Lecture (2)

Electricity and Magnetism II

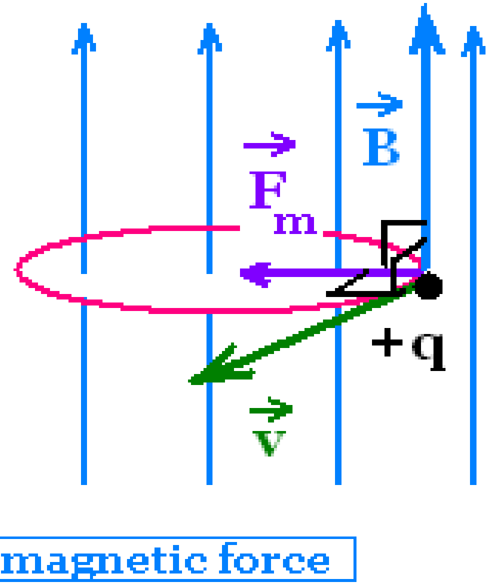
First Stage
Department of Physics
College of Science
Al-Muthanna University
2018-2019

Dr. Ahmed Almurshedi
Ph.D., M.Sc., B.Sc.
Medical Physics

Motion of a charged particle in a uniform magnetic field

Consider the motion of a charged particle in a uniform magnetic field, i.e. a magnetic field having the same intensity and direction at all its points. The magnetic force which is given : $\boldsymbol{F = q\,v\,B}$ is perpendicular to the velocity, its effect is to change the direction of the velocity without changing its magnitude, resulting in a uniform circular motion as in the figure below and by using the equation of motion, we have:

$$\boldsymbol{F = \frac{mv^2}{r}} \qquad \boldsymbol{Centerpetal\ force}$$



and then

$$\boldsymbol{\frac{mv^2}{r} = qvB}$$

$$\boldsymbol{r = \frac{mv}{qB}}$$

Which give the radius of the circle described by the charged particle of mass (**m**).

Above equation tells us that the curvature of the path of a charged particle in a magnetic field depends on the energy of the particle. The larger the energy (or the momentum **p=mv**), the larger the radius of the path and the smaller the curvature.

By writing:

$$\boldsymbol{v = \omega r} \qquad \textit{\textbf{where } } \boldsymbol{\omega} \textit{\textbf{ is the angular velocity of the particle,}}$$

$$\boldsymbol{\omega = \frac{q}{m} B}$$

Therefore, the angular velocity $\boldsymbol{\omega}$ is independent on the linear velocity $\boldsymbol{v}$ and depend only on the ratio ($\boldsymbol{\frac{q}{m}}$) and the field $\boldsymbol{B}$.

Above expression gives the magnitude of $\boldsymbol{\omega}$ but not its direction. We recall the acceleration in a uniform circular motion may be written in vector form as:

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}$$

Therefore the equation of motion $\vec{\boldsymbol{F}} = \boldsymbol{m}\vec{\boldsymbol{a}}$ becomes

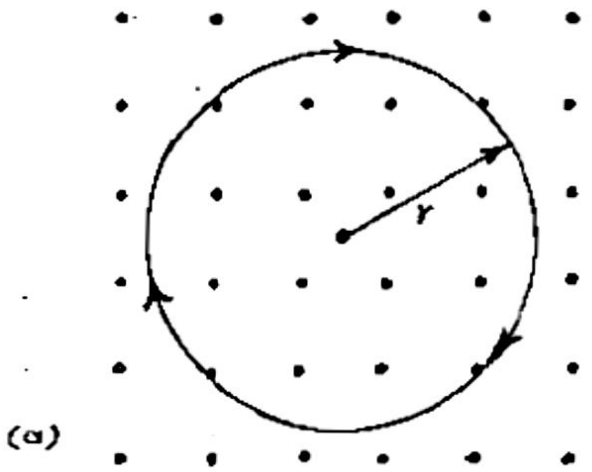
$$\boldsymbol{m}\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}} = \boldsymbol{q}\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$$

