Exponential functions, Trigonometric functions, Hyperbolic function:

1-Exponential functions and power

Exponential are defined by ρe^{iy} **as**

$$W = e^{z} = e^{x+iy} = e^{x}.e^{iy} = e^{x}(\cos y + i\sin y)$$

$$\operatorname{Re}(e^{z}) = e^{x}\cos y$$

$$\operatorname{Im}(e^{z}) = e^{x}\sin y$$

$$\left|e^{z}\right| = \left|e^{x}.e^{iy}\right| = \left|e^{x}\right|.\left|e^{iy}\right| = e^{x}$$

If a is any positive number, then we define

$$W = a^{z} = e^{\ln a^{z}} = e^{z\ln a}$$

Exercise 1:

Find the values of z that satisfies the following equations

(1) $e^{z} = -e^{2}$, (2) $e^{z} = \sqrt{e}\,\dfrac{(1+i)}{\sqrt{2}}$

(1)

$$e^{z} = e^{2}(-1+i0) = e^{2}(\cos\pi + i\sin\pi) = e^{2}.e^{i(\pi+2\pi n)} = e^{2+i(\pi+2\pi n)}$$

$$\Rightarrow z = 2 + i(\pi + 2\pi n)$$

(2)

$$e^z = e^{1/2}\frac{(1+i)}{\sqrt{2}} = e^{1/2}(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}) = e^{1/2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$$

$$\therefore e^z = e^{1/2}.e^{\frac{i\pi}{4}} = e^{\frac{1}{2}+i\frac{\pi}{4}} \Rightarrow\Rightarrow z = \frac{1}{2}+i\frac{\pi}{4}$$

<u>Exercise 2: Find the real part and imaginary part of</u> $[\frac{e}{2}(-1-i\sqrt{3}]^{3\pi}$.

$$[\frac{e}{2}(1-i\sqrt{3})]^{3\pi} = e^{3\pi\ln[\frac{e}{2}(-1-i\sqrt{3})]} = e^{3\pi\ln[e(-\frac{1}{2}-i\frac{\sqrt{3}}{2})]}$$

$$\ln[e(-\frac{1}{2}-i\frac{\sqrt{3}}{2}) = 1+\ln(-\frac{1}{2}-i\frac{\sqrt{3}}{2})$$

$$3\pi\ln[e(-\frac{1}{2}-i\frac{\sqrt{3}}{2}) = 3\pi+3\pi\ln(-\frac{1}{2}-i\frac{\sqrt{3}}{2})$$

$$Let \qquad z = (x+iy) = -\frac{1}{2}-i\frac{\sqrt{3}}{2} = r(\cos\theta+i\sin\theta)$$

$$x = r\cos\theta = -\frac{1}{2} \qquad , \qquad y = r\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow r=1$$

$$\theta = \tan^{-1}\sqrt{3} = -\frac{\pi}{3} \qquad , \qquad \theta \qquad must \qquad be \qquad \frac{4\pi}{3}$$

$$\therefore z = e^{i(\frac{4\pi}{3}+2\pi n)}$$

$$\ln z = \ln(-\frac{1}{2}-i\frac{\sqrt{3}}{2}) = i(\frac{4\pi}{3}+2\pi n)$$

$\therefore$

$$[\frac{e}{2}(1-i\sqrt{3})]^{3\pi} = e^{3\pi+3\pi\ln(-\frac{1}{2}-i\frac{\sqrt{3}}{2})} = e^{3\pi+i\pi^2(4+6n)}$$

Exercise 1: Find the principle values of $(1+i\sqrt{3})^{(1+i)}$

$$(1+i\sqrt{3})^{(1+i)} = e^{(1+i)\ln(1+\sqrt{3})}$$

$$let \qquad z = x+iy = (1+\sqrt{3}) = r(\cos\theta + i\sin\theta)$$

$$x = r\cos\theta = 1, \quad y = r\sin\theta = \sqrt{3}, \quad \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}, \quad r = 2$$

$$(1+i\sqrt{3}) = 2[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}] = 2e^{i\frac{\pi}{3}}$$

$$\ln(1+i\sqrt{3}) = \ln 2 + i(\frac{\pi}{3} + 2\pi n)$$

$$(1+i)\ln(1+\sqrt{3}) = (1+i)[\ln 2 + i(\frac{\pi}{3} + 2\pi n)]$$

$$= [\ln 2 + (\frac{\pi}{3} + 2\pi n) + i[\ln 2 + (\frac{\pi}{3} + 2\pi n)]$$

$$\therefore (1+i\sqrt{3})^{(1+i)} = e^{(1+i)\ln(1+\sqrt{3})} = e^{[\ln 2 - (\frac{\pi}{3} + 2\pi n)]}.e^{i[\ln 2 + (\frac{\pi}{3} + 2\pi n)]}$$

$$= e^{[\ln 2 - (\frac{\pi}{3} + 2\pi n)]}.\{\cos[\ln 2 + (\frac{\pi}{3} + 2\pi n)] + i\sin[\ln 2 + (\frac{\pi}{3} + 2\pi n)]\}$$

$$\therefore \mathrm{Re}[(1+i\sqrt{3})^{(1+i)}] = e^{[\ln 2 - (\frac{\pi}{3} + 2\pi n)]}.\cos[\ln 2 + (\frac{\pi}{3} + 2\pi n)]$$

$$\mathrm{Im}[(1+i\sqrt{3})^{(1+i)}] = e^{[\ln 2 - (\frac{\pi}{3} + 2\pi n)]}.\sin[\ln 2 + (\frac{\pi}{3} + 2\pi n)]$$

