Physical Pendulum (Compound Pendulum)

A compound/physical pendulum is a rigid body of any arbitrary shape capable of rotating in a vertical plane about an axis passing through the pendulum but not through the center of gravity of the pendulum.

The distance between the point of suspension and the center of gravity is called the length of the pendulum, When the pendulum is displaced through an angle θ from the mean position, a restoring torque comes into play which tries to bring the pendulum back to the mean position. However, the oscillation continues due to the inertia of restoring force.

As for the simple pendulum, the restoring force of the physical pendulum is the force of gravity. With the simple pendulum, the force of gravity acts on the center of the pendulum bob. In the case of the physical pendulum, the force of gravity acts on the center of mass (CM) of an object. The object oscillates about a point O. Consider an object of a generic shape as shown in Figure 19.

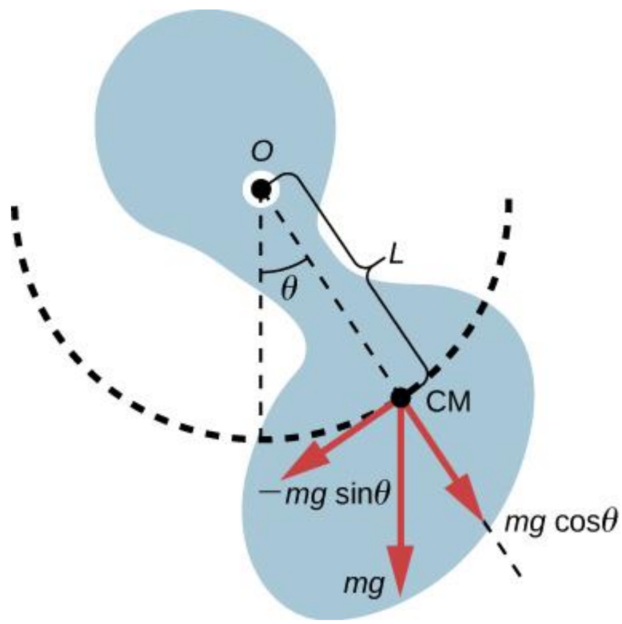

Figure 19: A physical pendulum

When a physical pendulum is hanging from a point but is free to rotate, it rotates because of the torque applied at the CM, produced by the component of the object's weight that acts tangent to the motion of the CM. Taking the counterclockwise direction to be positive, the component of the gravitational force that acts tangent to the motion is $-mg\ sin\theta$. The minus sign is the result of the restoring force acting in the opposite direction of the increasing angle. Recall that the torque is equal to $\vec{\tau} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$. The magnitude of the torque is equal to the length of the radius arm times the tangential component of the force applied, $|\tau| = rFsin\theta$. Here, the length L of the radius arm is the distance between the point of rotation and the CM.

To analyze the motion, start with the net torque. Recall from Fixed-Axis Rotation on rotation that the net torque is equal to the moment of inertia $I = \int r^2 dm$ times the angular Acceleration α where $\alpha = \frac{d^2\theta}{dt^2}$:

$$I\alpha = \tau_{\text{net}} = L\,(-mg)\sin\theta.$$

Using the small angle approximation $sin\theta \approx \theta$ and rearranging:

$$\begin{aligned} I\alpha &= -L\,(mg)\,\theta; \\ I\frac{d^2\theta}{dt^2} &= -L\,(mg)\,\theta; \\ \frac{d^2\theta}{dt^2} &= -\left(\frac{mgL}{I}\right)\theta. \end{aligned}$$

The solution is:

$$\theta(t) = \Theta\cos(\omega t + \phi)\,,$$

where Θ is the maximum angular displacement. The angular frequency is:

$$\omega = \sqrt{\frac{mgL}{I}}$$

The period is therefore:

$$T = 2\pi\sqrt{\frac{I}{mgL}}$$

Note that for a simple pendulum, the moment of inertia is $I = \int r^2 dm = mL^2$ and the period reduces to $T = 2\pi\sqrt{\frac{L}{g}}$.

Torsional Pendulum

A **torsional pendulum** consists of a rigid body suspended by a light wire or spring (Figure 20). When the body is twisted some small maximum angle Θ and released from rest, the body oscillates between $(\theta = +\Theta)$ and $(\theta = -\Theta)$. The restoring torque is supplied by the shearing of the string or wire.

