Chapter one

The Slop and The equation of a Straight line

1.The distance between two point

The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the following:

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} \qquad (1)$$

$$\Delta x = x_2 - x_1 , \; \Delta y = y_2 - y_1$$

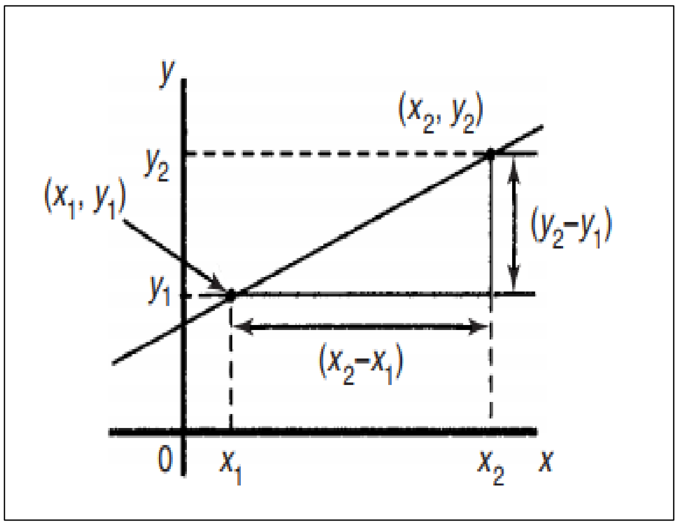


Figure 1

Example: Calculate the distance between the point (-1, 2) and (2, -2).

Solution:

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(2-(-1))^2 + (-2-2)^2} = \sqrt{(3)^2 + (4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

2- The Slop of a Straight Line

If a straight line is not parallel to the y-axis (x=0), and the straight line graph passing through co-ordinates $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ then the **slope** of a straight line is the ratio of the change in the value of y to the change in the value of x between any two points on the line and is given by:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{………………….. ……………………………} \quad (2)$$

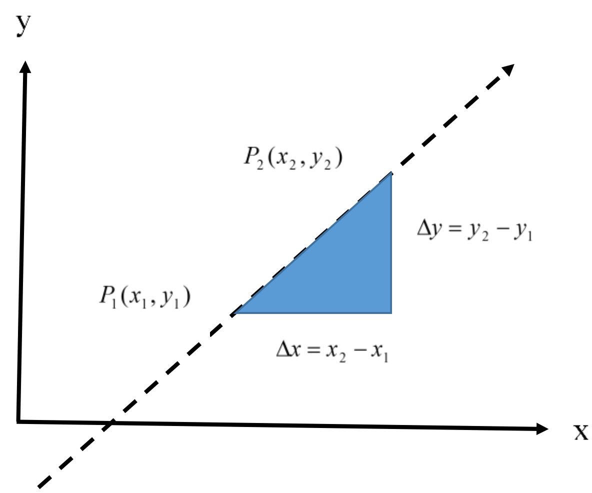


Figure 2

<u>Example:</u> Calculate the slope of the line through the points P$_1$(1,2) and P$_2$(3,8).

Solution:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-2}{3-1} = \frac{6}{2} = 3$$

If, as x increases, ($\rightarrow$), y also increases ($\rightarrow$), then the gradient (slope) is positive. If as x increases ($\rightarrow$), y decreases ($\downarrow$), then the slope is negative. See figure (3):

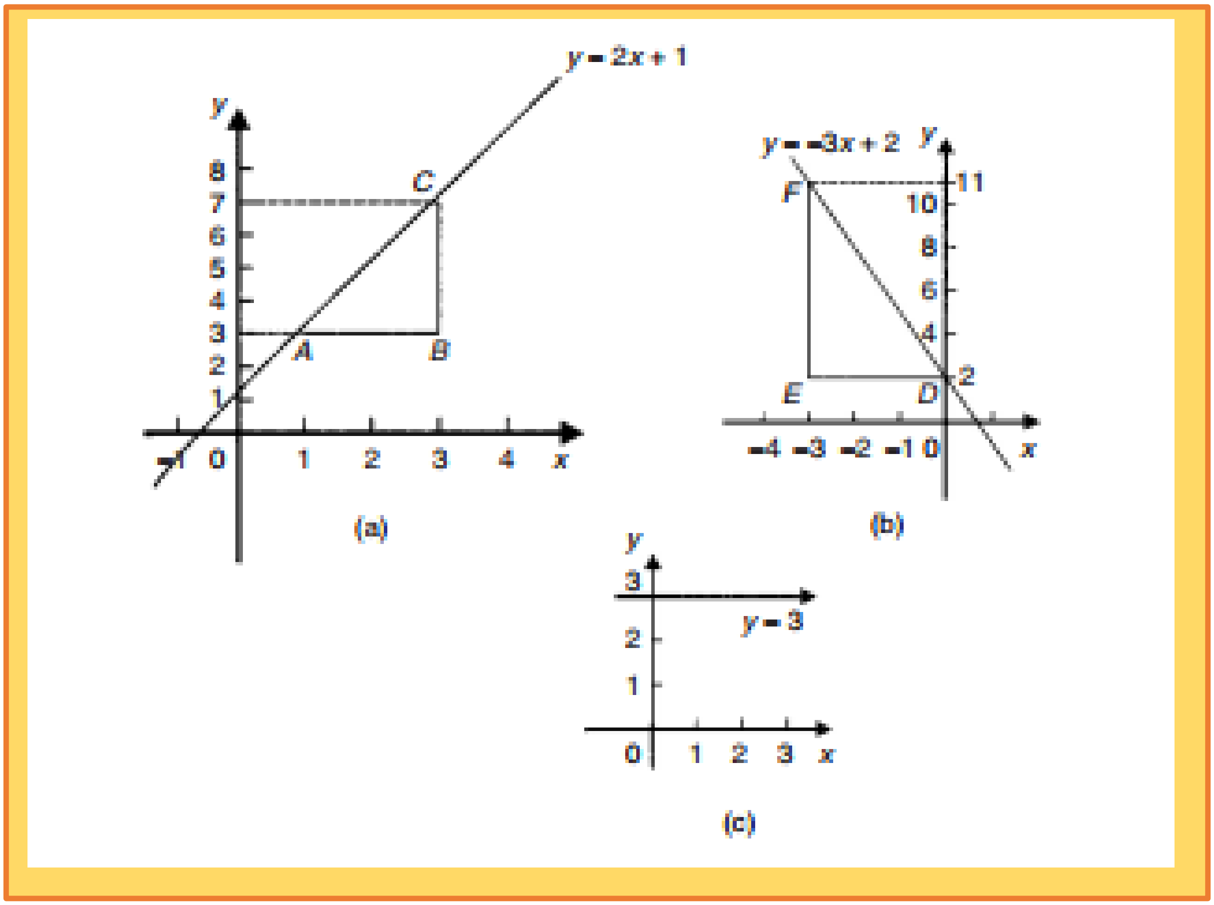


Figure (3)

In figure 3(a), the slope is 2

In figure 3(b), the slope is -3

Figure 3(c) shows a straight line graph y = 3. Since the straight line is horizontal the gradient (slope) is zero.

The value of y when x = 0 is called the *y*-axis intercept. In Fig. .3(a) the y-axis intercept is 1 and in Fig. .3(b) is 2.

3- Equation of Straight lines

From equation (2) [$m=\dfrac{\Delta y}{\Delta x}=\dfrac{y-y_1}{x-x_1}$ **], multiplying both side by (** $x-x_1$ **) gives us the more useful equation:**

$$y-y_1 = m(x-x_1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

$$y=mx-mx_1+y_1$$
$$y=mx+(y_1-mx_1)$$

$$y=mx+b\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}$$

Where m is the slope of the line and b is (y_1+mx_1) **, which is a constant in fact (0,b) is the point where the line crosses the y-axis. The number b is called the y-intercept of the line, and Eq. (4) is called the** ***slope- intercept equation of the line.***

Example: Write an equation for the line through the point (1,2) with slope $m=-\dfrac{3}{4}$**. Where does this line cross the y-axis? The x-axis?**

Solution:

$$y-y_1 = m(x-x_1)$$

$$y-2=-\frac{3}{4}(x-1)$$

$$y=-\frac{3}{4}x+\frac{3}{4}+2=-\frac{3}{4}x+\frac{11}{4}$$

To find where the **line cross the y-axis, we set x=0 in equation above and solve for y:**

$$y=-\frac{3}{4}(0)+\frac{11}{4}=\frac{11}{4}$$

To find where the **line cross the x-axis, we set y=0 in equation above and solve for x:**

$$0=-\frac{3}{4}x+\frac{11}{4}\Rightarrow\frac{3}{4}x=\frac{11}{4}\Rightarrow x=\frac{11}{3}$$

Example: $y=5x+2$**, represents a straight line of gradient (slope) 5 and y-axis intercept 2. Similarly,** $y=-3x-4$ **represents a straight line of gradient -3 and y-axis intercept -4.**

For a horizontal line the equation $y = mx + b$ **reduces to** $y = 0.x + b$

or $y = b$**, the equation** $y = -5$ **is the slope- intercept equation of the line that passes the through point (0,-5) with slope m=0.**

Chapter Two

Functions

Given two sets A and B**,** a set with elements that are ordered pairs ***(***x**,** y***),*** where x is an element of A and y is an element of B**,** is a relation from A to B**.** A relation from A to B defines a relationship between those two sets. A function is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the **input**; the element of the second set is called the **output**. Functions are used all the time in mathematics to describe relationships between two sets. For any function, when we know the input, the output is determined, so we say that the output is a function of the input. For example, the area of a square is determined by its side length, so we say that the area (the output) is a function of its side length (the input). The velocity of a ball thrown in the air can be described as a function of the amount of time the ball is in the air. Since functions have so many uses, it is important to have precise definitions and terminology to study them.

Definition

A function ***f*** consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the domain of the function. The set of outputs is called the range of the function.

The concept of a function can be visualized using **Figure 2.1**, **Figure 2.2**, and **Figure 2.3**.

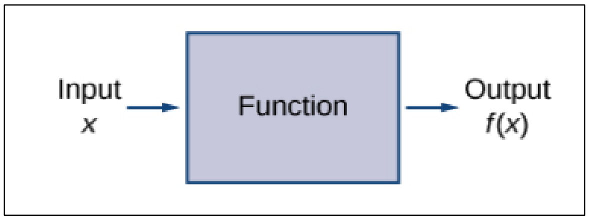


Figure 2.1 A function can be visualized as an input/output device.

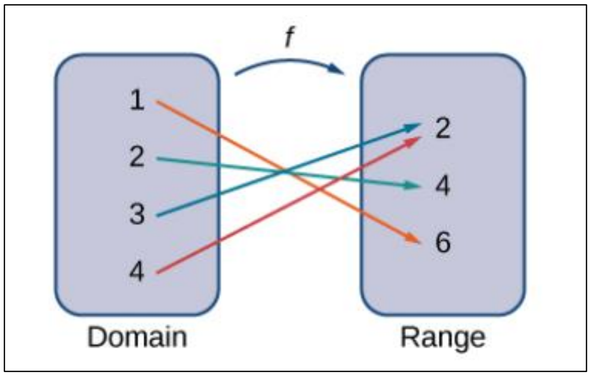


Figure 2.2 A function maps every element in the domain to exactly one element in the range.

