/I = 0$, since this module is P-generated. However, for the projective T-module P

we conclude from 51.3 that $PI \neq P$ for every proper left ideal $I \subset S$, and hence (by 12.11) $P/PI \simeq P \otimes_S S/I \neq 0$, a contradiction. Therefore $M \otimes_T P_S$ is faithfully flat.

For a possible application of 51.13 let us mention that, for example, semi-injective modules with *acc* on annihilator submodules and also finitely generated semi-projective modules with *dcc* on cyclic submodules have right perfect endomorphism rings (see 31.12, 31.10).

Literature: ALBU-NĂSTĂSESCU, STENSTRÖM;
Abrams, Albu-Wisbauer, Anh-Márki, Fuller [3], Fuller-Hullinger, Héaulme, Hullinger, Lenzing [4], Menini, Năstăsescu [4].

52 Functor Rings of $\sigma[M]$ and R-MOD

1.Properties of $\widehat{Hom}_R(U,-)$. 2.Properties of $\widehat{Hom}_R(\widetilde{U},-)$. 3.Properties of $-\otimes_R\widetilde{U}$ and $-\otimes_{\widetilde{T}}\widetilde{U}^*$. 4.Pure injective modules in MOD-R. 5.Functor rings and functor categories. 6.Flat modules and exact functors on f.p. modules. 7.Functor rings of regular modules. 8.Functor rings of semisimple modules. 9.Exercises.

Taking V as direct sum of a representing set of *all* finitely generated, resp. all finitely presented, modules in $\sigma[M]$, we obtain additional properties of $\widehat{Hom}(V,-)$ and $\widehat{End}(V)$ which will be the subject of this section.

For a left R-module M, let $\{U_\alpha\}_A$ be a representing set of the finitely generated modules in $\sigma[M]$.

We define $U = \bigoplus_A U_\alpha$, $T = \widehat{End}(U)$ and call T the *functor ring of the finitely generated modules* of $\sigma[M]$.

The reason for this notation will become clear in 52.5.

If P is a progenerator in $\sigma[M]$, i.e. $Hom(P,-) : \sigma[M] \to End(P)$-$MOD$ is an equivalence, we have $End(U) \simeq End(Hom(P,U))$ and

$$T = \widehat{End}(U) \simeq \widehat{End}(Hom(P,U)) \simeq \widehat{End}(\bigoplus\nolimits_A Hom(P,U_\alpha)).$$

Since $\{Hom_R(P,U_\alpha)\}_A$ is a representing set of the finitely generated $End(P)$-modules, in this case $\sigma[M]$ and $End(P)$-MOD have isomorphic functor rings.

For any M, U is a generator in $\sigma[M]$. U is M-projective if and only if all finitely generated modules in $\sigma[M]$ are projective, i.e. if M is semisimple (see 20.2).

Of course, T is a ring with enough idempotents and in the last paragraph we have already given a list of properties of $\widehat{Hom}(U,-)$ (see 51.2, 51.6, 51.7). Besides these, the following are of interest:

52.1 Properties of $\widehat{Hom}_R(U,-)$.
With the above notation we have:

(1) For $N \in \sigma[M]$, the following are equivalent:
(a) N is finitely generated;
(b) $\widehat{Hom}(U,N) \simeq Te$ for an idempotent $e \in T$;
(c) $\widehat{Hom}(U,N)$ is finitely generated and projective in T-MOD.

(2) If $X \in T$-MOD is finitely generated and projective, then $U \otimes_T X$ is also finitely generated.

(3) For every $N \in \sigma[M]$, $\widehat{Hom}(U,N)$ is flat in T-MOD.

(4) $\widehat{Hom}(U,N)$ is projective in T-MOD if and only if N is a direct summand of a direct sum of finitely generated modules.

(5) The weak global dimension of ${}_TT$ and T_T is ≤ 2.

(6) ${}_RU$ is a weak cogenerator and U_T is absolutely pure in MOD-T.

(7) M is locally noetherian if and only if ${}_TT$ is locally coherent.

(8) If Te is finitely cogenerated for every idempotent $e \in T$, then M is locally artinian.

(9) Assume $\sigma[M] = R$-MOD. Then ${}_RR$ is artinian if and only if Te is finitely cogenerated for every idempotent $e \in T$.

(10) $\widehat{Hom}(U,-)$ is an equivalence between the subcategories of the direct summands of direct sums of finitely generated modules in $\sigma[M]$ and the projective modules in T-MOD (with inverse $U \otimes_T -$).

Proof: (1) $(a) \Rightarrow (b)$ By definition of U, N is isomorphic to a direct summand of ${}_RU$.

$(b) \Rightarrow (c)$ is evident.

$(c) \Rightarrow (a)$ If $\widehat{Hom}(U,N)$ is finitely generated, then, by 51.5, the R-module $Tr(U,N) = N$ is also finitely generated.

(2) Since $X \simeq \widehat{Hom}(U, U \otimes X)$ by 51.6, the assertion follows from (1).

(3) If $\{N_\lambda\}_\Lambda$ are the finitely generated submodules of N, we have $N \simeq \varinjlim N_\lambda$. Since $\widehat{Hom}(U,-)$ preserves direct limits of monomorphisms (see 51.2), $\widehat{Hom}(U,N) \simeq \varinjlim \widehat{Hom}(U,N_\lambda)$ is a direct limit of projective modules (notice (1)) and therefore flat in *T-MOD*.

(4) follows from 51.6,(4).

(5) By 50.2, it is to show that for $h : P_1 \to P_0$, with P_0, P_1 finitely generated and projective in *T-MOD*, the module $Ke\,h$ is flat: From 51.7,(4) we have $Ke\,h \simeq \widehat{Hom}(U, U \otimes Ke\,h)$ and $Ke\,h$ is flat in *T-MOD* by (3).

(6) For the notion of a weak cogenerator see 48.1. Obviously, for every finitely generated submodule $K \subset U^{(\mathbb{N})}$, the factor module $U^{(\mathbb{N})}/K$ is cogenerated by U, and the second assertion follows by 51.1,(7).

(7) If ${}_TT$ is locally coherent, then Te is coherent for every idempotent $e \in T$ and, by 51.7,(8), Ue is coherent in $\sigma[M]$. Therefore all finitely generated modules are coherent in $\sigma[M]$ and M is locally noetherian (see 27.3).

Now let M be locally noetherian. It is sufficient to show that for every morphism $h : P_1 \to P_0$ between finitely generated, projective modules P_1, P_0 in *T-MOD*, $Ke\,h$ is finitely generated (see § 26).

As a submodule of the noetherian R-module $U \otimes P_1$, $U \otimes Ke\,h$ is finitely generated (notice (2)). Moreover, again by 51.7,(4), $Ke\,h$ is isomorphic to $\widehat{Hom}(U, U \otimes Ke\,h)$ and hence is finitely generated by (1).

(8) If Te is finitely cogenerated for every $e^2 = e \in T$, then every finitely generated module in $\sigma[M]$ is finitely cogenerated by 51.7,(8) and M is locally artinian (see 31.1).

(9) If ${}_RR$ is artinian, it has finite length and T is semiperfect by 51.7,(9). Then, by 51.8,(4), for every idempotent $e \in T$, the module Te is finitely cogenerated. The other implication is given by (8).

(10) The assertion follows from 51.6,(4), referring to the isomorphism $L \simeq \widehat{Hom}(U, U \otimes L)$ for projective modules $L \in T$-MOD (see 51.6,(2)) and then applying (4).

For an R-module M, now let $\{\widetilde{U}_\alpha\}_A$ be a representing set of all *finitely presented* modules in $\sigma[M]$.

We denote $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$, $\widetilde{T} = \widehat{End}(\widetilde{U})$, and call $\widetilde{T}$ the *functor ring of the finitely presented modules* of $\sigma[M]$.

Again we refer to 52.5 for the motivation of this name.

From the functorial isomorphism $\widehat{Hom}(\widetilde{U}, -) \simeq \bigoplus_A Hom(\widetilde{U}_\alpha, -)$ (see 51.2) we deduce:

$\widehat{Hom}(\widetilde{U}, -) : \sigma[M] \to AB$ is exact if and only if every finitely presented module in $\sigma[M]$ is M-projective, i.e. if M is regular in $\sigma[M]$ (see 37.2).

An exact sequence $0 \to K \to L \to N \to 0$ in $\sigma[M]$ is pure *in $\sigma[M]$ if and only if it remains exact under $\widehat{Hom}(\widetilde{U}, -)$ (see § 34).*

$\widehat{Hom}(\widetilde{U}, -)$ commutes with (all) direct limits in $\sigma[M]$.

Obviously $\widetilde{U}$ is equal to U, the direct sum of the finitely generated modules in $\sigma[M]$ (see page 506), if and only if M is locally noetherian.

In general, $\widetilde{U}$ is not necessarily a generator in $\sigma[M]$, and we will have to postulate this in several cases.

52.2 Properties of $\widehat{Hom}_R(\widetilde{U}, -)$.

Let ${}_RM$, $\widetilde{U}$ and $\widetilde{T} = \widehat{End}(\widetilde{U})$ be defined as above. Assume $\widetilde{U}$ is a generator in $\sigma[M]$. Then:

(1) A module ${}_{\widetilde{T}}L$ is flat in $\widetilde{T}$-MOD if and only if $L \simeq \widehat{Hom}_R(\widetilde{U}, N)$ for some $N \in \sigma[M]$. Then $L \simeq \widehat{Hom}_R(\widetilde{U}, \widetilde{U} \otimes_{\widetilde{T}} L)$.

(2) $\widehat{Hom}(\widetilde{U}, -)$ transforms pure sequences in $\sigma[M]$ to pure sequences in $\widetilde{T}$-MOD.

(3) For $N \in \sigma[M]$, the following are equivalent:
(a) N is finitely presented in $\sigma[M]$;
(b) $\widehat{Hom}_R(\widetilde{U}, N) \simeq \widetilde{T}e$ for an idempotent $e \in \widetilde{T}$;
(c) $\widehat{Hom}_R(\widetilde{U}, N)$ is finitely presented in $\widetilde{T}$-MOD.

(4) A module P is pure projective in $\sigma[M]$ if and only if $\widehat{Hom}_R(\widetilde{U}, P)$ is projective in $\widetilde{T}$-MOD.

(5) A module K is absolutely pure in $\sigma[M]$ if and only if $\widehat{Hom}_R(\widetilde{U}, K)$ is absolutely pure in $\widetilde{T}$-MOD.

(6) w.gl.dim $_{\widetilde{T}}\widetilde{T} \leq 2$ and gl.dim $_{\widetilde{T}}\widetilde{T} \leq 2 +$ p.gl.dim $\sigma[M]$.

(7) $_R\widetilde{U}$ is a weak cogenerator and $\widetilde{U}_{\widetilde{T}}$ is absolutely pure in MOD-$\widetilde{T}$.

(8) $_R\widetilde{U}$ is locally coherent if and only if $_{\widetilde{T}}\widetilde{T}$ is locally coherent.

(9) $\widehat{Hom}_R(\widetilde{U}, -)$ is an equivalence between the category $\sigma[M]$ and the category of flat modules in $\widetilde{T}$-MOD, with inverse $\widetilde{U} \otimes_{\widetilde{T}} -$.

Proof: Assuming that $\widetilde{U}$ is a generator in $\sigma[M]$, every module in $\sigma[M]$ is $\widetilde{U}$-supported (see 33.5) and we obtain (1), (2) and (4) directly from 51.10.

(3) $(a) \Rightarrow (b) \Rightarrow (c)$ is clear (by 52.1), $(c) \Rightarrow (a)$ is shown in 51.10,(4).

(5) If K is absolutely pure in $\sigma[M]$, then it is a pure submodule of its own M-injective hull $\widehat{K}$. Hence, by (2), $\widehat{Hom}(\widetilde{U}, K)$ is a pure submodule of the $\widetilde{T}$-injective module $\widehat{Hom}(\widetilde{U}, \widehat{K})$ (see 51.7) and therefore absolutely pure in $\widetilde{T}$-*MOD* (see 35.1).

On the other hand, let $\widehat{Hom}(\widetilde{U}, K)$ be absolutely pure, i.e. a pure submodule of the flat $\widetilde{T}$-module $\widehat{Hom}(\widetilde{U}, \widehat{K})$. Then $\widehat{Hom}(\widetilde{U}, \widehat{K})/\widehat{Hom}(\widetilde{U}, K)$ is flat in $\widetilde{T}$-*MOD* by 36.1, and therefore of the form $\widehat{Hom}(\widetilde{U}, N)$ with $N \in \sigma[M]$ (see (1)), i.e. we have an exact sequence

$$0 \to \widehat{Hom}_R(\widetilde{U}, K) \to \widehat{Hom}_R(\widetilde{U}, \widehat{K}) \to \widehat{Hom}_R(\widetilde{U}, N) \to 0\,.$$

Applying $\widetilde{U} \otimes_{\widetilde{T}} -$ we derive $N \simeq \widehat{K}/K$ (notice 51.7). From this we conclude that the sequence $0 \to K \to \widehat{K} \to \widehat{K}/K \to 0$ remains exact under $\widehat{Hom}(\widetilde{U}, -)$, i.e. is pure in $\sigma[M]$. Then K is a pure submodule of $\widehat{K}$ and hence is absolutely pure in $\sigma[M]$.

(6) Similarly to the proof of 52.1,(5), it can be shown that the kernels of morphisms between finitely generated, projective modules in T-*MOD* are flat modules. This yields the first inequality.

The second inequality follows from 50.6.

(7) Let K be a finitely generated submodule of $\widetilde{U}^{(\mathbb{N})}$. Then K is contained in a finite partial sum $\bigoplus_E \widetilde{U}_\alpha$, $\bigoplus_E \widetilde{U}_\alpha / K$ is finitely presented and hence cogenerated by $\widetilde{U}$.

Then, by 51.1,(7), $\widetilde{U}_{\widetilde{T}}$ is absolutely pure in $\widetilde{T}$-*MOD*.

(8) If ${}_{\widetilde{T}}\widetilde{T}$ is locally coherent, then, for every $e^2 = e \in \widetilde{T}$, $\widetilde{T}e$ is coherent and, by 51.7,(8), every $\widetilde{U}_\alpha$ is coherent in $\sigma[M]$.

Now let $\widetilde{U}$ be locally coherent. We have to show that in an exact sequence $0 \to K \to X_1 \to X_0$, with X_1, X_0 finitely generated, projective in $\widetilde{T}$-*MOD*, K is also finitely generated. In the exact sequence (notice 51.7,(1))

$$0 \to \widetilde{U} \otimes_{\widetilde{T}} K \to \widetilde{U} \otimes_{\widetilde{T}} X_1 \to \widetilde{U} \otimes_{\widetilde{T}} X_0 \,,$$

the $\widetilde{U} \otimes_{\widetilde{T}} X_i$ are finitely presented by (3), hence are coherent by assumption (notice $X_i \simeq \widehat{Hom}(\widetilde{U}, \widetilde{U} \otimes X_i)$). So $\widetilde{U} \otimes_{\widetilde{T}} K$ is also finitely presented in $\sigma[M]$. Since $K \simeq \widehat{Hom}(\widetilde{U}, \widetilde{U} \otimes K)$ by 51.7, K is finitely generated (presented) because of (3).

(9) The assertion follows from (1) and the isomorphism from 51.7,
$\widetilde{U} \otimes_{\widetilde{T}} \widehat{Hom}(\widetilde{U}, N) \simeq N$, for $N \in \sigma[M]$.

In the case $\sigma[M] = R$-*MOD*, in addition to the functors $\widetilde{U} \otimes_{\widetilde{T}} -$ and $\widehat{Hom}_R(\widetilde{U}, -)$ already encountered, there are two more interesting functors connected with $\widetilde{U}$:

Let $\{\widetilde{U}_\alpha\}_A$ be a representing set of finitely presented modules in R-*MOD*, $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$ and $\widetilde{T} = \widehat{End}(\widetilde{U})$ as above.

Denote $\widetilde{U}^* = \widehat{Hom}(\widetilde{U}, R)$. This is a $(\widetilde{T}, R)$-bimodule. Since R now is isomorphic to a direct summand of $\widetilde{U}$, there is an idempotent $e_o \in \widetilde{T}$ with

$$\begin{array}{lll} {}_{\widetilde{T}}\widetilde{U}^* = \widehat{Hom}(\widetilde{U}, R) \simeq \widetilde{T}e_o & \text{and} & \widetilde{U}_{\widetilde{T}} \simeq Hom(R, \widetilde{U}) \simeq e_o\widetilde{T}, \\ e_o\widetilde{U}^* \simeq R_R & \text{and} & \widetilde{U}e_o \simeq {}_RR. \end{array}$$

Consider the following two functors:

$$- \otimes_R \widetilde{U}_{\widetilde{T}} : MOD\text{-}R \to MOD\text{-}\widetilde{T}, \quad - \otimes_{\widetilde{T}} \widetilde{U}^*_R : MOD\text{-}\widetilde{T} \to MOD\text{-}R.$$

By 51.7, the canonical map $\mu_R : \widetilde{U} \otimes_{\widetilde{T}} \widetilde{U}^* \to R$, $u \otimes \varphi \mapsto (u)\varphi$, is an isomorphism. Together with the morphism

$$\kappa : \widetilde{U}^* \otimes_R \widetilde{U} \to \widetilde{T}, \quad \varphi \otimes u \mapsto (-)\varphi \cdot u,$$

we obtain the (functorial) isomorphism

$$id_M \otimes \mu_R : M \otimes_R \widetilde{U} \otimes_{\widetilde{T}} \widetilde{U}^*_R \longrightarrow M \otimes_R R \simeq M \,, \text{ for } M \in MOD\text{-}R,$$

and the (functorial) morphism

$$id_Y \otimes \kappa : Y \otimes_{\widetilde{T}} \widetilde{U}^* \otimes_R \widetilde{U}_{\widetilde{T}} \longrightarrow Y \otimes_{\widetilde{T}} \widetilde{T} \simeq Y\,, \text{ for } Y \in MOD\text{-}\widetilde{T},$$

which becomes an isomorphism in case $Y = \widetilde{U}_{\widetilde{T}}$.

52.3 Properties of $-\otimes_R \widetilde{U}$ and $-\otimes_{\widetilde{T}} \widetilde{U}^*$.
With the above notation we have:

(1) The functor $-\otimes_R \widetilde{U}_{\widetilde{T}} : MOD\text{-}R \to MOD\text{-}\widetilde{T}$ *is fully faithful, i.e. for all* $K, L \in MOD\text{-}R$,

$$Hom_R(K, L) \simeq Hom_{\widetilde{T}}(K \otimes_R \widetilde{U}_{\widetilde{T}}, L \otimes_R \widetilde{U}_{\widetilde{T}}).$$

(2) For every family $\{K_\lambda\}_\Lambda$ *of right R-modules,*

$$(\prod_\Lambda K_\lambda) \otimes_R \widetilde{U}_{\widetilde{T}} \simeq \prod_\Lambda^{\widetilde{T}} (K_\lambda \otimes_R \widetilde{U}_{\widetilde{T}}) \textit{ (product in } MOD\text{-}\widetilde{T}).$$

(3) An exact sequence $0 \to K \to L \to N \to 0$ *is pure in MOD-R if and only if the following sequence is exact (and pure) in* $MOD\text{-}\widetilde{T}$:

$$0 \to K \otimes_R \widetilde{U} \to L \otimes_R \widetilde{U} \to N \otimes_R \widetilde{U} \to 0\,.$$

(4) If the sequence $0 \to X \to Y \to Z \to 0$ *is pure exact in* $MOD\text{-}\widetilde{T}$, *then*

$$0 \to X \otimes_{\widetilde{T}} \widetilde{U}^*_R \to Y \otimes_{\widetilde{T}} \widetilde{U}^*_R \to Z \otimes_{\widetilde{T}} \widetilde{U}^*_R \to 0$$

is pure exact in MOD-R.

(5) A module $X_{\widetilde{T}}$ *is absolutely pure in* $MOD\text{-}\widetilde{T}$ *if and only if* $X_{\widetilde{T}} \simeq K \otimes_R \widetilde{U}_{\widetilde{T}}$ *for some* $K \in MOD\text{-}R$.

(6) A module K_R *is pure injective if and only if* $K \otimes_R \widetilde{U}_{\widetilde{T}}$ *is* $\widetilde{T}$*-injective.*

(7) A module K_R *is indecomposable if and only if* $K \otimes_R \widetilde{U}_{\widetilde{T}}$ *is indecomposable.*

Proof: (1) This is obtained from the isomorphism $\mu_R : \widetilde{U} \otimes_{\widetilde{T}} \widetilde{U}^* \to R$.

(2) Since each $\widetilde{U}_\alpha$ is finitely presented, we have, by 12.9, the isomorphisms (with the canonical idempotents $\{e_\alpha\}_A$):

$$\begin{array}{lll} (\prod_\Lambda K_\lambda) \otimes_R \widetilde{U}_{\widetilde{T}} & \simeq \bigoplus_A (\prod_\Lambda K_\lambda \otimes_R \widetilde{U} e_\alpha) & \simeq \\ \bigoplus_A \prod_\Lambda (K_\lambda \otimes_R \widetilde{U} e_\alpha) & \simeq \bigoplus_A (\prod_\Lambda (K_\lambda \otimes_R \widetilde{U})) e_\alpha & \simeq \prod_\Lambda^{\widetilde{T}} (K_\lambda \otimes_R \widetilde{U}). \end{array}$$

(3) If $0 \to K \to L \to N \to 0$ is pure exact, then
$0 \to K \otimes_R \widetilde{U} \to L \otimes_R \widetilde{U} \to N \otimes_R \widetilde{U} \to 0$ is also exact (see 34.5).

For every module $Q \in \widetilde{T}$-MOD, this sequence remains exact under $- \otimes_{\widetilde{T}} Q$ (notice $(K \otimes_R \widetilde{U}) \otimes_{\widetilde{T}} Q \simeq K \otimes_R (\widetilde{U} \otimes_{\widetilde{T}} Q)$), and therefore the sequence is pure in MOD-$\widetilde{T}$ by 49.4.

Now assume the second sequence to be exact. This means that the first sequence remains exact under $- \otimes_R \widetilde{U}_\alpha$ for every α. Hence this sequence is pure exact (see 34.5).

(4) This is proved similarly to (3), since the new sequence remains exact under $- \otimes_R P$, for every $P \in R$-MOD.

(5) Let $X_{\widetilde{T}}$ be absolutely pure in MOD-$\widetilde{T}$, i.e. weakly $\widetilde{T}$-injective (see 35.4). For a finitely presented module $P \in R$-MOD, there exists an exact sequence $R^k \to R^n \to P \to 0$ and we have the exact sequence

$$0 \to Hom_R(P, \widetilde{U}) \to Hom_R(R^n, \widetilde{U}) \to Hom_R(R^k, \widetilde{U}).$$

Since P is isomorphic to a direct summand of ${}_R\widetilde{U}$, $Hom_R(P, \widetilde{U})_{\widetilde{T}}$ is finitely generated. Hence $Hom_{\widetilde{T}}(-, X_{\widetilde{T}})$ is exact with respect to this sequence. Then we have the commutative exact diagram

$$\begin{array}{ccccccc}
Hom(\widetilde{U}, X) \otimes_R R^k & \to & Hom(\widetilde{U}, X) \otimes_R R^n & \to & Hom(\widetilde{U}, X) \otimes_R P & \to 0 \\
\downarrow \lambda_{R^k} & & \downarrow \lambda_{R^n} & & \downarrow \lambda_P & \\
Hom(Hom(R^k, \widetilde{U}), X) & \to & Hom(Hom(R^n, \widetilde{U}), X) & \to & Hom(Hom(P, \widetilde{U}), X) & \to 0,
\end{array}$$

where λ denotes the maps defined in 25.4 ($f \otimes p \to [g \mapsto f((p)g)]$). Since λ_{R^n} and λ_{R^k} are obviously isomorphisms, so too is λ_P. Then, for the finitely presented $\widetilde{U}_\alpha$ in R-MOD, we have

$$Hom_{\widetilde{T}}(\widetilde{U}, X) \otimes_R \widetilde{U}_\alpha \simeq Hom_{\widetilde{T}}(Hom_R(\widetilde{U}_\alpha, \widetilde{U}), X).$$

Forming the direct sum $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$ we obtain, with the canonical idempotents $\{e_\alpha\}_A$ (see 51.1),

$$Hom_{\widetilde{T}}(\widetilde{U}, X) \otimes_R \widetilde{U}_{\widetilde{T}} \simeq \bigoplus\nolimits_A Hom_{\widetilde{T}}(e_\alpha \widetilde{T}, X) \simeq \bigoplus\nolimits_A X e_\alpha \simeq X_{\widetilde{T}}.$$

With $K = Hom_{\widetilde{T}}(\widetilde{U}, X)$ this is exactly the assertion.

