Lecture (7)

Electricity and Magnetism II

First Stage
Department of Physics
College of Science
Al-Muthanna University
2018-2019

Dr. Ahmed Almurshedi
Ph.D., M.Sc., B.Sc.
Medical Physics

Self-Inductance

An inductor is a coil. The circuit symbol for an inductor is a coil. When an inductor is connected at a source, current will flow through it and this current will produce magnetic field inside the coil. That is there is magnetic flux crossing the loops of the coil due to its own magnetic field. If the current changes with time, then the magnetic field inside the coil will change with time which implies the magnetic flux crossing the coil will change with time. According to Faraday's law, this change in the magnetic flux will produce induced emf in the coil. This kind of induced emf is called **self induced emf ($\boldsymbol{\varepsilon_{self}}$)** because it is caused by the current in the coil itself.

Since the magnetic field due to a coil (solenoid) is proportional to the current flowing in the coil, the magnetic flux crossing the coils is proportional to the current. The constant of proportionality between the flux ($\emptyset_{self}$) and the current is defined to be the inductance (**L**) of the coil:

$$\emptyset_{self} = \boldsymbol{L\,I}$$

According to Faraday's law, the self-induced emf ($\boldsymbol{\varepsilon_{self}}$) is equal to the negative rate of change of this flux with time.

$$\boldsymbol{\varepsilon_{self} = -L\frac{dI}{dt}}$$

The negative sign indicates that the polarity of the self-induced emf is in such a way as to appose the cause for the change in flux, which is the rate of change of current with time.

The unit of measurement for inductance is **V/A** which is defined to be **Henry**, abbreviated as **H**.

The average induced emf in a given time interval Δt can be obtained by integrating ε_{self} with time interval Δt and then dividing by Δt:

$$\varepsilon_{self} = -\frac{L}{\Delta t}\int_{t}^{t+\Delta t}\frac{dI}{dt}\,dt = -\frac{L}{\Delta t}\int_{I}^{I+\Delta I}dI$$

and the average self induced emf may be given as

$$\varepsilon_{self} = -L\frac{\Delta I}{\Delta t} = -L\frac{(I_f - I_i)}{\Delta t}$$

Inductance of a solenoid in terms of its geometry

The inductance of an inductor depends on the geometry of the coil only. Consider a solenoid of length $\boldsymbol{l}$, radius **R** and number of turns **N**. The magnetic flux crossing the solenoid is **N** times the flux crossing a single turn:

$$\boldsymbol{\emptyset_{self} = NBA}$$

But, for a solenoid

$$\boldsymbol{A = \pi R^2} \qquad \boldsymbol{and} \qquad \boldsymbol{B = \frac{\mu_o N\, I}{l}}$$

Therefore,
$$\boldsymbol{L = \frac{N\,\emptyset_{self}}{I}}$$

The inductance of a solenoid in terms of its geometry is given as:

$$\boldsymbol{L = \frac{\mu_o N^2\,\pi R^2}{l}}$$

Magnetic energy Stored by an inductor

Consider an inductor connected to a source. According to Lenz's law, the self induced emf should oppose the source because it is the cause for the change of flux. Therefore the source has to do work to push a charge through the inductor. This work is stored by the inductor as magnetic energy. The work done by the source in pushing a charge **dq** across the inductor is

$$dw_{ext} = -dq\,\varepsilon_{self}$$

The negative sign is needed because the work done by the external force (source) is opposite to the work done by the self induced emf.

But

$$\varepsilon_{self} = -L\frac{dI}{dt}$$

$$dw_{ext} = -dq\left(-L\frac{dI}{dt}\right) = L\frac{dq}{dt}\,dI = LI\,dI$$

The amount of magnetic energy U_B stored by the inductor when the current is increased from zero to a value **I** is obtained by integration

$$U_B = \int dw_{ext} = \int_0^I LI\,dI$$

Therefore the amount of magnetic energy stored by an inductor when the current is **I** is given by:

$$U_B = \frac{1}{2}\,L\,I^2$$

Magnetic Energy Density Inside a Current Carrying Solenoid

Consider a solenoid of **N** turns, length **l** and cross sectional radius **R** carrying a current **I**.

