Lecture (9)

Electricity and Magnetism II

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Alternating Current Circuits

An alternating current circuit (ac) is a circuit where the voltage and the current vary with time typically like sine or cosine. A typical ac signal may be given as

$$\boldsymbol{v = V\sin(\omega t + \beta)}$$

$\boldsymbol{v}$ is the voltage at a given instant of time and is called instantaneous voltage. **V** is the maximum value of the voltage and is called the amplitude of the voltage. $\boldsymbol{\omega}$ is the number of radians executed per second and is called angular frequency of the voltage. It is related with frequency (**f**) as $\boldsymbol{\omega = 2\pi f}$ and with period (**T**) as $\boldsymbol{\omega = {}^{2\pi}/_{T}}$. $\boldsymbol{\beta}$ is called the phase angle of the voltage. Its effect is to shift the graph of $\sin \omega t$ either to the right (if negative) or to the left (if positive).

Phase angle of signal 2 minus the phase angle of signal 1 is called phase shift ($\boldsymbol{\theta}$) of signal 2 with respect to signal 1. The one with bigger phase angle is said to be leading the other and the one with a smaller phase angle is said to be lagging from other.

If $\boldsymbol{v_1 = V_1\sin(\omega t + \beta_1)}$ and $\boldsymbol{v_2 = V_2\sin(\omega t + \beta_2)}$

then $\boldsymbol{\theta = \beta_2 - \beta_1}$.

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A Resistor Connected to an ac Source

Let's consider a resistor of resistance **R** connected to an ac source whose voltage varies with time according to the equation $\boldsymbol{v = V\sin(\omega t)}$. Ohm's law applies to ac circuits instantaneously. Therefore the equation $\boldsymbol{v = iR}$ holds where $\boldsymbol{i}$ stands for the instantaneous current. This implies that,

$$\boldsymbol{i = \left({}^{V}\!/_{R}\right)\sin(\omega t) \quad where \quad v = V\sin(\omega t)}$$

This shows that, in a resistor, the phase shift ($\boldsymbol{\theta_R}$) between the voltage and the current is zero. In other words, the voltage and the current are in the phase.

$$\boldsymbol{\theta_R = 0}$$

Where $\boldsymbol{\theta_R = \beta_v - \beta_i}$ is the phase shift of voltage with respect to current for a resistor. Since $\boldsymbol{i = I\sin(\omega t) = \left({}^{V}\!/_{R}\right)\sin(\omega t)}$, Ohm's law also applies to the amplitudes of the voltage and the current.

$$\boldsymbol{V = IR}$$

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A Capacitor Connected to an ac Source

Lets consider a capacitor of capacitance **C** connected to an ac source whose potential difference varies with time according to the equation $\boldsymbol{v = V \sin(\omega t)}$. The instantaneous charge **q** of the capacitor is released with the instantaneous voltage by $\boldsymbol{q = vC}$. The instantaneous current is equal to the rate of change of the charge or the derivative of charge with respect to time in the language of calculus. Therefore, using calculus (because it cant be done algebraically),

$$i = \frac{dq}{dt} = C\frac{dv}{dt} = C\,\frac{d\{V\sin(\omega t)\}}{dt} = CV\omega \cos(\omega t)$$

But $\cos(\omega t) = \sin(\omega t + {}^{\pi}/_{2})$

$$i = CV\omega \sin\left(\omega t + {}^{\pi}/_{2}\right)$$

This implies that for a capacitor, the voltage lags from the current by ${}^{\pi}/_{2}$ $or\ 90°$.

$$\theta_C = \beta_v - \beta_i = -{}^{\pi}/_{2}$$

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Where $\boldsymbol{\theta_C}$ is the phase shift of the voltage with respect to current for a capacitor.