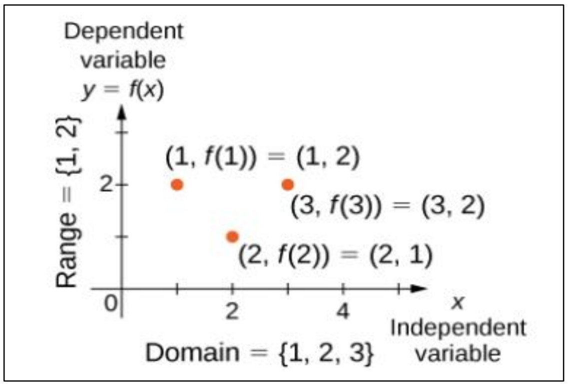


Figure 2.3 In this case, a graph of a function f has a domain of ***{1, 2, 3}*** and a range of ***{1, 2}*.** The independent variable is x and the dependent variable is y**.**

Every function is determined by two things: (1) the domain of the first variable x and (2) the Range or the rule or condition that the pairs (x, y) must satisfy to belong the function.

To graph a function, we carry out three steps.

1.Make a table of pairs from the function.

2.Plot enough of corresponding point to learn the shape of graph. Add more pairs to the table if necessary.

3.Complete the sketch by connecting the points.

Example:

Problem 1. Plot the graph $f(x) = y = 4x + 3$ in the range x = -3 to x = +4. From the graph, find (a) the value of y when x = 2.2, and (b) the value of x when y = -3

Solution:

The domain is taken to be the set of all real numbers *x* for which *f* (*x*) is a real number

D $f(x)$ =R

To find the range we set

$$f(x) = y = 4x + 3 \Rightarrow 4x = y - 3$$

$$\Rightarrow x = \frac{y-3}{4}$$

So that the range is taken to be the set of all real numbers y.

R $f(x)$ =R

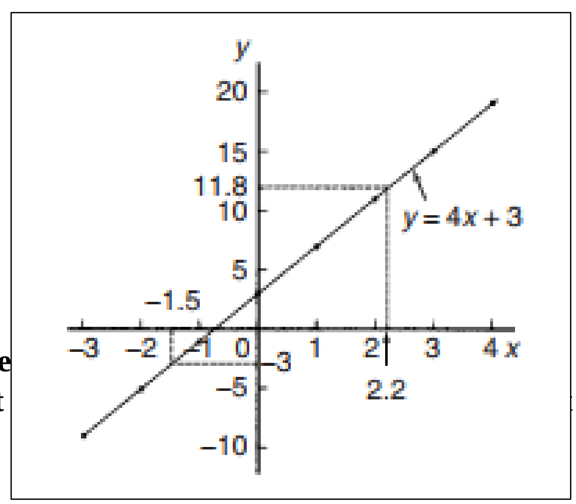
Whenever an equation is given and a graph is required, a table giving corresponding values of the variable is necessary. The table is achieved as follows:

When $x = -3$, $y = 4x + 3 = 4(-3) + 3$
$= -12 + 3 = -9$

When $x = -2$, $y = 4(-2) + 3$
$= -8 + 3 = -5$, and so on.

Such a table is shown below:

x	-3	-2	-1	0	1	2	3	4
y	-9	-5	-1	3	7	11	15	19



(a) when x = 2.2, $y = 11.8$**, and**
(b) when y D = -3, $x = -1.5$

Example: Consider the function

$$f(x) = y = \sqrt{x+3} + 1$$

find the following: 1. The domain. 2.The range. 3. Sketch a graph of $f(x)$

Solution:

1.To find the domain we set

$$\sqrt{x+3} \geq 0 \Rightarrow x+3 \geq 0 \Rightarrow x \geq -3$$

D $f(x) = \{x : x \geq -3\}$

2. To find the range we set

$$y = \sqrt{x+3} + 1$$
$$y - 1 = \sqrt{x+3}$$
$$(y-1)^2 = x+3$$
$$x = (y-1)^2 - 3$$

We just need to verify that x is in the domain of $f(x)$. Since the domain of f consists of all real numbers greater than or equal to -3, and

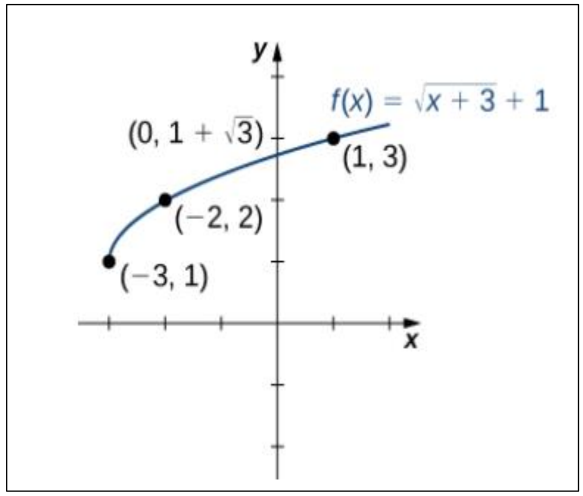
$$x = (y-1)^2 - 3 \geq -3$$

there does exist an x in the domain of $f(x)$. We conclude that the range of f is $\{y : y \geq -1\}$

R $f(x) = \{y : y \geq -1\}$

3. To graph this function, we make a table of values. Since we need x + 3 ≥ 0, we need to choose values of x ≥ -3. We choose values that make the square-root function easy to evaluate.

x	-3	-2	1
$f(x)$	1	2	3



Example: **Consider the function**

$f(x) = y = \sqrt{1-x^2}$

find the following: 1. The domain. 2.The range. 3. Sketch a graph of $f(x)$

Solution:

1.To find the domain we set

$$\sqrt{1-x^2} \geq 0 \Rightarrow 1-x^2 \geq 0$$
$$(1-x)(1+x) \geq 0$$
$$(1-x) \geq 0 \Rightarrow -x \geq -1 \Rightarrow\Rightarrow x < 1$$
$$or$$
$$(1+x) \geq 0 \Rightarrow\Rightarrow x \geq -1$$

So that

D $f(x) = \{x : -1 \leq x \leq 1\}$

2. To find the range we set

$$y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x = \sqrt{1-y^2}$$

$$\sqrt{1-y^2} \geq 0$$

$\sqrt{1-y^2} \geq 0 \Rightarrow 1-y^2 \geq 0$
$(1-y)(1+y) \geq 0$
$(1-y) \geq 0 \Rightarrow -y \geq -1 \Rightarrow\Rightarrow y < 1$
or
$(1+y) \geq 0 \Rightarrow\Rightarrow y \geq -1$

So that

R $f(x) = \{y : -1 \leq y \leq 1\}$

3. Exercise: (is left to the student as homework)

Example. Consider the function $f(x) = y = 1/\sqrt{3-x}$ **, find the following:**

1.The domain. 2.The range. 3. Sketch a graph of $f(x)$

1.To find the domain we set

$\sqrt{3-x} > 0 \Rightarrow -x > -3 \Rightarrow\Rightarrow x < 3$

The set of real number less than 3

D $f(x) = \{x : x < 3\}$

2. To find the range we set

$$y = \frac{1}{\sqrt{3-x}} \Rightarrow \sqrt{3-x} = \frac{1}{y} \Rightarrow 3-x = \frac{1}{y^2} \Rightarrow x = 3-\frac{1}{y^2}$$

We just need to verify that x is in the domain of $f(x)$. Since the domain of f consists of all real numbers **less than 3**, and

$$x = 3-\frac{1}{y^2} < 3$$

there does exist an x in the domain of $f(x)$. We conclude that the range of f is $\{y : y > 0\}$

Chapter Three

Exponential and Logarithmic Functions

1.Exponential

any function of the form $f(x) = b^x$, where $b > 0$, $b \neq 1$ is an exponential function with base b and exponent x . Exponential functions have constant bases and variable exponents. Note that a function of the form $f(x) = x^b$ for some constant b is not an exponential function but a power function. To see the difference between an exponential function and a power function, we compare the functions $y = x^2$ and $y = 2^x$ In Table 1, we see that both $y = 2^x$ and $y = x^2$ approach infinity as $x \to \infty$. Eventually, however, $y = 2^x$ becomes larger than $y = x^2$ and grows more rapidly as x → ∞.In the opposite direction, as x → -∞, $y = x^2 \to \infty$ whereas $y = 2^x \to 0$.

x	−3	−2	−1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
2^x	1/8	1/4	1/2	1	2	4	8	16	32	64

Table 1. Values of x^2 and 2^x

In Figure 1, we graph both $y = x^2$ and $y=2^x$ to show how the graphs differ.

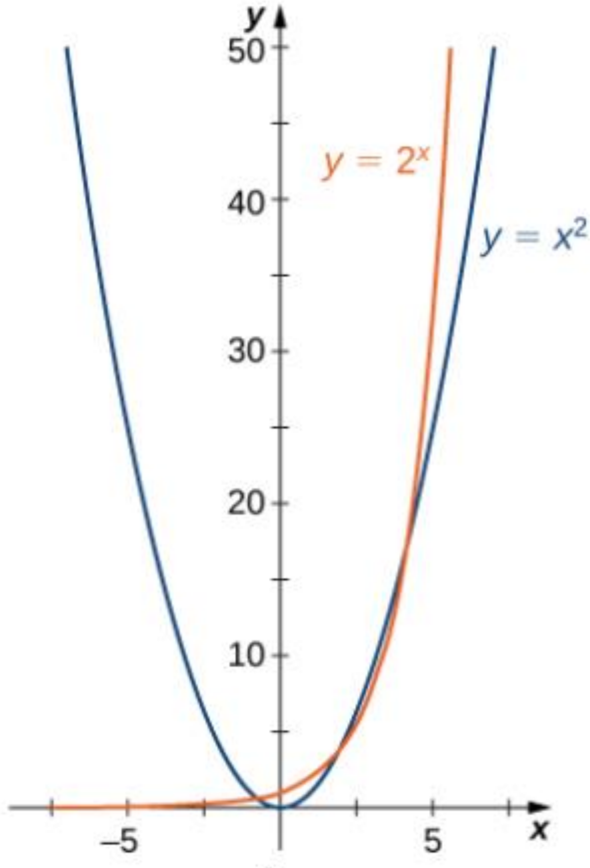


Figure 1. both $y = 2^x$ **and** $y = x^2$ **approach infinity as** $x \to \infty$**. Eventually, however,** $y = 2^x$ **becomes larger than** $y = x^2$ **and grows more rapidly as x** $\to \infty$**.In the opposite direction, as x** $\to -\infty$**,** $y = x^2 \to \infty$ **whereas** $y = 2^x \to 0$**.**

Rule: Laws of Exponents

For any constants $a > 0,\ b > 0,$ and for all x and y,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Use the laws of exponents to simplify each of the following expressions.

a. $\dfrac{\left(2x^{2/3}\right)^3}{\left(4x^{-1/3}\right)^2}$

b. $\dfrac{\left(x^3 y^{-1}\right)^2}{\left(xy^2\right)^{-2}}$

Solution

a. We can simplify as follows:

$$\frac{\left(2x^{2/3}\right)^3}{\left(4x^{-1/3}\right)^2} = \frac{2^3\left(x^{2/3}\right)^3}{4^2\left(x^{-1/3}\right)^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2 x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

b. We can simplify as follows:

$$\frac{\left(x^3 y^{-1}\right)^2}{\left(xy^2\right)^{-2}} = \frac{\left(x^3\right)^2\left(y^{-1}\right)^2}{x^{-2}\left(y^2\right)^{-2}} = \frac{x^6 y^{-2}}{x^{-2} y^{-4}} = x^6 x^2 y^{-2} y^4 = x^8 y^2.$$

The function $f(x) = e^x$ is the only exponential function b^x with b= $e \approx 2.718282$. we call the function $f(x) = e^x$ the *natural exponential function*.