The magnetic energy density $\boldsymbol{u_B}$ inside a solenoid is obtained as the ratio of the total magnetic energy ($\boldsymbol{U_B = \frac{1}{2}\, L\, I^2}$) to the volume of the solenoid ($\boldsymbol{V = \pi R^2 l}$).

$$\boldsymbol{u_B = \frac{U_B}{V} = \frac{L\, I^2}{2\,\pi R^2 l}}$$

But, the inductance of a solenoid is given as, $\boldsymbol{L = \frac{\mu_o N^2\, \pi R^2}{l}}$

and the magnetic field inside the solenoid is given as $\boldsymbol{B = \frac{\mu_o\, N\, I}{l}} \Rightarrow \boldsymbol{I = \frac{B\, l}{\mu_o\, N}}$

Now, substitutions for the inductance and the current, the magnetic energy density depends on the magnetic field according to the equation: $\boldsymbol{u_B = \frac{B^2}{2\,\mu_o}}$

Mutual Induction

When two coils carrying time dependent currents are in the vicinity of each other, they will induce on each other. This kind of induction is called **mutual induction**.

Consider two coils, coil 1 and coil 2, in the vicinity of each other. The mutual inductance of coil 2 with respect to coil 1 ($\boldsymbol{M_{21}}$) is defined to be the ratio between the flux crossing coil 2 due to the current in the coil 1 and the current in coil 1 ($\boldsymbol{I_1}$). If the number of turns of coil 2 is ($\boldsymbol{N_2}$) and the magnetic flux per coil crossing coil 2 is $\emptyset_{12}$, then

$$\boldsymbol{M_{21} = \frac{N_2 \emptyset_{21}}{I_1}}$$

Similarly, the mutual inductance of coil 1 with respect to coil 2 is defined to be

$$\boldsymbol{M_{12} = \frac{N_1 \emptyset_{12}}{I_2}}$$

It can be shown that $\boldsymbol{M_{21} = M_{12} = M}$

Where $\boldsymbol{M}$ is called the mutual inductance of the coils

$$\boldsymbol{M} = \frac{\boldsymbol{N_1 \emptyset_{12}}}{\boldsymbol{I_2}} = \frac{\boldsymbol{N_2 \emptyset_{21}}}{\boldsymbol{I_1}}$$

Applying Faraday's law the emf induced in coil 2 ($\boldsymbol{\varepsilon_2}$) is given by

$$\boldsymbol{\varepsilon_2 = -N_2 \frac{d\emptyset_{21}}{dt}}$$

But $\boldsymbol{\emptyset_{21} = \frac{M\,I_1}{N_2}}$ and it follows that

$$\boldsymbol{\varepsilon_2 = -M \frac{dI_1}{dt}}$$

The emf induced on coil 2 by coil 1 is directly proportional to the rate of change of the current flowing in coil 1. Similarly, the emf induced on coil 1 by coil 2 is given by:

$$\boldsymbol{\varepsilon_1 = -M \frac{dI_2}{dt}}$$

Mutual Inductance of Two Concentric Solenoids of the same Length

Consider two concentric solenoids with the inner solenoid being coil 1 and the outer being coil 2.

The magnetic flux induced on the solenoid 1 by solenoid 2 is

$$\emptyset_{12} = B_2\, A_1 = \left(\frac{\mu_o N_2 I_2}{l}\right) A_1$$

Therefore the mutual inductance of solenoid 1 with respect to solenoid 2 is

$$M = M_{12} = \frac{N_1\, \emptyset_{12}}{I_2}$$

Which simplifies to

$$M = \frac{\mu_o N_1\, N_2\, A_1}{l}$$

DR. AHMED ALMURSHEDI

Example 1: the current in an inductor of inductance 5 H varies with time according to equation $\boldsymbol{I = 2\sin 10t\ A}$ a) Give a formula for the self-induced emf as a function of time b) Calculate the value of the induced emf after $\frac{\pi}{40}$

Sol.