Now consider a module $K \in MOD$-R. Then $K \otimes_R \widetilde{U}_{\widetilde{T}}$ is a submodule of an injective module in MOD-$\widetilde{T}$. Using our result just proved there exists an exact sequence

$(*)$ $\quad 0 \longrightarrow K \otimes_R \widetilde{U}_{\widetilde{T}} \longrightarrow N \otimes_R \widetilde{U}_{\widetilde{T}}$, $N \in R$-MOD, $N \otimes \widetilde{U}_{\widetilde{T}}$ $\widetilde{T}$-injective.

Applying the functor $-\otimes_{\widetilde{T}}\widetilde{U}^*$, we obtain the commutative exact diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & K\otimes_R\widetilde{U}\otimes_{\widetilde{T}}\widetilde{U}^* & \longrightarrow & N\otimes_R\widetilde{U}\otimes_{\widetilde{T}}U^* \\
 & & \downarrow\simeq & & \downarrow\simeq \\
0 & \longrightarrow & K & \longrightarrow & N
\end{array} .$$

Applying $-\otimes_R\widetilde{U}$ we regain the sequence $(*)$, i.e. the sequence in the diagram remains exact and hence is pure by (3). Again by (3), $(*)$ is also pure. Therefore $K\otimes_R\widetilde{U}_{\widetilde{T}}$ is a pure submodule of an injective module and hence is absolutely pure in MOD-$\widetilde{T}$ (see 35.1).

(6) As we know from the preceding proof, for every $K\in MOD$-R there exists a pure sequence $0\to K\to N$, such that $0\to K\otimes_R\widetilde{U}_{\widetilde{T}}\to N\otimes_R\widetilde{U}_{\widetilde{T}}$ is exact and $N\otimes_R\widetilde{U}_{\widetilde{T}}$ is injective. If K is pure injective, then K is a direct summand of N and $K\otimes_R\widetilde{U}_{\widetilde{T}}$ is a direct summand of $N\otimes_R\widetilde{U}_{\widetilde{T}}$, and therefore is $\widetilde{T}$-injective.

Now assume $K\otimes_R U_{\widetilde{T}}$ is $\widetilde{T}$-injective. Then, for every pure sequence $0\to K\to N$ in MOD-R, the exact sequence $0\to K\otimes_R\widetilde{U}_{\widetilde{T}}\to N\otimes_R\widetilde{U}_{\widetilde{T}}$ splits. Hence also the given sequence splits. Thus K is pure injective (see 33.7).

(7) If $K=K_1\oplus K_2$ is a non-trivial decomposition, then

$$K\otimes\widetilde{U}_{\widetilde{T}}\simeq(K_1\otimes\widetilde{U}_{\widetilde{T}})\oplus(K_2\otimes\widetilde{U}_{\widetilde{T}}),$$

with non-zero summands (since $-\otimes_R\widetilde{U}\otimes_{\widetilde{T}}\widetilde{U}^*\simeq-\otimes_R R$).

On the other hand, let $K\otimes_R\widetilde{U}_{\widetilde{T}}=X_1\oplus X_2$ be a non-trivial decomposition in MOD-$\widetilde{T}$. Then X_1 and X_2 are absolutely pure in MOD-$\widetilde{T}$ (see 35.2) and, by (3), there exist non-zero K_1, K_2 in MOD-R, with $K_i\otimes_R\widetilde{U}_{\widetilde{T}}\simeq X_i$, $i=1,2$. Then $K\simeq K_1\oplus K_2$.

Some of the relations proved for $-\otimes_R\widetilde{U}$ and $-\otimes_{\widetilde{T}}\widetilde{U}^*_R$ can also be shown for $-\otimes_R U$ and $-\otimes_T\widehat{Hom}(U,R)$, with ${}_RU$ the sum of a representing set of all finitely generated R-modules. However, for example in 52.3,(3), it is necessary to demand the $\widetilde{U}_\alpha$ to be finitely presented.

The connection between the pure injective modules in MOD-R and the injective modules in MOD-$\widetilde{T}$ given by $-\otimes_R\widetilde{U}_{\widetilde{T}}$ enables us to transfer some known properties of injective modules to pure injective modules:

52.4 Properties of pure injective modules in MOD-R.

(1) Let K be a pure injective module in MOD-R. Then:

(i) $S = End_R(K_R)$ *is f-semiperfect and* $S/Jac(S)$ *is right self-injective.*

(ii) If K is indecomposable, then S is a local ring.

(2) For a module K in MOD-R the following are equivalent:

(a) $K^{(\Lambda)}$ *is pure injective for every set* Λ*, or just for* $\Lambda = I\!N$*;*

(b) $K^{(\Lambda)}$ *is a direct summand of* K^{Λ} *for every set* Λ*, or just for* $\Lambda = I\!N$*.*

Assume (a) (or (b)) holds for K in MOD-R. Then K is a direct sum of indecomposable modules, and if K is finitely generated, $End(K)$ *is a semiprimary ring.*

(3) (i) If ${}_RR$ *is coherent and* R_R *is perfect, then* $R_R^{(I\!N)}$ *is pure injective.*

(ii) If ${}_RR$ *is artinian, then* $R_R^{(I\!N)}$ *is pure injective.*

Proof: (1) If K_R is pure injective, $K \otimes_R \widetilde{U}_{\widetilde{T}}$ is $\widetilde{T}$-injective by 52.3, and $End_R(K_R) \simeq End_{\widetilde{T}}(K \otimes_R \widetilde{U}_{\widetilde{T}})$ is an endomorphism ring of a self-injective module. By 22.1, 42.11 and 19.9 it has the stated properties.

(2) $(a) \Rightarrow (b)$ is obvious, since $K^{(\Lambda)}$ is a pure submodule of K^{Λ} (see 33.9).

$(b) \Rightarrow (a)$ If $K^{(\Lambda)}$ is a direct summand of K^{Λ}, then $(K \otimes_R \widetilde{U}_{\widetilde{T}})^{(\Lambda)}$ is also a direct summand of $(K \otimes_R \widetilde{U}_{\widetilde{T}})^{\Lambda}$ (product in MOD-$\widetilde{T}$). As an absolutely pure module in MOD-$\widetilde{T}$, $K \otimes_R \widetilde{U}_{\widetilde{T}}$ is weakly $e\widetilde{T}$-injective for every idempotent $e \in \widetilde{T}$. Therefore, by 28.5, $(K \otimes_R \widetilde{U}_{\widetilde{T}})^{(\Lambda)} \simeq K^{(\Lambda)} \otimes_R \widetilde{U}_{\widetilde{T}}$ is injective in MOD-$\widetilde{T}$. Hence $K^{(\Lambda)}$ is pure injective (see 52.3,(6)). By 28.5,(2), we have only to consider the case $\Lambda = I\!N$.

By 28.6, $K \otimes_R \widetilde{U}_{\widetilde{T}}$ and K are direct sums of indecomposable modules. If K if finitely generated it satisfies the ascending chain condition for annihilator submodules by 28.5,(1). We conclude from 31.12 that $End_R(K) \simeq End_{\widetilde{T}}(K \otimes_R \widetilde{U}_{\widetilde{T}})$ is semiprimary.

(3)(i) If ${}_RR$ is coherent, then, by 26.6, $R_R^{I\!N}$ is flat. So $R^{I\!N}/R^{(I\!N)}$ is also a flat module in MOD-R (see 33.9, 36.6). Since R_R is perfect, flat modules in MOD-R are projective and the sequence

$$0 \longrightarrow R^{(I\!N)} \longrightarrow R_R^{I\!N} \longrightarrow R^{I\!N}/R^{(I\!N)} \longrightarrow 0$$

splits. Now we conclude from (2) that $R_R^{(I\!N)}$ is pure injective.

(ii) If ${}_RR$ is artinian, ${}_RR$ is noetherian and R_R (and ${}_RR$) is perfect.

The following description of functor categories apply for fairly arbitrary subcategories of R-MOD :

52.5 Functor rings and functor categories.
Let $\mathcal{C}$ be a full subcategory of finitely generated modules in R-MOD and $\{V_\alpha\}_A$ a representing set of the objects in $\mathcal{C}$.

Denote $V = \bigoplus_A V_\alpha$ and call $T = \widehat{End}(V)$ the functor ring *of $\mathcal{C}$. Then:*

(1) T-MOD is equivalent to the category of contravariant, additive functors $\mathcal{C} \to AB$ by the assignments

$$\begin{array}{lll} T\text{-}MOD \ni L & \sim\sim> & Hom_R(-,V) \otimes_T L, \\ T\text{-}MOD \ni f : L \to L' & \sim\sim> & id \otimes f : Hom(-,V) \otimes L \to Hom(-,V) \otimes L'. \end{array}$$

(2) MOD-T is equivalent to the category of covariant, additive functors $\mathcal{C} \to AB$ by the assignments

$$\begin{array}{lll} MOD\text{-}T \ni N & \sim\sim> & Hom_T(Hom_R(-,V), N) \\ MOD\text{-}T \ni f : N \to N' & \sim\sim> & Hom_T(Hom_R(-,V), f). \end{array}$$

Proof: It is easy to see that $id \otimes f$ and $Hom_T(Hom_R(-,U), f)$ determine functorial morphisms and the given assignments yield functors between the corresponding categories. We show that each functor has an inverse.

(1) Let $F : \mathcal{C} \to AB$ be a contravariant functor. We construct $\bigoplus_A F(V_\alpha)$ and denote by $\varepsilon_\alpha : V_\alpha \to V$ and $\pi_\alpha : V \to V_\alpha$ the canonical mappings.

For every $t \in T$, only a finite number of morphisms $\varepsilon_\alpha t \pi_\beta : V_\alpha \to V_\beta$ are non-zero and the morphisms $F(\varepsilon_\alpha t \pi_\beta) : F(V_\beta) \to F(V_\alpha)$ can be composed to a $\mathbb{Z}$-homomorphism

$$F(t) : \bigoplus_A F(V_\alpha) \to \bigoplus_A F(V_\alpha).$$

Writing this morphism on the left, $\bigoplus_A F(V_\alpha)$ becomes a left T-module with $t \cdot x = F(t)(x)$ for $t \in T$ and $x \in \bigoplus_A F(V_\alpha)$.

For $\beta \in A$ consider the idempotent $e_\beta = \pi_\beta \varepsilon_\beta \in T$. Then

$$e_\beta \cdot \bigoplus_A F(V_\alpha) = F(V_\beta) \subset \bigoplus_A F(V_\alpha)$$

and therefore $T(\bigoplus_A F(V_\alpha)) = \bigoplus_A F(V_\alpha)$, i.e. $\bigoplus_A F(V_\alpha)$ belongs to T-MOD.

If $\psi : F \to G$ is a functorial morphism, then we have a T-homomorphism $\bigoplus_A \psi_{V_\alpha} : \bigoplus_A F(V_\alpha) \to \bigoplus_A G(V_\alpha)$.

With the isomorphisms from 49.3, we have, for every V_β,

$$Hom_R(V_\beta, V) \otimes_T (\bigoplus_A F(V_\alpha)) \simeq e_\beta(\bigoplus_A F(V_\alpha)) = F(V_\beta),$$

and therefore a functorial isomorphism

$$Hom_R(-,V) \otimes_T (\bigoplus\nolimits_A F(V_\alpha)) \simeq F(-).$$

For every $L \in T\text{-}MOD$, the $\mathbb{Z}$-isomorphisms

$$\bigoplus\nolimits_A Hom_R(V_\alpha, V) \otimes_T L \simeq \bigoplus\nolimits_A e_\alpha L \simeq L$$

can also be viewed as T-isomorphisms. Therefore the assignments

$$L \;\sim\sim>\; Hom_R(-,V) \otimes_T L, \quad F \;\sim\sim>\; \bigoplus\nolimits_A F(V_\alpha),$$

determine an equivalence between the categories considered.

(2) Now let $F : \mathcal{C} \to AB$ be a covariant additive functor. Similarly to (1), we form the $\mathbb{Z}$-module $\bigoplus_A F(V_\alpha)$ which can be regarded as a right T-module with

$$(\bigoplus\nolimits_A F(V_\alpha))T = \bigoplus\nolimits_A F(V_\alpha) \in MOD\text{-}T.$$

By 49.3, we have, for every V_β,

$$\begin{aligned} Hom_T(Hom_R(V_\beta, V), \textstyle\bigoplus_A F(V_\alpha)) &\simeq Hom_T(e_\beta T, \textstyle\bigoplus_A F(V_\alpha)) \\ &\simeq (\textstyle\bigoplus_A F(V_\alpha))e_\beta = F(V_\beta), \end{aligned}$$

and therefore the functorial isomorphism

$$Hom_T(Hom_R(-,V), \bigoplus\nolimits_A F(V_\alpha)) \simeq F(-).$$

For $N \in MOD\text{-}T$, we obtain the isomorphisms

$$\bigoplus\nolimits_A Hom_T(Hom_R(V_\alpha, V), N) \simeq \bigoplus\nolimits_A Hom(e_\alpha T, N) \simeq \bigoplus\nolimits_A N e_\alpha \simeq N.$$

Then the assignments

$$N \;\sim\sim>\; Hom_T(Hom_R(-,V), N), \quad F \;\sim\sim>\; \bigoplus\nolimits_A F(V_\alpha),$$

define an equivalence between the categories considered.

For the subcategory of all finitely presented modules in $R\text{-}MOD$, the preceding assertions can be extended:

52.6 Flat modules and exact functors on f.p. modules.

Let $\mathcal{U}$ be the full subcategory of finitely presented modules in R-MOD, $\{\widetilde{U}_\alpha\}_A$ a representing set of objects in $\mathcal{U}$, $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$ and $\widetilde{T} = \widehat{End}(\widetilde{U})$. Then:

(1) Under the equivalence between $\widetilde{T}$-MOD and the additive contravariant functors $\quad \mathcal{U} \to AB, \quad L \sim\!\sim> Hom_R(-,\widetilde{U}) \otimes_{\widetilde{T}} L$ (see 52.5),
flat modules in $\widetilde{T}$-MOD correspond exactly to left exact functors.

(2) Under the equivalence between MOD-$\widetilde{T}$ and the additive covariant functors $\quad \mathcal{U} \to AB, \quad N \sim\!\sim> Hom_{\widetilde{T}}(Hom_R(-,\widetilde{U}), N)$ (see 52.5),
absolutely pure modules in MOD-$\widetilde{T}$ correspond exactly to right exact functors.

Proof: (1) If ${}_{\widetilde{T}}L$ is flat in $\widetilde{T}$-*MOD*, then $-\otimes_{\widetilde{T}} L : MOD\text{-}\widetilde{T} \to AB$ is an exact functor and hence $Hom_R(-,\widetilde{U}) \otimes_{\widetilde{T}} L$ converts cokernels to kernels.

On the other hand, let F be a contravariant left exact functor. Then there is a functorial isomorphism $F(-) \simeq Hom_R(-, F(R))$ (see 45.7). By the assignment given in 52.5, F corresponds to the $\widetilde{T}$-module

$$\bigoplus_A F(\widetilde{U}_\alpha) \simeq \bigoplus_A Hom_R(\widetilde{U}_\alpha, F(R)) \simeq \widehat{Hom}_R(\widetilde{U}, F(R)).$$

By 52.2, this is a flat module in $\widetilde{T}$-*MOD*.

(2) Let $K' \to K \to K'' \to 0$ be an exact sequence in $\mathcal{U}$. Then

$$0 \to Hom_R(K'',\widetilde{U}) \to Hom_R(K,\widetilde{U}) \to Hom_R(K',\widetilde{U})$$

is an exact sequence of finitely generated, projective modules in *MOD*-$\widetilde{T}$. If $N_{\widetilde{T}}$ is absolutely pure (FP-injective) in *MOD*-$\widetilde{T}$, then $Hom_{\widetilde{T}}(-, N_{\widetilde{T}})$ is exact with respect to this sequence and so $Hom_{\widetilde{T}}(Hom_R(-,\widetilde{U}), N)$ is right exact.

For a right exact functor $F : \mathcal{U} \to AB$, we have a functorial isomorphism $F(-) \simeq F(R) \otimes_R -$ (see 45.7). By 52.5, to this we assign the $\widetilde{T}$-module

$$\bigoplus_A F(\widetilde{U}_\alpha) \simeq \bigoplus_A F(R) \otimes_R \widetilde{U}_\alpha \simeq F(R) \otimes_R (\bigoplus_A \widetilde{U}_\alpha) = F(R) \otimes_R \widetilde{U}_{\widetilde{T}}.$$

It was shown in 52.3 that this module is absolutely pure in *MOD*-$\widetilde{T}$.

We finally want to consider the functor rings of two special types of modules. Recall our notation: For an R-module M, $\{U_\alpha\}_A$ denotes a representing set of finitely generated modules in $\sigma[M]$, $U = \bigoplus_A U_\alpha$, and $T = \widehat{End}(U)$.

$\{\widetilde{U}_\alpha\}_A$ denotes be a representing set of the finitely presented modules in $\sigma[M]$, $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$ and $\widetilde{T} = \widehat{End}(\widetilde{U})$.

52.7 Functor rings of regular modules.

For the R-module M assume $\widetilde{U}$ to be a generator in $\sigma[M]$. The following assertions are equivalent:

(a) M is regular in $\sigma[M]$;

(b) $\widehat{Hom}_R(\widetilde{U}, -) : \sigma[M] \sim\sim> \widetilde{T}$-MOD is an equivalence (an exact functor);

(c) $\widetilde{T}$ is a regular ring;

(d) $\widetilde{T}$ is a left semihereditary ring;

(e) every flat module is absolutely pure (=FP-injective) in $\widetilde{T}$-MOD.

Proof: $(a) \Leftrightarrow (b)$ The functor $\widehat{Hom}(\widetilde{U}, -)$ is exact (an equivalence, see 51.11) if and only if $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$ is projective in $\sigma[M]$, i.e. all finitely presented modules are M-projective. This characterises the regularity of M in $\sigma[M]$ (see 37.3).

$(a) \Rightarrow (c)$ is obvious from $(a) \Leftrightarrow (b)$. $(c) \Rightarrow (d), (e)$ is trivial.

$(d) \Rightarrow (a)$ Let K be a finitely generated submodule of a finitely presented module $\widetilde{U}_\alpha \in \sigma[M]$. We have an exact sequence of $\widetilde{T}$-modules

$$0 \to \widehat{Hom}_R(\widetilde{U}, K) \to \widehat{Hom}_R(\widetilde{U}, \widetilde{U}_\alpha) \to \widehat{Hom}_R(\widetilde{U}, \widetilde{U}_\alpha/K).$$

Since, by (d), every finitely generated submodule of $\widehat{Hom}(\widetilde{U}, \widetilde{U}_\alpha/K)$ is projective, we see that $\widehat{Hom}(\widetilde{U}, K)$ is a direct summand in $\widehat{Hom}(\widetilde{U}, \widetilde{U}_\alpha)$. Hence K is a direct summand of $\widetilde{U}_\alpha$, i.e. $\widetilde{U}_\alpha$ is regular in $\sigma[M]$. As a consequence, every finitely presented module in $\sigma[M]$ is $\widetilde{U}$-projective, i.e. projective in $\sigma[M]$.

$(e) \Rightarrow (a)$ For every $K \in \sigma[M]$, the flat module $\widehat{Hom}(\widetilde{U}, K)$ (see 52.2) is absolutely pure in $\widetilde{T}$-*MOD*. Hence K is absolutely pure in $\sigma[M]$ by 52.2,(5), and M is regular in $\sigma[M]$ by 37.2.

52.8 Functor rings of semisimple modules.

Let M be an R-module with functor rings T and $\widetilde{T}$ (notation as in 52.7). The following assertions are equivalent:

(a) M is a semisimple module;

(b) $\widehat{Hom}_R(U, -) : \sigma[M] \sim\sim> T$-MOD is an equivalence (an exact functor);

(c) T is a (left) semisimple ring;

(d) T is a regular ring;

(e) T is a left semihereditary ring;

(f) every projective module is injective in T-MOD;

(g) $\widetilde{U}$ *is a generator in* $\sigma[M]$ *and* $\widetilde{T}$ *is a (left) semisimple ring;*
(h) every cogenerator is a generator in T-MOD;
(i) for every idempotent $e \in T$*, the module* Te *is finitely cogenerated and* T*-injective;*
(j) Te *is* T*-injective for every idempotent* $e \in T$*.*

Proof: The equivalence of the conditions (a) to (e) can be obtained with the same proof as in 52.7.

The implications $(c) \Rightarrow (f) \Rightarrow (j)$ and $(c) \Rightarrow (h)$ are clear.

$(a) \Rightarrow (g)$ Since the semisimple module M is locally noetherian, we get $T = \widetilde{T}$ and the assertion follows from $(a) \Leftrightarrow (c)$.

$(g) \Rightarrow (a)$ We have seen in 52.7 that $\widetilde{T}$ semisimple implies that M is regular in $\sigma[M]$. Also, for every $N \in \sigma[M]$, the $\widetilde{T}$-module $\widehat{Hom}(\widetilde{U}, N)$ is projective by (g), and hence N is pure projective (see 52.2). Therefore every module is projective in $\sigma[M]$, i.e. M is semisimple.

$(h) \Rightarrow (i)$ This can be shown with the proof of $(b) \Rightarrow (c)$ in 48.11.

$(i) \Rightarrow (a)$ Since Te is finitely cogenerated for every $e^2 = e \in T$, the module ${}_RM$ is locally artinian by 52.1,(8). Every simple module $E \in \sigma[M]$ is M-injective since $\widehat{Hom}(U, E)$ is T-injective (see 51.7), i.e. M is co-semisimple (see 23.1). These two conditions imply that M is semisimple.

Of course, $(i) \Rightarrow (j)$ is trivial. However, for the next implication we need a theorem which we did not prove in this book:

$(j) \Rightarrow (a)$ For every finitely generated (cyclic) module $K \in \sigma[M]$, the projective module $\widehat{Hom}(U, K)$ is injective, and hence K has to be M-injective (see 51.7). By a result of Osofsky-Smith, this implies that M is semisimple (see remark to 20.3).

52.9 Exercises.

(1) Let the ring R be a left artinian Morita ring with dual ring S (see 47.15). Prove: The functor ring of the finitely generated left R-modules is isomorphic to the functor ring of the finitely generated right S-modules. (Fuller-Hullinger)

(2) Let R be a left artinian ring and T the functor ring of the finitely generated left R-modules. Prove that the following are equivalent:
(a) R is a left Morita ring;
(b) $Soc\, T_T \trianglelefteq T_T$ *and contains only finitely many non-isomorphic types of simple modules;*
(c) there exists a finitely generated, faithful module in T-MOD.

(Fuller-Hullinger)

Literature: Baer, Camillo [2], Facchini [4], Fuller-Hullinger, Gruson-Jensen, Shkhaied, Wisbauer [16], Zimmermann [5], Zimmermann-Huisgen [3,4], Zimmermann-Huisgen-Zimmermann [1].

53 Pure semisimple modules and rings

1.Submodules of V-pure projective modules. 2.Properties of locally noetherian modules. 3.Left noetherian functor rings (Kulikov Property). 4.Pure semisimple modules. 5.Properties of pure semisimple modules. 6.Left pure semisimple rings. 7.Right pure semisimple rings. 8.Exercises.

Functor rings are very useful to study and to describe global decomposition properties of a category. This will be the subject of this section. First we give some definitions.

Let M be an R-module and $\{V_\alpha\}_A$ a family of finitely generated modules in $\sigma[M]$. With $\mathcal{P} = \{V_\alpha \,|\alpha \in A\}$ a purity in $\sigma[M]$ in defined (see § 33) which is completely determined by the module $V = \bigoplus_A V_\alpha$.

Adapting the notation in § 33, a short exact sequence in $\sigma[M]$ is said to be *V-pure* if it remains exact under $\widehat{Hom}_R(V, -)$. A V-generated module is *V-pure projective* if it is a direct summand of a direct sum of V_α's (see 33.6). We will use this definition to point out the analogy of our results with the assertions about the purity considered in § 33.

For a representing set $\{\widetilde{U}_\alpha\}_A$ of all finitely presented modules in $\sigma[M]$, $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$ determines the ordinary purity in $\sigma[M]$. Then the property $\widetilde{U}$*-pure projective* is equivalent to *pure projective* (§ 34).