2. Logarithmic Functions

The exponential function $f(x) = b^x$ is one-to-one, with domain $(-\infty, \infty)$ and range $(0, \infty)$. Therefore, it has an inverse function, called the ***logarithmic function with base*** b. For any b > 0, b ≠ 1, the logarithmic function with base ***b***, denoted Log_b, has domain $(0, \infty)$ and range $(-\infty, \infty)$, and satisfies

$$Log_b(x) = y \text{ if and only if } b^y = x$$

For example,

$$
\begin{array}{lll}
\log_2(8) = 3 & \text{since} & 2^3 = 8, \\
\log_{10}\left(\frac{1}{100}\right) = -2 & \text{since} & 10^{-2} = \frac{1}{10^2} = \frac{1}{100}, \\
\log_b(1) = 0 & \text{since} & b^0 = 1 \text{ for any base } b > 0.
\end{array}
$$

Furthermore, since $y = \log_b(x)$ and $y = b^x$ are inverse functions,

$$\log_b(b^x) = x \text{ and } b^{\log_b(x)} = x.$$

The most commonly used logarithmic function is the function $\log_e$. Since this function uses natural e as its base, it is called the **natural logarithm**. Here we use the notation $\ln(x)$ or $\ln x$ to mean $\log_e(x)$. For example,

$$\ln(e) = \log_e(e) = 1,\ \ln\left(e^3\right) = \log_e\left(e^3\right) = 3,\ \ln(1) = \log_e(1) = 0.$$

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

Rule: Properties of Logarithms

If $a, b, c > 0, b \neq 1,$ and r is any real number, then

1. $\log_b(ac) = \log_b(a) + \log_b(c)$ (Product property)
2. $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$ (Quotient property)
3. $\log_b(a^r) = r\log_b(a)$ (Power property)

Solving Equations Involving Exponential Functions

Solve each of the following equations for x.

a. $5^x = 2$

b. $e^x + 6e^{-x} = 5$

Solution

a. Applying the natural logarithm function to both sides of the equation, we have

$$\ln 5^x = \ln 2.$$

Using the power property of logarithms,

$$x \ln 5 = \ln 2.$$

Therefore, $x = \ln 2 / \ln 5$.

b. Multiplying both sides of the equation by e^x, we arrive at the equation

$$e^{2x} + 6 = 5e^x.$$

Rewriting this equation as

$$e^{2x} - 5e^x + 6 = 0,$$

we can then rewrite it as a quadratic equation in e^x:

$$(e^x)^2 - 5(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

$$(e^x - 3)(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

Homework: Prove the following

Rule: Change-of-Base Formulas

Let $a > 0, b > 0,$ and $a \neq 1, b \neq 1.$

1. $a^x = b^{x\log_b a}$ for any real number x.

   If $b = e,$ this equation reduces to $a^x = e^{x\log_e a} = e^{x\ln a}$.

2. $\log_a x = \dfrac{\log_b x}{\log_b a}$ for any real number $x > 0$.

   If $b = e,$ this equation reduces to $\log_a x = \dfrac{\ln x}{\ln a}$.

Chapter Four

1.Trigonometric Functions

To define the trigonometric functions, first consider the unit circle centered at the origin and a point P = (x, y) on the unit circle. Let θ be an angle with an initial side that lies along the positive x -axis and with a terminal side that is the line segment OP. An angle in this position is said to be in ***standard position*** (**Figure 1**). We can then define the values of the six trigonometric functions for θ in terms of the coordinates x and y.

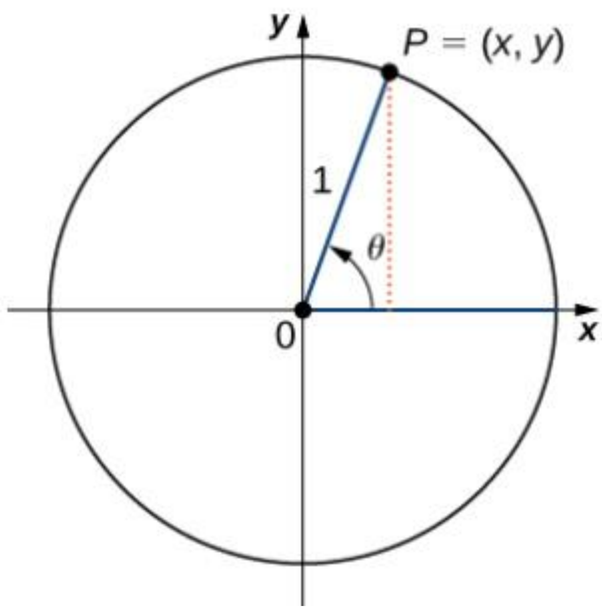


Figure 1. The angle θ is in standard position. The values of the trigonometric functions for θ are defined in terms of the coordinates x and y.

Definition:

Let P = ***(***x**,** y***)*** be a point on the unit circle centered at the origin O**.** Let θ be an angle with an initial side along the positive x -axis and a terminal side given by the line segment OP**.** The **trigonometric functions** are then defined as

$$\sin\theta = y \qquad \csc\theta = \frac{1}{y}$$
$$\cos\theta = x \qquad \sec\theta = \frac{1}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

If $x = 0$, $\sec\theta$ and $\tan\theta$ are undefined. If $y = 0$, then $\cot\theta$ and $\csc\theta$ are undefined.

We can see that for a point P = (x, y) on a circle of radius r with a corresponding angle θ, the coordinates x and y satisfy

$$\cos\theta = \frac{x}{r}$$
$$x = r\cos\theta$$
$$\sin\theta = \frac{y}{r}$$
$$y = r\sin\theta.$$

The values of the other trigonometric functions can be expressed in terms of x, y, and r

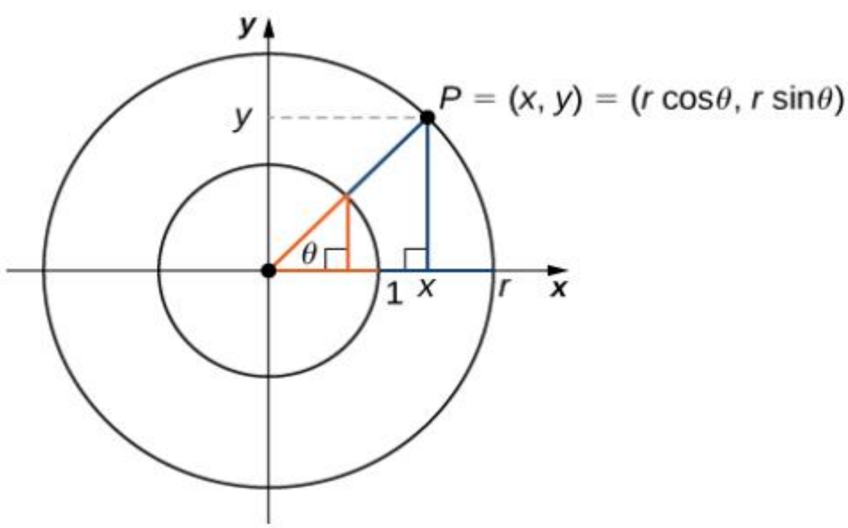


Figure 2. For a point P = ***(***x, y***)*** on a circle of radius r**,** the coordinates x and y satisfy x = r ***cos***θ and y = r***sin***θ***.***

Table 1.9 shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of sin$\boldsymbol{\theta}$ and cos$\boldsymbol{\theta}$.

$\boldsymbol{\theta}$	**sin**$\boldsymbol{\theta}$	**cos**$\boldsymbol{\theta}$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Table 1. Values of sin$\boldsymbol{\theta}$ and cos$\boldsymbol{\theta}$ at Major Angles $\boldsymbol{\theta}$ in the First Quadrant

Rule: Trigonometric Identities

Reciprocal identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Addition and subtraction formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double-angle formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

Solving Trigonometric Equations

For each of the following equations, use a trigonometric identity to find all solutions.

a. $1 + \cos(2\theta) = \cos\theta$

b. $\sin(2\theta) = \tan\theta$

Solution

a. Using the double-angle formula for $\cos(2\theta)$, we see that θ is a solution of

$$1 + \cos(2\theta) = \cos\theta$$

if and only if

$$1 + 2\cos^2\theta - 1 = \cos\theta,$$

which is true if and only if

$$2\cos^2\theta - \cos\theta = 0.$$

To solve this equation, it is important to note that we need to factor the left-hand side and not divide both sides of the equation by cosθ. The problem with dividing by cosθ is that it is possible that cosθ is zero. In fact, if we did divide both sides of the equation by cosθ, we would miss some of the solutions of the original equation. Factoring the left-hand side of the equation, we see that θ is a solution of this equation if and only if.