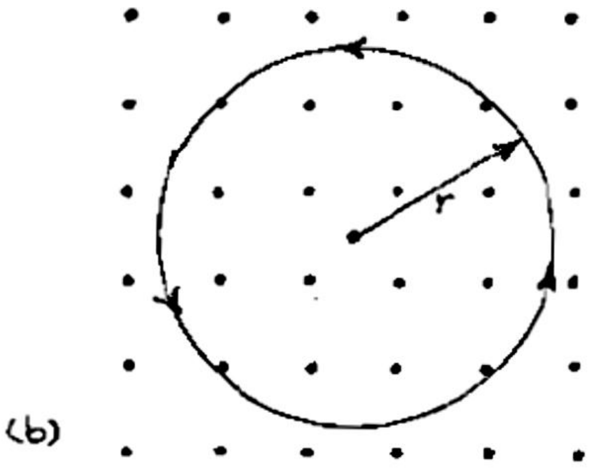
On reversing the vector product on the right hand side and dividing by (**m**), we get:

$$\vec{\boldsymbol{\omega}} = -\left(\frac{\boldsymbol{q}}{\boldsymbol{m}}\right)\vec{\boldsymbol{B}}$$

Which gives ($\vec{\boldsymbol{\omega}}$) in both magnitude and direction. The minus sign indicates that ($\vec{\boldsymbol{\omega}}$) has the opposite direction $\vec{B}$ for a positive charge, and the same direction for a negative charge, we call $\vec{\boldsymbol{\omega}}$ “cyclotron frequency”.

- It is customary to represent a **field** perpendicular to the paper by a dot (•) if it is directed toward the reader and by a cross (x) if it is directed into the page.
- Figure below represent the path of a positive and negative field perpendicular to the page. In (a) $\boldsymbol{\omega}$ is directed into the page and in (b) toward the reader.





q positive: $\vec{B}$ upward, $\vec{\omega}$ downward **q negative: $\vec{B}$ and $\vec{\omega}$ upward**

Figure circular path of negative and positive charge in uniform magnetic field

If a charged particle moves in a direction that is not perpendicular to the magnetic field, we may separate the velocity into its parallel and perpendicular components relative to the magnetic field. The parallel component remains unaffected and the perpendicular component changes continuously in direction but not in magnitude. The motion is then the resultant of a uniform motion parallel to the field and a circular motion around the field, with angular velocity given by:

$$\boldsymbol{\omega = \frac{q}{m} B}$$

The path is a helix, as shown in the figure below for a positive ion.

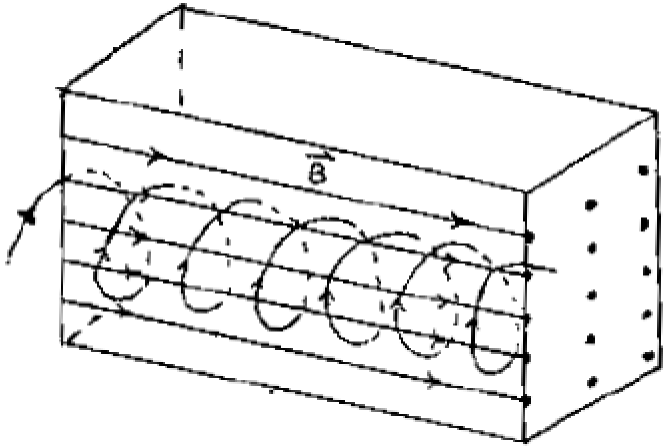


Figure helical path of a positive ion moving obliquely to a uniform magnetic field

Motion of a charged particle in a non-uniform magnetic field

We shall now consider the case when a particle moves in a magnetic field which is not uniform.

We learn from

$$r = \frac{mv}{qB}$$

that the larger the magnetic field, the smaller the radius of the path of the charged particle. Therefore, if the magnetic field is not uniform, the path is not circular.

Figure below shows a magnetic field directed left to right with its strength increasing in that direction. Thus a charged particle injected at the left hand side of the field describes a helix whose radius decreases continuously.

