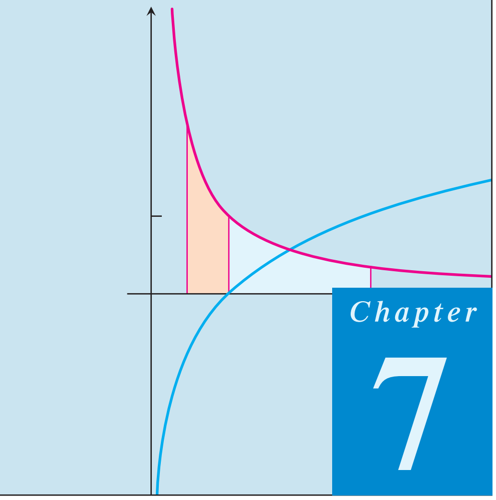


Chapter 7 Transcendental Functions

OVERVIEW Functions can be classified into two broad groups (see Section 1.4). Polynomial functions are called *algebraic*, as are functions obtained from them by addition, multiplication, division, or taking powers and roots. Functions that are not algebraic are called *transcendental.* The trigonometric, exponential, logarithmic, and hyperbolic functions are transcendental, as are their inverses.

Transcendental functions occur frequently in many calculus settings and applications, including growths of populations, vibrations and waves, efficiencies of computer algorithms, and the stability of engineered structures. In this chapter we introduce several important transcendental functions and investigate their graphs, properties, derivatives, and integrals.

7.1 Inverse Functions and Their Derivatives

A function that undoes, or inverts, the effect of a function f is called the *inverse* of f. Many common functions, though not all, are paired with an inverse. Important inverse functions often show up in formulas for antiderivatives and solutions of differential equations. Inverse functions also play a key role in the development and properties of the logarithmic and exponential functions, as we will see in Section 7.3.

One-to-One Functions

A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and $+1$; the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called one-to-one. These functions take on any one value in their range exactly once.

DEFINITION One-to-One Function

A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

EXAMPLE 1 Domains of One-to-One Functions

(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. The sine *is* one-to-one on $[0, \pi/2]$, however, because it is a strictly increasing function on $[0, \pi/2]$. ■

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If it intersects the line more than once, it assumes the same y-value more than once, and is therefore not one-to-one (Figure 7.1).

> **The Horizontal Line Test for One-to-One Functions**
> A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

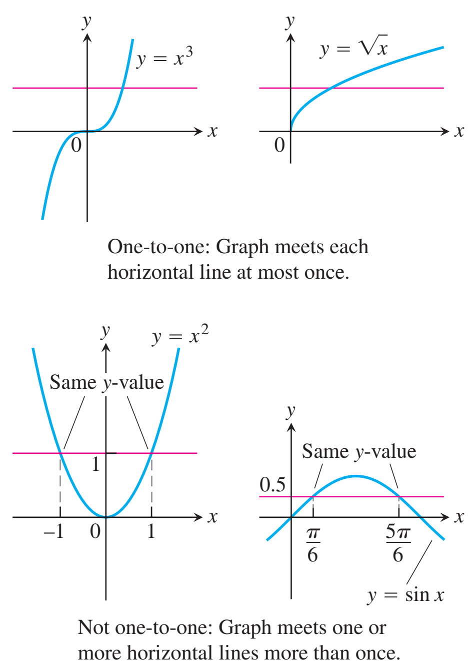


FIGURE 7.1 Using the horizontal line test, we see that $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$, but $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send an output back to the input from which it came.

DEFINITION Inverse Function

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The domains and ranges of f and f^{-1} are interchanged. The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent: $f^{-1}(x)$ does not mean $1/f(x)$.

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$ we get right back to x, just where we started. Similarly, if we take some number y in the range of f, apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y with which we began. Composing a function and its inverse has the same effect as doing nothing.