2-Trigonometric function

We define trigonometric function or circular function as fallows:

1. $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

2. $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

3. $\tan z = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$

4. $\csc z = \frac{1}{sinz} = \frac{2i}{e^{iz} - e^{-iz}}$

5. $\csc z = \frac{1}{cosz} = \frac{2}{e^{iz} + e^{-iz}}$

6. $sin^2 z + cos^2 z = 1$

7. $1 + tan^2 z = sec^2 z$

8. $1 + cot^2 z = csc^2 z$

9. $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

10. $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

11. $\tan(z_1 \pm z_2) = \frac{tanz_1 \pm tanz_2}{1 \mp tanz_1 tanz_2}$

Prove (1)

$e^{iz} = cosz + i\ sinz$

$e^{-iz} = cosz - i\, sinz$

$e^{iz} - e^{iz} \ = 2i\, sin$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Prove (6)

$$sin^2 z + cos^2 z = (\frac{e^{iz} - e^{-iz}}{2i})^2 + (\frac{e^{iz} + e^{-iz}}{2})^2$$

$$sin^2 z + cos^2 z = -(\frac{e^{2iz} - 2 + e^{-2iz}}{4}) + \frac{e^{2iz} + 2 + e^{-2iz}}{4}$$

$$sin^2 z + cos^2 z = (\frac{-e^{2iz} + 2 - e^{-2iz} + e^{2iz} + 2 + e^{-2iz}}{4})$$

$$sin^2 z + cos^2 z = \frac{4}{4} = 1$$

3-Hyperbolic function

1. $\sinh z = \frac{e^z - e^{-z}}{2}$

2. $\cosh z = \frac{e^z + e^{-z}}{2}$

3. $\tanh z = \frac{e^z - e^{-z}}{(e^z + e^{-z})}$

4. $cosh^2 z - sinh^2 z = 1$

5. $1 - tanh^2 z = sech^2 z$

6. $coth^2 z - 1 = csch^2 z$

7. $\sin(\mathrm{i}\, z) = i\ sinhz$

8. $\cos(\mathrm{i}\, z) = coshz$

9. $\tan(\mathrm{i}\, z) = i\ tanhz$

10. $\tanh(\mathrm{i}\, z) = i\ tanz$

4. **Inverse of Trigonometric and Hyperbolic function functions**

1. $\sin^{-1} z = -i\, ln[iz + \sqrt{1 - z^2}\,]$

2. $\cos^{-1} z = -i\, ln[z + \sqrt{z^2 - 1}\,]$

3. $\tan^{-1} z = \frac{i}{2} \ln[\,\frac{(i+z)}{(i-z)}\,]$

4. $\sinh^{-1} z = ln[z + \sqrt{z^2 + 1}\,]$

5. $\cosh^{-1} z = \ ln[z + \sqrt{z^2 - 1}\,]$

6. $\tanh^{-1} z = \frac{1}{2} \ln[\,\frac{(1+z)}{(1-z)}\,]$

Prove (1)

$z = sinw = \frac{1}{2i}(e^{iw} - e^{-iw})$……………………… 1

By multiplying both side of equation (1) by e^{iw} and rearranged the equation (1) we get:

$e^{2iw} - 2iz\, e^{iw} - 1 = 0$ ……………………..(2)

We can solve equation (2) by using quadratic formula we get:

$$e^{iw} = \frac{2iz \pm \sqrt{-4z^2 + 4}}{2}$$

Taking the + sine we get:

$$iw = \ln(iz + \sqrt{1 - z^2})$$

$$w = \frac{1}{i}\ln(iz + \sqrt{1 - z^2}) = -i\ \ln(iz + \sqrt{1 - z^2})$$

<u>**Homework**</u>

1. $\cos^{-1} z = -i\ ln[z + \sqrt{z^2 - 1}\ ]$

2. $\tan^{-1} z = \frac{i}{2} \ln[\ \frac{(i+z)}{(i-z)}\ ]$

3. $\sinh^{-1} z = ln[z + \sqrt{z^2 + 1}\ ]$

4. $\cosh^{-1} z = \ ln[z + \sqrt{z^2 - 1}\ ]$

5. $\tanh^{-1} z = \frac{1}{2} \ln[\ \frac{(1+z)}{(1-z)}\ ]$

6. $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

7. $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

8. $\tan(z_1 \pm z_2) = \frac{tanz_1 \pm tanz_2}{1 \mp tanz_1 tanz_2}$

9. $\sin(\mathrm{i}\, z) = i\ sinhz$

10. $\cos(\mathrm{i}\, z) = coshz$

11. $\tan(\mathrm{i}\, z) = i\ tanhz$

12. $\tanh(\mathrm{i}\, z) = i\ tanz$