Interference of Waves

Waves do interact with the boundaries of the medium, and all or part of the wave can be reflected.

Reflection and Transmission

When a wave propagates through a medium, it reflects when it encounters the boundary of the medium. The wave before hitting the boundary is known as the **incident wave**.

The wave after encountering the boundary is known as the **reflected wave**.

How the wave is reflected at the boundary of the medium depends on the boundary conditions; waves will react differently if the boundary of the medium is fixed in place or free to move (Figure 7). A **fixed boundary condition** exists when the medium at a boundary is fixed in place so it cannot move. A **free boundary condition** exists when the medium at the boundary is free to move.

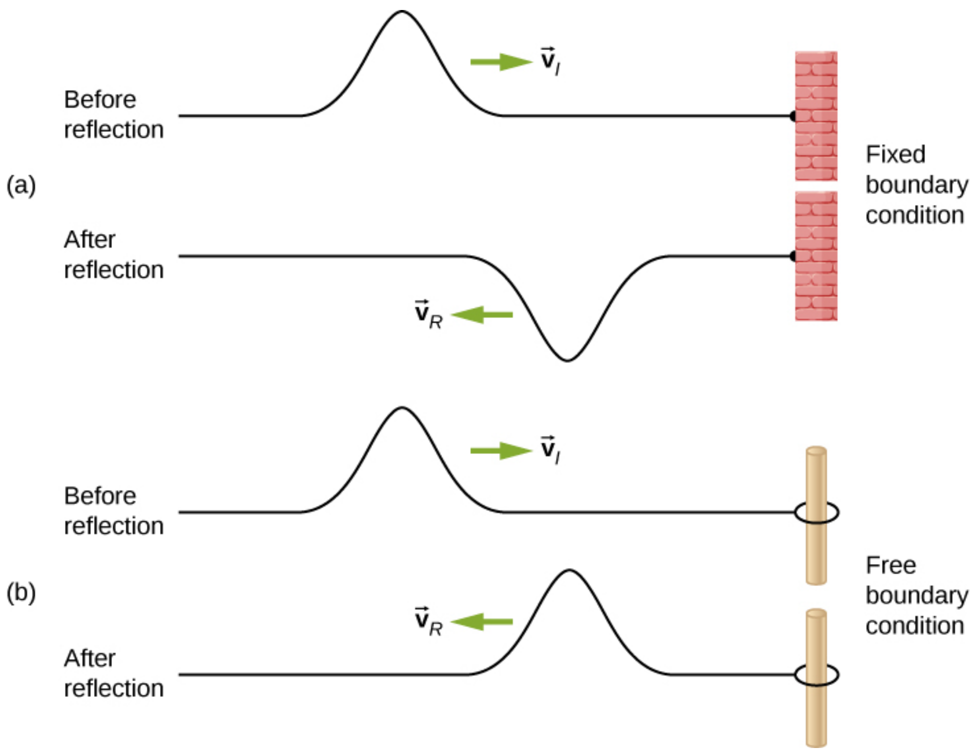


Figure (7)

