Wave Speed on a Stretched String

The speed of a wave depends on the characteristics of the medium. For example, in the case of a guitar, the strings vibrate to produce the sound. The speed of the waves on the strings, and the wavelength, determine the frequency of the sound produced. The strings on a guitar have different thickness but may be made of similar material. They have different linear densities, where the linear density is defined as the mass per length,

$$\mu = \frac{mass\ of\ string}{lenght\ of\ string} = \frac{m}{l} \qquad (7)$$

Wave Speed on a String under Tension

To see how the speed of a wave on a string depends on the tension and the linear density, consider a pulse sent down a taut string (Figure 5). When the taut string is at rest at the equilibrium position, the tension in the string F_T is constant.

Figure (5)

The speed of a pulse or wave on a string under tension can be found with the equation

$$|v| = \sqrt{\frac{F_T}{\mu}} \qquad (8)$$

Where F_T is the tension in the string and μ is the mass per length of the string.

Example.5// The Wave Speed of a Guitar Spring

On a six-string guitar, the high E string has a linear density of $\mu_{High\ E}=3.09*10^{-4}$ kg/m, and the low E string has a linear density of $\mu_{Low\ E}=5.78*10^{-3}$ kg/m (a) If the high E string is plucked, producing a wave in the string, what is the speed of the wave if the tension of the string is 56.40 N? (b) The linear density of the low E string is approximately 20 times greater than that of the high E string. For waves to travel through the low E string at the same wave speed as the high E, would the tension need to be larger or smaller than the high E string? What would be the approximate tension? (c) Calculate the tension of the low E string needed for the same wave speed.

Solution

a.

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{56.40\ \text{N}}{3.09 \times 10^{-4}\ \text{kg/m}}} = 427.23\ \text{m/s}.$$

b. The tension would need to be increased by a factor of approximately 20. The tension would be slightly less than 1128 N.

c.

$$F_T = \mu v^2 = 5.78 \times 10^{-3}\text{kg/m}(427.23\ \text{m/s})^2 = 1055.00\ \text{N}.$$

Speed of Compression Waves in a Fluid

The speed of a wave on a string depends on the square root of the tension divided by the mass per length, the linear density. In general, the speed of a wave through a medium depends on the elastic property of the medium and the inertial property of the medium.

$$|v| = \sqrt{\frac{elastic\ property}{inertial\ property}}$$

The elastic property describes the tendency of the particles of the medium to return to their initial position when perturbed. The inertial property describes the tendency of the particle to resist changes in velocity.

The speed of a longitudinal wave through a liquid or gas depends on the density of the fluid and the bulk modulus of the fluid,

$$v = \sqrt{\frac{B}{\rho}} \qquad (9)$$

Here the bulk modulus is defined as $B = -\frac{\Delta P}{\Delta V / V_\circ}$ where ΔP is the change in the pressure and the denominator is the ratio of the change in volume to the initial volume, and $\rho = \frac{m}{V}$ is the mass per unit volume.
For example, sound is a mechanical wave that travels through a fluid or a solid. The speed of sound in air with an atmospheric pressure of $1.013*10^5$ Pa and a temperature of 20 °C is $v_s \approx 343$ m/s because the density depends on temperature, the speed of sound in air depends on the temperature of the air.

Energy and Power of a Wave

All waves carry energy; the amount of energy in a wave is related to its amplitude and its frequency.
The energy of the wave depends on both the amplitude and the frequency. If the energy of each wavelength is considered to be a discrete packet of energy, a high-frequency wave will deliver more of these packets per unit time than a low-frequency wave. The average rate of energy transfer in mechanical waves is proportional to both the square of the amplitude and the square of the frequency.
If two mechanical waves have equal amplitudes, but one wave has a frequency equal to twice the frequency of the other, the higher frequency wave will have a rate of energy transfer a factor of four times as great as the rate of energy transfer of the lower-frequency wave.