Since

$$\boldsymbol{i = I\,sin\left(\omega t + {}^{\pi}/_{2}\right) = CV\omega\,sin\left(\omega t + {}^{\pi}/_{2}\right)}$$

$$\boldsymbol{I = CV\omega}$$

Where $\boldsymbol{I}$ is the amplitude of the current. The ratio between the amplitude of the voltage across a capacitor and the amplitude of the current across a capacitor is defined to be the **capacitive reactance** ($\boldsymbol{X_C}$) of the capacitor.

$$\boldsymbol{X_C = V/I}$$

The unit of measurement for capacitive reactance is ohm. Replacing $\boldsymbol{I}$ by $\boldsymbol{CV\omega}$, it follows that the capacitive reactance of a capacitor is inversely proportional to the frequency of the voltage (current).

$$\boldsymbol{X_C = \frac{1}{\omega C}}$$

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An Inductor Connected to an ac Source

Lets consider an inductor of inductance **L** connected to an ac source where the current in the circuit varies with time according to the equation $\boldsymbol{i = I\, sin\, (\omega t)}$. From Kirchhoff's loop rule, the voltage of the source is the negative of the self-induced voltage (so that they add up to zero): $\boldsymbol{v = -E_{self}}$ but $\boldsymbol{E_{self} = -L\frac{di}{dt}}$.

Hence, $\boldsymbol{v = L\frac{di}{dt} = L\frac{d}{dt}\{I\, sin\, (\omega t)\} = L\omega I\, cos\, (\omega t)}$

and $\boldsymbol{cos\, (\omega t) = sin\, \left(\omega t + \frac{\pi}{2}\right)}$

$\boldsymbol{v = L\omega I\, sin\, \left(\omega t + \frac{\pi}{2}\right)}$ when $\boldsymbol{i = I\, sin\, (\omega t)}$

This means, for an inductor, the voltage leads the current by $\boldsymbol{{}^{\pi}/_{2}\ or\ 90°}$.

$$\boldsymbol{\theta_L = {}^{\pi}/_{2}}$$

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Where $\boldsymbol{\theta_L = \beta_v - \beta_i}$ is the phase shift of the voltage with respect to the current.

Since, $\boldsymbol{v = V\ sin\ \left(\omega t + \frac{\pi}{2}\right) = L\omega I\ sin\ \left(\omega t + \frac{\pi}{2}\right)}$

$$\boldsymbol{V = L\omega I}$$

The ratio between the amplitude of the voltage across an inductor and the current across inductor is called the **inductive reactance** ($\boldsymbol{X_L}$) of the inductor.

$$\boldsymbol{X_L = V/I}$$

The unit of measurement for inductive reactance is the ohm. Replacing $\boldsymbol{V}$ by $\boldsymbol{L\omega I}$, it is seen that the inductive reactance of an inductor is directly proportional to the frequency of the signal.

$$\boldsymbol{X_L = \omega L}$$

Series Combination of a Resistor, an Inductor and a Capacitor Connected to an ac Source

Lets consider a series combination of a resistor of resistance **R**, an inductor of inductance **L** and a capacitor of capacitance **C** connected to an ac source where the current in the circuit varies with time according the equation $\boldsymbol{i = I\,sin\,(\omega t)}$. The currents through the resistor ($\boldsymbol{i_R}$), the inductor ($\boldsymbol{i_L}$) and the capacitor ($\boldsymbol{i_C}$) are equal and they are equal to the current in the circuit.

$$\boldsymbol{i_R = i_L = i_C = i = I\,sin\,(\omega t)}$$

The net instantaneous potential difference ($\boldsymbol{v}$) is equal to the sum of the instantaneous potential difference across the resistor ($\boldsymbol{v_R}$), the inductor ($\boldsymbol{v_L}$) and capacitor ($\boldsymbol{v_C}$).

$$\boldsymbol{v = v_R + v_L + v_C}$$

In a resistor, the voltage and the current are in phase; that is, their phase angles are equal. Therefore, since $\boldsymbol{i = I \sin(\omega t)}$, the instantaneous potential difference across the inductor is given by

$$\boldsymbol{v_R = V_R \sin(\omega t)}$$

Where $\boldsymbol{V_R = IR}$.

In an inductor, the voltage leads the current by $\boldsymbol{\pi/2}$, that is, the phase angle of the voltage is $\boldsymbol{\frac{\pi}{2}}$ more than the phase angle of the current. Since the current is given as $\boldsymbol{i = I \sin(\omega t)}$, the instantaneous potential difference across the inductor is given by:

$$\boldsymbol{v_L = V_L \sin\left(\omega t + \frac{\pi}{2}\right)}$$

Where $\boldsymbol{V_L = IX_L}$.