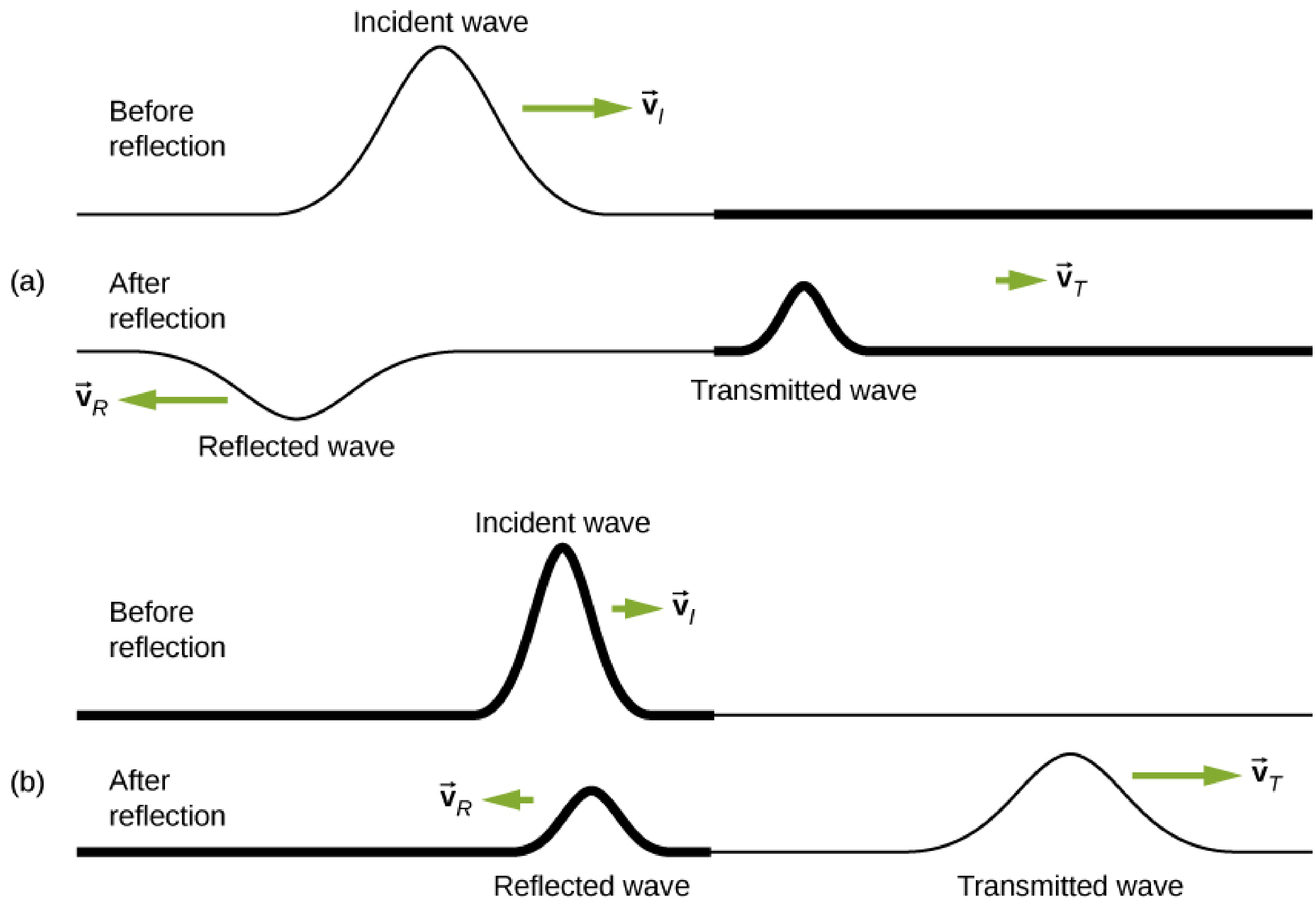


Figure (8)

In some situations, the boundary of the medium is neither fixed nor free. Consider Figure. 8(a), where a low-linear mass density string is attached to a string of a higher linear mass density. In this case, **the reflected wave is out of phase with respect to the incident wave. There is also a transmitted wave that is in phase with respect to the incident wave. Both the transmitted and the reflected waves have amplitudes less than the amplitude of the incident wave.**

Part (b) of the figure shows a high-linear mass density string is attached to a string of a lower linear density. In this case, **the reflected wave is in phase with respect to the incident wave. There is also a transmitted wave that is in phase with**

respect to the incident wave. Both the incident and the reflected waves have amplitudes less than the amplitude of the incident wave.

Superposition and Interference

Most interesting mechanical waves consist of a combination of two or more traveling waves propagating in the same medium. The principle of superposition can be used to analyze the combination of waves.

Consider two simple pulses of the same amplitude moving toward one another in the same medium, as shown in Figure (9) . Eventually, the waves overlap, producing a wave that has twice the amplitude, and then continues unaffected by the encounter. The pulses are said to interfere, and this phenomenon is known as **interference**.