Power in Waves

Consider a sinusoidal wave on a string that is produced by a string vibrator, as shown in Figure (6).
The **string vibrator** is a device that vibrates a rod up and down. A string of uniform linear mass density is attached to the rod, and the rod oscillates the string, producing a sinusoidal wave. The rod does work on the string, producing energy that propagates along the string. Since the string has a

constant linear density $\mu = \frac{\Delta m}{\Delta x}$,each mass element of the string has the mass $\Delta m = \mu \Delta x$.

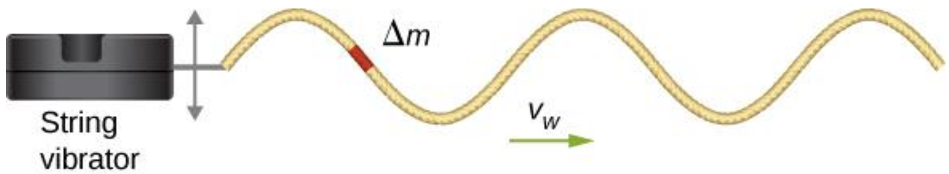

Figure (6)

The total mechanical energy of the wave is the sum of its kinetic energy and potential energy. The kinetic energy $\boldsymbol{K = \frac{1}{2} m v^2}$ of each mass element of the string of length Δx is $\boldsymbol{\Delta K = \frac{1}{2} (\Delta m) v_y^2}$ as the mass element oscillates perpendicular to the direction of the motion of the wave. Using the constant linear mass density, the kinetic energy of each mass element of the string with length is Δx.

$$\Delta K = \frac{1}{2} (\mu \Delta x) v_y^2$$

A differential equation can be formed by letting the length of the mass element of the string approach zero,

$$dK = \frac{1}{2} (\mu dx) v_y^2$$

Since the wave is a sinusoidal wave with an angular frequency ω , the position of each mass element may be modeled as $\boldsymbol{y(x,t) = A \sin(kx - \omega t)}$. Each mass element of the string oscillates with a velocity $\boldsymbol{v_y = \frac{\partial y(x,t}{\partial t} = -A\omega \cos(kx - \omega t)}$.

The kinetic energy of each mass element of the string becomes

$$\begin{aligned} dK &= \frac{1}{2}(\mu dx)\,(-A\omega \cos(kx - \omega t))^2, \\ &= \frac{1}{2}(\mu dx)\, A^2 \omega^2 \cos^2(kx - \omega t). \end{aligned}$$

The wave can be very long, consisting of many wavelengths. To standardize the energy, consider the kinetic energy associated with a wavelength of the wave. This kinetic energy can be integrated over the wavelength to find the energy associated with each wavelength of the wave:

$$
\begin{aligned}
dK &= \tfrac{1}{2}(\mu dx)\, A^2\omega^2 \cos^2(kx)\,, \\
\int_0^{K_\lambda} dK &= \int_0^{\lambda} \frac{1}{2}\mu A^2\omega^2 \cos^2(kx)\, dx = \tfrac{1}{2}\mu A^2\omega^2 \int_0^{\lambda} \cos^2(kx)\, dx, \\
K_\lambda &= \tfrac{1}{2}\mu A^2\omega^2 \left[\tfrac{1}{2}x + \tfrac{1}{4k}\sin(2kx)\right]_0^{\lambda} = \tfrac{1}{2}\mu A^2\omega^2 \left[\tfrac{1}{2}\lambda + \tfrac{1}{4k}\sin(2k\lambda) - \tfrac{1}{4k}\sin(0)\right], \\
K_\lambda &= \tfrac{1}{4}\mu A^2\omega^2 \lambda.
\end{aligned}
$$