We have seen in 39.8 that for modules M which are hereditary in $\sigma[M]$, submodules of projective modules in $\sigma[M]$ are again projective. Similarly we may ask in which case submodules of V-pure projective modules in $\sigma[M]$ are again V-pure projective. In contrast to the result mentioned above, the answer to this question cannot be obtained by merely looking at internal properties of the module M.

53.1 Submodules of *V*-pure projective modules.

Let M be an R-module and $\{V_\alpha\}_A$ a family of finitely generated modules in $\sigma[M]$ with the following properties:

(i) $V = \bigoplus_A V_\alpha$ is a generator in $\sigma[M]$,

(ii) factor modules of any V_α are submodules of direct sums of V_α's.

Denoting $T = \widehat{End}(V)$ the following assertions are equivalent:

(a) every submodule of a V-pure projective module is V-pure projective;

(b) the global dimension of ${}_TT$ is ≤ 2;

(c) if $\widehat{Hom}(V, N)$ is a submodule of a projective module in T-MOD, then $\widehat{Hom}(V, N)$ is projective in T-MOD.

Proof: $(a) \Rightarrow (b)$ We have to show that the kernel of morphisms between projective modules in T-MOD is also projective (see 50.3).

Since the projective modules in T-MOD are of the form $\widehat{Hom}(V,P)$, with some V-pure projective $P \in \sigma[M]$ (see 51.6,(4)), in an exact sequence

$$0 \to X \to \widehat{Hom}_R(V,P_1) \to \widehat{Hom}_R(V,P_0)\,,$$

with V-pure projective P_1, P_0 in $\sigma[M]$, the T-module X has to be projective. By 51.7, $X \simeq \widehat{Hom}(V, V \otimes X)$, where $V \otimes X$ is a submodule of P_1 and hence is V-pure projective by assumption. So X is projective by 51.6,(4).

$(b) \Rightarrow (c)$ Let $\widehat{Hom}(V,N) \subset \widehat{Hom}(V,P)$, with V-pure projective P in $\sigma[M]$. By 18.2, it suffices to show that $\widehat{Hom}(V,N)$ is Q-projective, for any injective Q in T-MOD. Consider the following diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & \widehat{Hom}_R(V,N) & \longrightarrow & \widehat{Hom}_R(V,P) \\ & & \downarrow g & & \\ Q & \longrightarrow & W & \longrightarrow & 0 \end{array} \quad .$$

If we can extend g to a morphism $\widehat{Hom}(V,P) \to W$, then, by projectivity of $\widehat{Hom}(V,P)$, we obtain a morphism $\widehat{Hom}(V,P) \to Q$, and then a morphism $\widehat{Hom}(V,N) \to Q$ with the desired properties.

Consider the set (compare the proof of 16.2)

$$\mathcal{F} = \{h : \widehat{Hom}_R(V,L) \to W \mid N \subset L \subset P \text{ and } h|_{\widehat{Hom}(V,N)} = g\}.$$

$\mathcal{F}$ can be ordered by

$$\begin{array}{c} [h_1 : \widehat{Hom}_R(V,L_1) \to W] < [h_2 : \widehat{Hom}_R(V,L_2) \to W] \\ \Leftrightarrow \;\; L_1 \subset L_2 \text{ and } h_2|_{\widehat{Hom}(V,L_1)} = h_1. \end{array}$$

Since each V_α is finitely generated, $\widehat{Hom}(V,-)$ preserves unions (see 51.2). Therefore $\mathcal{F}$ is inductively ordered and, by Zorn's Lemma, there exists a maximal element $h_o : \widehat{Hom}(V,L_o) \to W$ in $\mathcal{F}$. Assume $L_o \neq P$.

Since V is a generator, there exists $\gamma : V_\alpha \to P$ with $Im\,\gamma \not\subset L_o$.

$$(L_o + Im\,\gamma)/L_o \simeq Im\,\gamma/(L_o \cap Im\,\gamma)$$

is a factor module of V_α and hence, by (ii), is contained in a finite sum $V_{\alpha_1} \oplus \cdots \oplus V_{\alpha_k}$, and we have an exact diagram

$$\begin{array}{ccccccc} 0 \to & \widehat{Hom}_R(V,L_o) & \to & \widehat{Hom}_R(V,L_o + Im\,\gamma) & \to & \bigoplus_{i\le k}\widehat{Hom}_R(V,V_{\alpha_i}) & \\ & \downarrow h_o & & & & & \\ (Q \to) & W & \to & \widehat{W} & \to & \widehat{W}/W & \to 0\,, \end{array}$$

where $\widehat{W}$ denotes the T-injective hull of W. Since *gl.dim* ${}_TT \leq 2$, $\widehat{W}/W$ is injective (see 50.3). By the injectivity of $\widehat{W}$ and $\widehat{W}/W$, we obtain two vertical morphisms yielding a commutative diagram.

Since $\bigoplus_{i\leq k}\widehat{Hom}(V, V_{\alpha_i})$ is projective, there exists a homomorphism $\bigoplus_{i\leq k}\widehat{Hom}(V, V_{\alpha_i}) \to \widehat{W}$ and finally (see Homotopy Lemma) a morphism $\widehat{Hom}(V, L_o + Im\,\gamma) \to W$ extending h_o. This contradicts the maximality of h_o and yields $L_o = P$.

$(c) \Rightarrow (a)$ is obvious.

The following observations will be useful:

53.2 Properties of locally noetherian modules.

(1) Let M be a locally noetherian R-module and N in $\sigma[M]$. Assume $V = \bigoplus_\Lambda V_\lambda$ is a submodule of N and N/V is finitely generated. Then there exists a finitely generated submodule $K \subset N$ and a finite subset $\Lambda_o \subset \Lambda$, such that

$$N = (\bigoplus\nolimits_{\Lambda\setminus\Lambda_o} V_\lambda) \oplus (K + \sum\nolimits_{\Lambda_o} V_\lambda).$$

(2) If, for an R-module M,

(i) every simple module in $\sigma[M]$ is finitely presented in $\sigma[M]$, and

(ii) every non-zero module in $\sigma[M]$ contains a simple submodule,

then M is locally noetherian.

Proof: (1) Since N/V is finitely generated, there exists a finitely generated submodule $K \subset N$ with $V + K = N$. Choose a submodule $W \subset N$ maximal with respect to $W \cap V = 0$ (complement). Then $W \oplus V$ is essential in N (see 17.6) and for the injective hulls we have (see 27.3)

$$\widehat{N} = \widehat{W} \oplus \widehat{V} = \widehat{W} \oplus (\bigoplus\nolimits_\Lambda \widehat{V}_\lambda).$$

Now there exists a finite subset $\Lambda_o \subset \Lambda$ with $K \subset \widehat{W} \oplus (\bigoplus_{\Lambda_o} \widehat{V}_\lambda)$, and so $N = (\bigoplus_{\Lambda\setminus\Lambda_o} V_\lambda) \oplus (K + \sum_{\Lambda_o} V_\lambda)$.

(2) We show that every absolutely pure module K in $\sigma[M]$ is injective. Then M is locally noetherian by 35.7.

Assume K is not equal to its injective hull $\widehat{K}$ in $\sigma[M]$. Then, by (ii), there is a simple submodule $E \subset \widehat{K}/K$. For a suitable module $K \subset L \subset \widehat{K}$ we have the exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow E \longrightarrow 0.$$

This sequence is pure and splits because of (i). This is a contradiction to K being essential in $\widehat{K}$ and L. Hence $K = \widehat{K}$, i.e. K is injective.

Again let $U = \bigoplus_A U_\alpha$, resp. $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$, denote the sum of a representing set of all finitely generated, resp. finitely presented, modules in $\sigma[M]$. We obtain assertions about the functor rings $T = \widehat{End}(U)$ and $\widetilde{T} = \widehat{End}(\widetilde{U})$:

53.3 Left noetherian functor rings (Kulikov property).

For an R-module M, the following assertions are equivalent:

(a) *M is locally noetherian, and every submodule of a pure projective module is pure projective in $\sigma[M]$;*

(b) *M is locally noetherian and gl.dim ${}_TT \leq 2$;*

(c) *${}_TT$ is locally noetherian;*

(d) *$\widetilde{U}$ is a generator in $\sigma[M]$ and ${}_{\widetilde{T}}\widetilde{T}$ is locally noetherian.*

Proof: $(a) \Leftrightarrow (b)$ The assertion follows from 53.1, by taking as $\{V_\alpha\}_A$ a representing set of all finitely generated (= finitely presented) modules in $\sigma[M]$.

$(a) \Rightarrow (c), (d)$ For a locally noetherian M, $U = \widetilde{U}$ and $T = \widetilde{T}$.

It remains to show that every T-module $\widehat{Hom}(U, N)$, with finitely generated $N \in \sigma[M]$, is noetherian. Let $X \subset {}_T\widehat{Hom}(U, N)$. Since, by 52.2,(6), *w.gl.dim* ${}_TT \leq 2$ and flat modules in $\widetilde{T}$-*MOD* are of the form $\widehat{Hom}(U, Q)$, for a suitable $Q \in \sigma[M]$, we have an exact sequence

$$0 \longrightarrow \widehat{Hom}_R(U, Q) \longrightarrow \widehat{Hom}_R(U, P) \longrightarrow X \longrightarrow 0\,,$$

where $Q \subset P$, P is pure projective in $\sigma[M]$, and $0 \to Q \to P \to N$ is an exact sequence. Hence, by (a), P/Q is finitely generated and Q is pure projective, i.e. $Q \oplus Q' \simeq \bigoplus_\Lambda P'_\lambda$ with finitely presented P_λ's and some Q' in $\sigma[M]$ (see 33.6).

Since we can replace the sequence $0 \to Q \to P \to P/Q \to 0$ by the sequence $0 \to Q \oplus Q' \to P \oplus Q' \to P/Q \to 0$, we may assume without loss of generality that $Q = \bigoplus_\Lambda P_\lambda \subset P$, with P_λ finitely presented in $\sigma[M]$.

By 53.2, there exists a finite subset $\Lambda_o \subset \Lambda$ and a finitely generated submodule $K \subset P$ with $P = (\bigoplus_{\Lambda \setminus \Lambda_o} P_\lambda) \oplus L$, where $L = K + \sum_{\Lambda_o} P_\lambda$, i.e. L is finitely generated. Therefore we have the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \widehat{Hom}_R(U, \bigoplus_{\Lambda \setminus \Lambda_o} P_\lambda) & \to & \widehat{Hom}_R(U, P) & \to & \widehat{Hom}_R(U, L) & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \widehat{Hom}_R(U, Q) & \to & \widehat{Hom}_R(U, P) & \to & X & \to & 0
\end{array}\,.$$

Since $\widehat{Hom}(U, L)$ is finitely generated (see 52.2,(3)), X is also finitely generated and hence $\widehat{Hom}(U, N)$ is noetherian.

$(d) \Rightarrow (c)$ Since $\widetilde{U}$ is a generator in $\sigma[M]$, we obtain from 51.7,(8) that, for ${}_{\widetilde{T}}\widetilde{T}$ locally noetherian, each ${}_R\widetilde{U}_\alpha$ is also noetherian. Then $\sigma[M]$ is locally noetherian (by 27.3) and $\widetilde{T} = T$.

$(c) \Rightarrow (b)$ By 51.7, for locally noetherian ${}_TT$, $\sigma[M]$ is also locally noetherian (see $(d) \Rightarrow (c)$). By 52.1, *w.gl.dim* ${}_TT \leq 2$ and in 50.4 we have shown that, for ${}_TT$ locally noetherian, also *gl.dim* ${}_TT \leq 2$.

Pure projective $\mathbb{Z}$-modules are just direct summands of direct sums of finitely generated $\mathbb{Z}$-modules (see 33.6). It was observed by L. Kulikov (Mat. Sbornik 16, 1945) that submodules of pure projective $\mathbb{Z}$-modules are again pure projective, i.e. ${}_{\mathbb{Z}}\mathbb{Z}$ satisfies the conditions considered in 53.3. Hence modules of this type are said to have the *Kulikov property.*

As we have seen in 20.3, an R-module M is *semisimple* if and only if every exact sequence in $\sigma[M]$ splits.

If M is a direct sum of finitely generated M-projective modules, then M is *perfect* in $\sigma[M]$ if and only if every pure exact sequence

$$0 \longrightarrow K \longrightarrow M^{(\mathbb{N})} \longrightarrow N \longrightarrow 0$$

splits, since then flat factor modules of $M^{(\mathbb{N})}$ are projective in $\sigma[M]$ (see 51.4). The class of modules we are going to consider now lies between these two cases:

We call an R-module M *pure semisimple* if every pure exact sequence $0 \to K \to L \to N \to 0$ in $\sigma[M]$ splits.

$U = \bigoplus_A U_\alpha$ and $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$ denote again the sum of a representing set of all finitely generated, resp. finitely presented, modules in $\sigma[M]$.

53.4 Pure semisimple modules. Characterizations.

For an R-module M the following assertions are equivalent:

(a) M is pure semisimple;

(b) every module in $\sigma[M]$ is pure projective in $\sigma[M]$;

(c) every module in $\sigma[M]$ is pure injective in $\sigma[M]$;

(d) $U^{(\mathbb{N})}$ is pure injective in $\sigma[M]$;

(e) the functor ring $T = \widehat{End}(U)$ is left perfect;

(f) every module in $\sigma[M]$ is a direct summand of a direct sum of finitely generated modules;

(g) $\widetilde{U}$ is a generator in $\sigma[M]$ and

(i) the functor ring $\widetilde{T} = \widehat{End}(\widetilde{U})$ is left perfect, or

(ii) every indecomposable module is finitely presented in $\sigma[M]$;

(h) every module in $\sigma[M]$ is a direct sum of finitely presented (and indecomposable) modules in $\sigma[M]$.

Proof: The equivalence of (a), (b) and (c) follows directly from the definitions, $(c) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (e)$ By 49.9, we have to show that T has the descending chain condition for cyclic right ideals:

Let $s_1T \supset s_1s_2T \supset \cdots$ be a descending chain with $s_i \in T$. We apply 43.3 to the sequence $\{s_i : U \to U\}_{I\!N}$. With the notation of 43.3 we have that $Im\, g \simeq U^{(I\!N)}$ is pure injective. As a direct limit of direct summands (see 43.3), $Im\, g$ is a pure submodule (by 33.8) and therefore a direct summand in $U^{(I\!N)}$. By 43.3,(3), the chain considered is finite (see proof of $(d.i) \Rightarrow (d.iv)$ in 51.4).

$(e) \Rightarrow (f)$ For every module $N \in \sigma[M]$, $\widehat{Hom}(U,N)$ is a flat T-module (see 52.1,(3)) and therefore is projective because of (e). Then, by 51.6, N is a direct summand of direct sums of copies of U.

$(f) \Rightarrow (b)$ By 27.5 together with 8.10, $\sigma[M]$ is locally noetherian. Then finitely generated modules are finitely presented and every module is pure projective in $\sigma[M]$.

$(b) \Rightarrow (g)(i)$ Flat modules in $\widetilde{T}$-*MOD* are of the form $\widehat{Hom}(\widetilde{U},N)$ for some N in $\sigma[M]$ (see 52.2,(1)). Since every N is pure projective in $\sigma[M]$, every flat module in $\widetilde{T}$-*MOD* is projective (see 52.2) and ${}_{\widetilde{T}}\widetilde{T}$ is perfect by 49.9.

$(g)(i) \Leftrightarrow (ii)$ For an indecomposable module $N \in \sigma[M]$, $\widehat{Hom}(\widetilde{U},N)$ is indecomposable and flat. If ${}_{\widetilde{T}}\widetilde{T}$ is perfect, $\widehat{Hom}(\widetilde{U},N)$ is projective and hence finitely generated by 42.5. Then, by 52.2,(3), N is finitely presented in $\sigma[M]$.

On the other hand, every indecomposable, flat module in $\widetilde{T}$-*MOD* is of the form $\widehat{Hom}(\widetilde{U},N)$, for some indecomposable $N \in \sigma[M]$. If all these N's are finitely presented, then every indecomposable flat $\widetilde{T}$-module is projective in $\widetilde{T}$-*MOD*, and then, by 49.9,(e), ${}_{\widetilde{T}}\widetilde{T}$ is perfect.

$(g)(i) \Rightarrow (h)$ If ${}_{\widetilde{T}}\widetilde{T}$ is perfect, then, for every $N \in \sigma[M]$, we have by 42.5, $\widehat{Hom}(\widetilde{U},N) \simeq \bigoplus_\Lambda \widetilde{T}e_\lambda$ with (primitive) idempotents $e_\lambda \in \widetilde{T}$ and

$$N \simeq \widetilde{U} \otimes_{\widetilde{T}} \widehat{Hom}(\widetilde{U},N) \simeq \bigoplus\nolimits_\Lambda \widetilde{U} \otimes_{\widetilde{T}} \widetilde{T}e_\lambda .$$

Each $\widetilde{U} \otimes_{\widetilde{T}} \widetilde{T}e_\lambda \simeq \widetilde{U}e_\lambda$ is (indecomposable and) finitely presented.

$(h) \Rightarrow (b)$ follows from 33.6.

53.5 Properties of pure semisimple modules.

Let M be a pure semisimple module, $U = \bigoplus_A U_\alpha$ the sum of all non-isomorphic finitely generated modules in $\sigma[M]$ and $T = \widehat{E}nd(U)$. Then:

(1) ${}_RM$ and ${}_TT$ are locally noetherian.

(2) Every flat module is projective in $\sigma[M]$.

(3) Every self-projective module P in $\sigma[M]$ is perfect in $\sigma[P]$ and $\widehat{E}nd_R(P)$ is left perfect.

(4) For any finitely generated, self-projective module P in $\sigma[M]$, $End_R(P)$ is left artinian.

(5) Every finitely generated, self-projective self-generator in $\sigma[M]$ is artinian.

(6) If M is linearly compact, then U_T is injective in MOD-T.

(7) If M is locally artinian, then every direct sum $U_T^{(\Lambda)}$ is injective in MOD-T.

Proof: (1) From the proof of $(f) \Rightarrow (b)$ in 53.4 we obtain that ${}_RM$ is locally noetherian. This is also a direct consequence of the fact that any direct sum of injective modules is absolutely pure (see 35.2) and therefore injective (see 35.7). By 53.3, ${}_TT$ is locally noetherian.

(2) By definition, flat modules which are pure projective are projective in $\sigma[M]$ (see § 36).

(3) Let P be a self-projective module in $\sigma[M]$. As a direct sum of finitely generated modules, P is in fact projective in $\sigma[P]$. Moreover, P is pure semisimple (in $\sigma[P]$) and therefore, by (2), every P-generated flat module is projective in $\sigma[P]$. Then, by 51.4, P is perfect in $\sigma[P]$ and $\widehat{E}nd_R(P)$ is left perfect.

(4) Let P be finitely generated and self-projective. Then P is noetherian (M is locally noetherian) and $End_R(P)$ is left noetherian (follows from 18.4). By (3), $End_R(P)$ is left perfect. Hence the radical of $End_R(P)$ is a nil ideal and therefore nilpotent by 4.2. From 31.4 we conclude that $End_R(P)$ is left artinian.

(5) If P is a finitely generated, self-projective self-generator, then $\sigma[P]$ is equivalent to $End_R(P)$-*MOD* and, because of (4), P is artinian.

(6) ${}_RU$ is obviously a cogenerator in $\sigma[M]$ (by 53.4) and each U_α is linearly compact (see 29.8,(3)). Then it follows from 47.8 that, for $S = End_R(U)$, the module U_S is $Hom_R(U_\alpha, U)$-injective. Since $Hom_R(U_\alpha, U) = e_\alpha S = e_\alpha T$ (with $e_\alpha = \pi_\alpha \varepsilon_\alpha$) and T-homomorphisms are exactly the S-homomorphisms, U_T is $e_\alpha T$-injective, for every $\alpha \in A$, and therefore injective in *MOD-T* by 16.2 (since $T = \bigoplus_A e_\alpha T$).

(7) By 28.4,(2), we have to show that, for $B = End(U_T)$ and $\alpha \in A$, the set

$$\mathcal{K}(e_\alpha T, U_T) = \{Ke(X) \mid X \subset Hom_T(e_\alpha T, U_T) \simeq {}_BU_\alpha\}$$

is noetherian. There is an order reversing bijection (see 28.1) between this set and the set

$$\mathcal{A}(e_\alpha T, U_T) = \{An(K) \mid K \subset e_\alpha T\} = \{{}_BHom_T(e_\alpha T/K, U_T) \mid K \subset e_\alpha T\}.$$

But this is a set of B-submodules of $Hom_T(e_\alpha T, U_T) \simeq {}_BU_\alpha$. Since U is a generator in $\sigma[M]$, the B-submodules of U_α are exactly the R-submodules (see 15.7, 15.8). Since U_α is artinian we have the desired condition.

Let us call the ring R *left pure semisimple*, if ${}_RR$ is a pure semisimple module. The preceding results can be summerized as follows:

53.6 Left pure semisimple rings. Characterizations.

For the ring R let $U = \bigoplus_A U_\alpha$ be the sum of a representing set of all finitely generated R-modules and $T = \widehat{End}({}_RU)$.

(1) The following assertions are equivalent:

(a) R is left pure semisimple;

(b) every module in R-MOD is pure projective (or pure injective);

(c) ${}_TT$ is perfect;

(d) every indecomposable module is finitely presented in R-MOD;

(e) every module in R-MOD is a direct sum of finitely generated (and indecomposable) modules.

(2) If R is left pure semisimple, then:

(i) R is left artinian.

(ii) Every self-projective module P in R-MOD is perfect in $\sigma[P]$.

(iii) ${}_TT$ is locally noetherian, and every sum $U_T^{(\Lambda)}$ is injective in MOD-T.

(iv) R_R and all projective right R-modules are pure injective (since $R^{(\Lambda)} \otimes_R U_T \simeq U_T^{(\Lambda)}$ is T-injective, notice 52.3).

Of course, similar characterizations hold for *right* pure semisimple modules – in connection with the functor ring of the finitely presented *right* modules. It is remarkable that the functor ring of the finitely presented left R-modules also allows (further) characterizations of right pure semisimple rings R. We use the functors $- \otimes_R \widetilde{U}$ and $- \otimes_{\widetilde{T}} \widetilde{U}^*$ considered in 52.3.

53.7 Right pure semisimple rings. Characterizations.
Let R be a ring, $\{\widetilde{U}_\alpha\}_A$ a representing set of the finitely presented left R-modules, $\widetilde{U} = \bigoplus_A \widetilde{U}$ and $\widetilde{T} = \widehat{End}(\widetilde{U})$.
The following properties are equivalent:

(a) *R is right pure semisimple (in MOD-R);*

(b) *every module in MOD-R is a direct sum of indecomposable modules;*

(c) *every pure injective module in MOD-R is a direct sum of indecomposable modules;*

(d) *every direct sum of pure injective modules in MOD-R is pure injective;*

(e) *$\widetilde{T}_{\widetilde{T}}$ is locally noetherian;*

(f) *$\widetilde{U}_{\widetilde{T}}$ is noetherian.*

Proof: Recalling the characterizations of pure semisimple rings given in 53.6, the implications $(a) \Rightarrow (b) \Rightarrow (c)$ and $(a) \Rightarrow (d)$ are obvious.

$(c) \Rightarrow (e)$ By 52.3, every injective module X in *MOD*-$\widetilde{T}$ is of the form $K \otimes_R \widetilde{U}_{\widetilde{T}}$, with K some pure injective in *MOD-R*. If $K = \bigoplus_\Lambda K_\lambda$, with indecomposable $K_\lambda \in$*MOD-R*, then $X \simeq \bigoplus_\Lambda (K_\lambda \otimes_R \widetilde{U}_{\widetilde{T}})$, with $K_\lambda \otimes_R \widetilde{U}_{\widetilde{T}}$ also indecomposable (see 52.3).

Hence in *MOD*-$\widetilde{T}$ every injective module is a direct sum of indecomposables and, by 27.5, $\widetilde{T}_{\widetilde{T}}$ is locally noetherian.

$(d) \Rightarrow (e)$ Let $\{X_\lambda\}_\Lambda$ be injective modules in *MOD*-$\widetilde{T}$. Then by 52.3, $X_\lambda \simeq K_\lambda \otimes_R \widetilde{U}_{\widetilde{T}}$, with pure injective K_λ in *MOD-R*.