$$\cos\theta(2\cos\theta - 1) = 0.$$

Since $\cos\theta = 0$ when

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \ldots,$$

and $\cos\theta = 1/2$ when

$$\theta = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi, \ldots \text{ or } \theta = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi, \ldots,$$

we conclude that the set of solutions to this equation is

$$\theta = \frac{\pi}{2} + n\pi, \; \theta = \frac{\pi}{3} + 2n\pi, \text{ and } \theta = -\frac{\pi}{3} + 2n\pi, \; n = 0, \pm 1, \pm 2, \ldots.$$

b. Using the double-angle formula for $\sin(2\theta)$ and the reciprocal identity for $\tan(\theta)$, the equation can be written as

$$2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}.$$

To solve this equation, we multiply both sides by $\cos\theta$ to eliminate the denominator, and say that if θ satisfies this equation, then θ satisfies the equation

$$2\sin\theta\cos^2\theta - \sin\theta = 0.$$

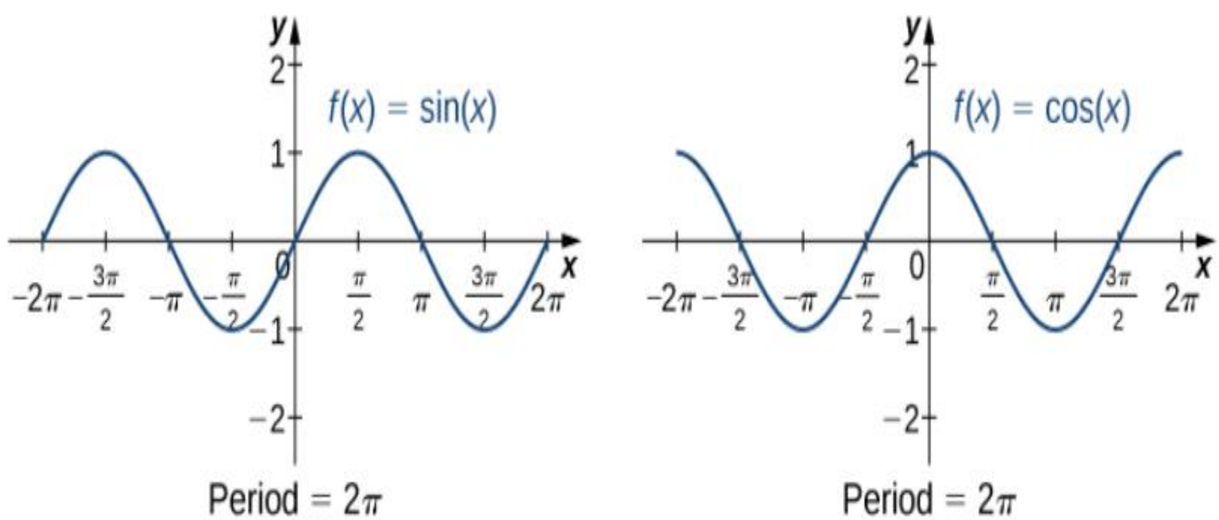
However, we need to be a little careful here. Even if θ satisfies this new equation, it may not satisfy the original equation because, to satisfy the original equation, we would need to be able to divide both sides of the equation by cosθ. However, if cosθ

= 0, we cannot divide both sides of the equation by cosθ. Therefore, it is possible that we may arrive at extraneous solutions. So, at the end, it is important to check for extraneous solutions. Returning to the equation, it is important that we factor sinθ out of both terms on the left-hand side instead of dividing both sides of the equation by sinθ. Factoring the left-hand side of the equation, we can rewrite this equation as

$$\sin\theta(2\cos^2\theta - 1) = 0.$$

Therefore, the solutions are given by the angles θ such that $\sin\theta = 0$ or $\cos^2\theta = 1/2$. The solutions of the first equation are $\theta = 0, \pm\pi, \pm 2\pi, \ldots.$ The solutions of the second equation are $\theta = \pi/4, (\pi/4) \pm (\pi/2), (\pi/4) \pm \pi, \ldots.$ After checking for extraneous solutions, the set of solutions to the equation is

$$\theta = n\pi \quad \text{and} \quad \theta = \frac{\pi}{4} + \frac{n\pi}{2}, n = 0, \pm 1, \pm 2, \ldots.$$



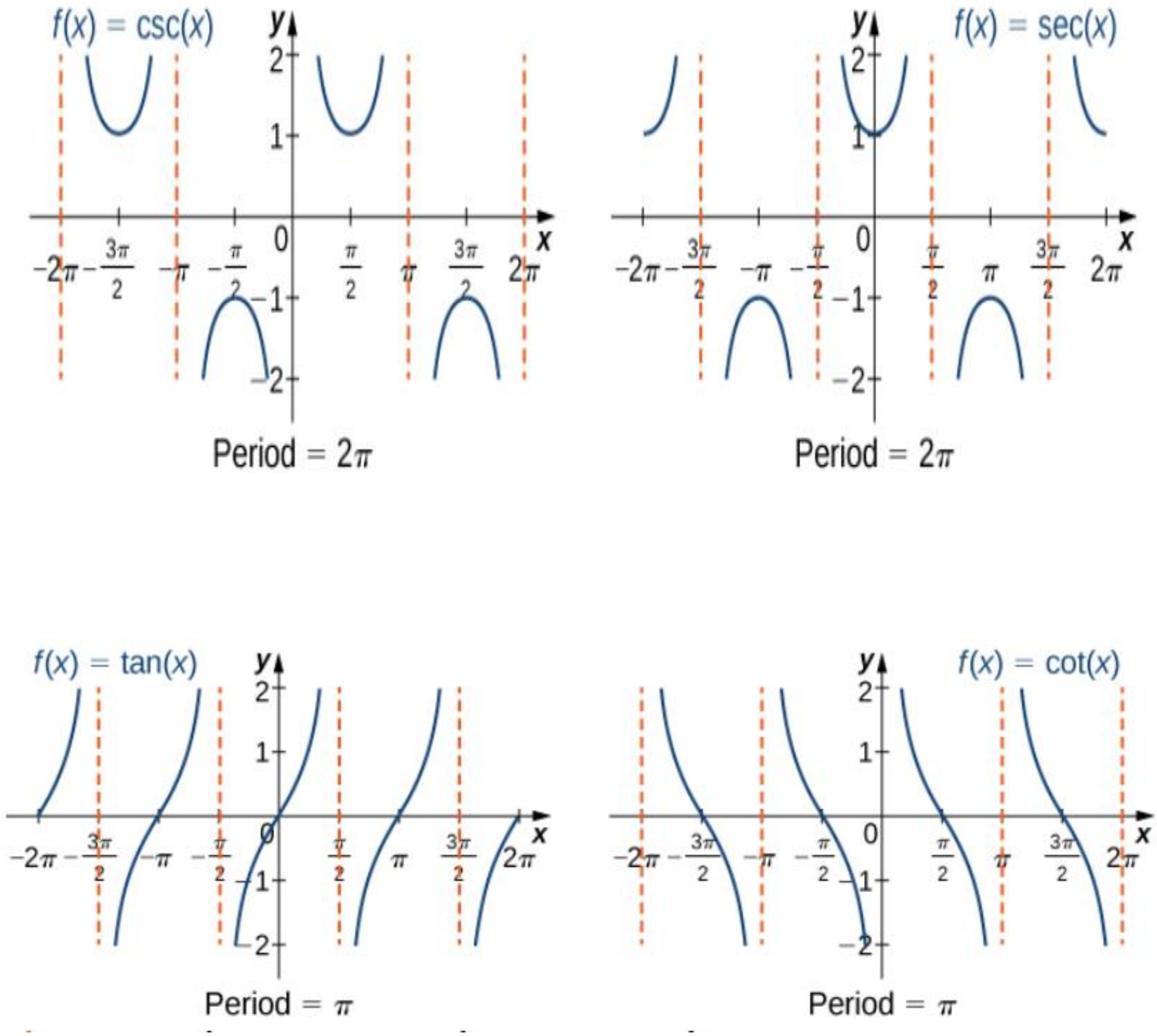


Figure 3 The six trigonometric functions are periodic.

Inverse Trigonometric Functions

Definition

The inverse sine function, denoted $\sin^{-1}$ or arcsin, and the inverse cosine function, denoted $\cos^{-1}$ or arccos, are defined on the domain $D = \{x| -1 \leq x \leq 1\}$ as follows:

$$\sin^{-1}(x) = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2};$$

$$\cos^{-1}(x) = y \text{ if and only if } \cos(y) = x \text{ and } 0 \leq y \leq \pi.$$

The inverse tangent function, denoted $\tan^{-1}$ or arctan, and inverse cotangent function, denoted $\cot^{-1}$ or arccot, are defined on the domain $D = \{x| -\infty < x < \infty\}$ as follows:

$$\tan^{-1}(x) = y \text{ if and only if } \tan(y) = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2};$$

$$\cot^{-1}(x) = y \text{ if and only if } \cot(y) = x \text{ and } 0 < y < \pi.$$

The inverse cosecant function, denoted $\csc^{-1}$ or arccsc, and inverse secant function, denoted $\sec^{-1}$ or arcsec, are defined on the domain $D = \{x||x| \geq 1\}$ as follows:

$$\csc^{-1}(x) = y \text{ if and only if } \csc(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2},\ y \neq 0;$$

$$\sec^{-1}(x) = y \text{ if and only if } \sec(y) = x \text{ and } 0 \leq y \leq \pi,\ y \neq \pi/2.$$

To graph the inverse trigonometric functions, we use the graphs of the trigonometric functions restricted to the domains defined earlier and reflect the graphs about the line y = x

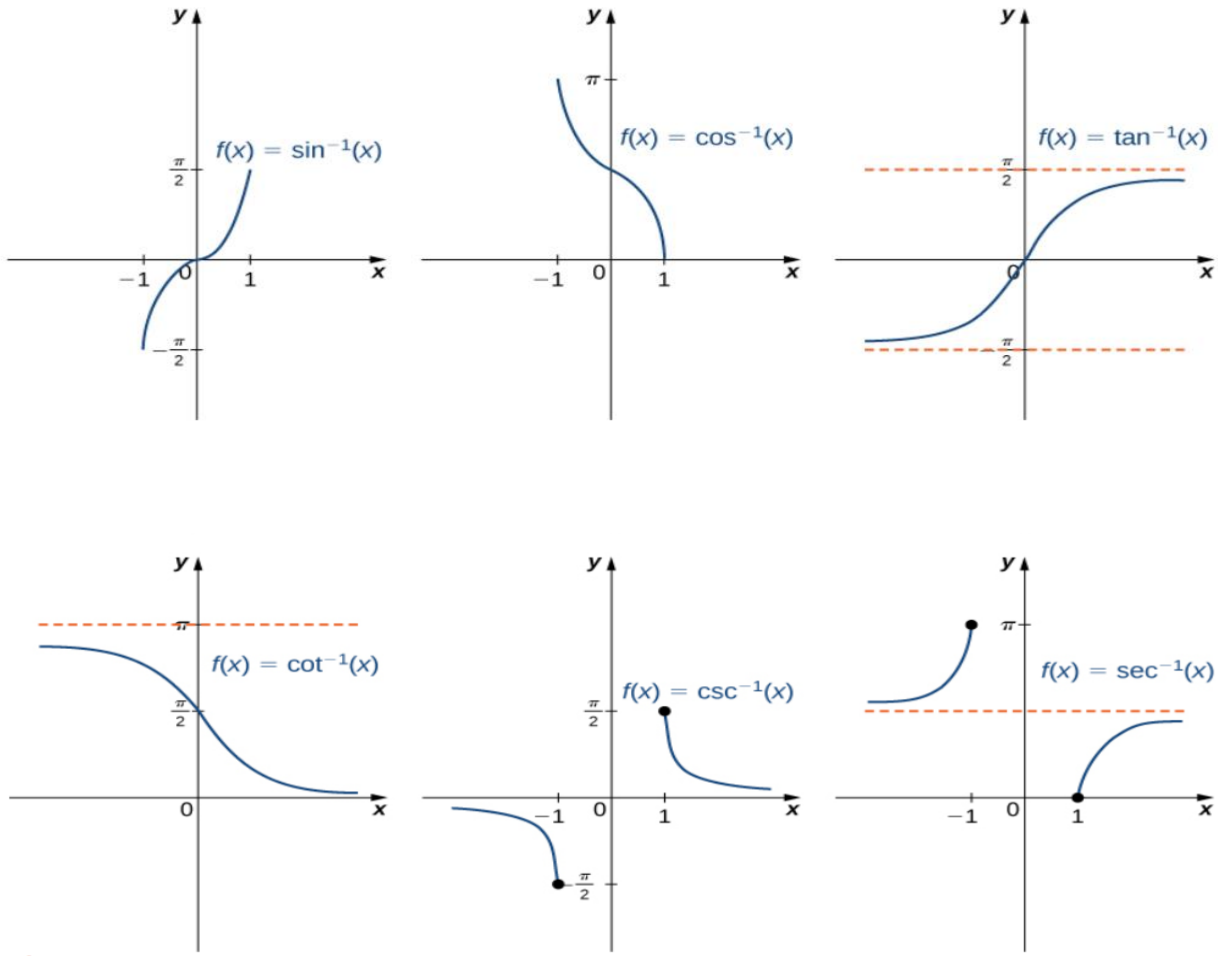


Figure 1. The graph of each of the inverse trigonometric functions is a reflection about the line y = x of the corresponding restricted trigonometric function.