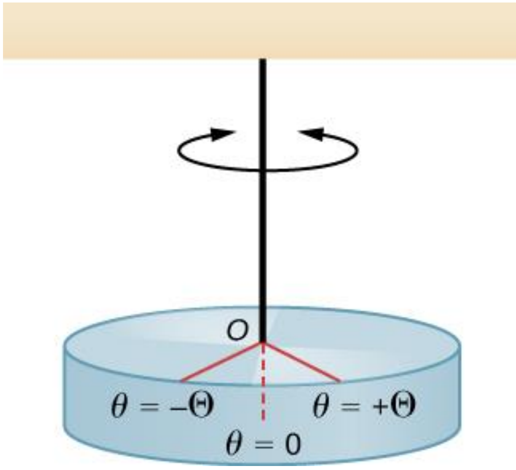

Figure.20: A torsional pendulum

The restoring torque can be modeled as being proportional to the angle:

$$\tau = -\kappa\theta$$

The variable kappa (κ)is known as the torsion constant of the wire or string. The minus sign shows that the restoring torque acts in the opposite direction to increasing angular displacement. The net torque is equal to the moment of inertia times the angular acceleration:

$$I\frac{d^2\theta}{dt^2} = -\kappa\theta;$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta.$$

This equation says that the second time derivative of the position (in this case, the angle) equals a negative
constant times the position. This looks very similar to the equation of motion for the SHM $\frac{d^2x}{dt^2} = -\frac{k}{m}x$, where the period was found to be $T = 2\pi\sqrt{\frac{m}{k}}$. Therefore, the period of the torsional pendulum can be found using

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

Example// A rod has a length of l=0.3m and a mass of 4 kg. A string is attached to the CM of the rod and the system is hung from the ceiling (Figure 21). The rod is displaced 10 degrees from the equilibrium position and released from rest. The rod oscillates with a period of 0.5 s. What is the torsion constant ?

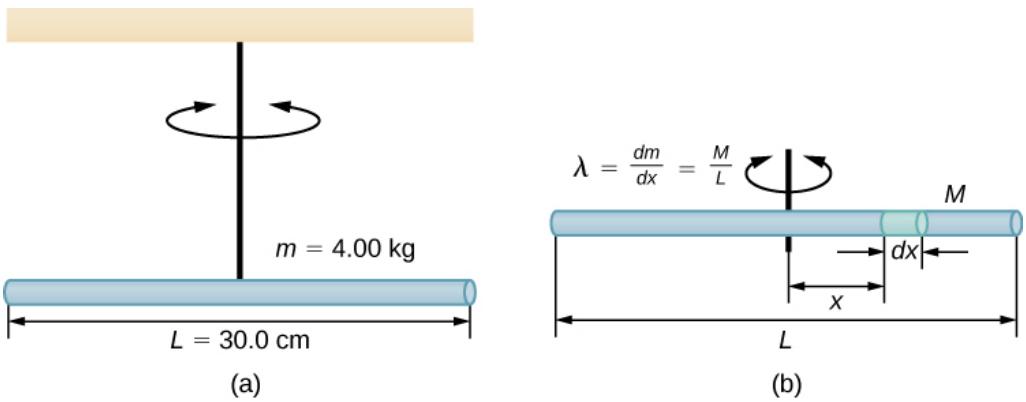

Figure 21 (a) A rod suspended by a string from the ceiling. (b) Finding the rod's moment of inertia.

Solution//

1. Find the moment of inertia for the CM:

$$I_{CM} = \int x^2 dm = \int_{-L/2}^{+L/2} x^2 \lambda dx = \lambda \left[\frac{x^3}{3}\right]_{-L/2}^{+L/2} = \lambda \frac{2L^3}{24} = \left(\frac{M}{L}\right)\frac{2L^3}{24} = \frac{1}{12}ML^2.$$

2. Calculate the torsion constant using the equation for the period:

$$T = 2\pi\sqrt{\frac{I}{\kappa}};$$

$$\kappa = I\left(\frac{2\pi}{T}\right)^2 = \left(\frac{1}{12}ML^2\right)\left(\frac{2\pi}{T}\right)^2;$$

$$= \left(\frac{1}{12}(4.00\,\text{kg})(0.30\,\text{m})^2\right)\left(\frac{2\pi}{0.50\,\text{s}}\right)^2 = 4.73\,\text{N}\cdot\text{m}.$$

Damped Oscillations

For a free oscillation, the energy remains constant. Hence oscillation continues indefinitely. However, in real fact, the amplitude of the oscillatory system gradually decreases due to experiences of damping force like friction and resistance of the media.

The oscillators whose amplitude, in successive oscillations goes on decreasing due to the presence of resistive forces are called damped oscillators, and oscillations are called damping oscillations.

The oscillations of a mass kept in water, charge oscillations in a LCR circuit are examples of damped oscillations. Let us assume that in addition to the elastic force F = -kx, there is a force that is opposed to the velocity, F = b v where b is a constant known as the resistive coefficient and it depends on the medium, shape of the body.