We observe that $\bigoplus_\Lambda X_\lambda = (\bigoplus_\Lambda K_\lambda) \otimes_R \widetilde{U}_{\widetilde{T}}$ is an injective $\widetilde{T}$-module, since, by (d), the sum $\bigoplus_\Lambda K_\lambda$ is pure injective. Now apply 27.3.

$(e) \Rightarrow (a)$ For every $K \in$ *MOD-R*, $K \otimes_R \widetilde{U}_{\widetilde{T}}$ is absolutely pure in *MOD*-$\widetilde{T}$ by 52.3. But for $\widetilde{T}_{\widetilde{T}}$ locally noetherian, the absolutely pure (= FP-injective) modules are injective (see 35.7). Therefore – again by 52.3 – K is a pure injective R-module. Now use the right hand version of 53.6,(1).

$(e) \Leftrightarrow (f)$ Since ${}_R\widetilde{U}$ is a generator in *R-MOD*, by 51.8, $\widetilde{U}_{\widetilde{T}}$ is finitely generated and *MOD*-$\widetilde{T} = \sigma[\widetilde{U}_{\widetilde{T}}]$. Now the assertion is evident.

53.8 Exercises.

Consider modules N in R-MOD with the following properties:

$(*)$ *Every pure submodule of N is a direct summand.*

$(**)$ *N contains no non-trivial pure submodule.*

Let $\{\widetilde{U}_\alpha\}_A$ be a representing set of the finitely presented modules in R-MOD, $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$ and *$\widetilde{T} = \widehat{End}(\widetilde{U})$. Prove:*

(1) Every R-module with $(*)$ *is a direct sum of R-modules with* $(**)$. *(Rososhek)*

(2) An R-module N has property $(*)$, *resp.* $(**)$, *if and only if* $\widehat{Hom}(\widetilde{U}, N)$ *has the corresponding property as a* $\widetilde{T}$*-module.*

(3) For a ring R we have the following two pairs of equivalent assertions:

(i) (a) Every injective left R-module has $(*)$;

(b) ${}_RR$ *is noetherian.*

(ii) (a) Every projective left R-module has $(*)$;

(b) ${}_RR$ *is perfect.*

Literature: Auslander [1,2], Brune, Fuller [3], Héaulme, Hullinger, Liu, Ishii, Ringel-Tachikawa, Rososhek [2], Rososhek-Turmanov, Simson [1-7], Wisbauer [9,11], Zimmermann [4], Zimmermann-Huisgen [2].

54 Modules of finite representation type

1.Morphisms between indecomposable modules. 2.Modules of finite type. 3.Rings of finite type. 4.Modules with the Kulikov property. 5.Modules noetherian over their endomorphism rings. 6.Left modules over right pure semisimple rings. 7.Exercises.

An R-module M is said to be of *finite (representation) type*, if M is locally of finite length and there are only finitely many non-isomorphic finitely generated indecomposable modules in $\sigma[M]$.

M is said to be of *bounded (representation) type*, if it is locally of finite length and there is a finite upper bound for the lengths of the finitely generated indecomposable modules in $\sigma[M]$.

For example, a semisimple module M with infinitely many non-isomorphic simple summands is of bounded type (the length of the indecomposable modules in $\sigma[M]$ is equal to 1), but not of finite type.

However, we will see in 54.2 that, for finitely generated R-modules M, both properties are equivalent. In particular, this gives an answer to the question, whether rings R of bounded type are also of finite type. This problem is well known as the *first Brauer-Thrall Conjecture* and could not be answered for a long time (see Ringel).

In addition to the results of the preceding paragraph we still need an assertion about chains of morphisms between indecomposable modules, the first part of which is the *Harada-Sai Lemma*:

54.1 Morphisms between indecomposable modules.
Let $\{N_\lambda\}_\Lambda$ be a family of indecomposable R-modules with $lg(N_\lambda) \leq b$ for every $\lambda \in \Lambda$ and some $b \in I\!N$.

(1) For every sequence of non-isomorphisms $\{f_r : N_{\lambda_r} \to N_{\lambda_{r+1}}\}_{I\!N}$, $\lambda_r \in \Lambda$, $f_1 \cdot f_2 \cdots f_k = 0$ for $k = 2^b - 1$.

(2) For $N = \bigoplus_\Lambda N_\lambda$ and $T = \widehat{End}(N)$,
(i) $T/Jac(T)$ is semisimple;
(ii) $Jac(T)$ is nilpotent.

Proof: (1) For simplicity let $\{f_r : N_r \to N_{r+1}\}_{I\!N}$ denote the sequence of non-isomorphisms, $lg(N_i) \leq b$. By induction on k ($\leq b$) we show that the length of the image of $f_1 \cdots f_{2^k-1}$ is $\leq b-k$. Since f_1 is not an isomorphism, this is obvious for $k = 1$.

Assume the assertion holds for k ($< b$). Then, for $f = f_1 \cdots f_{2^k-1}$ and $h = f_{2^k+1} \cdots f_{2^{k+1}-1}$, the lengths of $Im\, f$ and $Im\, h$ are $\leq b - k$.

If one of the two lengths is $\leq b-k-1$, then this holds also for the length of $Im(f_1 \cdots f_{2^{k+1}-1})$ and the assertion is verified.

Hence consider the case $lg(Im\,f) = lg(Im\,h) = b - k$. Denoting $g = f_{2^k}$ we have to show that $lg(Im\,fgh) \leq b - k - 1$. Assume $lg(Im\,fgh) = b - k$. Then

$$Im\,f \cap Ke\,gh = 0 \quad \text{and} \quad Im\,fg \cap Ke\,h = 0.$$

Since $lg(Im\,f) = b - k$ and $lg(Ke\,gh) = lg(N_{2^k}) - (b - k)$, we obtain from the first relation that $N_{2^k} = Im\,f \oplus Ke\,gh$. N_{2^k} being indecomposable we conclude $Ke\,gh = 0$ and g has to be monic.

Since $lg(Im\,fg) = b - k$ and $lg(Ke\,h) = lg(N_{2^k+1}) - (b - k)$, the second equation yields a decomposition $N_{2^k+1} = Im\,fg \oplus Ke\,h$, implying $Ke\,h = 0$ and g is epic. But then $g = f_{2^k}$ would be an isomorphism, contradicting our assumption.

(2) Since each N_λ is of finite length, $End(N_\lambda) \simeq End(Te_\lambda) \simeq e_\lambda T e_\lambda$ is a local ring (with $e_\lambda = \pi_\lambda \varepsilon_\lambda$, see 32.4). Therefore each Te_λ is a semiperfect T-module and $T/Jac(T) \simeq \bigoplus_\Lambda (Te_\lambda / Rad\,Te_\lambda)$ is semisimple.

Assume for some $s \in Jac(T)$ and $\lambda, \mu \in \Lambda$, $t := \varepsilon_\lambda s \pi_\mu : N_\lambda \to N_\mu$ is an isomorphism. Then $e_\lambda = \pi_\lambda t t^{-1} \varepsilon_\lambda = e_\lambda s \pi_\mu t^{-1} \varepsilon_\lambda \in Jac(T)$. However, $Jac(T)$ contains no non-zero idempotents (see 49.6) and hence, for any $s \in Jac(T)$, $\varepsilon_\lambda s \pi_\mu : N_\lambda \to N_\mu$ is not an isomorphism.

Given a sequence $\{s_i\}_{\mathbb{N}}$ of elements $s_i \in Jac(T)$ we obtain, with suitable finite sums $\sum e_\lambda$,

$$s_1 \cdot s_2 \cdots s_k = (\sum e_\lambda) s_1 (\sum e_\lambda) s_2 (\sum e_\lambda) \cdots (\sum e_\lambda) s_k (\sum e_\lambda).$$

Since, by (1), for some $k \in \mathbb{N}$ all products $e_{\lambda_0} s_1 e_{\lambda_1} s_2 \cdots e_{\lambda_{k-1}} s_k e_{\lambda_k}$ become zero, it follows that $s_1 \cdot s_2 \cdots s_k = 0$.

Again let $\{U_\alpha\}_A$ denote a representing set of the finitely generated modules in $\sigma[M]$, $U = \bigoplus_A U_\alpha$ and $T = \widehat{End}(U)$.

54.2 Modules of finite representation type.

For a finitely generated R-module M, the following are equivalent:

(a) *M is of finite representation type;*

(b) *M is of bounded representation type;*

(c) *${}_T T$ is locally of finite length;*

(d) *T is left and right perfect;*

(e) *$T/Jac(T)$ is semisimple and $Jac(T)$ is nilpotent;*

(f) *M is of finite length and $Jac(T)$ is nilpotent;*

(g) *there is a progenerator in $\sigma[M]$ and T_T is locally of finite length.*

If these assertions hold, then M is pure semisimple.

Proof: $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (f)$ Since M is of finite length, every finitely generated module in $\sigma[M]$ has finite length and hence is a direct sum of indecomposable modules. So we have $U = \bigoplus_\Lambda N_\lambda$, with indecomposable N_λ of bounded length. Since T is independent of the decomposition chosen for U, $T = \widehat{End}(\bigoplus_A U_\alpha) = \widehat{End}(\bigoplus_\Lambda N_\lambda)$ and, by 54.1, $Jac(T)$ is nilpotent.

$(b) \Rightarrow (e)$ also follows from 54.1.

$(e) \Rightarrow (d)$ is given by properties of perfect rings with local units (see 49.9).

$(d) \Rightarrow (c)$ We know from 53.4 and 53.5 that, if T is left perfect, then it is also left locally noetherian. T right perfect yields the descending chain condition for cyclic left ideals (see 49.9). Then, by 31.8, we also obtain the descending chain condition for finitely generated left ideals. Therefore ${}_TT$ is locally artinian and noetherian, i.e. locally of finite length.

$(c) \Rightarrow (a)$ By 52.1, ${}_TT$ locally noetherian and artinian yields that M is also locally noetherian and artinian. So M is of finite length. Only for the next step is it important that M is of finite length: By 32.4, there are only finitely many non-isomorphic simple modules $E_1, \ldots, E_k$ in $\sigma[M]$.

The functor $\widehat{Hom}(U, -)$ establishes an equivalence between the subcategory of direct summands of direct sums of finitely generated modules in $\sigma[M]$ and the subcategory of projective modules in *T-MOD* (see 52.1,(10)). Hereby finitely generated indecomposable modules in $\sigma[M]$ correspond to finitely generated indecomposable projective T-modules (by 51.7,(5)), which are in fact local and hence projective covers of simple T-modules (T is semiperfect, see 49.10).

Therefore $\widehat{Hom}(U, -)$ yields a bijection between a minimal representing set of finitely generated, indecomposable modules in $\sigma[M]$ and the set of projective covers of non-isomorphic simple modules in *T-MOD*.

For every finitely generated indecomposable module $L \in \sigma[M]$, there exists an epimorphism $L \xrightarrow{g} E_i$ for some $i \leq 1, \ldots, k$. Since U is a generator, $\widehat{Hom}(U, g) : \widehat{Hom}(U, L) \to \widehat{Hom}(U, E_i)$ is also non-zero and the simple factor module of $\widehat{Hom}(U, L)$ occurs as a composition factor of $\widehat{Hom}(U, E_i)$.

Now, because of (c), each $\widehat{Hom}(U, E_i)$ is of finite length. Hence there are only finitely many non-isomorphic simple modules in *T-MOD* and consequently there exists only a finite number of finitely generated, indecomposable modules in $\sigma[M]$.

$(c) \Leftrightarrow (g)$ By 51.13, there exists a finitely generated, projective generator P in $\sigma[M]$. The functor ring of the finitely generated left modules over $End_R(P)$ is isomorphic to T (see p. 506), i.e. T is the functor ring of the module category $End_R(P)$-MOD. As we will see in 54.3, in this case ${}_TT$ is locally of finite length if and only if this holds for T_T.

$(f) \Rightarrow (e)$ By 51.7,(9), T is a semiperfect ring.

A ring R is said to be of *left finite* or *bounded (representation) type* if ${}_RR$ is of corresponding type.

To describe these rings we combine the results just derived with the characterizations of pure semisimple rings in 53.6 and 53.7. With $U = \bigoplus_A U_\alpha$, the sum of a representing set of all finitely generated left R-modules, and $T = \widehat{E}nd(U)$ we obtain:

54.3 Rings of finite representation type.

For a ring R the following properties are equivalent:

(a) ${}_RR$ is of finite representation type;

(b) ${}_RR$ is of bounded representation type;

(c) ${}_TT$ is locally of finite length;

(d) T is left and right perfect;

(e) T_T is locally of finite length;

(f) U_T is of finite length;

(g) ${}_RR$ and R_R are pure semisimple;

(h) R_R is of finite representation type.

Proof: The equivalences of (a) to (d) are given by 54.2.

$(d) \Rightarrow (g)$ By 53.6, we have only to show that R_R is pure semisimple. Since T_T is perfect, $gl.dim\, T_T = w.gl.dim T_T \leq 2$ (see 52.1, 49.9). So the cokernels of morphisms between injective T-modules are again injective (see 50.3). For every $K \in MOD$-R we have, for suitable sets Λ', Λ, an exact sequence

$$R^{(\Lambda')} \otimes_R U_T \longrightarrow R^{(\Lambda)} \otimes_R U_T \longrightarrow K \otimes_R U_T \longrightarrow 0\,.$$

Since, by 53.6, $R^{(\Lambda)} \otimes U_T \simeq U_T^{(\Lambda)}$ is T-injective, $K \otimes_R U_T$ is also injective and therefore K_R is pure injective (see 52.3). Hence by 53.6,(1), R is right pure semisimple.

$(g) \Rightarrow (e)$ For ${}_RR$ pure semisimple, T is left perfect and hence satisfies the descending chain condition for finitely generated (cyclic) right ideals (see 53.6). Then in particular ${}_RR$ is noetherian (e.g. 53.6,(2)) and by 53.7, R_R pure semisimple implies that T_T is locally noetherian. Therefore T_T is locally artinian and noetherian.

$(e) \Leftrightarrow (f)$ is obvious, because U_T is finitely generated and $\sigma[U_T] = MOD$-T (see 51.8,(1)).

$(f) \Rightarrow (d)$ Let U_T be a module of finite length. Then, for every idempotent $e \in T$, eT is artinian (note $\sigma[U_T] = MOD\text{-}T$) and $eT/Rad\, eT$ is semisimple. Hence $T/Jac(T)$ is a left and right semisimple (see 49.6).

The modules $U_T Jac(T) \supset U_T Jac(T)^2 \supset \cdots$ form a descending chain of submodules of U_T. Hence, for some $n \in I\!N$, we obtain

$$U_T Jac(T)^n = (U_T Jac(T)^n) Jac(T).$$

By Nakayama's Lemma 49.7, we conclude $U_T Jac(T)^n = 0$ and consequently $Jac(T)^n = 0$ (U_T is faithful). So T is left and right perfect.

$(g) \Leftrightarrow (h)$ Changing sides this is proved similarly to $(a) \Leftrightarrow (g)$ with the functor ring of the finitely generated *right* R-modules.

For $(f) \Rightarrow (h)$ we give another short direct proof:

Let U_T be of finite length. Then, by (g), ${}_RR$ and R_R are pure semisimple. For every indecomposable module K_R, $K \otimes_R U_T$ is indecomposable and injective (see 52.3, 53.6) and hence it is an injective hull of a simple T-module (see 19.9). By 52.3, every injective hull X_T of a simple module in $MOD\text{-}T$ is of the form $K \otimes_R U_T$ with $K \in MOD\text{-}R$ indecomposable.

Hence the functor $- \otimes_R U_T$ gives a bijection between the isomorphism classes of indecomposable modules in $MOD\text{-}R$ and the isomorphism classes of injective hulls of simple modules in $MOD\text{-}T$. However, the latter modules are in one-to-one correspondence with the non-isomorphic simple modules in $MOD\text{-}T = \sigma[U_T]$. By 32.4, there are only finitely many of them. So R_R is of finite type.

We have seen above that a ring which is left and right pure semisimple is in fact of finite type. For Artin algebras it is known that left pure semisimple already implies finite type (see 54.7,(2)). We cannot prove that pure semisimple modules are of finite type. However we have the following related result for modules with the Kulikov property (see 53.3):

54.4 Modules with the Kulikov property.

Let M be a locally noetherian R-module and assume that there are only finitely many simple modules in $\sigma[M]$.

If M has the Kulikov property, then for every $n \in I\!N$ there are only finitely many indecomposable modules of length $\leq n$ in $\sigma[M]$.

Proof: Let $\{U_\alpha\}_A$ denote a representing set of the finitely generated modules in $\sigma[M]$, $U = \bigoplus_A U_\alpha$ and $T = \widehat{End}(U)$.

Choose $\{V_\lambda\}_\Lambda$ to be a representing set of the indecomposable modules of length $\leq n$ in $\sigma[M]$ and put $V = \bigoplus_\Lambda V_\lambda$.

By assumption, ${}_T T$ is locally noetherian. Since V is a direct summand of U we have $V = Ue$ for some idempotent $e \in End_R(U)$, and it is easy to see that the ring $\widehat{End}(V) \simeq eTe$ is also locally left noetherian.

For every $\lambda \in \Lambda$, there is an idempotent $e_\lambda \in \widehat{End}(V)$ with $\widehat{Hom}(V, V_\lambda) = \widehat{End}(V)e_\lambda$ and $e_\lambda \widehat{End}(V)e_\lambda \simeq End(V_\lambda)$ is a local ring (see 31.14). Hence the factor of $\widehat{End}(V)$ by its radical is left semisimple. Moreover, by the Harada-Sai Lemma 54.1, the Jacobson radical of $\widehat{End}(V)$ is nilpotent. Therefore $\widehat{End}(V)$ is a (left and) right perfect ring and enjoys the descending chain condition for finitely generated left ideals (see 49.9).

Combining the two properties we see that $\widehat{End}(V)$ has locally finite length on the left.

Similar to the proof of $(c) \Rightarrow (a)$ in 54.2 we observe that the functor $\widehat{Hom}(V, -)$ provides a bijection between the isomorphism classes of indecomposable summands of V and the projective covers of simple modules in $\widehat{End}(V)$-*MOD*.

Let $E_1, \ldots, E_k$ denote the simple modules in $\sigma[M]$. Then, for every indecomposable summand X of V, there is an epimorphism $g : X \to E_i$ for some $i \leq k$ and so $0 \neq \widehat{Hom}(V, g) : \widehat{Hom}(V, X) \to \widehat{Hom}(V, E_i)$. Hence the simple factor of $\widehat{Hom}(V, X)$ is a composition factor of $\widehat{Hom}(V, E_i)$.

Since any E_i occurs among the V_λ, the left $\widehat{End}(V)$-module $\widehat{Hom}(V, E_i)$ is of the form $\widehat{End}(V)e_i$ for a suitable idempotent e_i. Thus the fact that $\widehat{End}(V)$ has locally finite length on the left guarantees that $\widehat{Hom}(V, E_i)$ has finite length. Hence there are only finitely many non-isomorphic simple modules in $\widehat{End}(V)$-*MOD* and only finitely many non-isomorphic indecomposable direct summands of V.

As mentioned before it was shown by Kulikov that $\mathbb{Z}$ has the Kulikov property. However, $\mathbb{Z}$ does not satisfy the conditions of the above theorem.

It is easily seen from Kulikov's result (see p. 525) that also the $\mathbb{Z}$-modules $\mathbb{Z}_{p^\infty}$ (for prime numbers $p \in \mathbb{N}$) have the Kulikov property. Since there is only one simple module in $\sigma[\mathbb{Z}_{p^\infty}]$ our theorem applies.

Evidently, $\mathbb{Z}_{p^\infty}$ is not a pure semisimple $\mathbb{Z}$-module. Hence the functor ring of $\sigma[\mathbb{Z}_{p^\infty}]$ is left noetherian but not left perfect (see 53.4).

A description of pure semisimple $\mathbb{Z}$-modules is given in 56.11.

The following lemma will enable us to obtain properties of finitely presented left modules over a right pure semisimple module:

54.5 Modules noetherian over their endomorphism rings.

Consider an R-module $V = \bigoplus_\Lambda V_\lambda$ with finitely generated non-zero modules V_λ. Assume Λ to be infinite and consider non-zero elements $v_\lambda \in V_\lambda$. Suppose V is noetherian as a right $End(V)$-module.

Then there exist infinitely many distinct indices $\mu_0, \mu_1, \ldots$ in Λ and morphisms $f_k : V_{\mu_{k-1}} \to V_{\mu_k}$ such that, for all $n \in I\!N$,

$$(v_{\mu_0})f_1 f_2 \cdots f_n \neq 0.$$

Proof: For $S = End(V)$ consider the S-submodule of V generated by $\{v_\lambda\}_\Lambda$. Since V_S is noetherian, this is a finitely generated module, i.e. for suitable indices we have $v_{\lambda_0}S + \cdots + v_{\lambda_k}S = \sum_\Lambda v_\lambda S$.

Hence, for some $\mu_0 \in \{\lambda_1, \ldots, \lambda_k\}$, the set

$$\Lambda_1 = \{\lambda \in \Lambda \setminus \{\mu_0\} \mid v_{\mu_0} Hom(V_{\mu_0}, V_\lambda) \neq 0\}$$

is infinite. Now we choose a family of morphisms

$$f_\lambda^{(1)} \in Hom(V_{\mu_0}, V_\lambda) \ \text{ with } (v_{\mu_0})f_\lambda^{(1)} \neq 0, \ \lambda \in \Lambda_1.$$

As a direct summand of V, the module $W = \bigoplus_{\Lambda_1} V_\lambda$ is also noetherian over its endomorphism ring. Therefore we may repeat the above construction, with the v_λ replaced by $(v_{\mu_0})f_\lambda^{(1)}$, $\lambda \in \Lambda_1$, to find an index $\mu_1 \in \Lambda_1$, an infinite subset $\Lambda_2 \subset \Lambda_1 \setminus \{\mu_1\}$, and morphisms $f_\lambda^{(2)} \in Hom(V_{\mu_1}, V_\lambda)$ for which $(v_{\mu_0})f_{\lambda_1}^{(1)} f_\lambda^{(2)} \neq 0$ for all $\lambda \in \Lambda_2$.

Continuing this process we get the desired sequence of morphisms.

54.6 Left modules over right pure semisimple rings.

Assume R is a right pure semisimple ring. Then for every $n \in I\!N$ there are only finitely many indecomposable finitely presented left R-modules of length $\leq n$.

Proof: Let $\{U_\alpha\}_A$ denote a representing set of the finitely generated left R-modules, $U = \bigoplus_A U_\alpha$ and $T = \widehat{End}(U)$. Choose $\{U_\beta\}_B$, $B \subset A$, as a minimal representing set of the indecomposable finitely presented left R-modules of length $\leq n$. We know from 53.7 that U is noetherian over T and hence obviously also over $End(_RU)$. Since $\bigoplus_B U_\beta$ is a direct summand of U, we easily verify that it is again noetherian over its endomorphism ring.

If B is infinite then, by 54.6, there is an infinite subset $\{\beta_i \mid i \in I\!N\} \subset B$ and a sequence of homomorphisms $f_k : U_{\beta_{k-1}} \to U_{\beta_k}$ with $f_1 \cdots f_n \neq 0$ for every $n \in I\!N$. Since the f_k are non-isomorphisms this contradicts the Harada-Sai Lemma 54.1. Hence B has to be finite.

54.7 Exercises.

(1) Let K be a field. Prove that the rings

$$\begin{pmatrix} K & K \\ 0 & K \end{pmatrix} \quad \text{and} \quad K[X]/(X^n),\ n \in \mathbb{N} \text{ are of finite type.}$$

(2) A ring R is called an Artin algebra, *if the center C of R is artinian and R is finitely generated as a C-module.*

Let R be an Artin algebra and E the injective hull of $C/Jac\,C$ in C-MOD. Choose a representing set $\{U_\alpha\}_A$ of the finitely generated R-modules and denote $U = \bigoplus_A U_\alpha$, $T = \widehat{End}(U)$. For $N \in T$-MOD and $L \in$ MOD-T we define

$N^* = Hom_C(N,E) \cdot T, \quad L^* = T \cdot Hom_C(L,E).$

Prove:

(i) For every idempotent $e \in T$, the canonical mappings $eT \to (eT)^{**}$ and $Te \to (Te)^{**}$ are isomorphisms.

(ii) The functor $(-)^*$ defines dualities between the full subcategories of
– submodules of finitely generated modules in T-MOD and MOD-T;
– finitely presented modules in T-MOD and MOD-T.