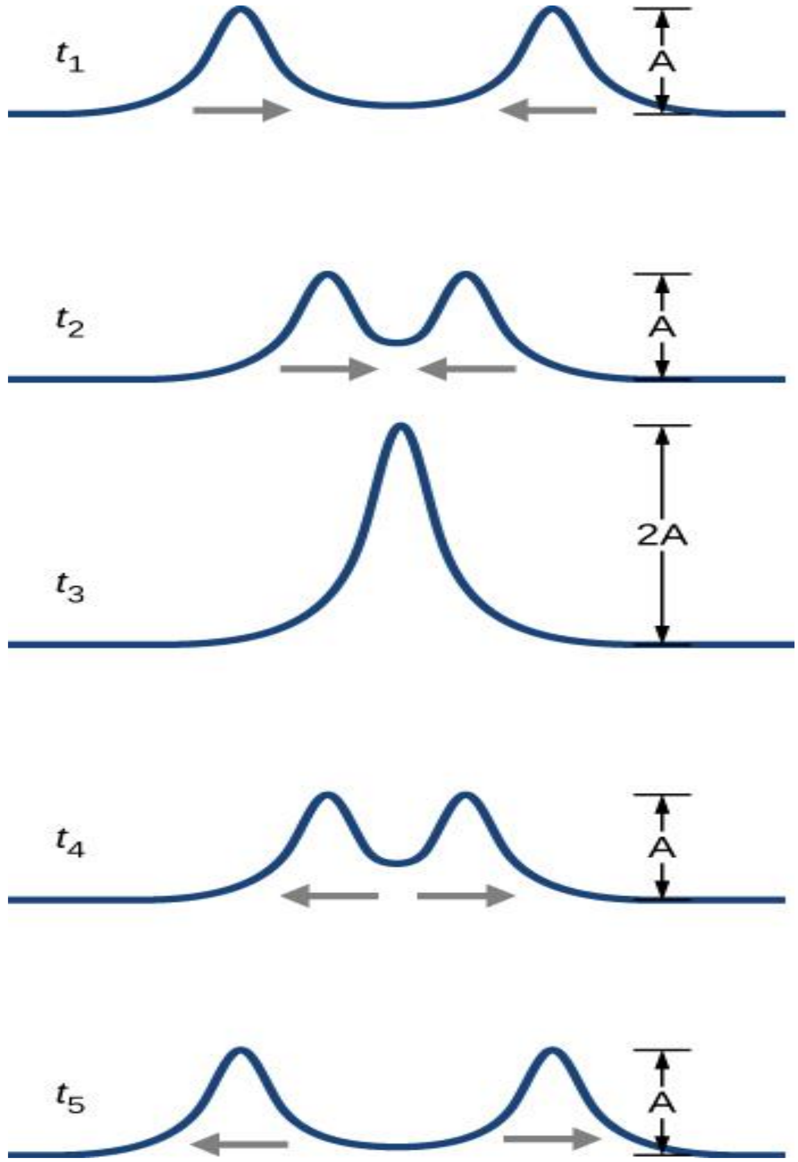
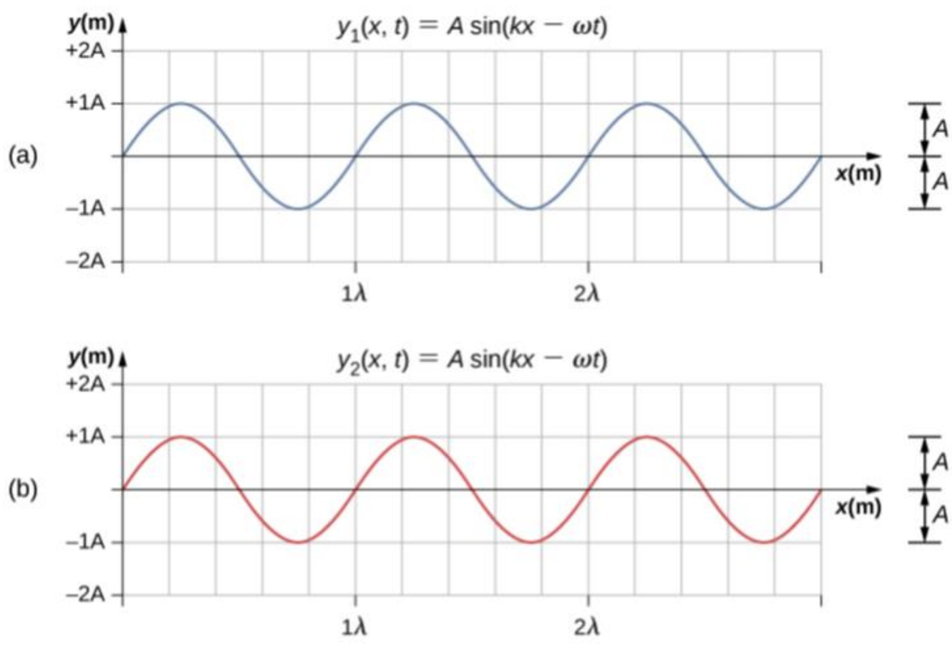


Figure (9)

To analyze the interference of two or more waves, we use the principle of superposition. **For mechanical waves, the principle of superposition states that if two or more traveling waves combine at the same point, the resulting position of the mass element of the medium, at that point, is the algebraic sum of the position due to the individual waves.** This property is exhibited by many waves observed, such as waves on a string, sound waves, and surface water waves. Electromagnetic waves also obey the superposition principle, but the electric and magnetic fields of the combined wave are added instead of the displacement of the medium. Waves that obey the superposition principle are **linear waves**; waves that do not obey the superposition principle are said to be **nonlinear waves**.

Mechanical waves that obey superposition are normally restricted to waves with amplitudes that are small with respect to their wavelengths. If the amplitude is too large, the medium is distorted past the region where the restoring force of the medium is linear.