$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in the domain of } f$$

$$(f \circ f^{-1})(y) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f)$$

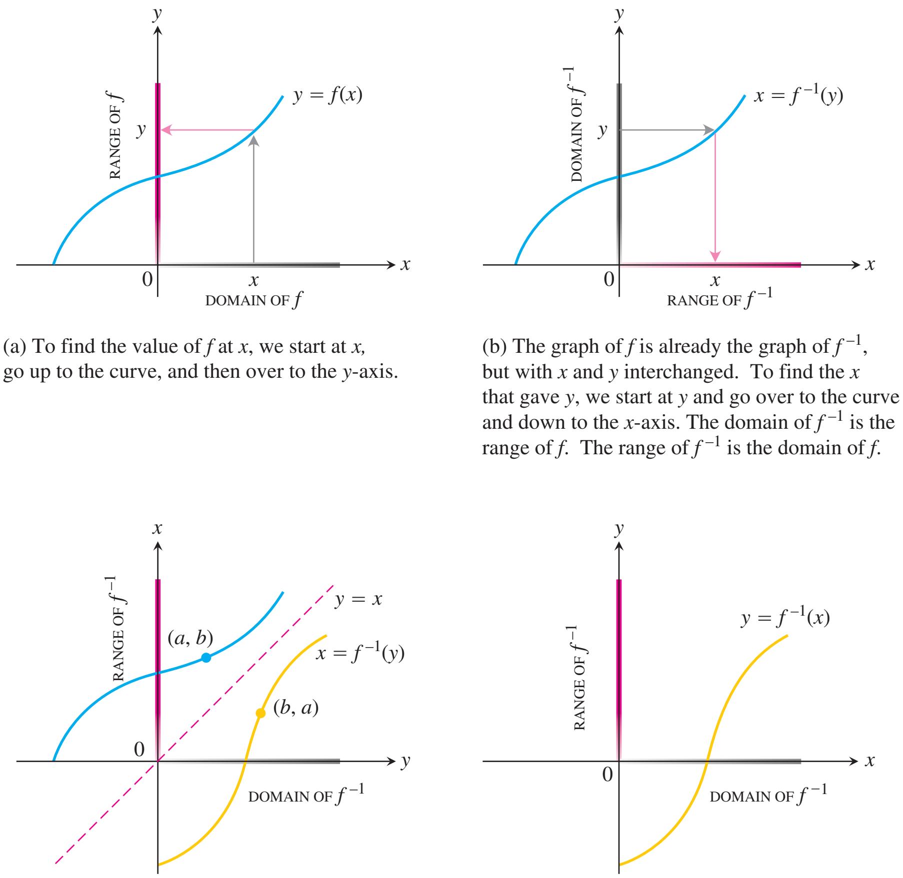
Only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2, then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

A function that is increasing on an interval, satisfying $f(x_2) > f(x_1)$ when $x_2 > x_1$, is one-to-one and has an inverse. Decreasing functions also have an inverse (Exercise 39). Functions that have positive derivatives at all x are increasing (Corollary 3 of the Mean Value Theorem, Section 4.2), and so they have inverses. Similarly, functions with negative derivatives at all x are decreasing and have inverses. Functions that are neither increasing nor decreasing may still be one-to-one and have an inverse, as with the function $\sec^{-1} x$ in Section 7.7.

Finding Inverses

The graphs of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x-axis, go vertically to the graph, and then move horizontally to the y-axis to read the value of y. The inverse function can be read from the graph by reversing this process. Start with a point y on the y-axis, go horizontally to the graph, and then move vertically to the x-axis to read the value of $x = f^{-1}(y)$ (Figure 7.2).

We want to set up the graph of f^{-1} so that its input values lie along the x-axis, as is usually done for functions, rather then on the y-axis. To achieve this we interchange the x and y axes by reflecting across the 45° line $y = x$. After this reflection we have a new graph that represents f^{-1}. The value of $f^{-1}(x)$ can now be read from the graph in the usual way, by starting with a point x on the x-axis, going vertically to the graph and then horizontally to the y-axis to get the value of $f^{-1}(x)$. Figure 7.2 indicates the relation between the graphs of f and f^{-1}. The graphs are interchanged by reflection through the line $y = x$.



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f is already the graph of f^{-1}, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

(c) To draw the graph of f^{-1} in the more usual way, we reflect the system in the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 7.2 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$.

The process of passing from f to f^{-1} can be summarized as a two-step process.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

EXAMPLE 2 Finding an Inverse Function

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution

1. *Solve for x in terms of y:*

$$\begin{aligned} y &= \frac{1}{2}x + 1 \\ 2y &= x + 2 \\ x &= 2y - 2. \end{aligned}$$

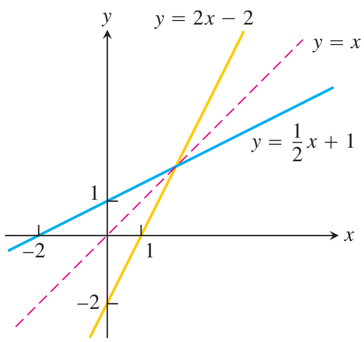


FIGURE 7.3 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other (Example 2).