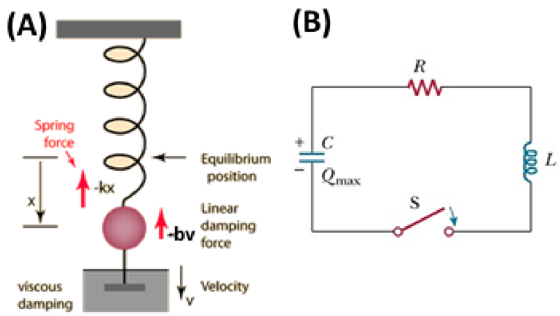

Fig.22. (A) Spring oscillation under damping is created by viscous liquid. (B) Equivalent LCR circuit in series

$$ma = -bv - kx.$$

Writing this as a differential equation in x, we obtain:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

This is a homogeneous, linear differential equation of second order.

The auxiliary equation is $D^2 + \frac{b}{m}D + \frac{k}{m} = 0$

The roots are $D_1 = -\frac{b}{2m} + \frac{1}{2m}\sqrt{b^2 - 4mk}$ and $D_2 = -\frac{b}{2m} - \frac{1}{2m}\sqrt{b^2 - 4mk}$

The solution can be derived as $x(t) = Ce^{-\left(\frac{b}{2m} - \frac{1}{2m}\sqrt{b^2-4mk}\right)t} + De^{-\left(\frac{b}{2m} + \frac{1}{2m}\sqrt{b^2-4mk}\right)t}$

This can be expressed as x(t) = $Ae^{-\frac{b}{2m}t}\cos(\omega t - \phi)$ where $\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$

$$A = \sqrt{C^2 + D^2} \qquad \phi = \tan^{-1}(D/C)$$

Recall that the angular frequency of a mass undergoing SHM is equal to the square root of the force constant divided by the mass. This is often referred to as the **natural angular frequency**, which is represented as $\omega_\circ = \sqrt{\frac{k}{m}}$. The angular freq uency for damped harmonic motion becomes $\omega = \sqrt{\omega_\circ^2 - (\frac{b}{2m})^2}$.

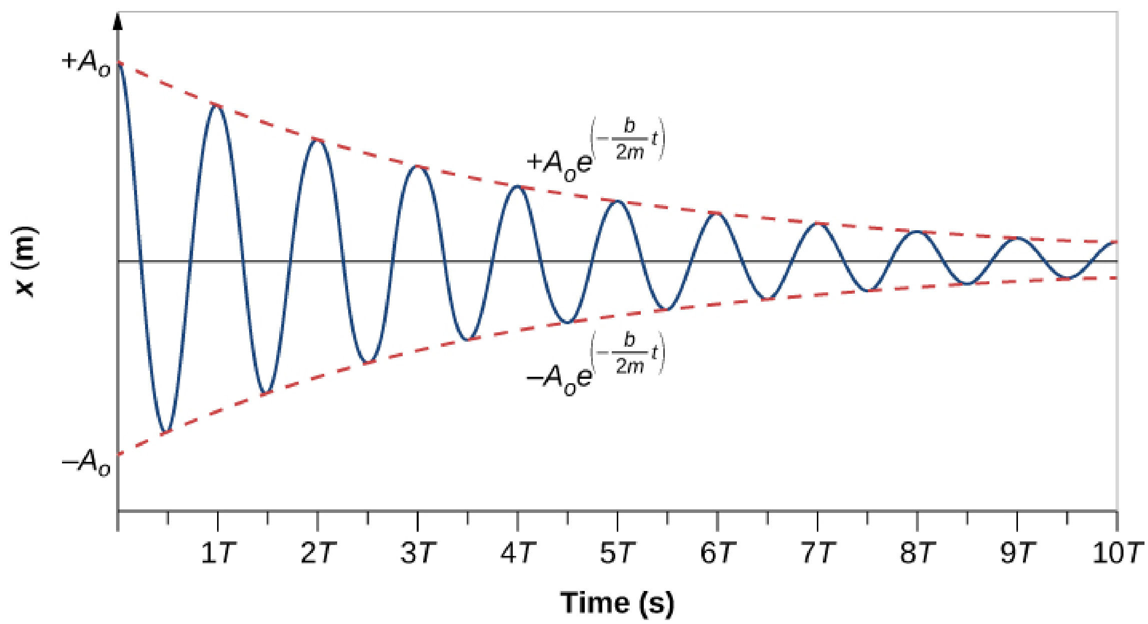

Figure 23 Position versus time for the mass oscillating on a spring in a viscous fluid.