(iii) A module N in T-MOD is finitely presented if and only if N^* is finitely generated.

(iv) The simple modules in T-MOD and MOD-T are finitely presented.

(v) R is left pure semisimple if and only if R is (left) of finite type.
Hint: 47.16, 52.1,(7), 54.3.
(Yamagata, Auslander)

Literature: Auslander [1,2,3], Gruson-Jensen, Liu, Ringel, Wisbauer [9,11,16], Yamagata [1], Zimmermann-Huisgen-Zimmermann [3].

55 Serial modules and rings

1.Uniserial modules. 2.Serial modules. 3.Left serial rings. 4.Relations with $\gamma(M)$. 5.Exchange property. 6.Serial modules with extension property. 7.When are all finitely M-presented modules serial? 8.Serial rings. 9.Modules with uniserial injective hulls. 10.When are all finitely M-generated modules serial? 11.Rings with every finitely generated module serial. 12.Hereditary modules and serial modules. 13.Uniserial modules and modules of finite length. 14.When are all modules in $\sigma[M]$ serial? 15.Serial modules and functor rings. 16.When are all R-modules serial? 17.Exercises.

In this section we want to find out under which circumstances finitely presented (finitely generated, all) modules in $\sigma[M]$ and *R-MOD* can be written as direct sums of uniserial modules. It is remarkable that this question is related to the structure of $End_R(M)$, resp. R, on the right hand side. In the first part of this section we shall not use functor rings.

An R-module N is called *uniserial* if its submodules are linearly ordered by inclusion. If $_RR$ (resp. R_R) is uniserial we call R *left (right) uniserial.* Note that left and right uniserial rings are in particular local rings.

Commutative uniserial rings are also known as *valuation rings.*

55.1 Uniserial modules. Characterizations and properties.

(1) For an R-module N the following are equivalent:

(a) N is uniserial;

(b) the cyclic submodules of N are linearly ordered;

(c) any submodule of N has at most one maximal submodule;

(d) for any finitely generated submodule $0 \neq K \subset N$, $K/Rad(K)$ is simple;

(e) for every factor module L of N, Soc L is simple or zero.

(2) Let N be a non-zero uniserial R-module. Then:

(i) Submodules and factor modules of N are again uniserial.

(ii) N is uniform, and finitely generated submodules of N are cyclic.

(iii) $Rad(N) \neq N$ if and only if N is finitely generated, $Soc(N) \neq 0$ if and only if N is finitely cogenerated.

(iv) If N is noetherian, there exists a possibly finite descending chain of submodules $N = N_1 \supset N_2 \supset \cdots$ with simple factors N_i/N_{i+1}.

(v) If N is artinian, there exists a possibly finite ascending chain of submodules $0 = S_0 \subset S_1 \subset S_2 \subset \cdots$ with simple factors S_{i+1}/S_i.

(vi) If N has finite length, there is a unique composition series in N.

(3) Let N be uniserial, M in R-MOD and $S = End(M)$.

(i) If M is self-projective, then ${}_S Hom(M,N)$ is uniserial.
(ii) If M is self-injective, then $Hom(N,M)_S$ is uniserial.
(iii) If N is finitely generated and M is weakly M-injective, then $Hom(N,M)_S$ is uniserial.

(4) (i) If N is uniserial and self-projective, $End(N)$ is left uniserial.
(ii) If N is uniserial and self-injective, $End(N)$ is right uniserial.

Proof: (1) $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (a)$ Let K, L be submodules of N with $K \not\subset L$ and $L \not\subset K$. Choosing $x \in K \setminus L$, $y \in L \setminus K$ we have, by (b), $Rx \subset Ry$ or $Ry \subset Rx$. In the first case we conclude $x \in Ry \subset L$, in the second case $y \in Rx \subset K$. Both are contradictions.

$(a) \Rightarrow (c)$ and $(a) \Rightarrow (d) \Rightarrow (e)$ are obvious (see $(2)(i)$).

$(d) \Rightarrow (b)$ Let us assume that we can find two cyclic submodules K, $L \subset N$ with $K \not\subset L$ and $L \not\subset K$. Then

$$(K + L) \,/\, (K \cap L) \simeq K/(K \cap L) \oplus L/(K \cap L)\,,$$

and the factor of $(K+L)\,/\,(K \cap L)$ by its radical contains at least two simple summands. Therefore the factor of $K+L$ by its radical also contains at least two simple summands. This contradicts (d).

$(e) \Rightarrow (d)$ We show that every non-zero finitely generated submodule $K \subset N$ contains only one maximal submodule: If V_1, $V_2 \subset K$ are different maximal submodules, then $K/(V_1 \cap V_2) \simeq K/V_1 \oplus K/V_2$ (see 9.12) is contained in the socle of $N/(V_1 \cap V_2)$. This is a contradiction to (e).

(2) (i) is evident.

(ii) Obviously every submodule is essential in N.

Assume $K = Rk_1 + \cdots + Rk_r \subset N$. Since the Rk_i's are linearly ordered, we have $K = Rk_j$ for some $j \le r$.

(iii) Here $Rad(N) \neq N$ implies $Rad(N) \ll N$ and $Soc(N) \neq 0$ means $Soc(N) \trianglelefteq N$. Now the assertions follow from 19.6 and 21.3.

(iv) For $N_1 = Rad\,N$, N/N_1 is simple. Since all submodules of N are finitely generated we define recursively $N_{i+1} = Rad(N_i)$ $(\neq N_i)$.

(v) Since N is artinian, put $S_0 := Soc(N) \neq 0$. All factor modules of N are artinian and we construct S_{i+1} by $S_{i+1}/S_i = Soc(N/S_i)$.

(vi) N is noetherian and artinian and the series in (iv) and (v) are finite and equal to each other.

(3) (i) For f, $g \in Hom(M,N)$ it is to show that $f \in Sg$ or $g \in Sf$:

Assume $Im\, g \subset Im\, f$. Then the diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow g & & \\ M & \xrightarrow{f} & Im\, f & \longrightarrow & 0 \end{array}$$

can be completed commutatively by some $s \in S$, i.e. $g = sf$.

(ii) Now consider $f,\, g \in Hom(N, M)$ and $Ke\, g \subset Ke\, f$. By factorizing suitably we obtain the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N/Ke\, g & \xrightarrow{g} & M \\ & & \downarrow f & & \\ & & M & & , \end{array}$$

and we can find some $s \in S$ with $f = gs$.

(iii) is obtained by the same proof as (ii).

(4) This can be derived from (3) for $M = N$.

Examples of uniserial $\mathbb{Z}$-modules are the modules $\mathbb{Z}/p^k\mathbb{Z}$ for any $k, p \in \mathbb{N}$, p a prime number. They have the unique composition series

$$\mathbb{Z}/p^k\mathbb{Z} \supset p\mathbb{Z}/p^k\mathbb{Z} \supset p^2\mathbb{Z}/p^k\mathbb{Z} \supset \cdots \supset p^{k-1}\mathbb{Z}/p^k\mathbb{Z} \supset 0 .$$

Also $\mathbb{Z}_{p^\infty}$, the $\mathbb{Z}$-injective hull of $\mathbb{Z}/p\mathbb{Z}$, p a prime number, is uniserial. As we have seen in 17.13, $\mathbb{Z}_{p^\infty}$ is artinian and uniserial, but not noetherian (not finitely generated).

We call an R-module N *serial* if it is a direct sum of uniserial modules. The ring R is called *left (right) serial* if ${}_RR$ (resp. R_R) is a serial module. We say R is *serial* if R is left and right serial.

In contrast to the case of uniserial modules, submodules and factor modules of serial modules need not be serial. However, we can state:

55.2 Serial modules. Properties.
Let N be a serial R-module, $M \in R$-MOD, and $S = End(M)$.

(1) Assume K is an $(R, End(N))$-submodule of N. Then K and N/K are serial modules.

(2) Assume N is finitely generated. Then:

(i) If M is self-projective, then ${}_SHom_R(M, N)$ is a serial S-module.

(ii) If M is weakly M-injective, then $Hom_R(N, M)_S$ is a serial S-module.

(3) If $N \in \sigma[M]$ is finitely generated and M-projective, then N is semiperfect in $\sigma[M]$ and every direct summand of N is serial.

(4) If N is finitely generated and self-projective, $End(N)$ is left serial.

(5) If N is finitely generated and weakly N-injective, then $End(N)$ is right serial.

Proof: Assume $N = \oplus_\Lambda N_\lambda$, with uniserial N_λ. Then there exist orthogonal idempotents $e_\lambda \in End(N)$ with $N_\lambda = Ne_\lambda$ (see 8.6).

(1) For any fully invariant submodule $K \subset N$, $K = \bigoplus_\Lambda Ke_\lambda$ and

$$N/K = (\bigoplus\nolimits_\Lambda Ne_\lambda)\,/\,(\bigoplus\nolimits_\Lambda Ke_\lambda) \simeq \bigoplus\nolimits_\Lambda (Ne_\lambda/Ke_\lambda).$$

As a submodule of Ne_λ, each Ke_λ is uniserial, and as a factor module of Ne_λ, each Ne_λ/Ke_λ is uniserial (see 55.1). Thus K and N/K are serial.

(2) The assertion follows directly from 55.1,(3), because $Hom(M,-)$ and $Hom(-,M)$ both preserve finite direct sums.

(3) The finitely generated, uniserial, M-projective modules are projective covers of simple modules (see 19.7). By 42.4, N is semiperfect in $\sigma[M]$. Every direct summand of N is also semiperfect, and hence is a direct sum of local modules P (see 42.5). These P's (as direct summands) are generated by the uniserial summands N_λ of N. So there exists an epimorphism $N_\lambda \to P$ for some N_λ and therefore P is uniserial.

(4) and (5) are given by (2) (for $M = N$).

For rings the results just proved have the following form:

55.3 Left serial rings. First properties.

(1) If R is a left serial ring, then R is semiperfect and

(i) For every ideal $I \subset R$, R/I is a left serial ring.

(ii) For idempotents $e \in R$, Re is serial and $End(Re) \simeq eRe$ is left serial.

(iii) If ${}_RR$ is absolutely pure, then R is also right serial.

(2) If R is a serial ring and $I \subset R$ an ideal, then R/I is serial.

Proof: (1) R is semiperfect by 55.2,(3).

(i) Two-sided ideals are fully invariant submodules of ${}_RR$ and the assertion follows from 55.2,(1).

(ii) is a consequence of 55.2,(3) and (4).

(iii) 'Absolutely pure' is equivalent to 'weakly R-injective' (or 'FP-injective') and the assertion follows from 55.2,(5).

(2) is given by (1)(i).

In studying serial and (M-) cyclic modules, the number of simple summands in semisimple factor modules gives useful information:

Let K be a finitely generated R-module with $K/Rad\,K$ semisimple and E a simple R-module. By $\gamma(K)$ we denote the *number of simple summands* in a decomposition of $K/Rad\,K$ and by $\gamma(K,E)$ the *number of summands isomorphic to* E in a decomposition of $K/Rad\,K$.

Then $\gamma(K)$ is exactly the length of $K/Rad\,K$ and $\gamma(K,E)$ the length of the E-generated, homogeneous component of $K/Rad\,K$.

55.4 Relations with $\gamma(M)$.
Assume M is a finitely generated R-module and $M/Rad\,M$ is semisimple.

(1) If $M = K \oplus L$, then, for every simple R-module E,
$$\gamma(M,E) = \gamma(K,E) + \gamma(L,E).$$

(2) If M is self-projective, then a finitely M-generated module K is M-cyclic if and only if $\gamma(K,E) \leq \gamma(M,E)$ for every simple module E.

(3) If M is serial, then, for every finitely M-generated submodule $K \subset M$, $\gamma(K) \leq \gamma(M)$.

Proof: (1) This follows immediately from $Rad\,M = Rad\,K \oplus Rad\,L$.

(2) For every M-generated module K, $K/Rad\,K$ is semisimple. If K is a factor module of M, we have an epimorphism $M/Rad\,M \to K/Rad\,K$ and the assertion is evident.

If, on the other hand, $\gamma(K,E) \leq \gamma(M,E)$ for every simple E, then there is an epimorphism $M/Rad\,M \to K/Rad\,K$ and the diagram

$$\begin{array}{ccccc} M & \longrightarrow & M/Rad\,M & \longrightarrow & 0 \\ & & \downarrow & & \\ K & \longrightarrow & K/Rad\,K & \longrightarrow & 0 \end{array}$$

can be completed commutatively by some morphism $M \to K$.

Since $Rad\,K \ll K$, this is in fact an epimorphism.

(3) We prove this by induction on the number of uniserial summands of M: If M is uniserial, then, for any non-zero finitely generated submodule $K \subset M$, $\gamma(K) = \gamma(M) = 1$ (see 55.1).

Now we assume that the assertion is true for n uniserial summands. Consider $M = M_1 \oplus \cdots \oplus M_{n+1}$, with M_i uniserial, and let $K \subset M$ be finitely M-generated. With $L = K \cap M_{n+1}$ we construct the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \to & K & \to & K/L & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & L/Rad\,L & \to & K/Rad\,K & \to & K/(L + Rad\,K) & \to & 0 \end{array} .$$

Since $K/L \subset M_1 \oplus \cdots \oplus M_n$, we have, by assumption, $\gamma(K/L) \leq n$ and $L/Rad\, L$ has at most one summand. Therefore $\gamma(K) \leq n+1$.

The following technical lemma will be useful in forthcoming proofs:

55.5 Exchange property.

Let $M = M_1 \oplus \cdots \oplus M_k$ be an R-module with each M_i indecomposable and $\pi_i : M \to M_i$ the canonical projections.

(1) For a submodule $L \subset M$ the following are equivalent:

(a) For some $i \leq k$, $\pi_i|_L : L \to M_i$ is an isomorphism;

(b) for some $i \leq k$, $M \simeq M_1 \oplus \cdots \oplus L \oplus \cdots \oplus M_k$, L in position i.

The projection $M \to L$ is then given by $\pi_i(\pi_i|_L)^{-1}$.

(2) Let P be an indecomposable direct summand of M. If

(i) M is self-injective, or

(ii) M is self-projective, finitely generated and semiperfect in $\sigma[M]$,

then $P \simeq M_j$ for some $j \leq k$ and $M \simeq M_1 \oplus \cdots \oplus P \oplus \cdots \oplus M_k$ with P in position j.

Proof: (1) $\pi_i|_L$ is monic if and only if $L \cap \bigoplus_{j\neq i} M_j = Ke\, \pi_i|_L = 0$. Since

$$(L)\pi_i = (L + \bigoplus\nolimits_{j\neq i} M_j)\pi_i = (L + \bigoplus\nolimits_{j\neq i} M_j) \cap M_i\,,$$

$\pi_i|_L$ is epic if and only if $M_i \subset L + \bigoplus_{j\neq i} M_j$, i.e. $L + \bigoplus_{j\neq i} M_j = M$.

(2) (i) If M is M-injective, an indecomposable direct summand P of M is M-injective and uniform. Since $\bigcap_{i\leq k} Ke(\pi_i|_P) = 0$, one of the $\pi_j|_P$ has to be monic and therefore an isomorphism, since the M_i's have no non-trivial direct summands. Now the assertion follows from (1).

(ii) In the given situation the M_i's are local modules. One of the $\pi_j|_P$ has to be epic, since otherwise $P \subset \bigoplus_{i\leq k} Rad\, M_i = Rad\, M$, which is impossible for direct summands. Then $\pi_j|_P$ also has to be monic, because it splits and P contains no non-trivial direct summands. The rest follows again from (1).

If M is a finitely generated, self-projective and serial R-module, then, by 55.2,(3), all finitely M-generated, M-projective modules are serial. Now we want to study the question, when certain factor modules of these modules are again serial. We need a new definition and a lemma:

We say that a module N has the *extension property for uniserial submodules* if any of these submodules is contained in a uniserial direct summand of N. With the definition used in 41.21 this means that every uniserial submodule *lies under a uniserial direct summand of N.*

55.6 Serial modules with extension property.

Let $M = M_1 \oplus \cdots \oplus M_k$ be a self-projective R-module with all M_i's cyclic and uniserial.

(1) If $k = 2$, then for any non-zero uniserial submodule $K \subset M = M_1 \oplus M_2$, the following are equivalent:

(a) K is contained in a uniserial direct summand of M;

(b) M/K is serial.

(2) For $k \geq 2$ the following are equivalent:

(a) every direct sum of copies of the M_i's has the extension property for M-cyclic uniserial submodules;

(b) if N is a finite direct sum of copies of the M_i's and K is a finitely M-generated submodule of N, then there exists a decomposition $N = P_1 \oplus \cdots \oplus P_r$, with uniserial P_i's, such that $K = \bigoplus_{i \leq r}(K \cap P_i)$.

Proof: (1) $(a) \Rightarrow (b)$ Let K be essential in a direct summand P of M, $M = P \oplus Q$. Then P and Q are uniserial modules (exchange property) and $M/K = (P/K) \oplus Q$ is serial.

$(b) \Rightarrow (a)$ If $K \ll M$, then $M \to M/K$ is a projective cover of M/K in $\sigma[M]$ and $\gamma(M/K) = \gamma(M) = 2$. Therefore $M/K \simeq L_1 \oplus L_2$ holds with uniserial modules $L_1, L_2 \neq 0$.

Let $p_1 : P_1 \to L_1$ and $p_2 : P_2 \to L_2$ be projective covers in $\sigma[M]$. Then $P_1 \oplus P_2 \to L_1 \oplus L_2 \simeq M/K$ is also a projective cover and we have $K \simeq Ke\,p_1 \oplus Ke\,p_2$. Since K is indecomposable, we may assume $Ke\,p_1 = 0$, i.e. L_1 is M-projective. Hence we obtain a decomposition $M = M' \oplus M''$ with $M' \simeq L_1$ and $K \subset M''$, with uniserial M''.

If $K \not\ll M$, then K contains a direct summand of M (M is amply supplemented) and therefore it is itself a direct summand.

(2) We only have to show $(a) \Rightarrow (b)$. Since M is semiperfect by assumption, we have for every finitely M-generated module K:

$K/Rad\,K$ is semisimple and $K = \bar{K} + L$ with finitely M-generated $\bar{K}$ and L, uniserial L and $\gamma(\bar{K}) = \gamma(K) - 1$.

Assume $N = N_1 \oplus \cdots \oplus N_r$, with each N_j isomorphic to some M_i, and $K \subset N$ is a finitely M-generated submodule. We give the proof by induction on $\gamma(K)$, where $\gamma(K) \leq \gamma(N) = r$ is given by 55.4. If $\gamma(K) = 1$, then K is uniserial, and the assertion follows from (a) by 55.5.

Assume the assertion is true for $k < r$, and let $K \subset N$ be M-generated with $\gamma(K) = k + 1$. We write $K = \bar{K} + L$, where $\bar{K}$ is a finitely M-generated submodule with $\gamma(\bar{K}) = k$, and L is M-cyclic and uniserial. Then there exists a decomposition $N = P_1 \oplus \cdots \oplus P_r$, with P_i uniserial and

$\bar{K} = \bigoplus_{i \le r}(\bar{K} \cap P_i)$. Let $\pi_i : N \to P_i$ denote the corresponding projections.

By (a), L is contained in a uniserial direct summand Q of N. One of the restrictions $\pi_i|_Q$ is an isomorphism by 55.5. Assume $\pi_1|_Q : Q \to P_1$ is an isomorphism. Then $(L)\pi_1 \subset \bar{K} \cap P_1$ or $\bar{K} \cap P_1 \subset (L)\pi_1$.

In the first case we conclude $(K)\pi_1 = (\bar{K} + L)\pi_1 \subset \bar{K} \cap P_1$ and obtain a decomposition

$$K = (K \cap P_1) \oplus (K \cap (P_2 \oplus \cdots \oplus P_r)).$$

Now $K \cap (P_2 \oplus \cdots \oplus P_r)$ is an M-generated submodule of $P_2 \oplus \cdots \oplus P_r$ with $\gamma(K \cap (P_2 \oplus \cdots \oplus P_r)) \le k$, and we can find a suitable decomposition for it by assumption.

In the second case, i.e. if $\bar{K} \cap P_1 \subset (L)\pi_1$, consider the decomposition $N = Q \oplus P_2 \oplus \cdots \oplus P_r$ (see 55.5). The corresponding projection $\pi : N \to Q$ is given by $\pi = \pi_1(\pi_1|_Q)^{-1}$ and hence $(\bar{K} \cap P_1)\pi \subset L$.

This means $(K)\pi = (\bar{K} + L)\pi = (\bigoplus_{i \le r}(\bar{K} \cap P_i) + L)\pi \subset L$ and

$$K = (K \cap Q) \oplus (K \cap (P_2 \oplus \cdots \oplus P_r)).$$

Again the second summand is M-generated with $\gamma(K \cap (P_2 \oplus \cdots \oplus P_r)) \le k$, and we obtain a decomposition by assumption.

Let us recall that an R-module N is *finitely M-presented* if there is an exact sequence $M^k \to M^n \to N \to 0$ with $k, n \in I\!N$ (see 46.9).

55.7 When are all finitely M-presented modules serial?

Let $M = M_1 \oplus \cdots \oplus M_k$ be a self-projective R-module with all M_i's cyclic and uniserial. Then the following assertions are equivalent:

(a) Every finitely M-presented R-module is serial;

(b) every direct sum of copies of the M_i's has the extension property for M-cyclic uniserial submodules;

(c) every factor module of $M \oplus M$ by a finitely M-generated submodule is serial;

(d) for $i, j \le k$, any diagram
$$\begin{array}{ccc} & & M_i \\ & \nearrow & \\ K & & \\ & \searrow & \\ & & M_j \end{array}$$
with M-cyclic uniserial module K, can be completed commutatively by $M_i \to M_j$ or $M_j \to M_i$;

(e) $End(M)$ is right (and left) serial.

Proof: $(b) \Rightarrow (a)$ Every finitely M-presented module is of the form N/K, where N is a direct sum of copies of the M_i's and $K \subset N$ a finitely M-generated submodule. By 55.6, we have a decomposition $N = P_1 \oplus \cdots \oplus P_r$ with uniserial P_i and $K = \bigoplus_{i \leq r}(K \cap P_i)$. Then $N/K \simeq \bigoplus_{i \leq r} P_i \,/\, (K \cap P_i)$ is serial.

$(a) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (d)$ If K' is an M-cyclic uniserial submodule of the (external) direct sum $M_i \oplus M_j$, then $(M_i \oplus M_j)/K'$ is isomorphic to the factor module of $M \oplus M$ by the M-generated submodule $(\bigoplus_{k \neq i} M_k) \oplus (\bigoplus_{k \neq j} M_k) \oplus K'$ and, by (c), it is serial.

Now let K be an M-cyclic uniserial R-module, and $f : K \to M_i$ and $g : K \to M_j$ given morphisms. With the injections ε_i, ε_j from M_i, M_j into $M_i \oplus M_j$ we construct the morphism $q^* = f\varepsilon_i - g\varepsilon_j : K \to M_i \oplus M_j$ and put $K' = Im\, q^*$. Denoting by $\bar{\varepsilon}_i$ the composition of ε_i with the canonical epimorphism $M_i \oplus M_j \to M_i \oplus M_j \,/\, K'$, we obtain the commutative diagram (pushout, 10.5)

$$\begin{array}{ccc} K & \xrightarrow{f} & M_i \\ \downarrow g & & \downarrow \bar{\varepsilon}_i \\ M_j & \xrightarrow{\bar{\varepsilon}_j} & M_i \oplus M_j \,/\, K' \end{array} .$$

By the above remarks, $(M_i \oplus M_j) \,/\, K'$ is serial, and it follows from 55.6,(1) that K' is contained in a uniserial direct summand Q of $M_i \oplus M_j$.

Assume $M_i \oplus M_j = Q \oplus L$ for some L. Then L is a uniserial (see 55.5), M-projective module and $(M_i \oplus M_j)/K' = (Q/K') \oplus L$.

The corresponding projection $\pi_L : M_i \oplus M_j/K' \to L$ yields

$$L = Im\, \bar{\varepsilon}_i \pi_L + Im\, \bar{\varepsilon}_j \pi_L \,,$$

and so $\bar{\varepsilon}_i \pi_L : M_i \to L$ or $\bar{\varepsilon}_j \pi_L : M_j \to L$ has to be epic and hence an isomorphism. The inverse mapping of $\bar{\varepsilon}_i \pi_L$, or $\bar{\varepsilon}_j \pi_L$, extends the diagram in the desired way.