2. *Interchange x and y:* $y = 2x - 2$.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

See Figure 7.3. ■

EXAMPLE 3 Finding an Inverse Function

You Try It

Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution We first solve for x in terms of y:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (Figure 7.4).

Notice that, unlike the restricted function $y = x^2, x \geq 0$, the unrestricted function $y = x^2$ is not one-to-one and therefore has no inverse. ■

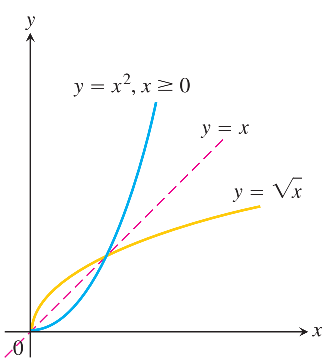


FIGURE 7.4 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 3).

Derivatives of Inverses of Differentiable Functions

If we calculate the derivatives of $f(x) = (1/2)x + 1$ and its inverse $f^{-1}(x) = 2x - 2$ from Example 2, we see that

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$

The derivatives are reciprocals of one another. The graph of f is the line $y = (1/2)x + 1$, and the graph of f^{-1} is the line $y = 2x - 2$ (Figure 7.3). Their slopes are reciprocals of one another.

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$ (Figure 7.5), the reflected line has slope $1/m$ (Exercise 36).

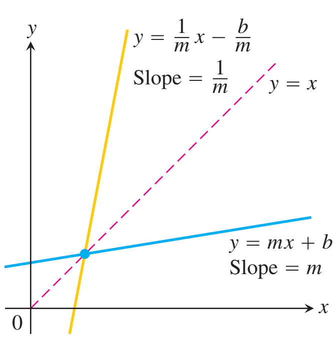
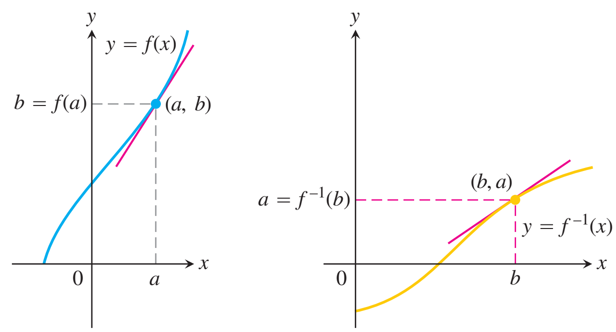


FIGURE 7.5 The slopes of nonvertical lines reflected through the line $y = x$ are reciprocals of each other.

The reciprocal relationship between the slopes of f and f^{-1} holds for other functions as well, but we must be careful to compare slopes at corresponding points. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$ and $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the point $(f(a), a)$ is the reciprocal $1/f'(a)$ (Figure 7.6). If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

If $y = f(x)$ has a horizontal tangent line at $(a, f(a))$ then the inverse function f^{-1} has a vertical tangent line at $(f(a), a)$, and this infinite slope implies that f^{-1} is not differentiable



The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

FIGURE 7.6 The graphs of inverse functions have reciprocal slopes at corresponding points.

at $f(a)$. Theorem 1 gives the conditions under which f^{-1} is differentiable in its domain, which is the same as the range of f.

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}} \tag{1}$$

The proof of Theorem 1 is omitted, but here is another way to view it. When $y = f(x)$ is differentiable at $x = a$ and we change x by a small amount dx, the corresponding change in y is approximately

$$dy = f'(a)\,dx.$$

This means that y changes about $f'(a)$ times as fast as x when $x = a$ and that x changes about $1/f'(a)$ times as fast as y when $y = b$. It is reasonable that the derivative of f^{-1} at b is the reciprocal of the derivative of f at a.