Waves can interfere **constructively** or **destructively**. Figure (10) shows two identical sinusoidal waves that arrive at the same point exactly in phase. Figure 10(a) and (b) show the two individual waves, and (c) shows the resultant wave that results from the algebraic sum of the two linear waves. The crests of the two waves are precisely aligned, as are the troughs. This superposition produces **constructive interference**. Because the disturbances add, constructive interference produces a wave that has twice the amplitude of the individual waves, but has the same wavelength.



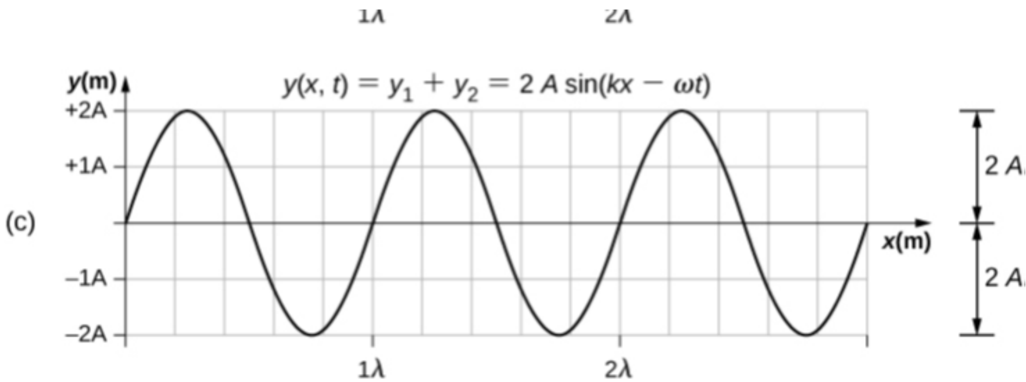
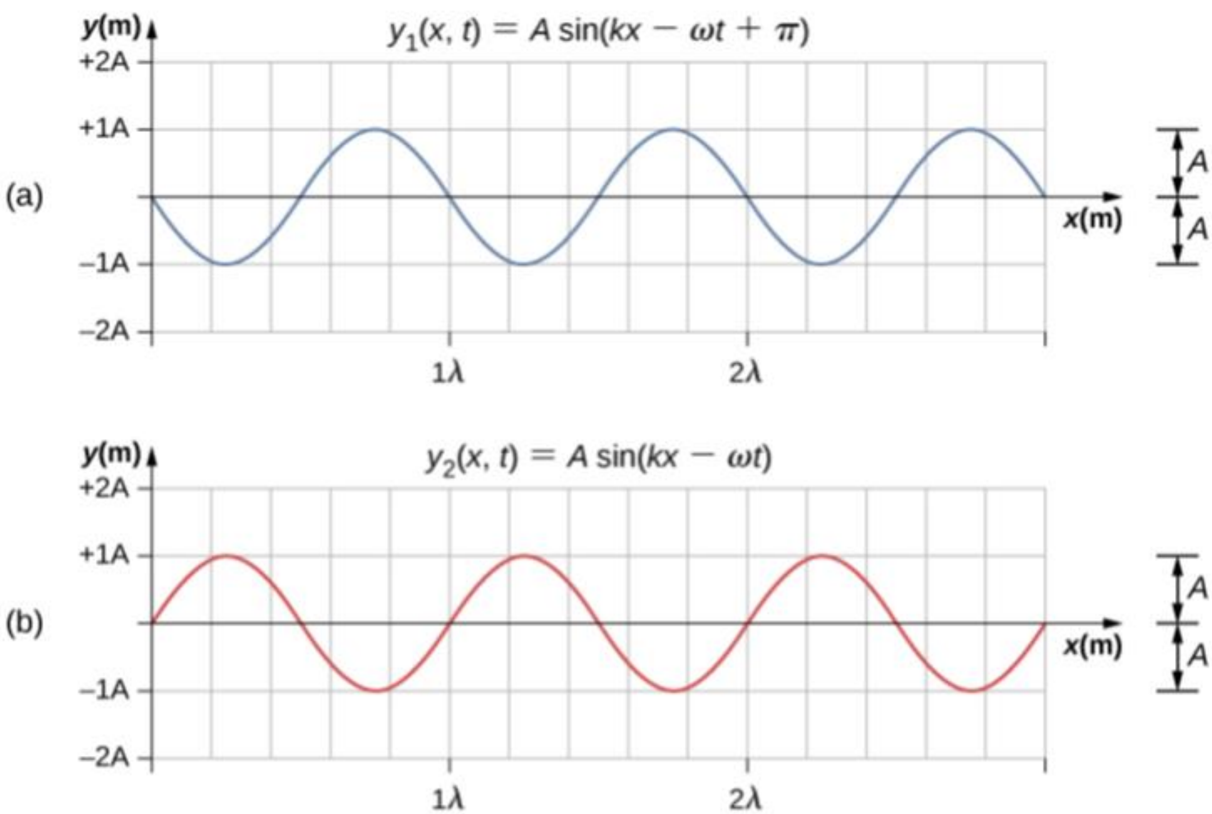


Figure 10 Constructive interference of two identical waves produces a wave with twice the amplitude, but the same wavelength.