When evaluating an inverse trigonometric function, the output is an angle. For example, to evaluate $\cos^{-1}\left(\frac{1}{2}\right)$, we need to find an angle θ such that $\cos\theta = \frac{1}{2}$. Clearly, many angles have this property. However, given the definition of $\cos^{-1}$, we need the angle θ that not only solves this equation, but also lies in the interval $[0, \pi]$. We conclude that $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

We now consider a composition of a trigonometric function and its inverse. For example, consider the two expressions $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $\sin^{-1}(\sin(\pi))$. For the first one, we simplify as follows:

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

For the second one, we have

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0.$$

Hyperbolic Functions

The hyperbolic functions are defined in terms of certain combinations of e^x and e^{-x}. These functions arise naturally in various engineering and physics applications, including the study of water waves and vibrations of elastic membranes. Another common use for a hyperbolic function is the representation of a hanging chain or cable, also known as a catenary (Figure 1.49). If we introduce a coordinate system so that the low point of the chain lies along the y -axis, we can describe the height of the chain in terms of a hyperbolic function. First, we define the **hyperbolic functions**.

Definition

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\cosh^2 t - \sinh^2 t = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = 1.$$

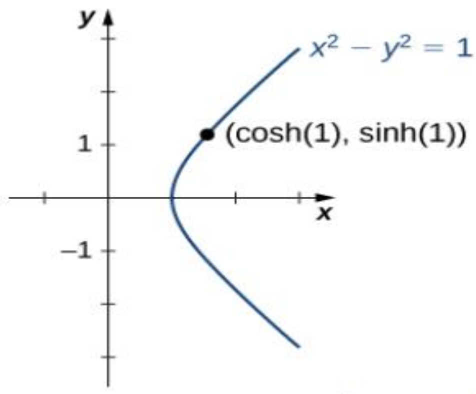
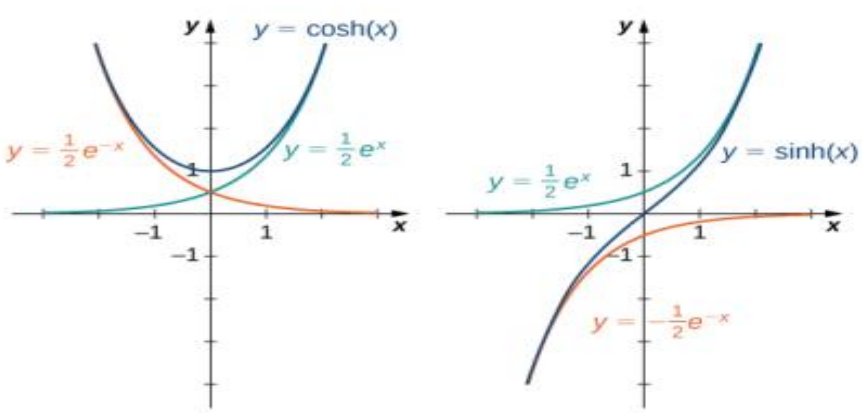


Figure 1. The unit hyperbola $\cosh^2 t - \sinh^2 t = 1$

Graphs of Hyperbolic Functions

To graph $\cosh x$ and $\sinh x$, we make use of the fact that both functions approach $(1/2)e^x$ as $x \to \infty$, since $e^{-x} \to 0$ as $x \to \infty$. As $x \to -\infty$, $\cosh x$ approaches $1/2e^{-x}$, whereas $\sinh x$ approaches $-1/2e^{-x}$. Therefore, using the graphs of $1/2e^x$, $1/2e^{-x}$, and $-1/2e^{-x}$ as guides, we graph $\cosh x$ and $\sinh x$. To graph $\tanh x$, we use the fact that $\tanh(0) = 1$, $-1 < \tanh(x) < 1$ for all x, $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$ as $x \to -\infty$. The graphs of the other three hyperbolic functions can be sketched using the graphs of $\cosh x$, $\sinh x$, and $\tanh x$ ([illegible]).



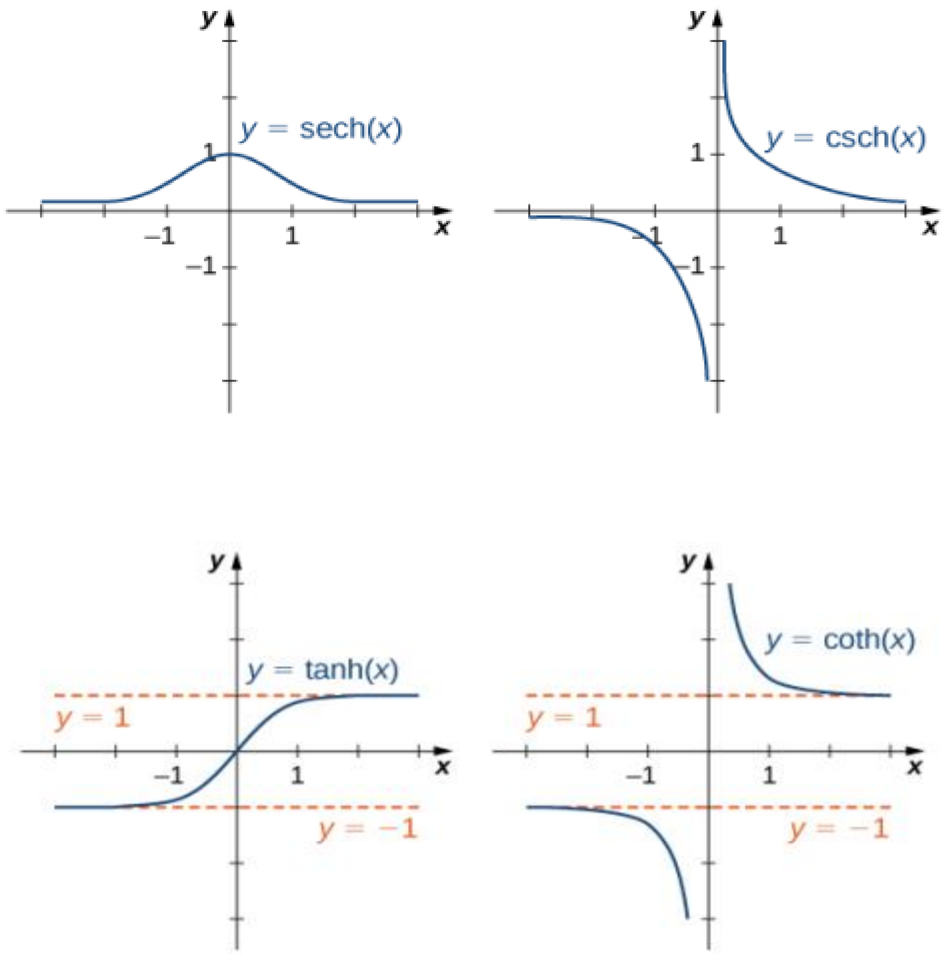


Figure 2. The hyperbolic functions involve combinations of e^x and e^{-x}.

Rule: Identities Involving Hyperbolic Functions

1. $\cosh(-x) = \cosh x$
2. $\sinh(-x) = -\sinh x$
3. $\cosh x + \sinh x = e^x$
4. $\cosh x - \sinh x = e^{-x}$

5. $\cosh^2 x - \sinh^2 x = 1$
6. $1 - \tanh^2 x = \operatorname{sech}^2 x$
7. $\coth^2 x - 1 = \operatorname{csch}^2 x$
8. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
9. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

Example

Evaluating Hyperbolic Functions

a. Simplify $\sinh(5 \ln x)$.

b. If $\sinh x = 3/4$, find the values of the remaining five hyperbolic functions.

Solution

a. Using the definition of the $\sinh$ function, we write

$$\sinh(5 \ln x) = \frac{e^{5 \ln x} - e^{-5 \ln x}}{2} = \frac{e^{\ln\left(x^5\right)} - e^{\ln\left(x^{-5}\right)}}{2} = \frac{x^5 - x^{-5}}{2}.$$

b. Using the identity $\cosh^2 x - \sinh^2 x = 1$, we see that

$$\cosh^2 x = 1 + \left(\frac{3}{4}\right)^2 = \frac{25}{16}.$$

Since $\cosh x \geq 1$ for all x, we must have $\cosh x = 5/4$. Then, using the definitions for the other hyperbolic functions, we conclude that $\tanh x = 3/5$, $\operatorname{csch} x = 4/3$, $\operatorname{sech} x = 4/5$, and $\coth x = 5/3$.

Inverse Hyperbolic Functions

Definition

Inverse Hyperbolic Functions

$$\sinh^{-1} x = \operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad \cosh^{-1} x = \operatorname{arccosh} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \operatorname{arccot} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1} x = \operatorname{arcsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \operatorname{arccsch} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

Let's look at how to derive the first equation. The others follow similarly. Suppose $y = \sinh^{-1} x$. Then, $x = \sinh y$ and, by the definition of the hyperbolic sine function, $x = \frac{e^y - e^{-y}}{2}$. Therefore,

$$e^y - 2x - e^{-y} = 0.$$

Multiplying this equation by e^y, we obtain

$$e^{2y} - 2xe^y - 1 = 0.$$

This can be solved like a quadratic equation, with the solution

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

Since $e^y > 0$, the only solution is the one with the positive sign. Applying the natural logarithm to both sides of the equation, we conclude that

$$y = \ln\left(x + \sqrt{x^2+1}\right).$$

Evaluating Inverse Hyperbolic Functions

Evaluate each of the following expressions.

$$\sinh^{-1}(2)$$

$$\tanh^{-1}(1/4)$$

Solution

$$\sinh^{-1}(2) = \ln\left(2 + \sqrt{2^2 + 1}\right) = \ln\left(2 + \sqrt{5}\right) \approx 1.4436$$

$$\tanh^{-1}(1/4) = \frac{1}{2}\ln\left(\frac{1 + 1/4}{1 - 1/4}\right) = \frac{1}{2}\ln\left(\frac{5/4}{3/4}\right) = \frac{1}{2}\ln\left(\frac{5}{3}\right) \approx 0.2554$$

Chapter Five

1.The Limit of a Function

We begin our exploration of limits by taking a look at the graphs of the functions

$$f(x) = \frac{x^2 - 4}{x - 2}, \quad g(x) = \frac{|x - 2|}{x - 2}, \quad \text{and } h(x) = \frac{1}{(x - 2)^2},$$

which are shown in Figure 1. In particular, let's focus our attention on the behavior of each graph at and around x = 2.