EXAMPLE 4 Applying Theorem 1

The function $f(x) = x^2, x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

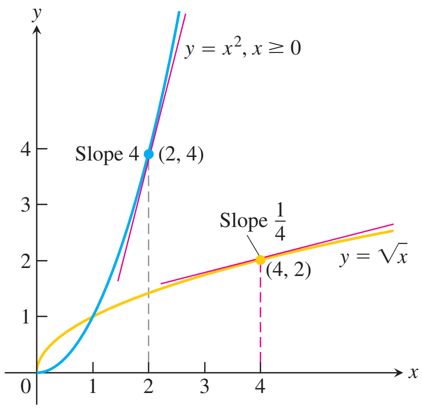


FIGURE 7.7 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 4).

Theorem 1 predicts that the derivative of $f^{-1}(x)$ is

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{2(f^{-1}(x))} \\ &= \frac{1}{2(\sqrt{x})}.\end{aligned}$$

Theorem 1 gives a derivative that agrees with our calculation using the Power Rule for the derivative of the square root function.

Let's examine Theorem 1 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 1 says that the derivative of f at 2, $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2x}\bigg|_{x=2} = \frac{1}{4}.$$

See Figure 7.7. ■

Equation (1) sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1}.

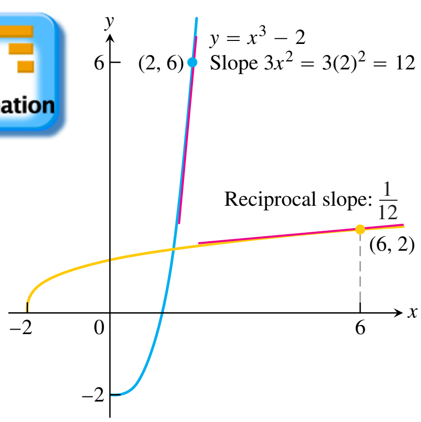


FIGURE 7.8 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 5).

Animation

EXAMPLE 5 Finding a Value of the Inverse Derivative

You Try It

Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution

$$\frac{df}{dx}\bigg|_{x=2} = 3x^2\bigg|_{x=2} = 12$$

$$\frac{df^{-1}}{dx}\bigg|_{x=f(2)} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=2}} = \frac{1}{12} \qquad \text{Eq. (1)}$$

See Figure 7.8. ■

Parametrizing Inverse Functions

We can graph or represent any function $y = f(x)$ parametrically as

$$x = t \quad \text{and} \quad y = f(t).$$

Interchanging t and $f(t)$ produces parametric equations for the inverse:

$$x = f(t) \quad \text{and} \quad y = t$$

(see Section 3.5).

For example, to graph the one-to-one function $f(x) = x^2, x \geq 0$, on a grapher together with its inverse and the line $y = x, x \geq 0$, use the parametric graphing option with

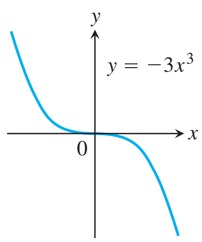
$$\begin{aligned}&\text{Graph of } f: && x_1 = t, && y_1 = t^2, && t \geq 0\\ &\text{Graph of } f^{-1}: && x_2 = t^2, && y_2 = t\\ &\text{Graph of } y = x: && x_3 = t, && y_3 = t\end{aligned}$$

EXERCISES 7.1

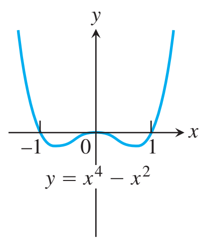
Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–6 are one-to-one, and which are not?

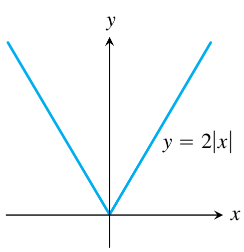
1.



2.



3.



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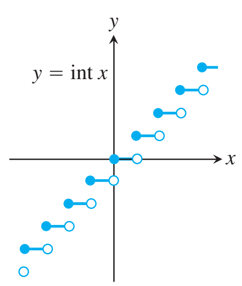

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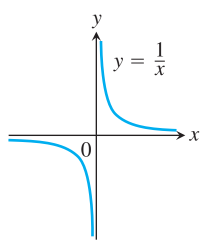
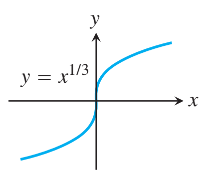


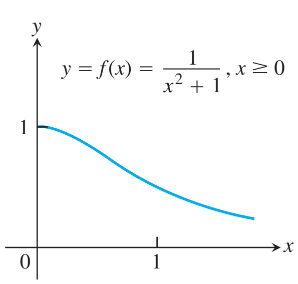
6.