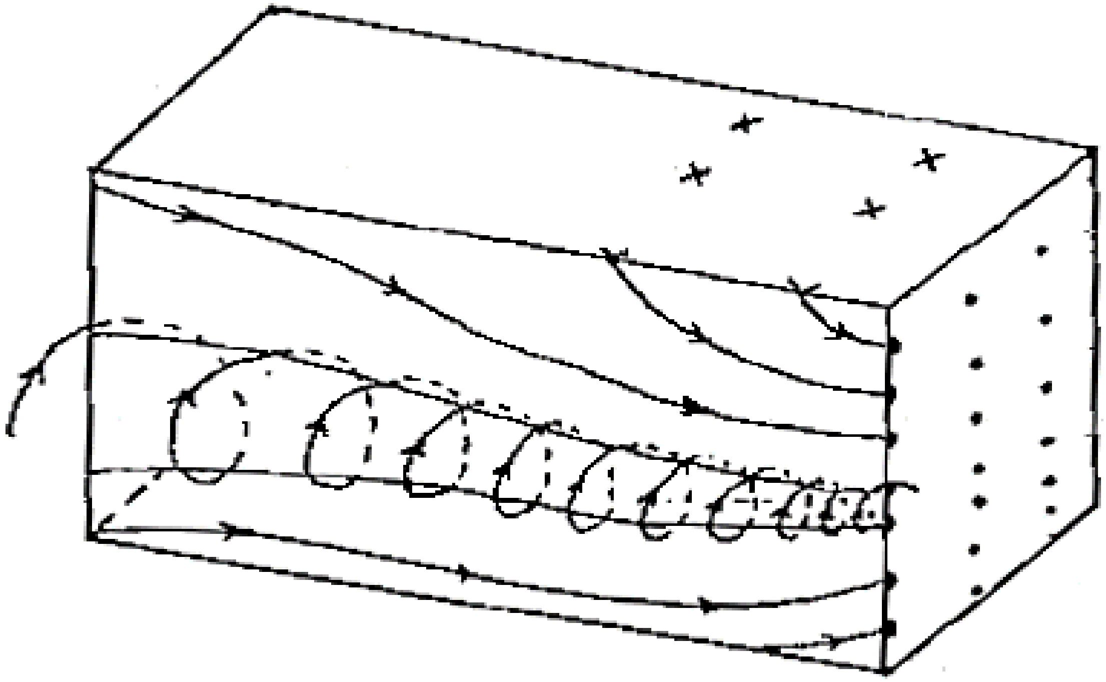


Figure path of a positive ion in a non-uniform magnetic field

A more detailed analysis, which we must omit here, would show that the component of the velocity parallel to the field does not remain constant but decreases (and therefore the pitch of the helix also decreases) as the particle moves in the direction of increasing field strength.

Magnetic force on a current carrying wire

Consider a long straight wire suspended in the region between the two magnetic poles.

The magnetic field points out the page and is represented with dots. It can be readily demonstrated that when a downward current passes through, the wire is deflected to the left. However, when the current is upward, the deflection is rightward, as shown in figure below.

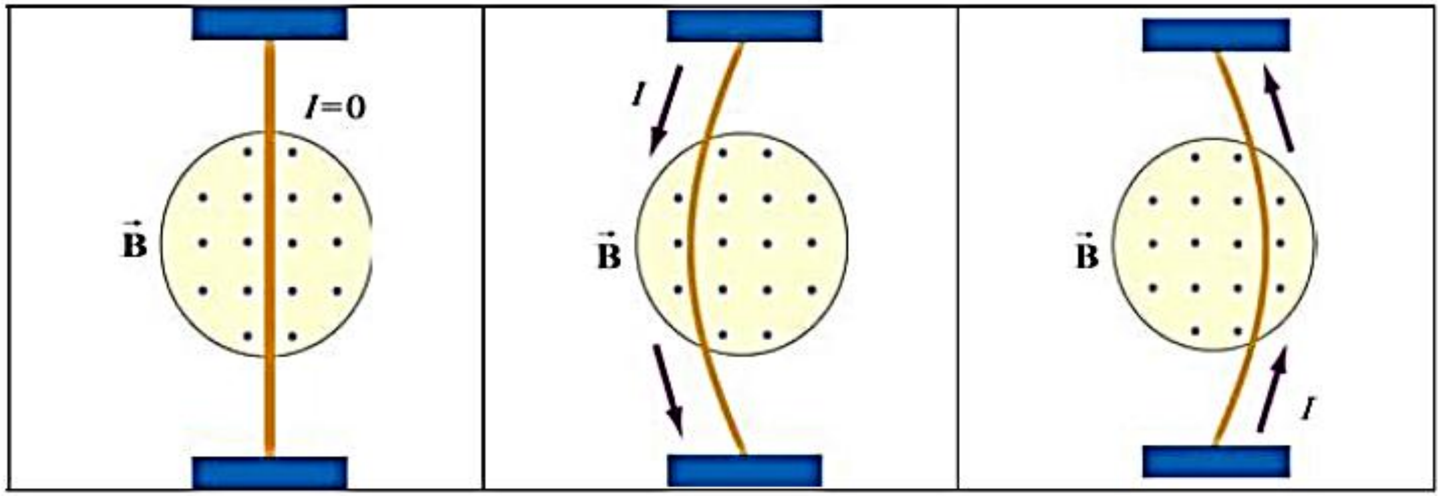


Figure deflection of current-carrying wire by magnetic force

DR. AHMED ALMURSHEDI

- To calculate the force exerted on the wire, consider a segment of wire of length $\boldsymbol{l}$ and cross sectional area **A**, as in the figure below. The magnetic field points into the page, and is represented with crosses.