Here, the term $Ae^{-\frac{b}{2m}t}$ represents the decreasing amplitude and (ωt-ϕ) represents phase.

Case 1: $b^2 > 4mk$ OVER DAMPING

Case 1: $b^2 < 4mk$ UNDER DAMPING

Case 1: $b^2 = 4mk$ CRITICAL DAMPING

Under damped: $b^2 > 4mk$

When the retarding force is less than k. A, the system oscillates with decreasing amplitude

Critically damped: $b^2 = 4mk$

When $\frac{b}{2m} = \omega_o$, the system does not oscillate

Over damped: $b^2 > 4mk$

When the retarding force is greater than k.A, $\frac{b}{2m} > \omega_o$

Over damping takes away the energy of the system and oscillations stop.

Critical damping is often desired, because such a system returns to equilibrium rapidly and remains at
equilibrium as well. In addition, a constant force applied to a critically damped system moves the system to a new equilibrium position in the shortest time possible without overshooting or oscillating about the new position.

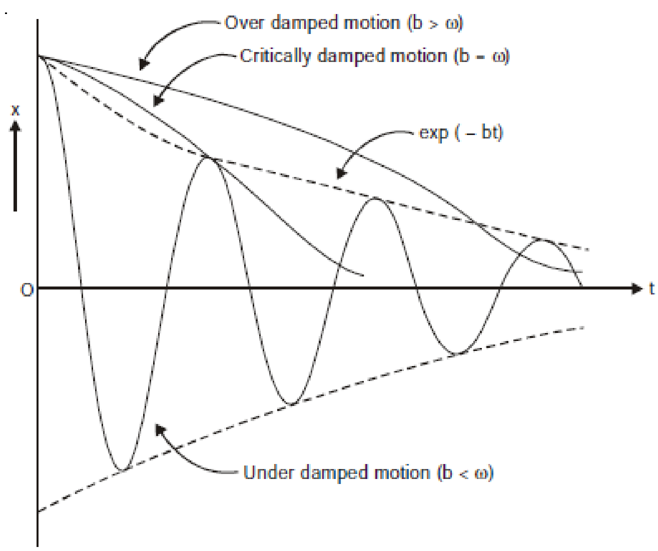

Figure .24

Forced Oscillations

The oscillations occur that under the action of an external periodic force are called forced oscillations. During forced oscillations the system oscillates with the frequency of the external periodic force. Examples for forced oscillations are Sonometer wire set to oscillations using a tuning fork or electromagnet. Resonance air column.

The forces acting on the system during forced oscillations
1. Restoring force acting in the direction opposite to the displacement.
2. Damping force due to the viscous medium.
3. External periodic force acting on the system.
Let F = Fo Cosωt be the oscillating applied force , The equation of motion is given by

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + F_0cos(\omega t)$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F_0cos(\omega t)$$

When an oscillator is forced with a periodic driving force, the motion may seem chaotic. The motions of the oscillator are known as transients. After the transients die out, the oscillator reaches a steady state, where the motion is periodic. After some time, the steady state solution to this differential equation is

$$x(t) = A\cos(\omega t + \phi)$$

the amplitude is equal to

$$A = \frac{F_\circ}{\sqrt{m^2(\omega^2 - \omega_\circ^2)^2 + b^2\omega^2}}$$

where $\omega_\circ = \sqrt{\frac{k}{m}}$ is the angular frequency of the driving force. Looking at the denominator of the equation for the amplitude, when the driving frequency is much smaller, or much larger, than the natural frequency, the square of the difference of the two angular frequencies $(\omega^2 - \omega_\circ^2)^2$ is positive and large, making the denominator large, and the result is a small amplitude for the oscillations of the mass. As the frequency of the driving force approaches the natural frequency of the system, the denominator becomes small and the amplitude of the oscillations becomes large. The maximum amplitude results when the frequency of the driving force equals the natural frequency of the system$A_{max} = \frac{F_\circ}{b\omega}$.

•A system's natural frequency is the frequency at which the system oscillates if not affected by driving or damping forces.

• A periodic force driving a harmonic oscillator at its natural frequency produces resonance. The system is said to resonate.

• The less damping a system has, the higher the amplitude of the forced oscillations near resonance. The more damping a system has, the broader response it has to varying driving frequencies.

Resonance: -

The amplitude of vibration becomes large for small damping and the maximum amplitude is inversely proportional to resistive term (b) hence called resonance. It is the phenomenon of a body setting a body into vibrations with its natural frequency by the application of a periodic force of the same frequency.

If the amplitude of oscillation is maximum when the driving frequency is the same as the natural frequency of an oscillator. (i.e., $\omega = \omega_0$).