Graphing Inverse Functions

Each of Exercises 7–10 shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use symmetry with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1}.) Identify the domain and range of f^{-1}.

7.



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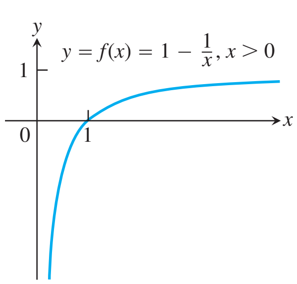

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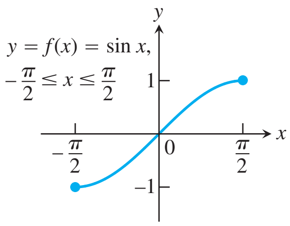
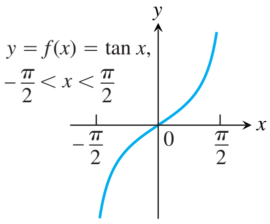


10.



11. a. Graph the function $f(x) = \sqrt{1 - x^2}$, $0 \le x \le 1$. What symmetry does the graph have?

b. Show that f is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \ge 0$.)

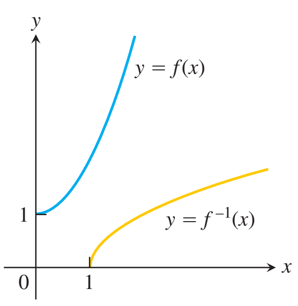
12. a. Graph the function $f(x) = 1/x$. What symmetry does the graph have?

b. Show that f is its own inverse.

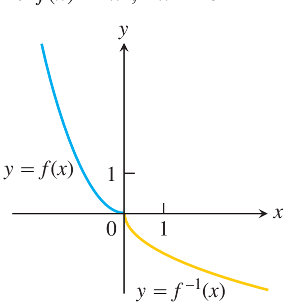
Formulas for Inverse Functions

Each of Exercises 13–18 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

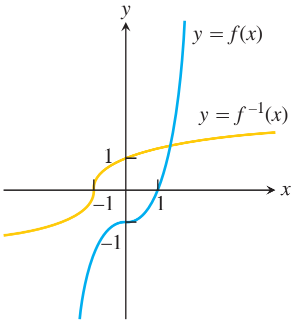
13. $f(x) = x^2 + 1, \quad x \ge 0$



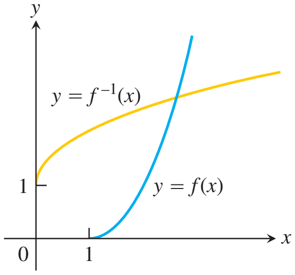
14. $f(x) = x^2, \quad x \le 0$



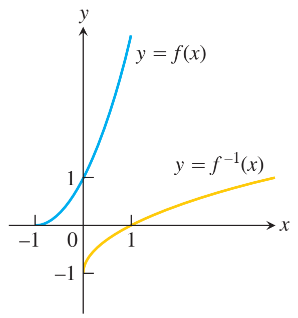
15. $f(x) = x^3 - 1$



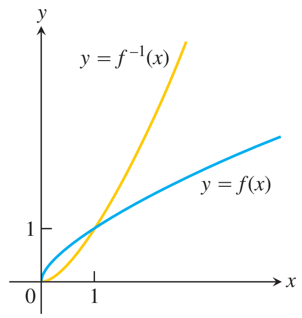
16. $f(x) = x^2 - 2x + 1, \quad x \ge 1$



17. $f(x) = (x + 1)^2, \quad x \geq -1$



18. $f(x) = x^{2/3}, \quad x \geq 0$



Each of Exercises 19–24 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

19. $f(x) = x^5$ **20.** $f(x) = x^4, \quad x \geq 0$

21. $f(x) = x^3 + 1$ **22.** $f(x) = (1/2)x - 7/2$

23. $f(x) = 1/x^2, \quad x > 0$ **24.** $f(x) = 1/x^3, \quad x \neq 0$

Derivatives of Inverse Functions

In Exercises 25–28:

a. Find $f^{-1}(x)$.

b. Graph f and f^{-1} together.

c. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

25. $f(x) = 2x + 3, \quad a = -1$ **26.** $f(x) = (1/5)x + 7, \quad a = -1$

27. $f(x) = 5 - 4x, \quad a = 1/2$ **28.** $f(x) = 2x^2, \quad x \geq 0, \quad a = 5$

29. a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

b. Graph f and g over an x-interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry about the line $y = x$.

c. Find the slopes of the tangents to the graphs of f and g at $(1, 1)$ and $(-1, -1)$ (four tangents in all).

d. What lines are tangent to the curves at the origin?