The charges move at an average drift velocity $\vec{\boldsymbol{v}}_{\boldsymbol{d}}$.

Since the total amount of charge in this segment is:

$$\boldsymbol{Q_{tot} = q\,(nAl)}$$

Where **n** is the number of charges per unit volume,

the total magnetic force on the segment is:

$$\vec{\boldsymbol{F}}_{\boldsymbol{B}} = \boldsymbol{Q_{tot}}\,\vec{\boldsymbol{v}}_{\boldsymbol{d}} \times \vec{\boldsymbol{B}} = \boldsymbol{qnAl}\left(\vec{\boldsymbol{v}}_{\boldsymbol{d}} \times \vec{\boldsymbol{B}}\right) = \boldsymbol{I}(\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}})$$

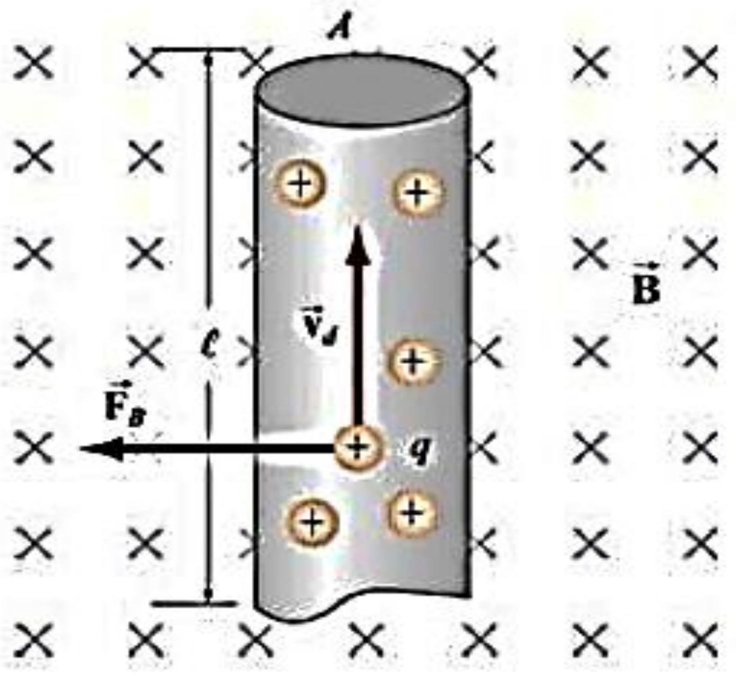


Figure magnetic force on a conducting wire

Where, $\boldsymbol{I = qn\,\vec{v}_d\,A}$ and $\vec{\boldsymbol{l}}$ is a length vector with a magnitude $\boldsymbol{l}$ and directed along the direction of the electric current.

- For a wire of arbitrary shape, the magnetic force can be obtained by summing over the forces acting on the small segments that make up the wire. Let the differential segment be denoted as $\boldsymbol{d\vec{S}}$.

The magnetic force acting on the segment is:

$$\boldsymbol{d\vec{F}_s = I\, d\vec{S} \times \vec{B}}$$

Thus the total force, $\boldsymbol{\vec{F}_s = I \int_a^b d\vec{S} \times \vec{B}}$

Where **a** and **b** represent the endpoints of the wire.

As an example, consider a curved wire carrying a current I in a uniform magnetic field $\boldsymbol{\vec{B}}$ as in the figure below:

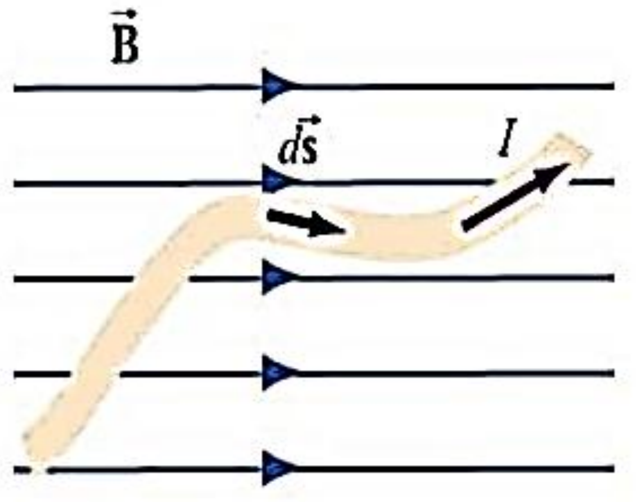


Figure current carrying wire placed in a magnetic field

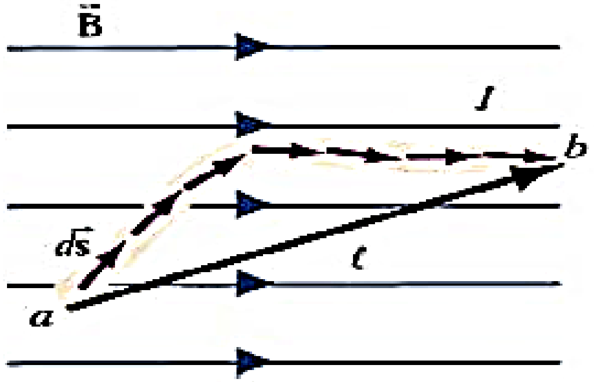


Figure a curved wire carrying a current I

DR. AHMED ALMURSHEDI

$$\vec{F}_B = I\left(\int_a^b d\vec{S}\right) \times \vec{B} = I\,\vec{l} \times \vec{B}$$

Where $\vec{l}$ is the length vector directed from **a** to **b**. However, if the wire forms a closed loop of arbitrary shape (as in the figure below) the force on the loop becomes.

$$\vec{F}_B = I\left(\oint d\vec{S}\right) \times \vec{B}$$

Since the set of differential length elements $d\vec{S}$ form a closed polygon, and their vector sum is zero.

i.e., $\oint d\vec{S} = 0$.

Then net magnetic force on a closed loop is $\vec{F}_B = 0.$

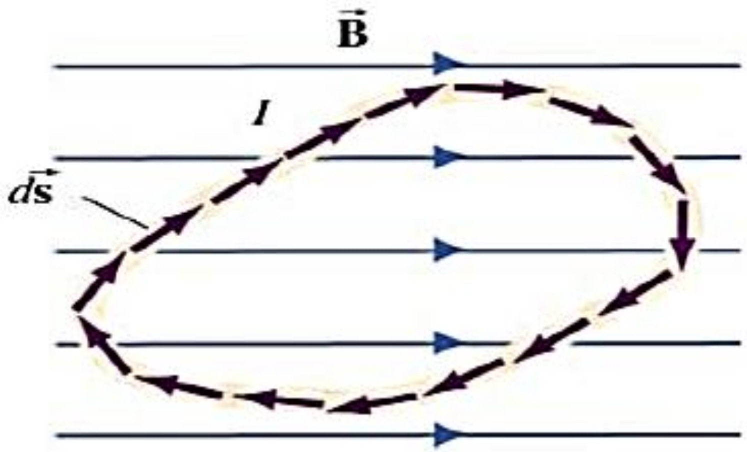


Figure a closed loop carrying a current I in a uniform magnetic field

Torque on a current loop

What happens when we place a rectangular loop carrying a current **I** in the xy plane and switch on a uniform magnetic field $\vec{B} = B\,\acute{\imath}$ which runs parallel to the plane of the loop, as in the figure below.