In a capacitor, the potential difference lags from the current by $\frac{\pi}{2}$; that is the phase angle of the voltage is $\frac{\pi}{2}$ less than the phase angle of the current. Since the current is given as $i = I\,sin\,(\omega t)$, the instantaneous voltage across the capacitor is given by:

$$v_C = V_C\,sin\,\left(\omega t - \frac{\pi}{2}\right)$$

Where $V_C = IX_C$.

The net instantaneous voltage is the sum of the instantaneous voltages across the resistor, inductor and capacitor

$$v = V_R\,sin\,(\omega t) + V_L\,sin\,\left(\omega t + \frac{\pi}{2}\right) + V_C\,sin\,\left(\omega t - \frac{\pi}{2}\right)$$

But $\boldsymbol{sin}\left(\boldsymbol{\omega t}+\frac{\pi}{2}\right)=\boldsymbol{cos}\,(\boldsymbol{\omega t})$ and $\boldsymbol{sin}\left(\boldsymbol{\omega t}-\frac{\pi}{2}\right)=-\boldsymbol{cos}\,(\boldsymbol{\omega t})$. Therefore,

$$\boldsymbol{v}=\boldsymbol{V_R}\,\boldsymbol{sin}\,(\boldsymbol{\omega t})+(\boldsymbol{V_L}-\boldsymbol{V_C})\,\boldsymbol{cos}\,(\boldsymbol{\omega t})$$

Also, if the phase shift of the net voltage with respect to the current is $\boldsymbol{\theta}$, then

$$\boldsymbol{v}=\boldsymbol{V}\,\boldsymbol{sin}\,(\boldsymbol{\omega t}+\boldsymbol{\theta})$$

This expression of $\boldsymbol{v}$ can be expressed in terms of $\boldsymbol{cos}\,(\boldsymbol{\omega t})$ and $\boldsymbol{sin}\,(\boldsymbol{\omega t})$ by expanding the sine: $\boldsymbol{v}=\boldsymbol{V}\,\boldsymbol{cos}\,(\boldsymbol{\theta})\,\boldsymbol{sin}\,(\boldsymbol{\omega t})+\boldsymbol{V}\boldsymbol{sin}\,(\boldsymbol{\theta})\,\boldsymbol{cos}\,(\boldsymbol{\omega t})$. The coefficients of $\boldsymbol{cos}\,(\boldsymbol{\omega t})$ and $\boldsymbol{sin}\,(\boldsymbol{\omega t})$ of both expressions of $\boldsymbol{v}$ can be equated because $\boldsymbol{cos}\,(\boldsymbol{\omega t})$ and $\boldsymbol{sin}\,(\boldsymbol{\omega t})$ are independent functions.

$$\boldsymbol{V_R}=\boldsymbol{V}\,\boldsymbol{cos}\,(\boldsymbol{\theta})\ \ldots\ldots\ldots(\mathbf{1})$$

$$\boldsymbol{V_L}-\boldsymbol{V_C}=\boldsymbol{V}\,\boldsymbol{sin}\,(\boldsymbol{\theta})\ \ldots\ldots\ldots(\mathbf{2})$$

An expression for the phase shift of the voltage with respect to the current can be obtained by dividing equation (2) by equation (1):

$$\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR} = \frac{X_L - X_C}{R}$$

Thus θ is given as follows: $\theta = \arctan\left\{\frac{X_L - X_C}{R}\right\} = \left\{\frac{\left\{\omega L - \frac{1}{\omega C}\right\}}{R}\right\}$

An expression for the amplitude of the net voltage can be obtained by squaring equations (1) and (2) and adding: $V_R{}^2 + (V_L - V_C)^2 = \{V\cos(\theta)\}^2 + \{V\sin(\theta)\}^2 = V^2$

Therefore, the amplitude of the net voltage is related with the amplitudes of the voltages across the resistor, inductor and capacitor as follows:

$$V = \sqrt{V_R{}^2 + (V_L - V_C)^2}$$

The total **impedance (Z)** of the series combination is defined to be the ratio between the amplitude of the net voltage and the amplitude of the current in the circuit.

$$Z = \frac{V}{I}$$

An expression for **Z** in terms of **R, L, C** can be obtained by expressing the voltages in terms of the current:

$$Z = \frac{V}{I} = Z = \frac{\sqrt{\{(IR)^2 + (IX_L - IX_C)^2\}}}{I}$$

$$Z = \sqrt{\{(R)^2 + (X_L - X_C)^2\}} = \sqrt{\left\{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right\}}$$