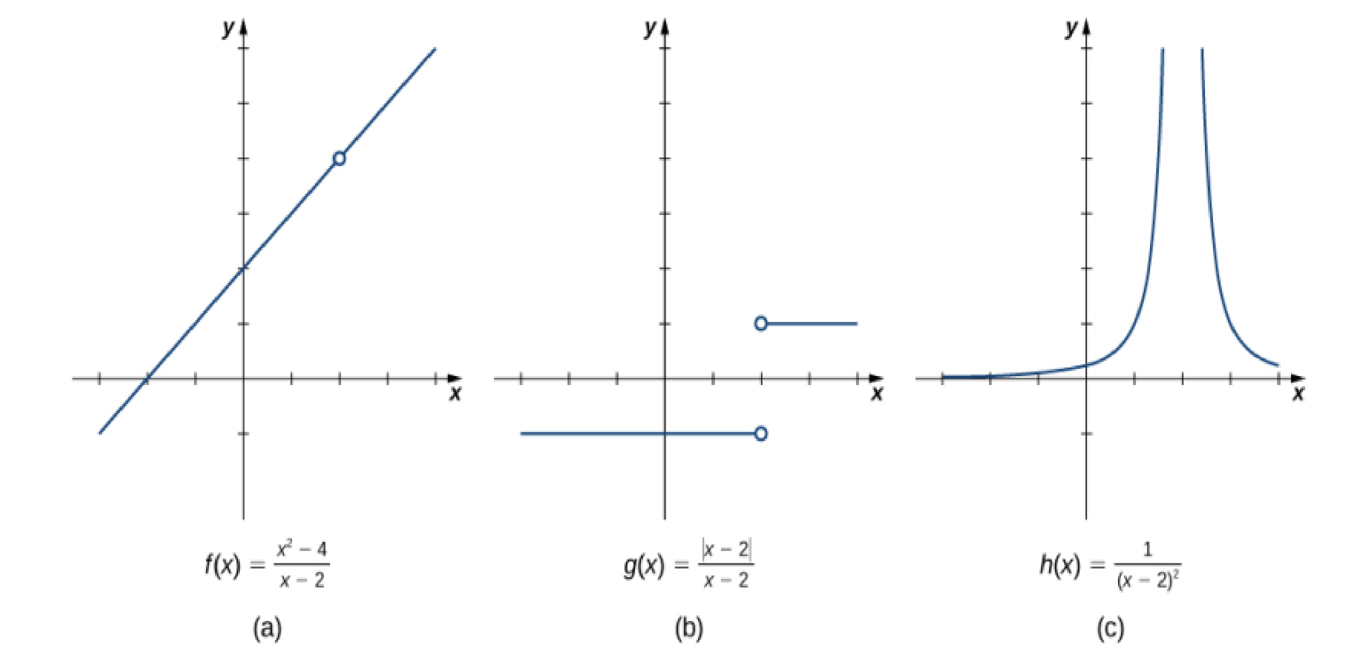


Figure 1. These graphs show the behavior of three different functions around x = 2.

Each of the three functions is undefined at x = 2, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of x = 2. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Intuitive Definition of a Limit $f(x) = (x^2 - 4)/(x - 2)$ behaves around x = 2 in ***Figure 1***. As the values of **x** approach 2 from either side of 2, the values of y = f (x) approach 4. Mathematically, we say that the limit of f (x) as **x** approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4.$$

From this very brief informal look at one limit, let's start to develop an intuitive definition of the limit. We can think of the limit of a function at a number a as being the one real number L that the functional values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following definition:

Definition

Let f ***(***x***)*** be a function defined at all values in an open interval containing **a**, with the possible exception of **a** itself, and let **L** be a real number. If **all** values of the function f***(***x***)*** approach the real number **L** as the values of x ***(≠*** a***)*** approach the number **a**, then we say that the limit of f ***(***x***)*** as **x** approaches **a** is **L**. (More succinct, as **x** gets closer to **a**, f ***(***x***)*** gets closer and stays close to **L**.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L.$$

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x \to a} f(x)$**,** we begin by completing a table of functional values. We should choose two sets of **x**-values—one set of values approaching **a** and less than **a**, and another set of values approaching **a** and greater than **a**. Table 1. demonstrates what your tables might look like.

x	$f(x)$		x	$f(x)$
$a - 0.1$	$f(a - 0.1)$		$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$		$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$		$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$		$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.			Use additional values as necessary.	

Table1. Table of Functional Values for $\lim_{x \to a} f(x)$

2. Next, let's look at the values in each of the f *(***x***)* **columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence f** *(***a** - *0.1),* **f** *(***a** - *0.01),* **f** *(***a** - *0.001).,* **f** *(***a** - *0.0001),* **and so on, and f** *(***a** + *0.1),* **f** *(***a** + *0.01),* **f** *(***a** + *0.001),* **f** *(***a** + *0.0001),* **and so on. (**Note**: Although we have chosen the** x**-values a** ± *0.1,* **a** ± *0.01,* **a** ± *0.001,* **a** ± *0.0001,* **and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)**

3. If both columns approach a common y**-value** L**, we state $\lim_{x \to a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.**

Example 1: Evaluate $\lim_{x \to 0} sinx/x$ using a table of functional values.

Solution
We have calculated the values $f(x) = \lim_{x \to 0} sinx/x$ for the values of x **listed in Table 2.**

x	$\frac{\sin x}{x}$		x	$\frac{\sin x}{x}$
−0.1	0.998334166468		0.1	0.998334166468
−0.01	0.999983333417		0.01	0.999983333417
−0.001	0.999999833333		0.001	0.999999833333
−0.0001	0.999999998333		0.0001	0.999999998333

Table 2. Table of Functional Values for $\lim_{x \to 0} sinx/x$

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output. As we read down each ($\sin x/x$) column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x \to 0}(sinx/x) = 1$. A calculator-or computer generated graph of $f(x) = \lim_{x \to 0} sinx/x$ would be similar to that shown in Figure 2**, and it confirms our estimate.**

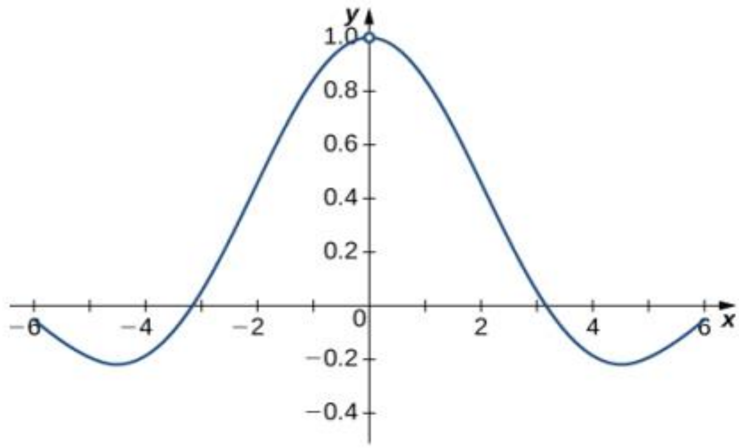


Figure 2. The graph of $f(x) = \lim_{x\to 0} sinx/x$ confirms the estimate from Table 2.

Example 2. Evaluate $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$ using a table of functional values.

Solution
As before, we use a table—in this case, Table 3.—to list the values of the function for the given values of x.

x	$\frac{\sqrt{x}-2}{x-4}$		x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869		4.1	0.248456731317
3.99	0.25015644562		4.01	0.24984394501
3.999	0.250015627		4.001	0.249984377
3.9999	0.250001563		4.0001	0.249998438
3.99999	0.25000016		4.00001	0.24999984

Table 3. Table of Functional Values for $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = 0.25$. We confirm this estimate using the graph of $f(x) = \lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$ shown in Figure 3.

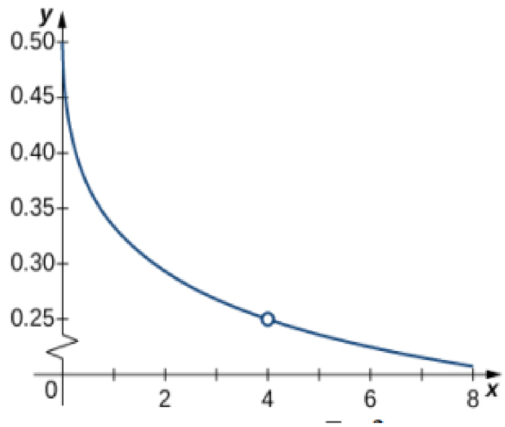


Figure 3. The graph of $f(x) = \frac{\sqrt{x}-2}{x-4}$ confirms the estimate from Table 3.

Definition

We define two types of one-sided limits.

Limit from the left: Let f (x) be a function defined at all values in an open interval of the form z, and let L be a real number. If the values of the function f (x) approach the real number L as the values of x (where x < a) approach the number a, then we say that L is the limit of f (x) as x approaches from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L.$$

Limit from the right: Let f (x) be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function f (x) approach the real number L as the values of x (where x > a) approach the number a, then we say that L is the limit of f (x) as x approaches from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L.$$

Example

For the function $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

x	$f(x) = x+1$		x	$f(x) = x^2 - 4$
1.9	2.9		2.1	0.41
1.99	2.99		2.01	0.0401
1.999	2.999		2.001	0.004001
1.9999	2.9999		2.0001	0.00040001
1.99999	2.99999		2.00001	0.0000400001

Table 4.

Table of Functional Values for $f(x) = \begin{cases} x+1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of f (x) does not exist at x = 2. *Figure 4* **shows a graph of f (x) and reinforces our conclusion about these limits.**

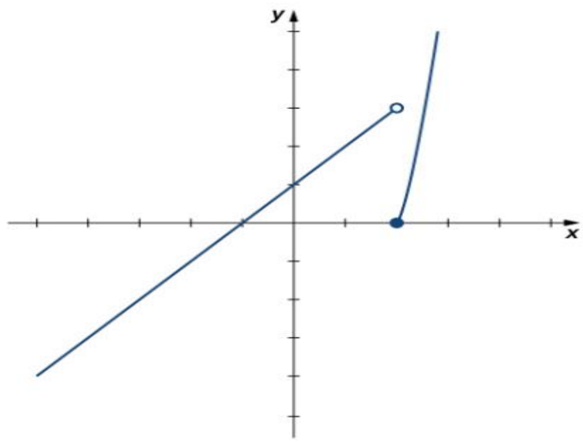


Figure 4.

The graph of $f(x) = \begin{cases} x + 1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$

Theorem 1: Two Important Limits

Let a be a real number and c be a constant.

i. $\lim\limits_{x \to a} x = a$

ii. $\lim\limits_{x \to a} c = c$

Theorem 2.: Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

$$\lim_{x \to a} f(x) = L. \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Definition

We define three types of infinite limits.