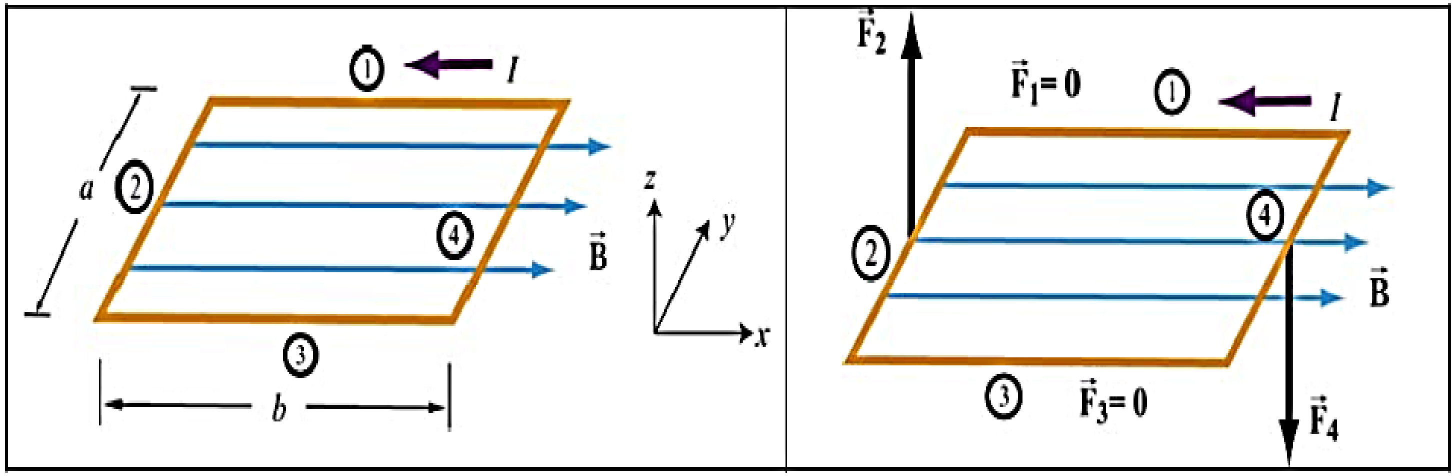


Figure (a) A rectangular current loop placed in a uniform magnetic field. (b) The magnetic forces acting on sides 2 and 4

From equation below, we see the magnetic forces acting on sides **1 and 3** vanish because the length vectors $\vec{\boldsymbol{l}}_1 = -\boldsymbol{b}\,\acute{\boldsymbol{i}}$ and $\vec{\boldsymbol{l}}_3 = \boldsymbol{b}\,\acute{\boldsymbol{i}}$ are parallel and anti parallel to $\vec{\boldsymbol{B}}$ and their cross products vanish. On the other hand, the magnetic forces acting on the segments **2 and 4** are non vanishing:

$$\vec{\boldsymbol{F}}_2 = \boldsymbol{I}\left(-\boldsymbol{a}\,\acute{\boldsymbol{j}}\right) \times (\boldsymbol{B}\,\acute{\boldsymbol{i}}) = \boldsymbol{IaB}\,\acute{\boldsymbol{k}}$$

$$\vec{\boldsymbol{F}}_4 = \boldsymbol{I}\left(\boldsymbol{a}\,\acute{\boldsymbol{j}}\right) \times (\boldsymbol{B}\,\acute{\boldsymbol{i}}) = -\boldsymbol{IaB}\,\acute{\boldsymbol{k}}$$

For $\vec{\boldsymbol{F}}_2$ pointing out of the page and $\vec{\boldsymbol{F}}_4$ into the page. Thus the net force on the rectangular loop is:

$$\vec{\boldsymbol{F}}_{net} = \vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{F}}_3 + \vec{\boldsymbol{F}}_4 = \boldsymbol{0}$$

As expected, even through the net force on the loop vanishes, the forces $\vec{F}_2$ and $\vec{F}_4$ will produce a torque which causes the loop to rotate about the y-axis. The torque with respect to the center of the loop is:

$$\vec{\tau} = \left(-\frac{b}{2}\,\acute{i}\right) \times \vec{F}_2 + \left(\frac{b}{2}\,\acute{i}\right) \times \vec{F}_4$$

$$\vec{\tau} = \left(-\frac{b}{2}\,\acute{i}\right) \times (IaB\,\acute{k}) + \left(\frac{b}{2}\,\acute{i}\right) \times (-IaB\,\acute{k})$$

$$\vec{\tau} = \left(\frac{IabB}{2} + \frac{IabB}{2}\right)\acute{j} = IabB\,\acute{j} = IAB\,\acute{j}$$

Where **A=ab** represents the area of the loop and the positive sign indicates that the rotation is clockwise about the y-axis. It is convenient to introduce the area vector $\vec{A} = A\,\acute{n}$ where $\acute{n}$ is a unit vector in the direction normal to the plane of the loop. The direction of the positive sense of $\acute{n}$ is set by the conventional right hand rule. In our case, we have $\acute{n} = +\acute{k}$. The above expression for torque can then be written as:

$$\vec{\tau} = I\vec{A} \times \vec{B}$$

- Notice that the magnitude of the torque is at maximum when $\vec{B}$ is parallel to the plane of the loop (or perpendicular to $\vec{A}$).
- Consider now the more general situation where the loop (or the area vector $\vec{A}$) makes an angle θ with respect to the magnetic field.

In above figure, the lever arms can be expressed by:

$$\vec{r}_2 = \frac{b}{2}\left(-\sin\theta\ \acute{\imath} + \cos\theta\ \acute{k}\right) = -\vec{r}_4$$

and the net torque is becomes:

$$\vec{\tau} = \vec{r}_2 \times \vec{F}_2 + \vec{r}_4 \times \vec{F}_4 = 2\vec{r}_2 \times \vec{F}_2$$

$$\vec{\tau} = 2\,.\frac{b}{2}\left(-\sin\theta\ \acute{\imath} + \cos\theta\ \acute{k}\right) \times (IaB\ \acute{k})$$