$(d) \Rightarrow (b)$ Let K be an M-cyclic uniserial submodule of $N = N_1 \oplus \cdots \oplus N_k$ with every $N_j = Rn_j$ isomorphic to some M_i.

Let $\pi_i : N \to N_i$, $\varepsilon_i : N_i \to N$ denote the corresponding projections, resp. injections, and $\varphi_i = \pi_i|_K : K \to N_i$. Assume without restriction $Ke\, \varphi_1 \subset Ke\, \varphi_2 \subset \cdots \subset Ke\, \varphi_k$.

Setting $K_i = (K)\varphi_i$ we obtain, by factorization, the morphisms

$\alpha_i : K_1 \to K_i$, $(u)\varphi_1 \mapsto (u)\varphi_i$, $2 \leq i \leq k$, and the diagrams

$$\begin{array}{ccccc} 0 & \longrightarrow & K_1 & \longrightarrow & N_1 = Rn_1 \\ & & \downarrow \alpha_i & & \\ 0 & \longrightarrow & K_i & \longrightarrow & N_i = Rn_i \end{array} .$$

By assumption, these diagrams can be extended commutatively by $N_1 \to N_i$ or $N_i \to N_1$. Choose elements $c_i \in R$ in the following way: If, for i, $\beta_i : N_1 \to N_i$ is a map as described above, then we put $c_i = 1$.

If, for j, the map is given by $\gamma_j : N_j \to N_1$, we see that (α_j and) γ_j has to be monic. Then we have a map $\beta_j = \gamma_j^{-1} : Im\,\gamma_j \to N_j$ and choose some $c_j \in R$ with $(c_j n_1)\beta_j = n_j$, i.e. $(n_j)\gamma_j = c_j n_1$.

One of the submodules $Rc_i n_1 \subset N_1$ is the smallest, i.e. there exists $r \leq k$ with $K_1 \subset Rc_r n_1 \subset Rc_i n_1$ for all $i \leq k$. If $Rc_i n_1 = N_1$ for all $i \leq k$, we take $r = 1$. Then the element

$$y = (c_r n_1)\varepsilon_1 + (c_r n_1)\beta_2\varepsilon_2 + \cdots + (c_r n_1)\beta_k\varepsilon_k$$

is in $\bigoplus_{i \leq k} N_i = N$, and $Ry \simeq Rc_r n_1$ is uniserial. For $u \in K$, by the definition of the φ_i, we have $u = u\varphi_1\varepsilon_1 + \cdots + u\varphi_k\varepsilon_k$, and there exists $a \in R$ with $u\varphi_1 = ac_r n_1$ yielding

$$u = (ac_r n_1)\varepsilon_1 + (ac_r n_1)\beta_2\varepsilon_2 + \cdots + (ac_r n_1)\beta_k\varepsilon_k = ay\ ,$$

i.e. $K \subset Ry$. Now, by the choice of r, $(c_r n_1)\beta_r = n_r$, implying

$$Ry + \sum_{i \neq r} N_i = N \ \text{ and } \ Ry \cap \sum_{i \neq r} N_i = 0\ ,$$

i.e. Ry is a direct summand of N.

$(d) \Rightarrow (e)$ We prove: $Hom(M_i, M)$ is right uniserial over $S = End(_R M)$ for all $i = 1, \ldots, k$. For f, $g \in Hom(M_i, M)$ assume without restriction $Ke\,f \subset Ke\,g$. Because of $(d) \Leftrightarrow (b)$, we know that $Im\,f \subset N_1$ and $Im\,g \subset N_2$ for suitable indecomposable direct summands N_1, N_2 of M, which are isomorphic to some M_i's (see 55.5). Therefore we have the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M_i/Ke\,f & \xrightarrow{f} & N_1 \\ & & \downarrow g & & \\ & & N_2 & & \end{array} .$$

By (d), there exist $s : N_1 \to N_2$ or $t : N_2 \to N_1$ completing the diagram commutatively. Regarding s and t as elements of S, we have $g = fs \in fS$

or $f = gt \in gS$. Then $Hom(M_i, M)$ is uniserial and $S = \bigoplus_{i \le k} Hom(M_i, M)$ is right serial.

With the general assumptions of the proposition it follows from 55.2 that S is also left serial.

$(e) \Rightarrow (d)$ Let K be an M-cyclic uniserial module and $f : K \to M_i$, $g : K \to M_j$ two morphisms. We may assume that there is an epimorphism $\alpha : M_1 \to K$ and consider αf and αg as elements of the uniserial right $End(M)$-module $Hom_R(M_1, M)$. Then there exists some $s \in End(M)$ with $\alpha f = \alpha g s$ or $t \in End(M)$ with $\alpha f t = \alpha g$. From these relations follows $f = gs$, resp. $g = ft$. By restriction and projection we now obtain the desired morphisms between M_i and M_j.

In particular for $M = R$ the above results yield assertions for rings which are left and right serial. We have called them *serial rings*.

55.8 Serial rings. Characterizations.
For a left serial ring R the following are equivalent:

(a) every finitely presented left R-module is serial;
(b) every finitely generated, projective left R-module has the extension property for cyclic, uniserial submodules;
(c) factor modules of ${}_RR \oplus R$ by finitely generated submodules are serial;
(d) every diagram
$$\begin{array}{ccc} & & Re \\ & \nearrow & \\ K & & \\ & \searrow & \\ & & Rf \end{array}$$
, with cyclic uniserial module K and primitive idempotents e, $f \in R$, can be commutatively completed by some $Re \to Rf$ or $Rf \to Re$;
(e) R is right serial.

The case, that all *finitely M-generated* modules are serial, will turn out to be closely connected to the question when uniform modules in $\sigma[M]$ are uniserial. The following assertions are useful for studying this problem:

55.9 Modules with uniserial injective hulls.
Let M be an R-module and $N_1, \ldots, N_k$ non-zero cyclic modules in $\sigma[M]$ with uniserial M-injective hulls $\widehat{N}_1, \ldots, \widehat{N}_k$. Then:

(1) $N = N_1 \oplus \cdots \oplus N_k$ has the extension property for uniserial submodules, and every uniserial submodule of N has a uniserial M-injective hull.

Attention: this is not correct, counter example $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$; may concern further assertions.

(2) For $i, j \leq k$, any diagram
$$\begin{tikzcd} & N_i \\ K \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & N_j \end{tikzcd}$$
with f, g monic, can be completed commutatively by some $N_i \to N_j$ or $N_j \to N_i$.

(3) If $N_1, \ldots, N_k$ are submodules of an R-module, then $N_1 + \cdots + N_k$ is a serial submodule.

Proof: (1) Let K be a uniserial submodule of N, $\pi_i : N \to N_i = Rn_i$ the canonical projections and $\varphi_i = \pi_i|_K$. We proceed similarly to the proof of 55.7,$(d) \Rightarrow (b)$. Assuming $Ke\,\varphi_1 \subset Ke\,\varphi_2 \subset \cdots \subset Ke\,\varphi_k$ we obtain, with the notation of 55.7, the diagrams

$$\begin{array}{ccccc} 0 & \longrightarrow & K_1 & \longrightarrow & N_1 = Rn_1 \\ & & \downarrow \alpha_i & & \\ 0 & \longrightarrow & K_i & \longrightarrow & N_i = Rn_i \quad \subset \widehat{N}_i \quad . \end{array}$$

For each $2 \leq i \leq k$ we can find a morphism $\beta_i : N_1 \to \widehat{N}_i$ yielding commutative diagrams. If $Im\,\beta_i \subset N_i$, we choose $c_i = 1$. If $N_j \subset Im\,\beta_j$, we take some $c_j \in R$ with $(c_j n_1)\beta_j = n_j$.

Again following the proof of 55.7,$(d) \Rightarrow (b)$, we can find a uniform direct summand Q of N with $K \subset Q$. Its M-injective hull $\widehat{Q}$, as an indecomposable direct summand of $\widehat{N} = \widehat{N}_1 \oplus \cdots \oplus \widehat{N}_k$, is uniserial (see 55.5).

(2) A diagram with non-zero K and monomorphisms

$$\begin{array}{ccc} K & \xrightarrow{f} & N_i \\ {\scriptstyle g}\downarrow & & \\ N_j & \subset & \widehat{N}_j \end{array}$$

can be extended commutatively by a monomorphism $h : N_i \to \widehat{N}_j$. If $Im\,h \subset N_j$, we are done. If $N_j \subset Im\,h$, then $h^{-1} : N_j \to N_i$ is the desired morphism.

(3) First we show that $N_1 + N_2$ is serial: The commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_1 \cap N_2 & \longrightarrow & N_1 & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & N_2 & \longrightarrow & N_1 + N_2 & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & Y & = & Y & & & & \end{array}$$

can, by (2), be extended commutatively by $N_1 \to N_2$ or $N_2 \to N_1$. In the first case the second row splits (Homotopy Lemma), in the second case the middle column splits. Hence in both cases, $N_1 + N_2$ is a direct sum of uniserial modules.

Now we prove the assertion by induction on the number k of summands in $N' = \sum_1^k N_i$. Without restriction we assume the sum $L = \sum_1^{k-1} N_i$ to be direct. Then $U = L \cap N_k$ is a uniserial submodule of L and, by (1), contained in a uniserial direct summand $\bar{U} \subset L$, i.e. $L = \bar{U} \oplus Q$ for some serial Q. We have

$Q + (\bar{U} + N_k) = L + N_k = N'$ and

$Q \cap (\bar{U} + N_k) = Q \cap ((\bar{U} + N_k) \cap L) = Q \cap \bar{U} = 0,$

i.e. $N' = Q \oplus (\bar{U} + N_k)$.

As shown above, $\bar{U}+N_k$ is serial, being a sum of two uniserial submodules (with uniserial $\widehat{N}_k$). Then N' is also serial.

In contrast to 55.7, in the next proposition it is not necessary to demand M to be self-projective:

55.10 When are all finitely *M*-generated modules serial?

For a finitely generated R-module M the following are equivalent:

(a) Every finitely M-generated module is serial;

(b) every factor module of $M \oplus M$ is serial;

(c) M is serial, and every indecomposable injective (uniform) module in $\sigma[M]$ is uniserial;

(d) M is serial, and every module, which is finitely generated by cyclic, uniserial modules in $\sigma[M]$, is serial.

If there is a generating set of cyclic uniserial modules in $\sigma[M]$, then (a) to (d) are also equivalent to:

(e) every finitely generated module in $\sigma[M]$ is serial.

Proof: $(a) \Rightarrow (b)$ and $(d) \Rightarrow (a)$ are obvious.

$(b) \Rightarrow (c)$ Let Q be an indecomposable, injective module in $\sigma[M]$. Then Q is M-generated (see 16.3) and every finitely generated submodule of Q is contained in a finitely M-generated submodule of Q. Since Q is uniform, we obtain from (b) that any sum of two M-cyclic submodules of Q is uniserial. Hence every finitely M-generated submodule of Q – and also Q itself – is uniserial.

$(c) \Rightarrow (d)$ If (c) holds, then in particular the injective hulls of the uniserial modules in $\sigma[M]$ are uniserial and the assertion follows from 55.9,(3).

$(d) \Leftrightarrow (e)$ Given a generating set as demanded, every finitely generated module in $\sigma[M]$ is a finite sum of cyclic uniserial modules.

For the ring R the preceding results yield:

55.11 Rings with every finitely generated module serial.
For a ring R, the following assertions are equivalent:
(a) Every finitely generated module in R-MOD is serial;
(b) every factor module of ${}_RR \oplus R$ is serial;
(c) R is left serial, and every indecomposable injective module in R-MOD is uniserial.

If these properties hold, R is also right serial.

If ${}_RR$ is noetherian, then (a)–(c) are equivalent to:
(d) R is (left and right) serial.

Proof: The equivalence of (a), (b) and (c) follows from 55.10. In particular, the finitely presented left modules are serial and hence, by 55.8, R is right serial.

If ${}_RR$ is noetherian, all finitely generated R-modules are finitely presented, and $(a) \Leftrightarrow (d)$ follows from 55.8.

If the module M is hereditary in $\sigma[M]$, then the factor modules of injective modules are again injective in $\sigma[M]$ (see 39.8). Therefore uniform modules in $\sigma[M]$ are uniserial in the following situation:

55.12 Hereditary modules and serial modules.
Let M be an R-module which is hereditary in $\sigma[M]$.

(1) If the M-injective hull $\widehat{M}$ is serial, then every indecomposable, injective module in $\sigma[M]$ is uniserial.

(2) If M is locally noetherian, then the following are equivalent:
(a) $\widehat{M}$ is serial;
(b) every indecomposable injective module in $\sigma[M]$ is uniserial;
(c) every injective module in $\sigma[M]$ is serial.

Proof: (1) Assume $\widehat{M} = \bigoplus_\Lambda M_\lambda$, with uniserial M-injective M_λ's, and let Q be an indecomposable injective module in $\sigma[M]$. Since the factor modules of the M_λ's are injective, any morphism $M_\lambda \to Q$ has to be epic or zero. Since $\widehat{M}$ generates every injective module in $\sigma[M]$, hence also Q, there is (at least) one epimorphism $M_\lambda \to Q$, and therefore Q is uniserial.

(2) $(a) \Rightarrow (b)$ has been shown in (1). $(c) \Rightarrow (a)$ is evident.

$(b) \Rightarrow (c)$ By Matlis' Theorem 27.4, every injective module is a direct sum of indecomposable modules.

We will need the next results to study those modules M, for which *all* modules in $\sigma[M]$ are serial.

55.13 Uniserial modules and modules of finite length.

Let M be an R-module and $\{N_\lambda\}_\Lambda$ a family of modules of finite length in $\sigma[M]$, whose lengths are bounded by some $n \in I\!N$.

(1) Assume $N = \sum_\Lambda N_\lambda$. Then:

(i) Every N-generated, uniserial module is of length $\leq n$.

(ii) If the M-injective hulls $\widehat{N}_\lambda$ are uniserial, then (at least) one of the N_λ's is N-injective.

(2) Assume $N = \bigoplus_\Lambda N_\lambda$. Then:

(i) Every uniserial module with non-zero socle which is cogenerated by N has length $\leq n$.

(ii) If the N_λ's have uniserial projective covers in $\sigma[M]$ of finite length, then (at least) one N_λ is N-projective.

(3) Assume N is a uniserial module of finite length in $\sigma[M]$. Then:

(i) If the M-injective hull $\widehat{N}$ is uniserial, then N is self-injective.

(ii) If N has an M-projective cover of finite length, then N is self-projective.

Proof: (1)(i) Let K be an N-generated uniserial module. Every finitely N-generated submodule K' of K is obviously N-cyclic, therefore a sum of factor modules of the N_λ's, in fact a factor module of one of the N_λ's. So $lg(K') \leq lg(N_\lambda) \leq n$, and also $lg(K) \leq n$.

(ii) Consider N_o with maximal length n_o among the N_λ's. The N-injective hull $\widetilde{N}_o$ of N_o is N-generated and, as a submodule of the uniserial M-injective hull $\widehat{N}_o$, it is also uniserial. By (i), we have $lg(\widetilde{N}_o) \leq n_o$ and consequently $N_o = \widetilde{N}_o$. Hence N_o is N-injective.

(2)(i) A uniserial module K with $Soc(K) \neq 0$ is cocyclic. If K is cogenerated by the N_λ's, then $K \subset N_\lambda$ for some $\lambda \in \Lambda$ (see 14.8). Therefore $lg(K) \leq n$.

(ii) For $\lambda \in \Lambda$, let P_λ denote the uniserial M-projective cover of N_λ, $lg(P_\lambda) \leq \infty$. Then $P = \bigoplus_\Lambda P_\lambda$ is a serial, projective module in $\sigma[M]$ and N is a factor module of P.

The module $L = Re(P,N) = \bigcap\{Ke\,f \mid f \in Hom(P,N)\}$ is a fully invariant submodule of P, and therefore $P/L = \bigoplus_\Lambda \bar{P}_\lambda$ is a self-projective module with uniserial $\bar{P}_\lambda = P_\lambda/(L \cap P_\lambda)$. By definition of L, P/L – and hence every $\bar{P}_\lambda$ – is cogenerated by N.

Because of (i), this means $lg(\bar{P}_\lambda) \leq n$ and $\bar{P}_\lambda \in \sigma[N]$ for every $\lambda \in \Lambda$, hence also $P/L \in \sigma[N]$.

On the other hand, N is also a factor module of P/L, i.e. $N \in \sigma[P/L]$, and the N_λ's are generated by $\{\bar{P}_\lambda\}_\Lambda$. For $N_o \in \{N_\lambda\}_\Lambda$, with maximal length n_o, there exists an epimorphism $\bar{P}_{\lambda_o} \to N_o$ for some $\lambda_o \in \Lambda$. Then $lg(\bar{P}_{\lambda_o}) = n_o$ and $\bar{P}_{\lambda_o} \simeq N_o$

As a direct summand of P/L, N_o is projective in $\sigma[P/L] = \sigma[N]$.

(3) follows immediately from (1) and (2).

55.14 When are all modules in $\boldsymbol{\sigma[M]}$ serial?

For an R-module M of finite length the following are equivalent:

(a) *every module in $\sigma[M]$ is serial;*

(b) *every finitely generated module in $\sigma[M]$ is serial;*

(c) *every finitely generated indecomposable module in $\sigma[M]$ is uniserial;*

(d) *every non-zero finitely generated module N in $\sigma[M]$ contains a non-zero N-projective (and N-injective) direct summand;*

(e) *every finitely generated indecomposable module in $\sigma[M]$ is self-injective and self-projective;*

(f) *for every finitely generated indecomposable module $K \in \sigma[M]$, $Soc\, K$ and $K/Rad\, K$ are simple;*

(g) *there is a progenerator $P \in \sigma[M]$, and, for any non-zero fully invariant submodule $K \subset P$, P/K contains a non-zero P/K-injective summand;*

(h) *there is a finitely generated, injective cogenerator Q in $\sigma[M]$, and every non-zero fully invariant submodule $L \subset Q$ has a non-zero L-projective factor module.*

Under these conditions M is of finite representation type.

Proof: $(a) \Rightarrow (b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$ Let every finitely generated, indecomposable module in $\sigma[M]$ be uniserial. Then obviously the indecomposable injective modules in $\sigma[M]$ are uniserial, and since they are M-generated, their length is bounded by the length of M. Therefore the lengths of all finitely generated, indecomposable modules in $\sigma[M]$ are bounded and, by 54.2, M is pure semisimple. Hence in $\sigma[M]$ every module is a direct sum of uniserial modules (see 53.4).

Also by 54.2, we see that there exists a progenerator in $\sigma[M]$.

$(a) \Rightarrow (d)$ We have shown in the proof $(c) \Rightarrow (a)$ that there exists a serial (semiperfect) progenerator in $\sigma[M]$. Then every finitely generated uniserial module has a uniserial projective cover. Since the injective hulls of uniserial modules are uniserial, the assertion follows from 55.13.

$(d) \Rightarrow (e)$ is obvious if we assume that there exists an N-injective and N-projective summand in N. If we only demand the existence of an N-projective direct summand, we have to argue differently (see $(d) \Rightarrow (h)$).

$(e) \Rightarrow (f)$ Self-injective indecomposable modules have a simple socle. Self-projective indecomposable modules K of finite length are local (see 32.4, 19.7), and hence $K/Rad\,K$ has to be simple.

$(f) \Rightarrow (c)$ Let N be an indecomposable injective module in $\sigma[M]$. Then, because of (f), for every finitely generated $K \subset N$, the factor module $K/Rad\,K$ is simple, and, by 55.1, N is uniserial.

Let L be an indecomposable module of finite length in $\sigma[M]$. Because of (f), $Soc(L)$ is simple, and L is a submodule of the injective hull of $Soc(L)$. We have already seen that this injective hull is uniserial.

$(a) \Rightarrow (g)$ In the proof of $(c) \Rightarrow (a)$ the existence of a progenerator P in $\sigma[M]$ was shown. The remaining assertion follows from (d).

$(g) \Rightarrow (e)$ Let N be a finitely generated, indecomposable module in $\sigma[M]$ and $K = Re(P, N) = \bigcap\{Ke\,f \mid f \in Hom(P, N)\}$.

K is a fully invariant submodule of P, P/K is self-projective, generates N, and $P/K \subset N^r$ for some $r \in I\!N$ (note that P has finite length).

By (g), P/K contains the P/K-injective hull $\widetilde{E}$ of some simple submodule $E \subset P/K$ which, as a direct summand, is also P/K-projective. Since $\widetilde{E}$ is cogenerated by N, $\widetilde{E}$ is isomorphic to a submodule of N (see 14.8) and hence $\widetilde{E} \simeq N$. So N is projective and injective in $\sigma[P/K] = \sigma[N]$.

$(d) \Rightarrow (h)$ We only assume that any non-zero finitely generated module N in $\sigma[M]$ has a non-zero N-projective factor module. Then the finitely generated submodules K of indecomposable, injective modules in $\sigma[M]$ are self-projective, and $K/Rad\,K$ is simple (see $(e) \Rightarrow (f)$). Consequently the indecomposable, injective modules in $\sigma[M]$ are uniserial (see 55.1), and, by 55.13,(1), their lengths are bounded. Since there are only finitely many non-isomorphic simple modules in $\sigma[M]$ (see 32.4), their M-injective hulls form an injective cogenerator of finite length in $\sigma[M]$.

$(h) \Rightarrow (e)$ Let N be a finitely generated, indecomposable module in $\sigma[M]$ and $L = Tr(N, Q)$, i.e. $L \in \sigma[N]$. N is finitely cogenerated by Q – hence also by L – and $\sigma[N] = \sigma[L]$.

As a fully invariant submodule of Q, L is self-injective and, because of (h), it has an L-projective, local factor module V (= L-projective cover of a simple factor module of L), which is also L-injective. This V is N-generated, hence a factor module of N. Since N is indecomposable we conclude $N \simeq V$. So N is projective and injective in $\sigma[L] = \sigma[N]$.

Further characterizations of the modules just considered are given via the functor ring:

55.15 Serial modules and functor rings.

Let M be a finitely generated R-module, $\{U_\alpha\}_A$ a representing set of the finitely generated modules in $\sigma[M]$, $U = \bigoplus_A U_\alpha$ and $T = \widehat{End}(U)$ (see § 52). The following are equivalent:

(a) M is of finite length, and every module in $\sigma[M]$ is serial;

(b) T is left and right perfect, and for every primitive idempotent $e \in T$, eT and Te are self-injective;

(c) T is left perfect and, for every primitive idempotent $e \in T$, Te and eT have simple, essential socles;

(d) $\sigma[M]$ has a finitely generated generator, T is semiperfect, and, for primitive idempotents $e \in T$, Te and eT have simple, essential socles.

Proof: $(a) \Rightarrow (b)$ We have seen in 55.14 that M is of finite type. Hence, by 54.2, T is left and right perfect and also right locally noetherian. The latter property yields that U_T is not only absolutely pure (see 52.1) but also injective in MOD-T (see 53.6).

For a primitive idempotent $e \in T$, there exists some finitely generated indecomposable module $K \in \sigma[M]$ with

$$Te \simeq \widehat{Hom}_R(U,K) \quad\text{and}\quad eT \simeq Hom_R(K,U).$$

Since K is self-injective (see 55.14), this also holds for $\widehat{Hom}_R(U,K)$ by 51.7.

K is also self-projective by 55.14, and we show that $Hom_R(K,U)$ is self-injective. Consider the exact diagram in MOD-T

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \xrightarrow{\varepsilon} & Hom_R(K,U) \\ & & \downarrow f & & \\ & & Hom_R(K,U) & & \end{array} .$$

As mentioned above, the functor $Hom_T(-,U_T) = (-)^*$ is exact, and hence we obtain the exact diagram

$$\begin{array}{ccccc} & & K \simeq Hom_R(K,U)^* & & \\ & & \downarrow f^* & & \\ K \simeq Hom_R(K,U)^* & \xrightarrow{\varepsilon^*} & X^* & \longrightarrow & 0 \,, \end{array}$$

which can be extended commutatively by some $h : K \to K$. Applying the functor $Hom_R(-,U) = (-)^*$ leads to the commutative diagram (see 45.10)

$$\begin{array}{ccccc} X & \xrightarrow{\Phi_X} & X^{**} & \xrightarrow{\varepsilon^{**}} & Hom_R(K,U) \\ & & \downarrow f^{**} & \swarrow h^* & \\ & & Hom_R(K,U) & & \end{array} ,$$

with $\varepsilon = \Phi_X \varepsilon^{**}$ and $f = \Phi_X f^{**}$ and the assertion is verified.