Infinite limits from the left: Let f (x) **be a function defined at all values in an open interval of the form** (b, a)

1- **If the values of** f (x) **increase without bound as the values of** *x* **(where** x < a) **approach the number** *a***, then we say that the limit as** *x* **approaches from the left is positive infinity and we write**

$$\lim_{x \to a^-} f(x) = +\infty.$$

2- If the values of *f* (*x*) decrease without bound as the values of *x* (where *x* < *a*) approach the number *a*, then we say that the limit as *x* approaches from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty.$$

Infinite limits from the right: Let f *(*x*)* **be a function defined at all values in an open interval of the form** *(***a**, **c***).*

1- If the values of f ***(***x***)*** increase without bound as the values of x (where x **>** a***)*** approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty.$$

2-If the values of f (x) decrease without bound as the values of ***x*** (where x > a) approach the number ***a***, then we say that the limit as ***x*** approaches from the left is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty.$$

<u>Two-sided infinite limit: Let f (x) be defined for all x ≠ a in an open interval containing a</u>.

1-If the values of f (x) increase without bound as the values of *x* (where x ≠ a) approach the number *a*, then we say that the limit as *x* approaches *a* is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty.$$

2- If the values of *f* (*x*) decrease without bound as the values of *x* (where *x* ≠ *a*) approach the number *a*, then we say that the limit as *x* approaches *a* is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty.$$

Example 5. Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = \frac{1}{x}$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \frac{1}{x}$

b. $\lim_{x \to 0^+} \frac{1}{x}$

c. $\lim_{x \to 0} \frac{1}{x}$

Solution
Begin by constructing a table of functional values.

x	$\frac{1}{x}$		x	$\frac{1}{x}$
−0.1	−10		0.1	10
−0.01	−100		0.01	100
−0.001	−1000		0.001	1000
−0.0001	−10,000		0.0001	10,000
−0.00001	−100,000		0.00001	100,000
−0.000001	−1,000,000		0.000001	1,000,000

Table 5. Table of Functional Values for $f(x) = \frac{1}{x}$

a. The values of 1/x decrease without bound as ***x*** approaches 0 from the left. We conclude that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

b. The values of 1/x increase without bound as ***x*** approaches 0 from the right. We conclude that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

c- Since $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$\lim_{x \to 0} \frac{1}{x}$ does not exis

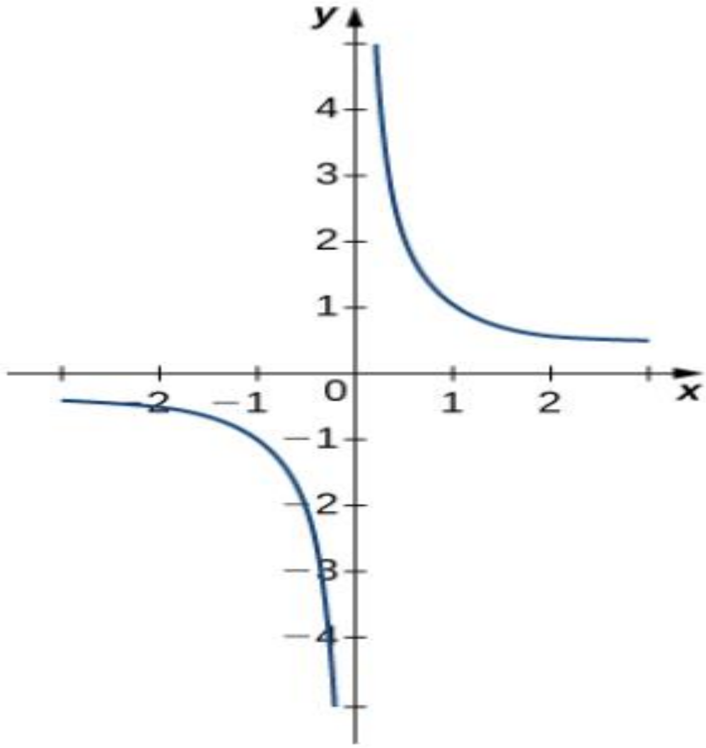


Figure 5. The graph of $f(x) = \frac{1}{x}$ confirms that the limit as **x** approaches 0 does not exist.

Theorem 3: Infinite Limits from Positive Integers

If n is a positive even integer, then

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If n is a positive odd integer, then

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

Example 6. Evaluate each of the following limits

a. $\lim_{x \to -3^-} \frac{1}{(x+3)^4}$

b. $\lim_{x \to -3^+} \frac{1}{(x+3)^4}$

c. $\lim_{x \to -3} \frac{1}{(x+3)^4}$

Solution

a. $\lim_{x \to -3^-} \frac{1}{(x+3)^4} = +\infty$

b. $\lim_{x \to -3^+} \frac{1}{(x+3)^4} = +\infty$

c. $\lim_{x \to -3} \frac{1}{(x+3)^4} = +\infty$

The Limit Laws

Theorem 4: Basic Limit Results

For any real number a and any constant c,

i. $\lim_{x \to a} x = a$

ii. $\lim_{x \to a} c = c$

Example 7. Evaluate each of the following limits using Basic Limit Results.

a. $\lim_{x \to 2} x$

b. $\lim_{x \to 2} 5$

Solution

a. The limit of x as x approaches a is a: $\lim_{x \to 2} x = 2$.

b. The limit of a constant is that constant: $\lim_{x \to 2} 5 = 5$.

Theorem 5: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

Sum law for limits: $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

Difference law for limits: $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$

Product law for limits: $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

Quotient law for limits: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every positive integer n.

Root law for limits: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even.

Example:

Use the limit laws to evaluate $\lim_{x \to -3}(4x + 2)$.

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \to -3}(4x + 2) &= \lim_{x \to -3} 4x + \lim_{x \to -3} 2 && \text{Apply the sum law.} \\ &= 4 \cdot \lim_{x \to -3} x + \lim_{x \to -3} 2 && \text{Apply the constant multiple law.} \\ &= 4 \cdot (-3) + 2 = -10. && \text{Apply the basic limit results and simplify.} \end{aligned}$$

Example:

Using Limit Laws Repeatedly

Use the limit laws to evaluate $\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$.

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4} = \frac{\lim_{x \to 2}(2x^2 - 3x + 1)}{\lim_{x \to 2}(x^3 + 4)}$$ Apply the quotient law, making sure that. $(2)^3 + 4 \neq 0$

$$= \frac{2 \cdot \lim_{x \to 2} x^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{\lim_{x \to 2} x^3 + \lim_{x \to 2} 4}$$ Apply the sum law and constant multiple law.

$$= \frac{2 \cdot \left(\lim_{x \to 2} x\right)^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{\left(\lim_{x \to 2} x\right)^3 + \lim_{x \to 2} 4}$$ Apply the power law.

$$= \frac{2(4) - 3(2) + 1}{(2)^3 + 4} = \frac{1}{4}.$$ Apply the basic limit laws and simplify.

Example:

Evaluate the $\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4}$.

Solution

Since 3 is in the domain of the rational function $f(x) = \frac{2x^2 - 3x + 1}{5x + 4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4} = \frac{10}{19}.$$

Example:

Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x \to 3} \frac{x^2 - 3x}{2x^2 - 5x - 3}$.

Solution

Step 1. The function $f(x) = \frac{x^2 - 3x}{2x^2 - 5x - 3}$ is undefined for $x = 3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x \to 3} \frac{x^2 - 3x}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)}$$

Step 2. For all $x \neq 3$, $\frac{x^2 - 3x}{2x^2 - 5x - 3} = \frac{x}{2x+1}$. Therefore,

$$\lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)} = \lim_{x \to 3} \frac{x}{2x+1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x \to 3} \frac{x}{2x+1} = \frac{3}{7}.$$

Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim_{x \to -1} \frac{\sqrt{x+2} - 1}{x+1}$.

Solution

Step 1. $\frac{\sqrt{x+2} - 1}{x+1}$ has the form $0/0$ at -1. Let's begin by multiplying by $\sqrt{x+2} + 1$, the conjugate of $\sqrt{x+2} - 1$, on the numerator and denominator:

$$\lim_{x \to -1} \frac{\sqrt{x+2} - 1}{x+1} = \lim_{x \to -1} \frac{\sqrt{x+2} - 1}{x+1} \cdot \frac{\sqrt{x+2} + 1}{\sqrt{x+2} + 1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:

$$= \lim_{x \to -1} \frac{x+1}{(x+1)\left(\sqrt{x+2} + 1\right)}.$$

Step 3. Then we cancel:

$$= \lim_{x \to -1} \frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x \to -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{2}.$$

Example:

Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim\limits_{x \to 1} \dfrac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$.

Solution

Step 1. $\dfrac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} = \lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} \cdot \frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x - 1)$:

$$= \lim_{x \to 1} \frac{2 - (x + 1)}{2(x - 1)(x + 1)}.$$

Step 3. Then, we simplify the numerator:

$$= \lim_{x \to 1} \frac{-x + 1}{2(x - 1)(x + 1)}.$$

Step 4. Now we factor out -1 from the numerator:

$$= \lim_{x \to 1} \frac{-(x - 1)}{2(x - 1)(x + 1)}.$$

Step 5. Then, we cancel the common factors of $(x - 1)$:

$$= \lim_{x \to 1} \frac{-1}{2(x + 1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x \to 1} \frac{-1}{2(x + 1)} = -\frac{1}{4}.$$

Example:

Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim_{x \to 0} \left(\frac{1}{x} + \frac{5}{x(x - 5)} \right)$.

Solution

Both $1/x$ and $5/x(x - 5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\begin{aligned} \frac{1}{x} + \frac{5}{x(x - 5)} &= \frac{x - 5 + 5}{x(x - 5)} \\ &= \frac{x}{x(x - 5)}. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{x \to 0} \left(\frac{1}{x} + \frac{5}{x(x - 5)} \right) &= \lim_{x \to 0} \frac{x}{x(x - 5)} \\ &= \lim_{x \to 0} \frac{1}{x - 5} \\ &= -\frac{1}{5}. \end{aligned}$$

Example:

Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim\limits_{x \to 3^-} \sqrt{x-3}$

b. $\lim\limits_{x \to 3^+} \sqrt{x-3}$

Solution

Figure 8. illustrates the function $f(x) = \sqrt{x-3}$ and aids in our understanding of these limits.

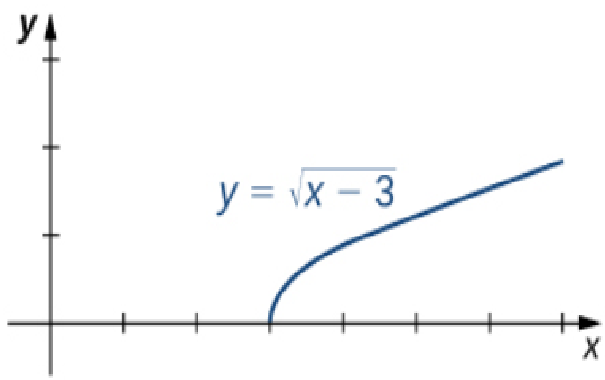


Figure 8. The graph shows the function $f(x) = \sqrt{x-3}$

a. The function $f(x) = \sqrt{x-3}$ is defined over the interval $[3, +\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim\limits_{x \to 3^-} \sqrt{x-3}$. In fact, since $f(x) = \sqrt{x-3}$ is undefined to the left of 3, $\lim\limits_{x \to 3^-} \sqrt{x-3}$ does not exist.

b. Since $f(x) = \sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim\limits_{x \to 3^+} \sqrt{x-3}$. By applying these limit laws we obtain $\lim\limits_{x \to 3^+} \sqrt{x-3} = 0$.