$$\vec{\tau} = IabB\ \sin\theta\ \acute{\jmath} = I\,\vec{A} \times \vec{B}$$

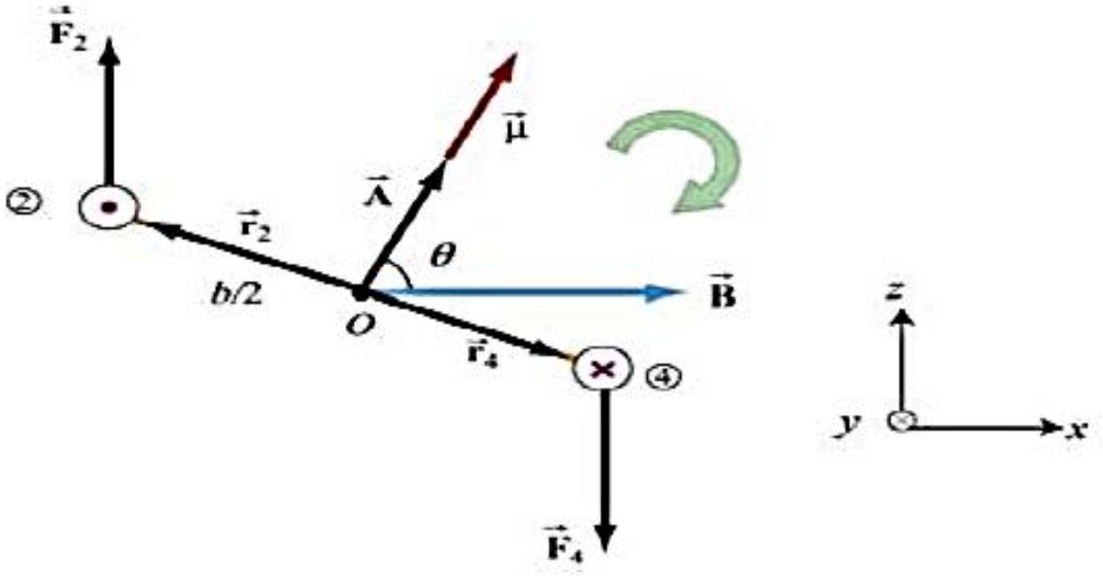


Figure Rotation of a rectangular current loop

For the loop consisting of **N** turns, the magnitude of the torque is:

$$\vec{\boldsymbol{\tau}} = \boldsymbol{NIAB}\ \sin\boldsymbol{\theta}$$

The quantity $\boldsymbol{NI\vec{A}}$ is called magnetic dipole moment $\vec{\boldsymbol{\mu}}$ **:** $\vec{\boldsymbol{\mu}} = \boldsymbol{NI\vec{A}}$

The direction of $\vec{\boldsymbol{\mu}}$ is the same as the area $\vec{\boldsymbol{A}}$ (perpendicular to the plane of the loop) and is determined by the right hand rule.

The SI unit of the magnetic dipole moment is ($Ampere.meter^2$).

Using the expression for $\vec{\boldsymbol{\mu}}$ the torque exerted on the current carrying loop can be written:

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$$

This expression in analogues to $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{p}} \times \vec{\boldsymbol{E}}$ the torque exerted on an electric dipole moment $\vec{\boldsymbol{p}}$ in the presence of an electric field $\vec{\boldsymbol{E}}$.

Magnetic force on a dipole

Consider the situation where a small dipole is placed along the symmetric axis of a bar magnet, as in the figure below:

The dipole experiences an attractive force by the bar magnet whose magnetic field is non uniform in the space. Thus, an external force must be applied to move the dipole to the right. The amount of force $\boldsymbol{F_{ext}}$ exerted by an external agent to move the dipole by a distance $\Delta \boldsymbol{x}$ is given by:

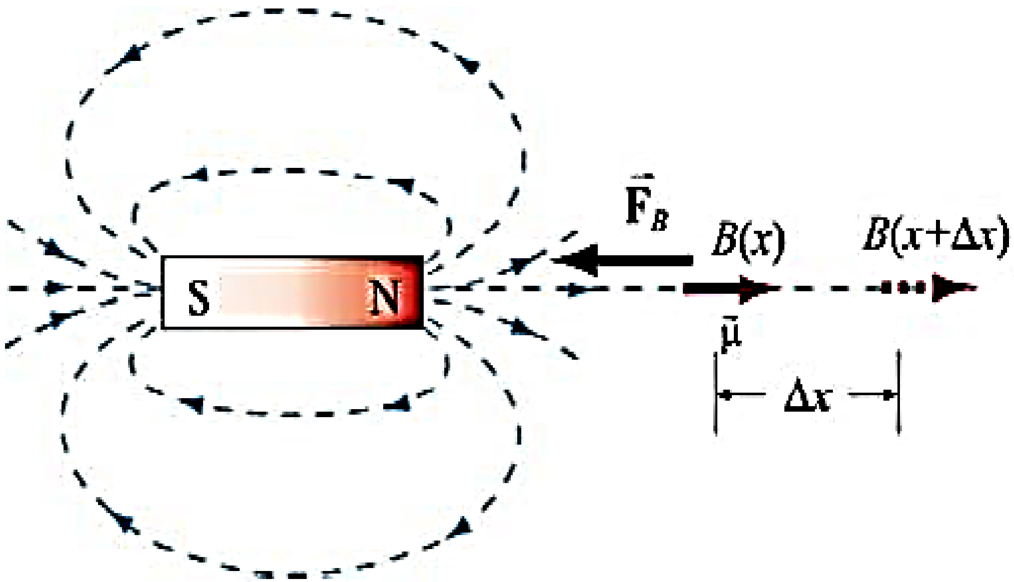


Figure a magnetic dipole near a bar magnet

$$\boldsymbol{F_{ext}\,\Delta x = W_{ext} = \Delta U = -\mu B(x + \Delta x) + \mu B(x)}$$

$$\boldsymbol{F_{ext}\,\Delta x = -\mu[B(x + \Delta x) - B(x)]}$$

For small $\Delta \boldsymbol{x}$, the external force may be obtained as:

$$\boldsymbol{F_{ext}} = -\boldsymbol{\mu} \frac{[\boldsymbol{B}(\boldsymbol{x} + \Delta \boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x})]}{\Delta \boldsymbol{x}} = -\boldsymbol{\mu} \frac{\boldsymbol{dB}}{\boldsymbol{dx}}$$

Which is positive quantity since $\frac{\boldsymbol{dB}}{\boldsymbol{dx}} < \boldsymbol{0}$, i.e., the magnetic field decreases with increasing $\boldsymbol{x}$. This is precisely the force needed to overcome the attractive force due to the bar magnet. Thus, we have:

$$\boldsymbol{F_B} = \boldsymbol{\mu} \frac{\boldsymbol{dB}}{\boldsymbol{dx}} = \frac{\boldsymbol{d}}{\boldsymbol{dx}} (\vec{\boldsymbol{\mu}}.\vec{\boldsymbol{B}})$$

More generally, the magnetic force experienced by a dipole $\vec{\boldsymbol{\mu}}$ placed in non uniform magnetic field $\vec{\boldsymbol{B}}$ can be written as:

$$\vec{\boldsymbol{F}}_{\boldsymbol{s}} = \boldsymbol{\nabla}(\vec{\boldsymbol{\mu}}.\vec{\boldsymbol{B}})$$

Where, $\boldsymbol{\nabla} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} \acute{\boldsymbol{\imath}} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}} \acute{\boldsymbol{\jmath}} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}} \acute{\boldsymbol{k}}$ is gradient Operator.

Example 1: A proton is moving in a circular orbit of radius 14 cm in a uniform 0.35 T magnetic field perpendicular to the velocity of the proton. Find the linear speed of the proton.

Sol.

Example 2: A particle with charge $q = 3.2 \times 10^{-19}\,C$ mass $m = 3 \times 10^{-27}\,kg$ and velocity $\vec{v} = 5 \times 10^{5}\,(\acute{\imath})\,m/s$, enters a region of uniform magnetic field $\vec{B} = 1.6\,(\acute{\jmath})T$. (a) compute magnitude and direction of the magnetic force (b) compute the radius of the resulting charge circular orbit, angular velocity and the frequency

Sol.

Example 3: In an experiment designed to measure the magnitude of a uniform electric field, electrons are accelerated from rest through a potential difference of 350 V. The electrons travel along a curved path because of the magnetic force exerted on them, and the radius of the path is measured by 7.5 cm. If the magnetic field is perpendicular to the beam. (a) what is the magnitude of the field? (b) what is the angular speed of the electrons?

- **Sol.**

H.W: two isotopes of uranium, ^{235}U (mass of $3.90 \times 10^{-25}\ kg$) and ^{238}U (mass of $3.95 \times 10^{-25}\ kg$), are sent into a mass spectrometer with a speed of $1.05 \times 10^{5}\ m/s$. Given that each isotope is singly ionized, and the strength of the magnetic field is 0.750 T, what is the separation between two isotopes after they complete half a circular orbit?