Figure (11) shows two identical waves that arrive exactly 180° out of phase, producing **destructive interference**. Figure 11(a) and (b) show the individual waves, and Figure 11(c) shows the superposition of the two waves. Because the troughs of one wave add the crest of the other wave, the resulting amplitude is zero for destructive interference—the waves completely cancel.



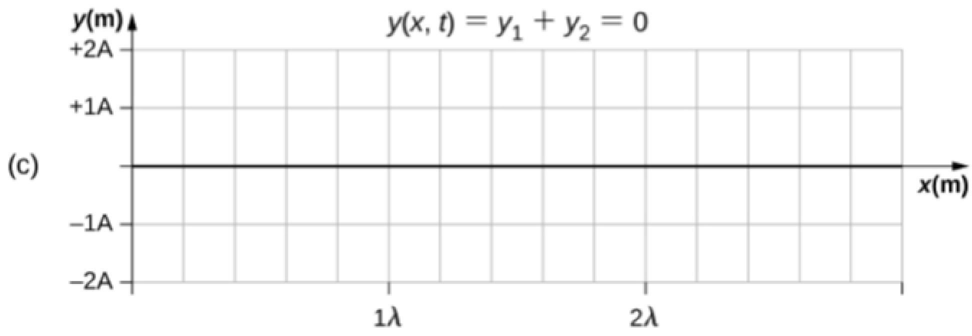


Figure 11 Destructive interference of two identical waves, one with a phase shift of 180°(π rad) , produces zero amplitude, or complete cancellation.

*When linear waves interfere, the resultant wave is just the algebraic sum of the individual waves as stated in the principle of superposition.

* When two linear waves in the same medium interfere, the height of a resulting wave is the sum of the heights of the individual waves, taken point by point.

*The superposition of most waves produces a combination of constructive and destructive interference and can vary from place to place and time to time.

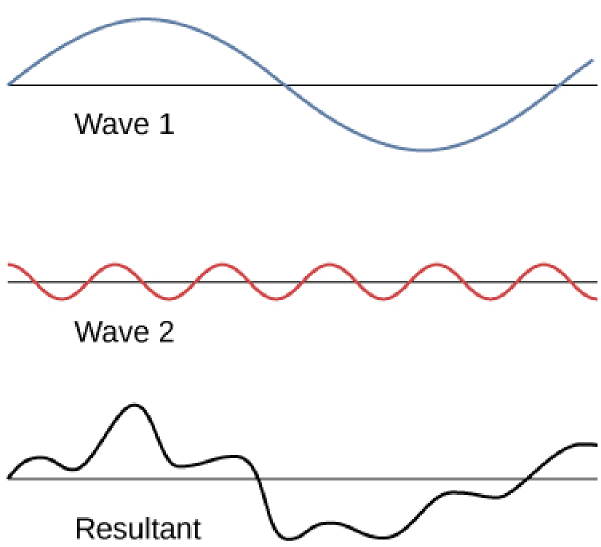


Figure 12 Superposition of nonidentical waves exhibits both constructive and destructive interference.

Superposition of Sinusoidal Waves that Differ by a Phase Shift

Many examples in physics consist of two sinusoidal waves that are identical in amplitude, wave number, and angular frequency, but differ by a phase shift:

$$y_1(x,t) = A\sin(kx - \omega t + \phi),$$
$$y_2(x,t) = A\sin(kx - \omega t).$$

When these two waves exist in the same medium, the resultant wave resulting from the superposition of the two individual waves is the sum of the two individual waves:

$$y_R(x,t) = y_1(x,t) + y_2(x,t) = A\sin(kx - \omega t + \phi) + A\sin(kx - \omega t)$$

by using the trigonometric identity:

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

where $u = kx - \omega t + \phi$ and $v = kx - \omega t$. The resulting wave becomes

$$\begin{aligned} y_R(x,t) &= y_1(x,t) + y_2(x,t) = A\sin(kx - \omega t + \phi) + A\sin(kx - \omega t) \\ &= 2A\sin\left(\frac{(kx-\omega t+\phi)+(kx-\omega t)}{2}\right)\cos\left(\frac{(kx-\omega t+\phi)-(kx-\omega t)}{2}\right) \\ &= 2A\sin\left(kx - \omega t + \frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right). \end{aligned}$$

This equation is usually written as

$$y_R(x,t) = \left[2A\cos(\frac{\phi}{2})\right]\sin\left(kx - \omega t + \frac{\phi}{2}\right) \quad (13)$$

The resultant wave has the same wave number and angular frequency, an amplitude of $A_R = \left[2A\cos(\frac{\phi}{2})\right]$,and a phase shift equal to half the original phase shift.