Example:

Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x - 3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim\limits_{x \to 2^-} f(x)$

b. $\lim\limits_{x \to 2^+} f(x)$

c. $\lim\limits_{x \to 2} f(x)$

Solution:

Figure 9. illustrates the function f ***(***x***)*** and aids in our understanding of these limits.

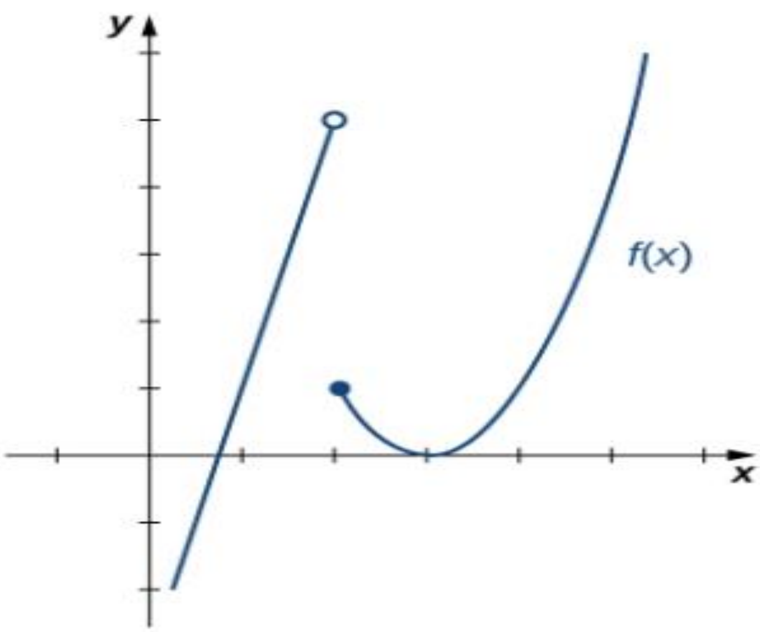


Figure 9. This graph shows a function f ***(***x***).***

a. Since $f(x) = 4x - 3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x - 3$ and apply the limit laws:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x - 3) = 5.$$

b. Since $f(x) = (x - 3)^2$ for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x - 3)^2$ and apply the limit laws:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (x - 3)^2 = 1.$$

c. Since $\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 1$, we conclude that $\lim_{x \to 2} f(x)$ does not exist.

2.<u>Continuity</u>

We begin our investigation of continuity by exploring what it means for a function to have continuity at a point. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in Figure 1. We see that the graph of f ***(***x***)*** has a hole at **a**. In fact, f ***(***a***)*** is undefined. At the very least, for f ***(***x***)*** to be continuous at **a**, we need the following condition:

i. $f(a)$ is defined

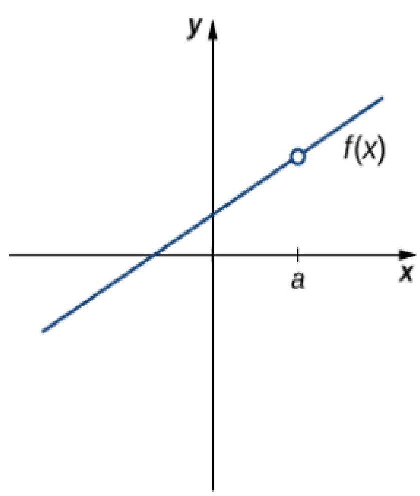


Figure 1. The function f ***(***x***)*** is not continuous at **a** because f ***(***a***)*** is undefined.

However, as we see in Figure 2. this condition alone is insufficient to guarantee continuity at the point a. Although ***f* (*a*)** is defined, the function has a gap at a. In this example, the gap exists because $\lim_{x \to a} f(x)$ does not exist. We must add another condition for continuity at a—namely,

$$\text{ii. } \lim_{x \to a} f(x) \text{ exists.}$$

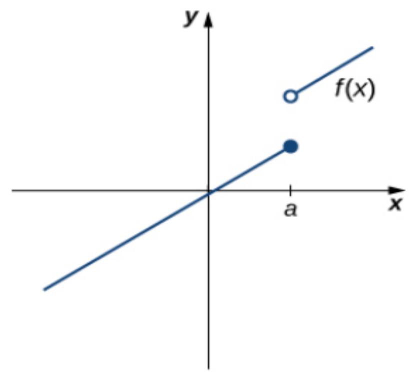


Figure 2. The function f *(*x*)* is not continuous at a **because** $\lim_{x \to a} f(x)$ **does not exist.**

However, as we see in Figure 3., these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a. We must add a third condition to our list:

$$\text{iii. } \lim_{x \to a} f(x) = f(a).$$

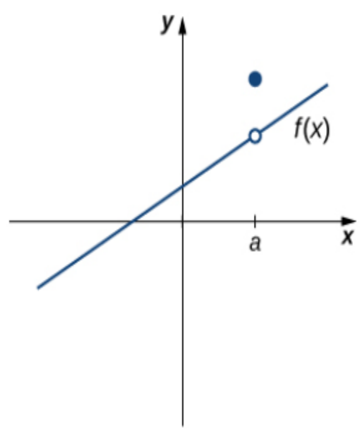


Figure 3. The function f (x) is not continuous at a because $\lim_{x \to a} f(x) \neq f(a)$

Now we put our list of conditions together and form a definition of continuity at a point.

Definition

A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

i. $f(a)$ is defined

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** a if it fails to be continuous at a.

Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If $\lim_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x \to a} f(x)$. If $\lim_{x \to a} f(x) \neq f(a)$, then the function is not continuous at a. If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at a.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

Example: <u>**Determining Continuity at a Point, Condition 1**</u>

Using the definition, determine whether the function $f(x) = (x^2 - 4)/(x - 2)$ is continuous at x = 2. Justify the conclusion.

Solution

Let's begin by trying to calculate f *(2).* We can see that $f(2) = 0/0$, which is undefined. Therefore, $f(x) = (x^2 - 4)/(x - 2)$ is discontinuous at 2 because f (2) is undefined. The graph of f (x) is shown in Figure 4.

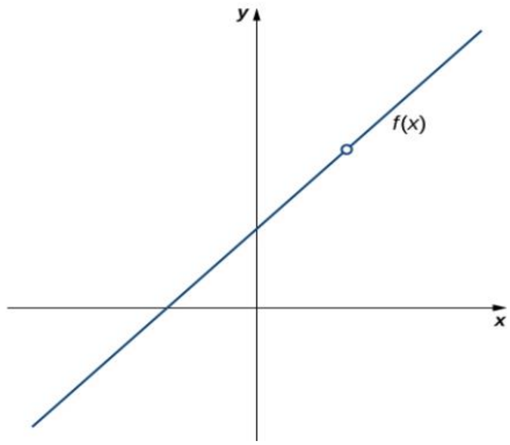


Figure 4. The function f *(*x*)* is discontinuous at 2 because f *(2)* is undefined.

Example:

Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \le 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(3)$.

$$f(3) = -(3^2) + 4 = -5.$$

Thus, $f(3)$ is defined. Next, we calculate $\lim_{x \to 3} f(x)$. To do this, we must compute $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$:

$$\lim_{x \to 3^-} f(x) = -(3^2) + 4 = -5$$

$$\lim_{x \to 3^+} f(x) = 4(3) - 8 = 4.$$

Therefore, $\lim_{x\to 3} f(x)$ does not exist. Thus, f (x) is not continuous at 3. The graph of f (x) is shown in Figure5.

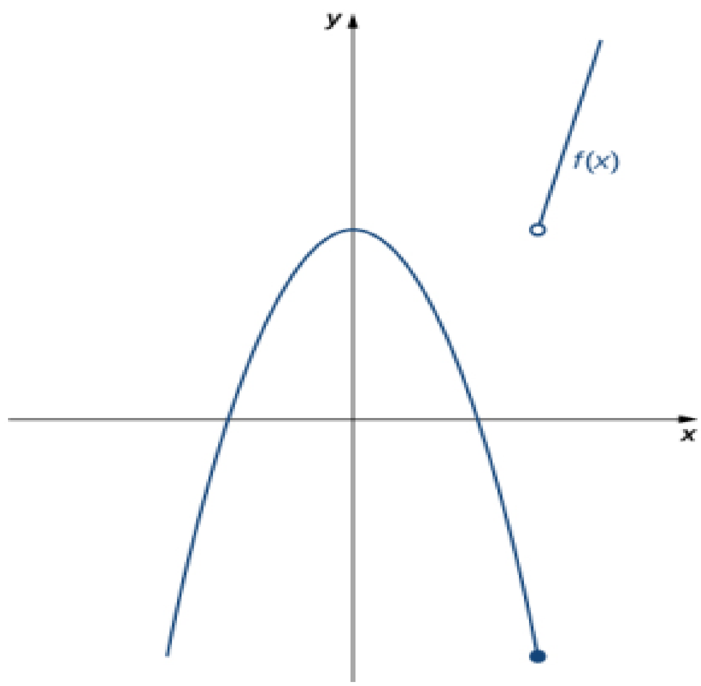


Figure 5. The function f *(x)* is not continuous at 3 because $\lim_{x\to 3} f(x)$ does not exist.

Example:

Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution

First, observe that

$$f(0) = 1.$$

Next,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Last, compare $f(0)$ and $\lim_{x \to 0} f(x)$. We see that

$$f(0) = 1 = \lim_{x \to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

Homework

Using the definition, determine whether the function $f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x+4 & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$.

Derivative Functions

If $f(x)$ is single-valued and continuous in some region of x , the derivative of $f(x)$, denoted by $f'(x)$, is defined as

2.1

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Where $\Delta x = x - x_0$

Example: Find the derivative of $f(x) = \sqrt{x}$

Solution:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \times \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(\sqrt{x+\Delta x})^2 + \sqrt{x}\sqrt{x+\Delta x} - \sqrt{x}\sqrt{x+\Delta x} - (\sqrt{x})^2}{\Delta x\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{x+\Delta x - x}{\Delta x\sqrt{x+\Delta x} + \sqrt{x}} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\sqrt{x+\Delta x} + \sqrt{x}} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\therefore f'(x) = \frac{1}{2\sqrt{x}}$$

Example: Find the derivative of the function $f(x) = x^2 - 2x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 2(x+\Delta x) - (x^2 - 2x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x - x^2 + 2x}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 - 2\Delta x}{\Delta x} = \frac{\Delta x(2x + \Delta x - 2)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} ((2x + \Delta x - 2) = 2x - 2$$