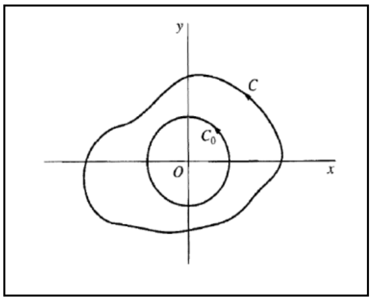
Exercises. Cauchy's integral formulas



If $f(z)$ is analytic function within and on a simple closed curve C

and z_0 is any interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\, dz \quad or \quad \oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) \quad \textbf{(5.1)}$$

$$\frac{d^n}{dz^2} f(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \quad or \quad \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{2\pi i \frac{d^n}{dz^2} f(z_0)}{n!}$$

Example3. Evaluate the integral $\oint_C \frac{z^3}{(z+1)^3} dz$, where C is simple closed curve (C is circle $|z-1| = 3$).

$|z-1| = 3 \Rightarrow \sqrt{(x-1)^2 + y^2} = 3$, **So C is enclosing** $z_0 = -1$ **or** $z_0 = -1$ **is inside C**

$$\oint_C \frac{z^3}{(z+1)^3} dz = \frac{2\pi i}{2} f^2(z_0), \qquad f(z) = z^3 \Rightarrow f'(z) = 3z^2 \Rightarrow \frac{d^2}{dz^2} f(z) = 6z \Rightarrow \frac{d^2}{dz} f^2(-1) = -6$$

$$\therefore \oint_C \frac{z^3}{(z+1)^3} dz = \frac{2\pi i}{2}\frac{d^2}{dz^2} f^2(-1) = -6\pi i$$

Example 2. Evaluate the value of $\oint_C \frac{zdz}{z-3}$ where C is $|z+i| = 4$

$|z+i| = 4 = \sqrt{x^2 + (y+1)} = 4$(circle with radius equal to 4 and center (0,-1)

$z_0 = 3,$ So as we see that z_0 is inside C

$$\oint_C \frac{zdz}{z-3} = 2\pi i f(z_o) = 6\pi i$$

Example 3. Calculate $f(\frac{\pi}{3})$, **where** $f(z)=\oint_C \frac{\tan z}{(z-a)^3}dz$. **Here C is the circle** $|z|=2$.

$$f(\frac{\pi}{3})=\oint_C \frac{\tan z}{(z-\frac{\pi}{3})^3}dz, \quad n+1=3 \Rightarrow \quad n=2. \quad f(z)=\tan z$$

$$\therefore \oint_C \frac{\tan z}{(z-\frac{\pi}{3})^3}dz=\frac{2\pi i}{2}\frac{d^2}{dz^2}f(\frac{\pi}{3})$$

$$f(z)=\tan z \Rightarrow f'(z)=\sec^2 z$$

$$\therefore \frac{d^2}{dz^2}f(z)=2\sec z[\sec z.\tan z]=2\sec z[\frac{\sin z}{\cos^2 z}]$$

$$\therefore \frac{d^2}{dz^2}f(\frac{\pi}{3})=2\sec(\pi/3).[\frac{\sin(\pi/3)}{\cos^2(\pi/3)}]=2\frac{1}{1/2}[\frac{\sqrt{3}/2}{1/4}]=8\sqrt{3}$$

$$\therefore f(\frac{\pi}{3})=\oint_C \frac{\tan z}{(z-\frac{\pi}{3})^3}dz=\frac{2\pi i}{2}\frac{d^2}{dz^2}f(\frac{\pi}{3})=i\pi 8\sqrt{3}$$