EULER'S FORMULA

For real θ, we know from Chapter 1 the power series for $\sin\theta$ and $\cos\theta$:

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots, \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots. \end{aligned} \tag{9.1}$$

From our definition (8.1), we can write the series for e to any power, real or imaginary. We write the series for $e^{i\theta}$, where θ is real:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} \cdots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right). \end{aligned} \tag{9.2}$$

(The rearrangement of terms is justified because the series is absolutely convergent.) Now compare (9.1) and (9.2); the last line in (9.2) is just $\cos\theta + i\sin\theta$. We then have the very useful result we introduced in Section 3, known as Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{9.3}$$

Thus we have justified writing any complex number as we did in (4.1), namely

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{9.4}$$

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}.$$

$$\begin{aligned} z_1 \cdot z_2 &= r_1\, e^{i\theta_1} \cdot r_2\, e^{i\theta_2} = r_1\, r_2\, e^{i(\theta_1+\theta_2)}, \\ z_1 \div z_2 &= \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}. \end{aligned}$$