Standing Waves

Under certain conditions, waves can bounce back and forth through a particular region, effectively becoming stationary. These are called **standing waves**.

In the case of standing waves, the relatively large amplitude standing waves are produced by the superposition of smaller amplitude component waves.

If the two waves have the same amplitude and wavelength, then they alternate between constructive and destructive interference. The resultant looks like a wave standing in place and, thus, is called a standing wave.`

Consider two identical waves that move in opposite directions. The first wave has a wave function of $\boldsymbol{y_1(x,t) = A \sin(kx - \omega t)}$ and the second wave has a wave function $\boldsymbol{y_2(x,t) = A \sin(kx + \omega t)}$. The waves interfere and form a resultant wave

$$y(x,t) = y_1(x,t) + y_2(x,t),$$
$$y(x,t) = A\sin(kx - \omega t) + A\sin(kx + \omega t)$$

This can be simplified using the trigonometric identity

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$

where $\alpha = kx$ and $\beta = \omega t$, giving us

$$y(x,t) = A[\sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t) + \sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega$$

which simplifies to

$$y(x,t) = [2A\sin(kx)]\cos(\omega t) \qquad (14)$$

Initially, at time t=0, the two waves are in phase, and the result is a wave that is twice the amplitude of the individual waves. The waves are also in phase at the time $t = \frac{T}{2}$ .In fact, the waves are in phase at any integer multiple of half of a period:

$$t = n\frac{T}{2} \text{ where } n = 0, 1, 2, 3.... \text{ (in phase).}$$

At other times, the two waves are 180° (π radians) out of phase, and the resulting wave is equal to zero. This happens at

$$t = \frac{1}{4}T, \frac{3}{4}T, \frac{5}{4}T, ..., \frac{n}{4}T \text{ where } n = 1, 3, 5.... \text{ (out of phase).}$$

Notice that some x-positions of the resultant wave are always zero no matter what the phase relationship is.
These positions are called **nodes**. Consider the solution to the sum of the two waves

$$y(x, t) = [2A \sin (kx)]\cos (\omega t)$$

Finding the positions where the sine function equals zero provides the positions of the nodes.

$$\begin{aligned} \sin (kx) &= 0 \\ kx &= 0, \pi, 2\pi, 3\pi, ... \\ \frac{2\pi}{\lambda}x &= 0, \pi, 2\pi, 3\pi, ... \\ x &= 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, ... = n\frac{\lambda}{2} \quad n = 0, 1, 2, 3, ... \end{aligned}$$

There are also positions where y oscillates between y=±A. These are the **antinodes**. We can find them by considering which values of x result in $\sin(kx) = \pm 1$.

$$\begin{aligned} \sin (kx) &= \pm 1 \\ kx &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ... \\ \frac{2\pi}{\lambda}x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ... \\ x &= \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, ... = n\frac{\lambda}{4} \quad n = 1, 3, 5, ... \end{aligned}$$

The resulting wave appears to be a sine wave with nodes at integer multiples of half wavelengths. The antinodes oscillate between y=±2A due to the cosine term, cos(ωt), which oscillates between ±1.
A common example of standing waves is the waves produced by stringed musical instruments.
The symmetrical boundary conditions (a node at each end) dictate the possible frequencies that can excite standing waves. Starting from a frequency of zero and slowly increasing the frequency, the first mode n=1 appears as shown in Figure 13. The first mode, also called the fundamental mode or the first harmonic, shows half of a wavelength has formed, so the wavelength is equal to twice the length between the nodes $\lambda_1 = 2L$. The **fundamental frequency**, or first harmonic frequency, that drives this mode is

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L},$$

where the speed of the wave is $v = \sqrt{\frac{F_T}{\mu}}$ .Keeping the tension constant and increasing the frequency leads to the second harmonic or the mode n=2. This mode is a full wavelength $\lambda_2 = L$ and the frequency is twice the fundamental frequency:

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2f_1 .$$

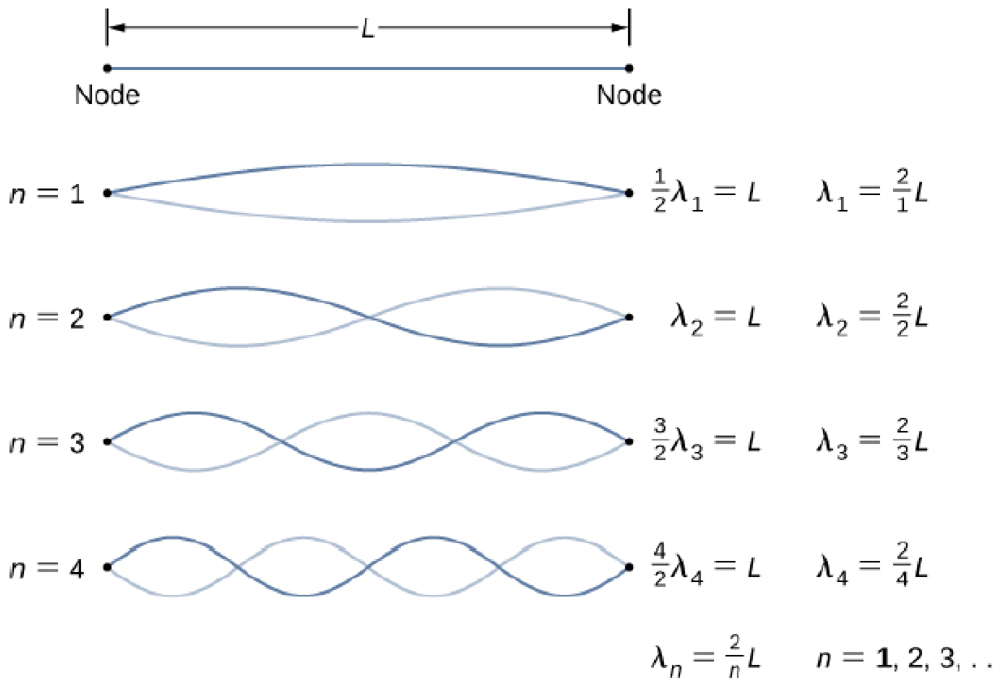


Figure 13 Standing waves created on a string of length L. A node occurs at each end of the string. The nodes are boundary conditions that limit the possible frequencies that excite standing waves. The standing wave patterns possible on the string are known as the normal modes.

The next two modes, or the third and fourth harmonics, have wavelengths of $\lambda_3 = \frac{2}{3}L$and $\lambda_4 = \frac{2}{4}L$ driven by frequencies of $f_3 = \frac{3v}{2L} = 3f_1$and $f_4 = \frac{4v}{2L} = 4f_1$ All frequencies above the frequency f_1 are known as the **overtones**. The equations for the wavelength and the frequency can be summarized as:

$$\lambda_n = \frac{2}{n}L \qquad n = 1,2,3,4,5 \ldots \qquad (15)$$

$$f_n = n\frac{v}{2L} = nf_1 \qquad n = 1,2,3,4,5 \ldots. \qquad (16)$$

The standing wave patterns that are possible for a string, the first four of which are shown in Figure 13, are known as the **normal modes**, with frequencies known as the normal frequencies. In summary, the first frequency to produce a normal mode is called the fundamental frequency (or first harmonic). Any frequencies above the fundamental frequency are overtones. The second frequency of the n=2 normal mode of the string is the first overtone (or second harmonic). The frequency of the n=3 normal mode is the second overtone (or third harmonic) and so on.

Note// The solutions shown as Eq. 15 and Eq 16 are for a string with the boundary condition of a node on each end. When the boundary condition on either side is the same, the system is said to have symmetric boundary conditions. Eq.15 and Eq.16 are good for any symmetric boundary conditions, that is, nodes at both ends or antinodes at both ends.

Example.7// **Standing Waves on a String**

Consider a string L=2m of attached to an adjustable-frequency string vibrator as shown in Figure 14. The waves produced by the vibrator travel down the string and are reflected by the fixed boundary condition at the pulley. The string, which has a linear mass density μ=0.006 kg/m of is passed over a frictionless pulley of a negligible mass, and the tension is provided by a 2.00-kg hanging mass. (a) What is the velocity of the waves on the string? (b) Draw a sketch of the first three normal modes of the standing waves that can be produced on the string and label each with the wavelength. (c) List the frequencies that the string vibrator must be tuned to in order to produce the first three normal modes of the standing waves.