Example 2: The current in a 2 mH inductor change from 10 A to 4 A in 0.2 seconds. Calculate the average self-induced emf in the inductor?

Sol.

Example 3: A solenoid of length 10 cm, number of turns 200 and cross sectional radius 2 cm is carrying a current 2A. a) Calculate the magnetic energy density inside the solenoid b) Calculate the total magnetic energy stored inside the solenoid

Sol.

(a)

A series combination of an inductor and a resistor connected to a dc source

Consider a battery of emf $\boldsymbol{\varepsilon}$ connected to a series combination of a resistor of a resistance **R** and an inductor of inductance **L**. An inductor behaves like a resistor when its current increases (gaining magnetic energy at the expense of electrical energy) and behaves like a source when current decreases (losing magnetic energy to create electrical energy). Therefore, both resistor and source sign conventions apply depending on the situation. If transversed in the direction of the current, the potential difference in both cases is $-\boldsymbol{L}\frac{\boldsymbol{dI}}{\boldsymbol{dt}}$.

The difference between resistor and source behavior is contained in the sign of $\frac{\boldsymbol{dI}}{\boldsymbol{dt}}$. Now applying Kirchhoff's rule in the direction of the current:

$$\boldsymbol{\varepsilon - L\frac{dI}{dt} - IR = 0}$$

$$\boldsymbol{\frac{dI}{dt} = \frac{\varepsilon - IR}{L} \qquad \Longrightarrow \qquad dI = \frac{\varepsilon - IR}{L}\, dt}$$

DR. AHMED ALMURSHEDI

Integrating, $$\int_0^I \frac{dI}{\varepsilon - IR} = \int_0^t \frac{dt}{L}$$

Let $\boldsymbol{u = \varepsilon - IR}$ Then $\boldsymbol{du = -R dI}$

$\boldsymbol{u(t = 0) = \varepsilon}$ and $\boldsymbol{u(t) = \varepsilon - IR}$

With this substitution the integral becomes $$-\frac{1}{R}\int_{\varepsilon}^{\varepsilon - IR} \frac{du}{u} = \frac{t}{L}$$

Which implies that, $$Ln\left(\frac{\varepsilon - IR}{\varepsilon}\right) = -\frac{R}{L}\, t$$

Or $$\varepsilon - IR = \varepsilon\, e^{-\frac{R}{L}t}$$

and solving for the current the following expression for the current as a function of time is obtained,

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

DR. AHMED ALMURSHEDI

The current attains its maximum value at infinity

$$I_{max} = \frac{\varepsilon}{R} \lim_{t\to\infty}\left(1 - e^{-\frac{R}{L}t}\right)$$

The current approaches its maximum value asymptotically with time. The expression $\frac{R}{L}$ determines how fast the current approaches its maximum value and is called the time constant of the circuit. The greater the time constant the faster the current approaches its maximum value. The potential difference across the resistor is equal to current times its resistance:

$$V_R(t) = \varepsilon\left(1 - e^{-\frac{R}{L}t}\right)$$

The potential difference across the resistor is zero initially and approaches the emf of the battery asymptotically as time approaches infinity. The self induced emf of the inductor can be obtained as

$$\boldsymbol{\varepsilon_{self} = -L\frac{dI}{dt} = -L\frac{d}{dt}\left\{\frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right\}}$$

Which gives the following expression for the dependence of the emf:

$$\boldsymbol{\varepsilon_{self}(t) = -\varepsilon\, e^{-\frac{R}{L}t}}$$

The self induced emf has its maximum value initially and approaches zero asymptotically as time approaches infinity. Adding the potential difference and the self induced voltage shows that it is always equal $\boldsymbol{-\varepsilon}$ as expected:

$$\boldsymbol{-V_R + \varepsilon_{self} = -\varepsilon\left(1 - e^{-\frac{R}{L}t}\right) - \varepsilon\, e^{-\frac{R}{L}t} = -\varepsilon}$$

Example 1: A series combination of a 1000 Ω resistor and a 20 H inductor is connected to a battery of emf 20 V. (a) calculate the maximum current (b) calculate the time taken for the current to reach a value one fourth of its maximum value

Sol. (a