For the last part of this proof we could also use general properties of the adjoint pair of functors $Hom_R(-, U)$, $Hom_T(-, U)$ (see 45.9, 45.10).

$(b) \Rightarrow (c)$ Let $e \in T$ be a primitive idempotent. Since ${}_TT$ and T_T are perfect, eT and Te have essential socles (see 49.9). As indecomposable, self-injective modules, Te and eT are uniform and so have simple socles.

$(c) \Rightarrow (a)$ For T left perfect, M is pure semisimple, in particular noetherian (see 53.4, 53.5). Every finitely generated, projective module in T-*MOD* is a direct sum of indecomposable modules (see 49.10) and hence, because of (c), finitely cogenerated. By 52.1, M is artinian.

Let K be a finitely generated, indecomposable module in $\sigma[M]$. Then $\widehat{Hom}_R(U, K)$ and $Hom_R(K, U)$ are indecomposable projective T-modules, have simple essential socles (because of (c)), and hence are uniform. Therefore the submodules $\widehat{Hom}_R(U, Soc(K))$, resp. $Hom_R(K/Rad\, K, U)$, are indecomposable and so are $Soc(K)$ and $K/Rad\, K$. Hence the latter modules are indecomposable and semisimple, i.e. they are simple. Now we derive from 55.14 that all modules in $\sigma[M]$ are serial.

$(a) \Rightarrow (d)$ We have seen in 55.14 that there exists a progenerator in $\sigma[M]$. The rest has been shown already in $(a) \Rightarrow (c)$.

$(d) \Rightarrow (a)$ Over the semiperfect ring T, every finitely generated, projective module in T-*MOD* is a direct sum of indecomposable modules (see 49.10) and, because of (d), is finitely cogenerated. By 52.1, M is artinian. Hence the finitely generated generator in $\sigma[M]$ is also artinian and, by 32.8, noetherian. The rest follows similarly to $(c) \Rightarrow (a)$.

Semiperfect rings T, whose indecomposable, projective left and right modules have simple and essential socles, are called *QF-2 rings* (see (d) in 55.15).

In 55.14 we *assumed* M to be of finite length. If there exists a progenerator P in $\sigma[M]$ and all modules in $\sigma[M]$ are serial, then it can be *concluded* that P is of finite length. Since in this case $\sigma[M]$ is equivalent to a full category of modules, this is contained in the following characterizations for rings.

55.16 When are all R-modules serial?

(1) For a ring R the following assertions are equivalent:

(a) every module in R-MOD is serial;

(b) R is a serial ring and left (and right) artinian;

(c) every module in MOD-R is serial;

(d) the functor ring T of the finitely generated R-modules is a QF-2 ring.

(2) For a left artinian ring R the following are equivalent:

(a) every (finitely generated) module in R-MOD is serial;

(b) every finitely generated, indecomposable module in R-MOD is uniserial;

(c) every non-zero finitely generated N in R-MOD has a non-zero N-projective factor module (N-injective submodule);

(d) every finitely generated, indecomposable left R-module is self-injective and self-projective;

(e) for every finitely generated, indecomposable $K \in R$-MOD, $Soc\,K$ and $K/Rad\,K$ are simple;

(f) every factor ring $\bar{R} \neq 0$ of R contains a non-zero $\bar{R}$-injective summand;

(g) R-MOD has a finitely generated, injective cogenerator Q, and every fully invariant submodule $L \subset Q$ has an L-projective factor module.

Proof: (1) $(a) \Rightarrow (b)$ By 55.11, R is left and right serial. Since every module in R-MOD is a direct sum of indecomposable modules, R is left pure semisimple by 53.6 and, in particular, left artinian (see 53.6).

Now we derive from 55.13 that the lengths of the indecomposable, i.e. uniserial modules in R-MOD are bounded (by the length of ${}_RR$). Therefore ${}_RR$ and R_R are of finite type (see 54.3) and R_R is artinian.

$(b) \Rightarrow (a)$ If R is a serial left artinian (hence left noetherian) ring, then, by 55.11, the finitely generated modules in R-MOD are serial. Hence it follows from 55.14 that all modules in R-MOD are serial.

$(b) \Leftrightarrow (c)$ is obtained symmetrically to $(a) \Leftrightarrow (b)$.

$(a) \Leftrightarrow (d)$ is a special case of 55.15.

(2) These are the characterizations of 55.14. Fully invariant submodules of ${}_RR$ are exactly two-sided ideals of R. This yields the formulation in (f).

55.17 Exercises.

(1) Consider a left R-module M. An element $m \in M$ is called singular *if $An_R(m) \trianglelefteq {}_RR$. M is called* non-singular *if it contains no non-trivial singular elements.*

Prove that for a left serial ring R we have the following two pairs of equivalent assertions:

(i) (a) ${}_RR$ is non-singular;

(b) ${}_RR$ is semihereditary.

(ii) (a) ${}_RR$ is non-singular and noetherian;

(b) ${}_RR$ is hereditary. (Warfield, Sandomierski, Kirichenko)

(2) Let R be a left serial ring and $J = Jac\,R$. Prove that the following are equivalent (Deshpande):
(a) $\bigcap_{\mathbb{N}} J^n = 0$;
(b) ${}_RR$ *is noetherian and, for essential left ideals* $I \subset R$, R/I *is artinian.*

(3) A ring R is called left πc-ring *if every cyclic left R-module is* π-*injective (see 41.20, 41.23,(1)). Prove:*

(i) If R is a left πc*-ring, and* $e, f \in R$ *are orthogonal primitive idempotents with* $eRf \neq 0$, *then* Re *and* Rf *are isomorphic minimal left ideals.*

(ii) For a semiperfect ring R the following are equivalent:
(a) R is left πc*-ring;*
(b) $R = A \times B$, *where* A *is an artinian semisimple ring and* B *is a direct sum of left uniserial rings. (Goel-Jain)*

(4) Try to show that for a ring R the following are equivalent:
(a) Every finitely generated left R-module is serial;
(b) R is (left and right) serial and the injective hulls of the simple left R-modules are uniserial. (Gregul-Kirichenko)

(5) An R-module M is called minimal faithful *if it is a direct summand in every faithful R-module. A ring R having a minimal faithful R-module is said to be a* QF-3-ring. *Prove:*

For a left artinian ring R with $J = Jac\,R$, *the following are equivalent:*
(a) R is serial;
(b) R/J^2 *is serial;*
(c) every factor ring of R is a left QF-3-ring.

(6) Let R be a left hereditary ring whose injective hull is flat in R-MOD. Show that every left artinian factor ring of R is serial. (Eisenbud-Griffith)

Literature: See references for section 56.

56 Homo-serial modules and rings

1.Homo-uniserial modules. 2.Homo-serial self-projective modules. 3.Left homo-serial rings. 4.Homo-serial self-injective modules. 5.All modules $N \in \sigma[M]$ flat over $End(N)$. 6.All modules $N \in \sigma[M]$ FP-injective over $End(N)$. 7.All modules homo-serial in $\sigma[M]$ (M of finite length). 8.All modules homo-serial in $\sigma[M]$ (with progenerator). 9.All R-modules homo-serial. 10.Homo-serial modules and functor rings. 11.Homo-serial $\mathbb{Z}$-modules. 12.Exercise.

Examples of serial modules are semisimple modules, the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ (see § 17), or the proper factor rings of Dedekind rings (see 40.6). All these cases have a characteristic property which we describe in the following definitions:

We call an R-module N *homogeneously uniserial*, or *homo-uniserial*, if for any non-zero finitely generated submodules K, $L \subset N$, the factor modules $K/Rad\, K$ and $L/Rad\, L$ are simple and isomorphic.

By 55.1, homo-uniserial modules N are uniserial and $N/Rad(N) \neq 0$ if and only if N is finitely generated.

N is called *homogeneously serial*, or *homo-serial*, if it is a direct sum of homo-uniserial modules. The ring R is called *left (right) homo-serial* if $_RR$ (resp. R_R) is a homo-serial module.

If $_RR$ is uniserial, then $_RR$ is always homo-uniserial (hence local), since in this case there is only one simple module ($= R/Jac\, R$) in R-*MOD*.

56.1 Homo-uniserial modules. Properties.
Let M and N be left R-modules. Then:

(1) N is homo-uniserial if and only if for any submodules K, $L \subset N$ for which N/K and N/L have non-zero socles, these socles are simple and isomorphic.

(2) Assume N is homo-uniserial and $S = End(_RM)$.

(i) If M is self-projective, finitely generated and semiperfect in $\sigma[M]$, then $_SHom_R(M,N)$ is homo-uniserial.

(ii) If M is self-injective and finitely cogenerated, then $Hom_R(N,M)_S$ is homo-uniserial.

(3) Assume N is homo-uniserial.

(i) If N is self-projective and finitely generated, N is a generator in $\sigma[N]$.

(ii) If N is self-injective and finitely cogenerated, then N is a cogenerator in $\sigma[N]$.

Proof: (1) By 55.1, any homo-serial module has the desired property. On the other hand, if N has this property, then N is uniserial by 55.1, and for any finitely generated submodule $K' \subset N$, we have $K'/RadK' \simeq Soc(N/Rad\,K')$. Now the assertion is evident.

(2)(i) Let N be homo-uniserial. By 55.1, ${}_SHom(M,N)$ is uniserial. Besides, for every pair $f,\, g \in Hom(M,N)$, $Mf/Rad(Mf) \simeq Mg/Rad(Mg)$. From the exact sequence

$$0 \longrightarrow Rad(Mf) \longrightarrow Mf \longrightarrow Mf/Rad(Mf) \longrightarrow 0$$

we obtain, with the functor $Hom(M,-)$, the isomorphism

$$Hom(M,Mf)/Hom(M,Rad(Mf)) \simeq Hom(M,Mf/Rad(Mf)) \ .$$

M is finitely generated, self-projective and semiperfect, hence it is a good module and we have (see 18.4)

$$Hom(M,Mf) \simeq Sf, \quad Hom(M,Rad(Mf)) \simeq Jac(S)f \ ,$$

and S is semiperfect (see 42.12). From this we conclude $Rad(Sf) \simeq Jac(S)f$ and the above isomorphism yields

$$Sf/Rad(Sf) \simeq Hom(M,Mf/Rad(Mf)) \ ,$$

and for all $f,\, g \in S$, $Sf/Rad(Sf) \simeq Sg/Rad(Sg)$.

(ii) Again by 55.1, we know that $Hom(N,M)_S$ is uniserial. By (1), we obtain, for any $f,\, g \in Hom(N,M)$,

$$Soc(Ng) \simeq Soc(N/Ke\,g) \simeq Soc(N/Ke\,f) \simeq Soc(Nf).$$

From the exact sequence

$$0 \to Soc(Nf) \to Nf \to Nf/Soc(Nf) \to 0$$

we obtain the isomorphism

$$Hom(Nf,M)/Hom(Nf/Soc(Nf),M) \simeq Hom(Soc(Nf),M).$$

Since M is self-injective and finitely cogenerated, S is semiperfect (see 22.1) and $Hom(Nf,M) \simeq Hom(N/Ke\,f,M) \simeq fS$. From the diagram

$$\begin{array}{ccccc} N & \xrightarrow{f} & (Nf+Soc\,M)/Soc\,M & \subset & M/Soc\,M \\ & & \downarrow & & \\ & & M & & \end{array}$$

we conclude by 22.1,(5) and 23.3,

$$Hom(Nf/Soc(Nf), M) \simeq f Hom(M/Soc\, M, M) \simeq f Jac(S) \simeq Rad(fS).$$

Then $fS/Rad(fS) \simeq Hom(Soc(Nf), M)$ and $fS/Rad(fS) \simeq gS/Rad(gS)$ for any $f, g \in Hom(N, M)$. Hence $Hom(N, M)_S$ is homo-uniserial.

(3) (i) Since there is only one simple module in $\sigma[N]$, and this is a factor module of N ($\simeq N/Rad\, N$), N is a generator in $\sigma[N]$ by 18.5.

(ii) is dual to (i).

56.2 Homo-serial self-projective modules.

For a finitely generated self-projective R-module M, the following assertions are equivalent:

(a) M is homo-serial;

(b) M is semiperfect in $\sigma[M]$ and every finitely generated submodule of M is M-cyclic;

(c) M is a self-generator and End(M) is left homo-serial;

(d) M is a self-generator, End(M) is semiperfect, and every finitely generated left ideal in End(M) is cyclic;

(e) M is a self-generator and End(M) is isomorphic to a finite product of matrix rings over left uniserial rings;

(f) $\sigma[M]$ is equivalent to a category S-MOD, with S a finite product of left uniserial rings.

Proof: $(a) \Rightarrow (b)$ As a direct sum of local modules, M is semiperfect in $\sigma[M]$ (see 55.2). The simple modules in $\sigma[M]$ are simple subfactors of the homo-uniserial summands of M and are generated by these (see 56.1,(3)). Therefore M generates every simple module in $\sigma[M]$ and hence, by 18.5, is a generator in $\sigma[M]$.

Let $M = M_1 \oplus \cdots \oplus M_k$ be a decomposition of M such that each M_i consists of those homo-uniserial summands of M with isomorphic simple factor modules. Then every $\sigma[M_i]$ contains exactly one simple module (up to isomorphism) and $Hom(M_i, M_j) = 0$ for $i \neq j$.

Hence, for every finitely generated submodule $K \subset M$, we obtain

$$K = Tr(M, K) = \bigoplus\nolimits_{i \leq k} K_i \quad \text{with } K_i = Tr(M_i, K) \subset M_i.$$

Therefore it suffices to show that every K_i is a factor module of M_i. Since M_i is a generator in $\sigma[M_i]$, K_i is finitely M_i-generated.

By 55.4,(3), $\gamma(K_i) \leq \gamma(M_i)$. Since there is (up to isomorphism) only one simple module in $\sigma[M_i]$, K_i is M_i-cyclic by 55.4,(2).

$(b) \Rightarrow (a)$ Under the conditions of (b), M is a generator in $\sigma[M]$ and a direct sum of local modules. Let $M = P \oplus Q$ with P local. Assume P is not uniserial. Then, by 55.1, there exists a finitely generated submodule $K \subset P$, for which the semisimple module $K/Rad\,K$ is not simple. Hence $\gamma(K) \geq 2$ and

$$\gamma(K \oplus Q) = \gamma(K) + \gamma(Q) > 1 + \gamma(Q) = \gamma(M).$$

By 55.4, $K \oplus Q$ cannot be M-cyclic, contradicting (b).

Now assume P is not homo-uniserial. Then there exists a finitely generated submodule $L \subset P$ with $E := L/Rad\,L \not\simeq P/Rad\,P$, $\gamma(M, E) = \gamma(Q, E)$ and $\gamma(L \oplus Q, E) = 1 + \gamma(M, E)$. Again by 55.4, this yields that $L \oplus Q$ cannot be M-cyclic, also a contradiction to (b).

$(a) \Rightarrow (c)$ In $(a) \Rightarrow (b)$ we have seen that M is a generator in $\sigma[M]$. By 56.1, we conclude that $End(M)$ is left homo-serial.

$(c) \Rightarrow (d)$ This follows from the proof of $(a) \Rightarrow (b)$, replacing M by the ring $End(M)$ considered as left module.

$(d) \Rightarrow (b)$ M is semiperfect in $\sigma[M]$ if and only if $End(M)$ is semiperfect (see 42.12). Every finitely M-generated submodule $K \subset M$ can be written as $K = MI$, with I a finitely generated left ideal of $End(M)$. By (d), $I = End(M)t$ for some $t \in End(M)$ and $K = MI = Mt$ is M-cyclic.

$(a) \Rightarrow (e)$ Let $M = M_1 \oplus \cdots \oplus M_k$ be a decomposition of M, as in the proof of $(a) \Rightarrow (b)$. Then $M_i = N_1 \oplus \cdots \oplus N_r$ is a direct sum of homo-uniserial M-projective modules N_j with $N_j/Rad\,N_j \simeq N_1/Rad\,N_1$ for all $j \leq r$. From this we see that N_j is a projective cover of N_1 and hence $N_j \simeq N_1$ for every $j \leq r$.

Therefore $End(M_i) \simeq End(N_1)^{(r,r)}$ with $End(N_1)$ a left uniserial ring (see 55.1). Since $Hom(M_i, M_j) = 0$ for $i \neq j$, $End(M)$ is isomorphic to the product of the rings $End(M_i)$.

$(e) \Rightarrow (d)$ Obviously it suffices to show that in a matrix ring over a uniserial ring D, finitely generated left ideals are cyclic. Of course, finitely generated left ideals in D are cyclic.

By the equivalence $Hom_D(D^r, -) : D\text{-}MOD \rightarrow D^{(r,r)}\text{-}MOD$, finitely generated left ideals in $D^{(r,r)}$ correspond exactly to finitely generated left ideals in D, and hence they are factor modules of $Hom_D(D^r, D)$, i.e. they are cyclic.

$(a) \Rightarrow (f)$ Again let $M = M_1 \oplus \cdots \oplus M_k$ be the decomposition of M from the proof of $(a) \Rightarrow (b)$. In every M_i we choose a homo-uniserial summand $\widetilde{M_i}$. Then $\widetilde{M} = \widetilde{M_1} \oplus \cdots \oplus \widetilde{M_k}$ is a projective generator in $\sigma[M]$.

Since $Hom(\widetilde{M}_i, \widetilde{M}_j) = 0$ for $i \neq j$, $S = End(M) \simeq \prod_{i \leq k} End(\widetilde{M}_i)$ with left uniserial rings $End(\widetilde{M}_i)$ (see 55.1).

$Hom_R(\widetilde{M}, -)$ is an equivalence between $\sigma[M]$ and S-MOD (see 46.2).

$(f) \Rightarrow (a)$ Let P be a progenerator in $\sigma[M]$ and $S = End(P)$ a finite product of left uniserial rings. From the implication $(e) \Rightarrow (a)$ already shown applied to P, we obtain that P is a homo-serial module. Then every finitely generated, projective module in $\sigma[M]$ is homo-serial, in particular M is homo-serial.

For rings, the results in 56.2 yield:

56.3 Left homo-serial rings.

For a ring R the following assertions are equivalent:

(a) ${}_RR$ is homo-serial;

(b) R is semiperfect, and every finitely generated left ideal in R is cyclic;

(c) R is isomorphic to a finite product of matrix rings over left uniserial rings;

(d) R is Morita equivalent to a ring S, which is a product of left uniserial rings.

Rings, whose finitely generated left ideals are cyclic, are called *left Bezout rings*. Observe that 56.3 only concerns *semiperfect* left Bezout rings.

Dually to 56.2 we obtain:

56.4 Homo-serial self-injective modules.

For a finitely cogenerated, self-injective R-module M, the following are equivalent:

(a) M is homo-serial;

(b) every finitely cogenerated factor module of M is isomorphic to a submodule of M;

(c) M is a self-cogenerator and $End(M)$ is a right Bezout ring.

Proof: $(a) \Rightarrow (c)$ By 56.1,(3), M cogenerates every simple module in $\sigma[M]$ and, by 16.5, M is a cogenerator in $\sigma[M]$. From 56.1,(2), we obtain that $End(M)$ is right homo-serial and hence is a right Bezout ring (see 56.2).

$(c) \Rightarrow (b)$ A finitely cogenerated factor module L of M is of the form $L \simeq M/Ke\, I$, for I some finitely generated right ideal of $End(M)$ and $Ke\, I = \bigcap\{Ke\, f \mid f \in I\}$. By (c), $I = tEnd(M)$ for some $t \in End(M)$ and hence $L \simeq M/Ke\, t \simeq Mt \subset M$.

$(b) \Rightarrow (a)$ Dually to $\gamma(M)$ considered in 55.4, we are now interested in the length of the socle:

For a finitely cogenerated module L and a simple module E, let $\delta(L,E)$ be the number of summands in a decomposition of $Soc(L)$ which are isomorphic to E. Since M is self-injective we note (dually to 55.4,(2)) that $L \in \sigma[M]$ is isomorphic to a submodule of M if and only if $\delta(L,E) \leq \delta(M,E)$ for every simple module E.

Let Q be an indecomposable direct summand of M, i.e. $M = Q \oplus V$ for some V.Then Q is uniserial: Assume for some $K \subset Q$, $Soc(Q/K) \neq 0$ is not simple (see 55.1). Then the length of $Soc(M/K) = Soc((Q/K) \oplus V)$ is greater than the length of $Soc(M)$ and hence M/K cannot be isomorphic to a submodule of M.

Q is homo-uniserial: Assume, for some $L \subset Q$, $E := Soc(Q/L) \neq 0$ is not isomorphic to $Soc(Q)$. Then

$$\delta(M/L,E) = \delta(Q/L \oplus V, E) = 1 + \delta(M,E)$$

and M/L is not isomorphic to a submodule of M. This contradicts (b).

In 15.9 we have shown that a module M is flat over $End(M)$ if and only if the kernels of morphisms $M^k \to M^n$, k, $n \in I\!N$, are M-generated. This is not yet sufficient for M to be a generator in $\sigma[M]$. However, if *all* modules in $\sigma[M]$ have the corresponding property we have:

56.5 All modules $N \in \sigma[M]$ flat over $End(N)$.

(1) For an R-module M the following assertions are equivalent:

(a) every module $N \in \sigma[M]$ is flat over $End(N)$;

(b) every self-injective module $N \in \sigma[M]$ is flat over $End(N)$;

(c) every module $N \in \sigma[M]$ is a generator in $\sigma[N]$.

(2) If the conditions given in (1) hold for M, then:

(i) Every non-zero module in $\sigma[M]$ contains a maximal submodule.

(ii) Every module in $\sigma[M]$ has a superflous radical.

(iii) Finitely generated, self-projective modules in $\sigma[M]$ are self-injective.

Proof: (1) $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$ are obvious.

$(b) \Rightarrow (c)$ By 15.5, M is a generator in $\sigma[M]$ if it generates every (cyclic) submodule $K \subset M^l$, $l \in I\!N$.

The direct sum of the M-injective hulls $\widehat{M^l}$ and $\widehat{M^l/K}$ is a self-injective module. The kernel of the morphism

$$f : \widehat{M^l} \oplus (\widehat{M^l/K}) \longrightarrow \widehat{M^l} \oplus (\widehat{M^l/K}), \quad (x,y) \mapsto (0, x+K),$$

is isomorphic to $K \oplus \widehat{M^l/K}$ and, by (b), is generated by the M-generated modules $\widehat{M}^l$ and $\widehat{M^l/K}$. Hence K is generated by M and M is a generator in $\sigma[M]$.

For $N \in \sigma[M]$, we consider similar constructions with N-injective hulls, and the same proof yields that N is a generator in $\sigma[N]$.

(2) (i) Every non-zero $N \in \sigma[M]$ generates a simple module E in $\sigma[N]$. Thus there is an epimorphism $h : N \to E$ and $Ke\, h$ is maximal in N.

(ii) We show that every proper submodule $K \subset N$, $N \in \sigma[M]$, is contained in a maximal submodule: By (i), in N/K there exists a maximal submodule L/K with $K \subset L \subset N$. Then L is maximal in N.

(iii) Let N be finitely generated and self-projective with N-injective hull $\widehat{N}$. Then $\widehat{N}$ is a generator in $\sigma[\widehat{N}] = \sigma[N]$, and N is a direct summand of $\widehat{N}^k$, $k \in I\!N$, i.e. it is N-injective.

A module M is FP-injective over $End(M)$ if and only if the cokernels of morphisms $M^k \to M^n$, k, $n \in I\!N$, are cogenerated by M (see 47.7). From this we derive:

56.6 All modules $N \in \sigma[M]$ FP-injective over $End(N)$.

For an R-module M the following properties are equivalent:

(a) every module $N \in \sigma[M]$ is FP-injective over $End(N)$;

(b) every module $N \in \sigma[M]$ is a cogenerator in $\sigma[N]$.

If there is a progenerator of finite length in $\sigma[M]$, then (a), (b) are equivalent to the following:

(c) every self-projective module $N \in \sigma[M]$ is FP-injective over $End(N)$.

Proof: $(a) \Rightarrow (c)$ and $(b) \Rightarrow (a)$ are obvious.

