The Linear Wave Equation

We have determined the velocity of the medium at a position x by taking the partial derivative, with respect to time, of the position y. For a transverse wave, this velocity is perpendicular to the direction of propagation of the wave. We found the acceleration by taking the partial derivative, with respect to time, of the velocity, which is the second time derivative of the position:

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2}{\partial t^2}(A\sin(kx-\omega t+\phi)) = -A\omega^2 \sin(kx-\omega t+\phi).$$

Now consider the partial derivatives with respect to the other variable, the position x, holding the time constant. The first derivative is the slope of the wave at a point x at a time t,

$$\text{slope} = \frac{\partial y(x,t)}{\partial x} = \frac{\partial}{\partial x}(A\sin(kx-\omega t+\phi)) = Ak\cos(kx-\omega t+\phi).$$

The second partial derivative expresses how the slope of the wave changes with respect to position—in other words, the curvature of the wave, where

$$\text{curvature} = \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{\partial^2}{\partial x^2}(A\sin(kx-\omega t+\phi)) = -Ak^2 \sin(kx-\omega t+\phi).$$

The ratio of the acceleration and the curvature leads to a very important relationship in physics known as the **linear wave equation**. Taking the ratio and using the equation $v = \frac{\omega}{k}$ yields the linear wave equation (also known simply as the wave equation or the equation of a vibrating string),

$$\frac{\frac{\partial^2 y(x,t)}{\partial t^2}}{\frac{\partial^2 y(x,t)}{\partial x^2}} = \frac{-A\omega^2 \sin(kx-\omega t+\phi)}{-Ak^2 \sin(kx-\omega t+\phi)}$$

$$= \frac{\omega^2}{k^2} = v^2,$$

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}.$$

This equation is the **linear wave equation**, which is one of the most important equations in physics and engineering. We derived it here for a transverse wave, but it is equally important when investigating longitudinal waves. This

relationship was also derived using a sinusoidal wave, but it successfully describes any wave or pulse that has the form $y(x,t) = f(x \pm vt)$.
An interesting aspect of the linear wave equation is that if two wave functions are individually solutions to the linear wave equation, then the sum of the two linear wave functions is also a solution to the wave equation.
Consider two transverse waves that propagate along the x-axis, occupying the same medium. Assume that the individual waves can be modeled with the wave functions $y_1(x,t) = f(x \pm vt)$ and $y_2(x,t) = g(x \pm vt)$ which are solutions to the linear wave equations and are therefore linear wave functions. The sum of the wave functions is the wave function

$$y_1(x,t) + y_2(x,t) = f(x \mp vt) + g(x \mp vt) .$$

Consider the linear wave equation:

$$\frac{\partial^2 (f+g)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 (f+g)}{\partial t^2}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} = \frac{1}{v^2} \left[\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 g}{\partial t^2} \right] .$$

This has shown that if two linear wave functions are added algebraically, the resulting wave function is also linear.

Q// Wind gusts create ripples on the ocean that have a wavelength of 5cm and propagate at 2m/s.What is their frequency?

Example.4/Interference of Waves on a String
Consider a very long string held taut by two students, one on each end. Student A oscillates the end of the string producing a wave modeled with the wave function $y_1(x,t)$=A sin $(kx - \omega t)$ and student B oscillates the string producing at twice the frequency, moving in the opposite direction. Both waves move at the same speed $v = \frac{\omega}{k}$ The two waves interfere to form a resulting wave whose wave function is $y_R(x,t)=y_1(x,t)+y_2(x,t)$.
Find the velocity of the resulting wave using the linear wave equation.

Solution

1. Write the wave function of the second wave: $y_2(x,t) = A\sin(2kx + 2\omega t)$.
2. Write the resulting wave function:
$$y_R(x,t) = y_1(x,t) + y(x,t) = A\sin(kx - \omega t) + A\sin(2kx + 2\omega t).$$
3. Find the partial derivatives:
$$\begin{aligned}
\frac{\partial y_R(x,t)}{\partial x} &= -Ak\cos(kx - \omega t) + 2Ak\cos(2kx + 2\omega t), \\
\frac{\partial^2 y_R(x,t)}{\partial x^2} &= -Ak^2\sin(kx - \omega t) - 4Ak^2\sin(2kx + 2\omega t), \\
\frac{\partial y_R(x,t)}{\partial t} &= -A\omega\cos(kx - \omega t) + 2A\omega\cos(2kx + 2\omega t), \\
\frac{\partial^2 y_R(x,t)}{\partial t^2} &= -A\omega^2\sin(kx - \omega t) - 4A\omega^2\sin(2kx + 2\omega t).
\end{aligned}$$
4. Use the wave equation to find the velocity of the resulting wave:

$$\begin{aligned}
\frac{\partial^2 y(x,t)}{\partial x^2} &= \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}, \\
-Ak^2\sin(kx - \omega t) - 4Ak^2\sin(2kx + 2\omega t) &= \frac{1}{v^2}\left(-A\omega^2\sin(kx - \omega t) - 4A\omega^2\sin(2kx + 2\omega t)\right. \\
k^2\left(-A\sin(kx - \omega t) - 4A\sin(2kx + 2\omega t)\right) &= \frac{\omega^2}{v^2}\left(-A\sin(kx - \omega t) - 4A\sin(2kx + 2\omega t)\right), \\
k^2 &= \frac{\omega^2}{v^2},\ |v| = \frac{\omega}{k}.
\end{aligned}$$

H.W/ A travelling harmonic wave on string is described by y(x,t)=7.5sin(0.0050x +12t +π/4)

a) What are the displacement and velocity of oscillation of a point at x=1 cm and t=1 s?

b)Is this velocity equal to the velocity of wave propagation ?