Functions of a complex Variable

1. Functions

If to each of set of complex numbers z, there corresponds one or more value of variable w then w is called a function of the complex variable z, written $w = f(z)$. in general, we can write

$$f(z) = f(x+iy) = u(x,y) + iv(x,y), \tag{1.1}$$

Example. Consider a simple function of z, namely $f(z) = z^2$. We may write

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy = u(x,y) + iv(x,y),$$

where $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.

2. Analytic functions

If $f(z)$ is single-valued and continuous in some region of z plane the derivative of $f(z)$, denoted by $f'(z)$, is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{2.1}$$

Provided the limit exists independent of the manner in which $\Delta z \to 0$. If the limit exists for $z = z_0$, then the function $f(z)$ is called analytic at z_0. if the limit exists for all z values in a region R, then $f(z)$ is called analytic function in R.

Example 1. Show that $(d/dz)(z^2) = 2z$. By (2.1) we have

$$\frac{d}{dz}(z^2) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z}$$
$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$

We see that the result is independent of *how* Δz tends to zero; thus z^2 is an analytic function. By the same method it follows that $f'(z) = \frac{d}{dz}(z)^n = nz^{n-1}$ if *n* is a positive integer.