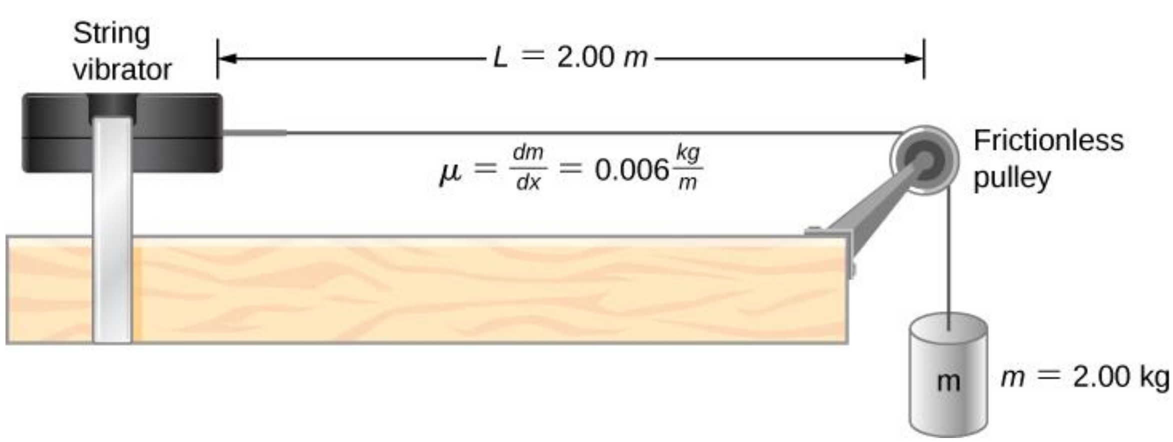


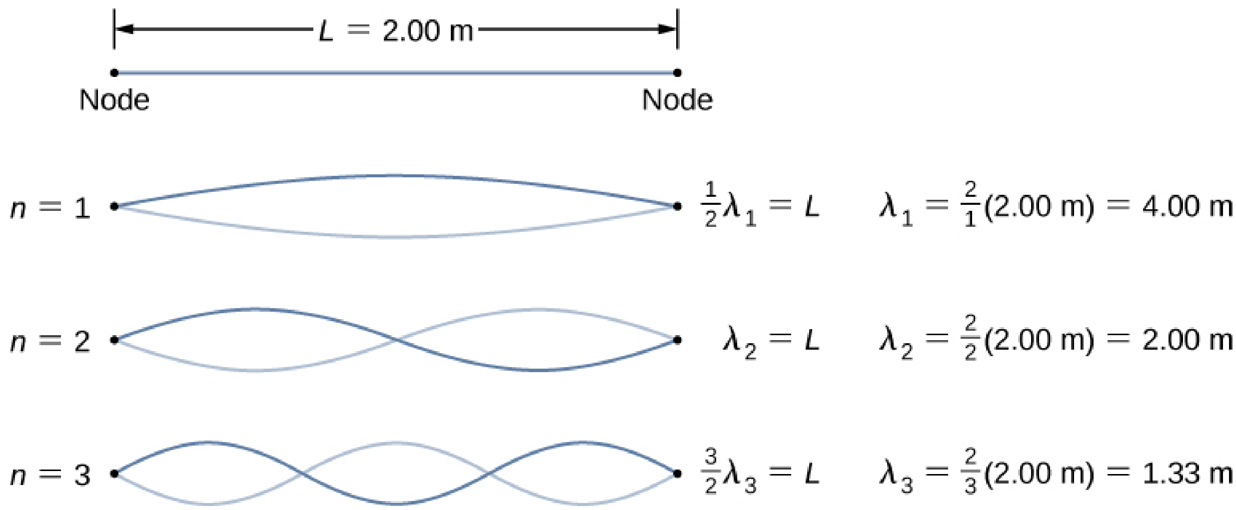
Figure 14: string attached to an adjustable-frequency string vibrator.

Solution

a. Begin with the velocity of a wave on a string. The tension is equal to the weight of the hanging mass. The linear mass density and mass of the hanging mass are given:

$$v=\sqrt{\frac{F_T}{\mu}}=\sqrt{\frac{mg}{\mu}}=\sqrt{\frac{2\,\text{kg}\left(9.8\,\frac{\text{m}}{\text{s}}\right)}{0.006\,\frac{\text{kg}}{\text{m}}}}=57.15\text{ m/s}$$

b. The first normal mode that has a node on each end is a half wavelength. The next two modes are found by adding a half of a wavelength.



c. The frequencies of the first three modes are found by using $f=\frac{v\omega}{\lambda}$

$$f_1=\frac{v_w}{\lambda_1}=\frac{57.15\text{ m/s}}{4.00\text{ m}}=14.29\text{ Hz}$$
$$f_2=\frac{v_w}{\lambda_2}=\frac{57.15\text{ m/s}}{2.00\text{ m}}=28.58\text{ Hz}$$
$$f_3=\frac{v_w}{\lambda_3}=\frac{57.15\text{ m/s}}{1.333\text{ m}}=42.87\text{ Hz}$$

H.w //Consider the experimental setup shown below. The length of the string between the string vibrator and the pulley is L=1 m. The linear density of the string is μ=0.006 kg/m. The string vibrator can oscillate at any frequency. The hanging mass is 2.00 kg. (a)What are the wavelength and frequency of n=6 mode? (b) The string oscillates the air around the string. What is the wavelength of the sound if the speed of the sound is v=343 m/s.