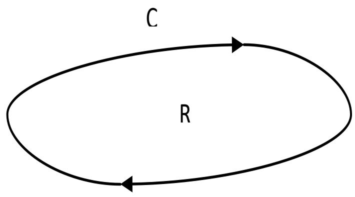
4.Cauchy's theorem

Let C be a simple closed curve. If $f(z)$ is analytic function within the region bounded by C as well as on C then we have Cauchy's theorem that



4.1 $$\int_C f(z)dz \equiv \oint_C f(z)dz = 0$$

Proof:

$$\oint_C f(z)dz = \oint_C (u+iv)(dx+idv) = \oint_C udx - vdy + i\oint_C vdx + udy$$

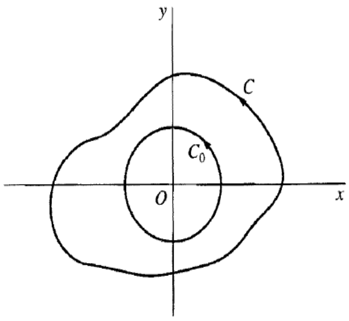
By using Green's theorem $\oint_C pdx + Qdy = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$, we get:

$$\oint_C udx - vdy = \iint_R (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dxdy \quad , \quad \oint_C vdx - udy = \iint_R (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dxdy$$

Where R is the region (simply-connected) bounded by C. Since $f(z)$ is analytic function within the region bounded by C, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, and so the above integrals are zero.

Then $\int_C f(z)dz \equiv \oint_C f(z)dz = 0$**, assuming $f'(z)$ to be continuous.**

<u>Theorem 2</u> : If $f(z)$ is analytic function within and on the boundary of region bounded by two closed curves C_0 and C_1, then;



4.2 $$\oint_{C0} f(z)dz = \oint_{C_1} f(z)dz$$

Prove that

5. Cauchy's integral formulas

If $f(z)$ is analytic function within and on a simple closed curve C and z_0 is any interior to C, then

5.1 $$f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

Or

5.2 $$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

Also, the *n*th derivative of $f(z)$ at $z=z_0$ is given by

5.3 $$\frac{d^n}{dz}f(z_0)=\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

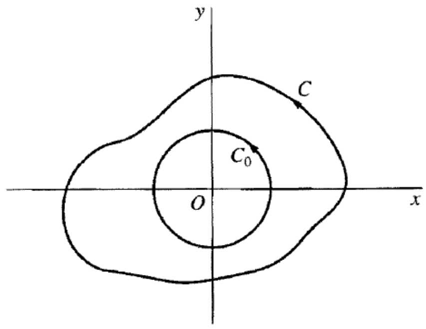
Or

5.4 $$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i\frac{d^n}{dz}f(z_0)}{n!}$$

Example 1.

$$\oint_C \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i f^2(z_0)}{2!} = \pi i\frac{d^2}{dz}f(z_0)$$

Prove (5.1)



Let C_0 be a circle of radius *r* having center at $z = z_0$.

Since $f(z)$ is analytic function within and on the boundary of region bounded by two closed curves C and C_0, then from equation (4.2) we have

$$\oint_C \frac{f(z)}{z - z_0}\, dz = \oint_{C_0} \frac{f(z)}{z - z_0}\, dz$$ **, To evaluate this last integral, not that on C_0,**

$|z - z_0| = r$ **or** $z - z_0 = re^{i\theta}$**, and** $dz = ire^{i\theta} d\theta$**. The integral equals to**

$$\oint_C \frac{f(z)}{z - z_0}\, dz = \oint \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\, ire^{i\theta} d\theta = i\oint f(z_0 + re^{i\theta}) d\theta$$

As $r \to 0$, $z \to z_0$

$$\oint_C \frac{f(z)}{z - z_0}\, dz = if(z_0)\int_0^{2\pi} d\theta = 2\pi i f(z_0)$$

Example3. Evaluate the integral $\oint_C \frac{z^3}{(z+1)^3} dz$, where C is simple closed curve enclosing $z_0 = -1$

$\oint_C \frac{z^3}{(z+1)^3} dz = \frac{2\pi i}{2} f^2(z_0)$, $\qquad f(z) = z^3 \Rightarrow f'(z) = 3z^2 \Rightarrow f^2(z) = 6z \Rightarrow f^2(-1) = -6$

$\therefore \oint_C \frac{z^3}{(z+1)^3} dz = \frac{2\pi i}{2} f^2(-1) = -6\pi i$

Example 2. Calculate $f(\frac{\pi}{3})$**, where** $f(z)=\oint_C \frac{\tan z}{(z-a)^3}dz$**. Here C is the circle** $|z|=2$**.**

$$f(\frac{\pi}{3})=\oint_C \frac{\tan z}{(z-\frac{\pi}{3})^3}dz, \quad n+1=3 \Rightarrow \quad n=2. \quad f(z)=\tan z$$

$$\therefore \oint_C \frac{\tan z}{(z-\frac{\pi}{3})^3}dz=\frac{2\pi i}{2}\frac{d^2}{dz}f(\frac{\pi}{3})$$

$$f(z)=\tan z \Rightarrow f'(z)=\sec^2 z$$

$$\therefore \frac{d^2}{dz}f(z)=2\sec z[\sec z.\tan z]=2\sec z[\frac{\sin z}{\cos^2 z}]$$

$$\therefore \frac{d^2}{dz}f(\frac{\pi}{3})=2\sec(\pi/3).[\frac{\sin(\pi/3)}{\cos^2(\pi/3)}]=2\frac{1}{1/2}[\frac{\sqrt{3}/2}{1/4}]=8\sqrt{3}$$

$$\therefore f(\frac{\pi}{3})=\oint_C \frac{\tan z}{(z-\frac{\pi}{3})^3}dz=\frac{2\pi i}{2}f''(\frac{\pi}{3})=i\pi 8\sqrt{3}$$