$(a) \Rightarrow (b)$ By 17.12, M is a cogenerator in $\sigma[M]$ if it cogenerates the M-injective hulls $\widehat{E}$ of simple modules $E \in \sigma[M]$. Since $\widehat{E}$ is M-generated, there exists an epimorphism $f : M^{(\Lambda)} \to \widehat{E}$. Consider the morphism

$$h : Ke\, f \oplus M^{(\Lambda)} \longrightarrow Ke\, f \oplus M^{(\Lambda)}, \quad (x, y) \mapsto (0, x),$$

for which

$$Coke\, h \simeq Ke\, f \oplus (M^{(\Lambda)}/Ke\, f) \simeq Ke\, f \oplus \widehat{E}$$

is cogenerated by $Ke\, f \oplus M^{(\Lambda)} \subset M^{(\Lambda)} \oplus M^{(\Lambda)}$. Then $\widehat{E}$ is cogenerated by M and M is a cogenerator in $\sigma[M]$.

The same proof (with N instead of M) shows that every $N \in \sigma[M]$ is a cogenerator in $\sigma[N]$.

$(c) \Rightarrow (a)$ Again let $\widehat{E}$ be the injective hull of a simple module $E \in \sigma[M]$. The progenerator of finite length is perfect in $\sigma[M]$, and hence there exists an M-projective cover $f : P \to \widehat{E}$.

As in the proof of $(a) \Rightarrow (b)$, we obtain that $Ke\, f \oplus \widehat{E}$ is cogenerated by P. Now P, as a projective module, is isomorphic to a submodule of $M^{(\Omega)}$. Hence $\widehat{E}$ is cogenerated by M and M is a cogenerator in $\sigma[M]$.

Since in every $\sigma[N]$, $N \in \sigma[M]$, there exists a progenerator (see 32.8), the same proof holds for N.

Now we turn to the question when all modules in $\sigma[M]$ are homo-serial. First we will assume that M is of finite length and later on we will demand the existence of a progenerator in $\sigma[M]$.

56.7 All modules homo-serial in $\sigma[M]$ (M of finite length).
For an R-module M of finite length, the following are equivalent:

(a) Every module in $\sigma[M]$ is homo-serial;

(b) every finitely generated module in $\sigma[M]$ is homo-serial;

(c) every finitely generated, indecomposable module in $\sigma[M]$ is homo-uniserial;

(d) every finitely generated, indecomposable module $N \in \sigma[M]$ is self-projective and a cogenerator in $\sigma[N]$;

(e) every self-injective module $N \in \sigma[M]$ is projective in $\sigma[N]$;

(f) there is a finitely generated, injective cogenerator Q in $\sigma[M]$, and every fully invariant submodule $L \subset Q$ is projective (and a generator) in $\sigma[L]$.

In this case there is a progenerator in $\sigma[M]$ and M is of finite type.

Proof: The equivalence of (a), (b) and (c) is derived from the proof of 55.14, replacing 'uniserial' by 'homo-uniserial'.

$(a) \Rightarrow (d)$ An indecomposable module $N \in \sigma[M]$ is of course homo-uniserial and, by 55.14, self-projective and self-injective. By 56.1, this implies that N is a cogenerator in $\sigma[N]$.

$(d) \Rightarrow (c)$ Let $N \in \sigma[M]$ be finitely generated and indecomposable. Being a cogenerator in $\sigma[N]$, N contains an injective hull of any simple module in $\sigma[N]$. In fact, N has to be isomorphic to all these injective hulls. Hence N is self-injective, has simple socle, and (up to isomorphism) there is only one simple module in $\sigma[N]$.

Consequently, all finitely generated, indecomposable modules in $\sigma[N]$ are self-injective and self-projective, i.e. N is uniserial by 55.14. By the above observation, N is in fact homo-uniserial.

$(a) \Rightarrow (e)$ Let $N \in \sigma[M]$ be self-injective. By 55.14, there exists a progenerator in $\sigma[M]$ and hence also an artinian progenerator P in $\sigma[N]$ (see 32.8), $P = \bigoplus_{i \leq k} P_i$, with homo-uniserial P_i. By 55.14, any P_i is self-injective and, since $Hom(P_i, P_j) = 0$ or $P_i \simeq P_j$ (see proof of 56.2,$(a) \Rightarrow (e)$), P_i is also P_j-injective for $i \neq j \leq k$. Hence P is also self-injective. Now by 56.4, P is a cogenerator in $\sigma[P] = \sigma[N]$, P is a QF module and all injectives in $\sigma[N]$ are projective (see 48.14).

$(e) \Rightarrow (d)$ Let $N \in \sigma[M]$ be finitely generated and indecomposable and $\widetilde{N}$ the N-injective hull of N. By assumption, $\widetilde{N}$ is projective in $\sigma[\widetilde{N}] = \sigma[N]$ and hence is a submodule of some direct sum $N^{(\Lambda)}$. If E is a simple submodule of $\widetilde{N}$, then its N-injective hull $\widetilde{E} \subset \widetilde{N}$ is N-projective and hence a submodule of N (see 18.4,(2), 14.8). Therefore $N \simeq \widetilde{E}$ is self-injective and self-projective.

For every $K \in \sigma[N]$ with N-injective hull $\widetilde{K}$, the sum $N \oplus \widetilde{K}$ is self-injective and hence projective in $\sigma[N \oplus \widetilde{K}] = \sigma[N]$. Therefore $\widetilde{K}$ is a projective module in $\sigma[N]$ and hence cogenerated by N (see 18.4). This means that N is a cogenerator in $\sigma[N]$.

$(a) \Rightarrow (f)$ We know, from 55.14, that there exists a finitely generated cogenerator Q in $\sigma[M]$. Any fully invariant submodule $L \subset Q$ is self-injective and hence self-projective (since $(a) \Rightarrow (e)$).

By 56.2, the self-projective, finitely generated, homo-serial module L is a generator in $\sigma[L]$.

$(f) \Rightarrow (d)$ Let $N \in \sigma[M]$ be finitely generated and indecomposable. Then $L = Tr(N, Q)$ is an N-generated, fully invariant submodule of Q, i.e. it is self-injective and, by (f), self-projective. Since N is finitely cogenerated by L, $\sigma[N] = \sigma[L]$.

As a self-projective module of finite length, L is semiperfect in $\sigma[N]$, i.e. it has a local, N-projective direct summand P. This P is N-generated and hence is a factor module of N. Thus $N \simeq P$ and N is N-projective.

It is easy to verify that L is a cogenerator in $\sigma[N]$. As a projective module in $\sigma[N]$, L is a submodule of some direct sum $N^{(\Lambda)}$ and hence N is also a cogenerator in $\sigma[N]$.

Any of the conditions in 56.7 together with the finite length of M implies the existence of a progenerator in $\sigma[M]$. If we already know that a progenerator exists, we may ask for properties which imply the finiteness condition:

56.8 All modules homo-serial in $\sigma[M]$ (with progenerator).
Let M be an R-module. Assume there exists a progenerator P in $\sigma[M]$.

Then the following are equivalent:

(a) Every module in $\sigma[M]$ is homo-serial;

(b) every module $N \in \sigma[M]$ is a cogenerator in $\sigma[N]$;

(c) for every fully invariant $K \subset P$, P/K is a cogenerator in $\sigma[P/K]$;

(d) for every fully invariant $K \subset P$, P/K is a noetherian QF module;

(e) P is noetherian and, for every fully invariant $K \subset P$, P/K is self-injective;

(f) P is noetherian and every (self-injective) module $N \in \sigma[M]$ is a generator in $\sigma[N]$;

(g) P is noetherian and every self-projective module in $\sigma[M]$ is self-injective;

(h) P is artinian and every finitely generated, indecomposable module $N \in \sigma[M]$ is a generator and a cogenerator in $\sigma[N]$;

(i) P is artinian, there exists a finitely generated cogenerator Q in $\sigma[M]$, and every fully invariant submodule $L \subset Q$ is a generator in $\sigma[L]$;

(j) P is artinian, and left ideals and right ideals in $End(P)$ are cyclic.

Proof: $(a) \Rightarrow (b),(f)$ Since $\sigma[M]$ is equivalent to $End(P)$-MOD we conclude from (a) (see 55.16) that $End(P)$ and P have finite length.

For any $N \in \sigma[M]$, let $\bar{P}$ denote the direct sum of the projective covers of the non-isomorphic simple modules in $\sigma[N]$, i.e. $\bar{P} = P_1 \oplus \cdots \oplus P_k$, with P_i homo-uniserial and $Hom(P_i, P_j) = 0$ for $i \neq j$. By 55.14, the P_i are self-injective and hence $\bar{P}$ is also self-injective.

Now we obtain from 56.4 that $\bar{P}$ is a cogenerator in $\sigma[N]$. As an injective and projective module in $\sigma[N]$, $\bar{P}$ is generated and cogenerated by N (see 16.3, 18.4) and hence N is a generator and a cogenerator in $\sigma[N]$.

$(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (d)$ By assumption $\bar{P} = P/Rad\,P$ is a cogenerator in $\sigma[\bar{P}]$. Therefore, by 23.1, $\bar{P}$ is cosemisimple, i.e. all simple modules are $\bar{P}$-injective. Being submodules of $\bar{P}$, the simple modules are direct summands and hence are $\bar{P}$-projective. By 20.3, this implies that $\bar{P}$ is semisimple (and finitely generated) and so there are only finitely many simple modules in $\sigma[P]$. We conclude from 48.11 that P is finitely cogenerated.

Now we show that P is noetherian. Any submodule $K \subset P$ is finitely cogenerated and hence $K = K_1 \oplus \cdots \oplus K_r$, with finitely cogenerated, indecomposable modules K_i. Consider the fully invariant submodules of P

$$U_i := Re(P, K_i) = \bigcap \{Kef \mid f \in Hom(P, K_i)\}.$$

P/U_i is self-projective and, by assumption, a cogenerator in $\sigma[P/U_i]$. From the above proof – with P replaced by P/U_i – we see that P/U_i is also finitely cogenerated and hence a submodule of K_i^n, for some $n \in I\!N$. This means that K_i is a cogenerator in $\sigma[P/U_i] = \sigma[K_i]$ and hence contains the K_i-injective hull $\widetilde{E}$ of a simple module $E \in \sigma[K_i]$, i.e. $K_i \simeq \widetilde{E}$.

As a direct summand of the finitely generated cogenerator P/U_i, $\widetilde{E}$ is finitely generated and hence $K = K_1 \oplus \cdots \oplus K_r$ is also finitely generated, i.e. N is noetherian.

$(d) \Rightarrow (a)$ As a noetherian QF module, P is also artinian and hence has finite length (see 48.14). We conclude from 55.14,(g), that every module in $\sigma[M] = \sigma[P]$ is serial. Similar to part of the proof $(c) \Rightarrow (d)$ we can see that finitely cogenerated indecomposable modules in $\sigma[M]$ are self-injective and self-cogenerators. Hence they are homo-uniserial.

$(d) \Leftrightarrow (e)$ follows from the characterizations of noetherian QF modules in 48.14.

$(f) \Rightarrow (e)$ Let K be a fully invariant submodule of P. Then P/K is self-projective and, by (f), is generated by the P/K-injective hull $\widehat{P/K}$. Therefore P/K is a direct summand of $\widehat{P/K}^n$, $n \in I\!N$, and hence P/K-injective.

$(a) \Rightarrow (g)$ Let N be a self-projective module in $\sigma[M]$. Then, as a direct sum of finitely generated modules, N is projective in $\sigma[N]$. In the proof $(a) \Rightarrow (b)$ we have seen that there exists a noetherian injective generator $\bar{P}$ in $\sigma[N]$. Hence N is also N-injective.

$(g) \Rightarrow (e)$ is obvious.

$(a) \Rightarrow (h)$ is clear from what we have proved so far (see $(a) \Rightarrow (b), (e)$).

$(h) \Rightarrow (a)$ By 56.7, it is enough to show that any finitely generated, indecomposable $N \in \sigma[M]$ is self-projective and a cogenerator in $\sigma[N]$. If (h) is given, it remains to verify that N is self-projective: N generates the N-projective cover of a simple module in $\sigma[N]$ and so is isomorphic to it.

$(a) \Rightarrow (i)$ This was shown in 56.7.

$(i) \Rightarrow (h)$ Let $N \in \sigma[M]$ be finitely generated and indecomposable. For $L = Tr(N, Q)$, we have $\sigma[L] = \sigma[N]$ (see proof of $(f) \Rightarrow (d)$ in 56.7). By (i), L is a generator in $\sigma[N]$. Then N is also a generator in $\sigma[N]$ and therefore it is N-projective (see $(h) \Rightarrow (a)$). N is generated by the N-injective module L and hence it is also N-injective. Since there is only one simple module up to isomorphism in $\sigma[N]$ (for which N is the N-projective cover), N is a cogenerator in $\sigma[N]$.

$(a) \Rightarrow (j)$ P is homo-serial, self-injective (see (d)) and self-projective. By 56.2 and 56.4, finitely generated left and right ideals of $End(P)$ are cyclic. We conclude from 48.14 that $End(P)$ is left and right artinian (and noetherian).

$(j) \Rightarrow (a)$ Since P is artinian, $End(P)$ is semiperfect. By 56.3, $End(P)$ is left and right homo-serial and 56.2 implies that P is homo-serial.

Now we see, from 55.8, that all finitely generated modules in $\sigma[M]$ are serial and, in fact, homo-serial since P is a homo-serial generator. By 56.7, we conclude that all modules in $\sigma[M]$ are homo-serial.

Let us compile some characterizations of rings resulting from the preceding theorems:

56.9 All *R*-modules homo-serial.

For a ring R the following assertions are equivalent:

(a) Every module in R-MOD is homo-serial;

(b) every $N \in$ R-MOD is FP-injective over $End(N)$;

(c) every $N \in$ R-MOD is a generator in $R/An_R(N)$-MOD;

(d) every $N \in$ R-MOD is finitely generated and flat (projective) over $End(N)$;

(e) ${}_RR$ is noetherian and every (self-injective) $N \in$ R-MOD is flat over $End(N)$;

(f) ${}_RR$ is noetherian, and every self-projective module is self-injective;

(g) ${}_RR$ is artinian, and every self-injective $N \in$ R-MOD is projective in $\sigma[N]$;

(h) ${}_RR$ is artinian, and every finitely generated, indecomposable $N \in$ R-MOD is a generator and a cogenerator in $\sigma[N]$;

(i) every factor ring $\bar{R}$ of R is a cogenerator in $\bar{R}$-MOD;

(j) every factor ring $\bar{R}$ of R is a noetherian QF ring;

(k) ${}_RR$ is a noetherian QF module and, for any two-sided ideal $I \subset R$, ${}_RI$ is self-projective (or a generator in $\sigma[{}_RI]$);

(l) ${}_RR$ (and R_R) is artinian, and left and right ideals in R are cyclic.

The same characterizations also apply for right R-modules.

If R has these properties, R is of finite representation type.

Proof: Except for (c) and (d) all properties are immediately derived from 56.5 to 56.8.

$(a) \Rightarrow (c)$ We know from 56.8 that every $N \in$ *R-MOD* is a generator in $\sigma[N]$ and the ring $\bar{R} := R/An_R(N)$ is left artinian. Since N is a faithful $\bar{R}$-module, $\bar{R} \subset N^k$ for some $k \in I\!N$, and hence $\bar{R}$-$MOD = \sigma[N]$.

$(c) \Rightarrow (d)$ Since N is a generator in the full module category $\bar{R}$-*MOD*, N is finitely generated and projective over $End(N)$ (see 18.8).

$(d) \Rightarrow (i)$ Let Q be a cogenerator in R-*MOD*. Then Q is a faithful R-module. If Q is finitely generated over $End(Q)$, then $R \in \sigma[Q]$ (by 15.4) and Q is a generator in $\sigma[Q] = R$-*MOD* (see 56.5). Hence all cogenerators in R-*MOD* are also generators and R is a cogenerator in R-*MOD* (by 48.12).

Similarly we see that every factor ring $\bar{R}$ of R is also a cogenerator in $\bar{R}$-*MOD*.

Let us now describe the rings under consideration by means of their functor rings:

56.10 Homo-serial modules and functor rings.
Let R be a ring, $\{\widetilde{U}_\alpha\}_A$ a representing set of all finitely presented left R-modules, $\widetilde{U} = \bigoplus_A \widetilde{U}_\alpha$ and $\widetilde{T} = \widehat{End}(\widetilde{U})$ (see 52.2). The following assertions are equivalent:

(a) Every module in R-MOD is homo-serial;

(b) every flat module X in $\widetilde{T}$-MOD is FP-injective over $End(X)$;

(c) ${}_{\widetilde{T}}\widetilde{T}$ is perfect and every projective module X in $\widetilde{T}$-MOD is FP-injective over End(X);

(d) R_R is noetherian and every FP-injective module Y in MOD-$\widetilde{T}$ is flat over $End(Y)$;

(e) $\widetilde{T}_{\widetilde{T}}$ is locally noetherian and every injective module Y in MOD-$\widetilde{T}$ is flat over $End(Y)$.

Proof: $(a) \Rightarrow (b)$ By 52.2, flat modules in $\widetilde{T}$-*MOD* are of the form $\widehat{Hom}(\widetilde{U}, N)$ with $N \in R$-*MOD*. We have to show that, for any morphism

$$g : \widehat{Hom}(\widetilde{U}, N)^k \to \widehat{Hom}(\widetilde{U}, N)^l, \quad k, l \in I\!N,$$

$Coke\, g$ is cogenerated by $\widehat{Hom}(\widetilde{U}, N)$ (see 47.7). We know from 56.9,(b), that $\widetilde{U} \otimes Coke\, g$ is cogenerated by N, i.e. $\widetilde{U} \otimes Coke\, g \subset N^\Lambda$ for some Λ. Then

$$Coke\, g \subset \widehat{Hom}(\widetilde{U}, \widetilde{U} \otimes_{\widetilde{T}} Coke\, g) \subset \widehat{Hom}(\widetilde{U}, N)^\Lambda,$$

proving our assertion.

$(b) \Rightarrow (a)$ According to 56.9 we have to show that, for any $N \in R$-*MOD*, the cokernel of $f : N^k \to N^l$, k, $l \in I\!N$, is cogenerated by N. Because of (b), $Coke\, \widehat{Hom}(\widetilde{U}, f) \subset \widehat{Hom}(\widetilde{U}, N^\Lambda)$ for some Λ, and

$$Coke\, f \subset \widetilde{U} \otimes_{\widetilde{T}} Coke\, \widehat{Hom}(\widetilde{U}, f) \subset \widetilde{U} \otimes_{\widetilde{T}} \widehat{Hom}(\widetilde{U}, N^\Lambda) \simeq N^\Lambda.$$

$(a) \Rightarrow (c)$ Since R is of finite type, $\widetilde{T}$ is left (and right) perfect (see 54.3). The remaining assertion follows from $(a) \Rightarrow (b)$.

$(c) \Rightarrow (b)$ is clear since, for a left perfect $\widetilde{T}$, flat modules in $\widetilde{T}$-*MOD* are projective.

$(a) \Rightarrow (d)$ FP-injective modules in *MOD*-$\widetilde{T}$ are of the form $K \otimes_R \widetilde{U}_{\widetilde{T}}$ with K in *MOD-R* (see 35.4, 52.3). It is to prove that, for any morphism $g : K^k \otimes \widetilde{U} \to K^l \otimes \widetilde{U}$, $k, l \in \mathbb{N}$, $Ke\, g$ is generated by $K \otimes \widetilde{U}$ (see 15.9).

We know from 56.9 that $Ke\, g \otimes \widehat{Hom}(\widetilde{U}, R)$ is generated by K. $Ke\, g$ is a factor module of $Ke\, g \otimes_{\widetilde{T}} \widehat{Hom}(\widetilde{U}, R) \otimes_R \widetilde{U}$ (construct a suitable diagram), and hence is also generated by $K \otimes_R \widetilde{U}$.

$(d) \Rightarrow (a)$ By 56.9, we have to verify that, for any K in *MOD-R*, the kernel of $f : K^k \to K^l$, $k, l \in \mathbb{N}$, is generated by K. Because of (d), the kernel of $f \otimes id_{\widetilde{U}}$ is generated by $K \otimes_R \widetilde{U}$. Then $Ke\, f$ is generated by

$$K \otimes_R \widetilde{U} \otimes_{\widetilde{T}} \widehat{Hom}(\widetilde{U}, R) \simeq K \ .$$

$(a) \Rightarrow (e)$ R is of finite type and hence $\widetilde{T}_{\widetilde{T}}$ is locally noetherian (see 54.3). The remaining assertion follows from $(a) \Rightarrow (d)$.

$(e) \Rightarrow (d)$ By 53.7, R_R is pure semisimple and hence noetherian (in fact artinian) by 53.6. The other assertion is clear since for locally noetherian $\widetilde{T}_{\widetilde{T}}$, FP-injective modules in *MOD*-$\widetilde{T}$ are injective.

It was show by I.V. Bobylev that the rings considered in 56.9 and 56.10 can also be characterized by the condition

Every (self-injective) module N in R-MOD is projective over $End(N)$.

Applying 56.10, certain properties of the functor ring can also be used to describe when all modules in $\sigma[M]$ are homo-serial (compare 55.15).

Before turning to homo-serial modules over $\mathbb{Z}$, let us recall that every $\mathbb{Z}$-torsion module M is a direct sum of its p-submodules $p(M)$ (see 15.10).

56.11 Homo-serial $\mathbb{Z}$-modules.

(1) For any $\mathbb{Z}$-module M the following are equivalent:

(a) M is a torsion module;

(b) every finitely generated module in $\sigma[M]$ is (homo-) serial.

(2) For any $\mathbb{Z}$-module M the following are equivalent:

(a) M is a torsion module and, for every prime number p, the p-submodule $p(M)$ of M is a (direct) sum of uniserial modules of bounded length;

(b) every module in $\sigma[M]$ is a direct sum of cyclic, homo-uniserial

modules;
(c) M is pure semisimple.

Proof: (1) $(a) \Rightarrow (b)$ If M is a torsion module, then every finitely generated $N \in \sigma[M]$ is a torsion module and $I = An_{\mathbb{Z}}(N) \neq 0$. As a module over the artinian principal ideal ring $\mathbb{Z}/I$, N is homo-serial (see 56.9).

$(b) \Rightarrow (a)$ The cyclic submodules of M are serial and hence cannot be isomorphic to $\mathbb{Z}$.

(2) $(a) \Rightarrow (b)$ Every module $N \in \sigma[M]$ is a torsion module and, for every prime number p, we easily verify (using 15.10) that $p(N) \in \sigma[p(M)]$.

In $\sigma[p(M)]$ any finitely generated, indecomposable module K is (homo-) uniserial (see (1)). Then its injective hull $\widetilde{K}$ in $\sigma[p(M)]$ is also uniserial and the length of K (in fact of $\widetilde{K}$) is bounded by the bound of the lengths of the uniserial summands of $p(M)$ (see 55.13). Hence, by 54.2, $p(M)$ is pure semisimple and, by (1) and 53.4, every module in $\sigma[p(M)]$, in particular $p(N)$, is a direct sum of cyclic homo-uniserial modules. Now this is also true for $N = \bigoplus p(N)$.

$(b) \Rightarrow (c)$ is clear by 53.4.

$(c) \Rightarrow (a)$ Let M be pure semisimple. First we observe that M is a torsion module: If not, we conclude $\mathbb{Z} \in \sigma[M]$ and $\sigma[M] = \mathbb{Z}$-*MOD*. However, $\mathbb{Z}$ is not pure semisimple. Hence (by 53.4) M is a direct sum of cyclic uniserial (torsion) modules of type $\mathbb{Z}/(p^n)$, with $p, n \in \mathbb{N}$, p prime. It remains to show that that there exists a bound for the lengths of the uniserial summands of $p(M)$:

Assume for every $k \in \mathbb{N}$ we can find a uniserial direct summand $N_k \subset p(M)$ with $lg(N_k) \geq k$. Then we have an epimorphism $N_k \to \mathbb{Z}/(p^k)$ and finally an epimorphism

$$\bigoplus_{\mathbb{N}} N_k \to \varinjlim \mathbb{Z}/(p^k) \simeq \mathbb{Z}_{p^\infty}.$$

Therefore $\mathbb{Z}_{p^\infty}$ is generated by $p(M)$, i.e. it belongs to $\sigma[p(M)] \subset \sigma[M]$. However, $\mathbb{Z}_{p^\infty}$ is indecomposable but not finitely presented. This yields a contradiction to M being pure semisimple (see 53.4).

56.12 Exercise.

Let R be a commutative local ring with maximal ideal $J = Jac\,R$. Show that the following properties are equivalent:
(a) The R-injective hull of R/J is uniserial;

(b) R is uniserial and, for every ideal $I \neq 0$, R/I is linearly compact;
(c) every indecomposable injective R-module is uniserial;
(d) every finitely generated R-module is serial.

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