In Oscillations, the potential energy stored in a spring with a linear restoring force is equal to $\boldsymbol{U = \frac{1}{2} k_s x^2}$,where the equilibrium position is defined as x=0 m. When a mass attached to the spring oscillates in simple harmonic motion, the angular frequency is equal to $\boldsymbol{\omega = \sqrt{\frac{k_s}{m}}}$ .As each mass element oscillates in simple harmonic motion, the spring constant is equal to $\boldsymbol{k_s = \Delta m\omega^2}$.The potential energy of the mass element is equal to

$$\Delta U = \frac{1}{2}k_s x^2 = \frac{1}{2}\Delta m\omega^2 x^2$$

Note that k_s is the spring constant and not the wave number $\boldsymbol{k = \frac{2\pi}{\lambda}}$ .This equation can be used to find the energy over a wavelength. Integrating over the wavelength, we can compute the potential energy over a wavelength:

$$
\begin{aligned}
dU &= \tfrac{1}{2}k_s x^2 = \tfrac{1}{2}\mu\omega^2 x^2 dx, \\
U_\lambda &= \tfrac{1}{2}\mu\omega^2 A^2 \int_0^{\lambda} \sin^2(kx)\, dx = \tfrac{1}{4}\mu A^2\omega^2 \lambda
\end{aligned}
$$

The potential energy associated with a wavelength of the wave is equal to the kinetic energy associated with a wavelength.

The total energy associated with a wavelength is the sum of the potential energy and the kinetic energy:

$$E_\lambda = U_\lambda + K_\lambda,$$
$$E_\lambda = \frac{1}{4}\mu A^2\omega^2\lambda + \frac{1}{4}\mu A^2\omega^2\lambda = \frac{1}{2}\mu A^2\omega^2\lambda$$

The time-averaged power of a sinusoidal mechanical wave, which is the average rate of energy transfer associated with a wave as it passes a point, can be found by taking the total energy associated with the wave divided by the time it takes to transfer the energy. For a sinusoidal mechanical wave, the time-averaged power is therefore the energy associated with a wavelength divided by the period of the wave. The wavelength of the wave divided by the period is equal to the velocity of the wave,

$$P_{ave} = \frac{E_\lambda}{T} = \frac{1}{2}\mu A^2\omega^2\frac{\lambda}{T} = \frac{1}{2}\mu A^2\omega^2 v \qquad (10)$$

Example.6// Power Supplied by a String Vibrator

Consider a two-meter-long string with a mass of 70.00 g attached to a string vibrator.

The tension in the string is 90.0 N. When the string vibrator is turned on, it oscillates with a frequency of 60 Hz and produces a sinusoidal wave on the string with an amplitude of 4.00 cm and a constant wave speed. What is the time-averaged power supplied to the wave by the string vibrator?

Solution

We need to calculate the linear density to find the wave speed:

$$\mu = \frac{m_s}{L_s} = \frac{0.070\text{ kg}}{2.00\text{ m}} = 0.035\text{ kg/m}.$$

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{90.00\text{ N}}{0.035\text{ kg/m}}} = 50.71\text{ m/s}.$$

$$\omega = 2\pi f = 2\pi\left(60\text{ s}^{-1}\right) = 376.80\text{ s}^{-1}$$

Calculate the time-averaged power:

$$P = \frac{1}{2}\mu A^2 \omega^2 v = \frac{1}{2}\left(0.035\frac{\text{kg}}{\text{m}}\right)(0.040\ \text{m})^2\left(376.80\ \text{s}^{-1}\right)^2\left(50.71\frac{\text{m}}{\text{s}}\right) = 201.59\ \text{W}.$$

Note// Another important characteristic of waves is the **intensity** of the waves. Waves can also be concentrated or spread out. All these pertinent factors are included in the definition of intensity (I) as power per unit area:

$$I = \frac{P}{A} \qquad (11)$$

where P is the power carried by the wave through area A. The definition of intensity is valid for any energy in transit, including that carried by waves. The SI unit for intensity is watts per square meter (W/m^2). Many waves are spherical waves that move out from a source as a sphere.

As a spherical wave moves out from a source, the surface area of the wave increases as the radius increases ($\boldsymbol{A = 4\pi r^2}$) .The intensity for a spherical wave is therefore

$$I = \frac{P}{4\pi r^2} \qquad (12)$$