30. a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.

b. Graph h and k over an x-interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.

c. Find the slopes of the tangents to the graphs at h and k at $(2, 2)$ and $(-2, -2)$.

d. What lines are tangent to the curves at the origin?

31. Let $f(x) = x^3 - 3x^2 - 1, x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

32. Let $f(x) = x^2 - 4x - 5, x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.

33. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of df^{-1}/dx at $x = 4$.

34. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.

Inverses of Lines

35. a. Find the inverse of the function $f(x) = mx$, where m is a constant different from zero.

b. What can you conclude about the inverse of a function $y = f(x)$ whose graph is a line through the origin with a nonzero slope m?

36. Show that the graph of the inverse of $f(x) = mx + b$, where m and b are constants and $m \neq 0$, is a line with slope $1/m$ and y-intercept $-b/m$.

37. a. Find the inverse of $f(x) = x + 1$. Graph f and its inverse together. Add the line $y = x$ to your sketch, drawing it with dashes or dots for contrast.

b. Find the inverse of $f(x) = x + b$ (b constant). How is the graph of f^{-1} related to the graph of f?

c. What can you conclude about the inverses of functions whose graphs are lines parallel to the line $y = x$?

38. a. Find the inverse of $f(x) = -x + 1$. Graph the line $y = -x + 1$ together with the line $y = x$. At what angle do the lines intersect?

b. Find the inverse of $f(x) = -x + b$ (b constant). What angle does the line $y = -x + b$ make with the line $y = x$?

c. What can you conclude about the inverses of functions whose graphs are lines perpendicular to the line $y = x$?

Increasing and Decreasing Functions

39. As in Section 4.3, a function $f(x)$ increases on an interval I if for any two points x_1 and x_2 in I,

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) > f(x_1).$$

Similarly, a function decreases on I if for any two points x_1 and x_2 in I,

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) < f(x_1).$$

Show that increasing functions and decreasing functions are one-to-one. That is, show that for any x_1 and x_2 in I, $x_2 \neq x_1$ implies $f(x_2) \neq f(x_1)$.

Use the results of Exercise 39 to show that the functions in Exercises 40–44 have inverses over their domains. Find a formula for df^{-1}/dx using Theorem 1.

40. $f(x) = (1/3)x + (5/6)$ **41.** $f(x) = 27x^3$

42. $f(x) = 1 - 8x^3$ **43.** $f(x) = (1 - x)^3$

44. $f(x) = x^{5/3}$

Theory and Applications

45. If $f(x)$ is one-to-one, can anything be said about $g(x) = -f(x)$? Is it also one-to-one? Give reasons for your answer.

46. If $f(x)$ is one-to-one and $f(x)$ is never zero, can anything be said about $h(x) = 1/f(x)$? Is it also one-to-one? Give reasons for your answer.

47. Suppose that the range of g lies in the domain of f so that the composite $f \circ g$ is defined. If f and g are one-to-one, can anything be said about $f \circ g$? Give reasons for your answer.

48. If a composite $f \circ g$ is one-to-one, must g be one-to-one? Give reasons for your answer.

49. Suppose $f(x)$ is positive, continuous, and increasing over the interval $[a, b]$. By interpreting the graph of f show that

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a).$$

50. Determine conditions on the constants a, b, c, and d so that the rational function

$$f(x) = \frac{ax + b}{cx + d}$$

has an inverse.

51. If we write $g(x)$ for $f^{-1}(x)$, Equation (1) can be written as

$$g'(f(a)) = \frac{1}{f'(a)}, \quad \text{or} \quad g'(f(a)) \cdot f'(a) = 1.$$

If we then write x for a, we get

$$g'(f(x)) \cdot f'(x) = 1.$$

The latter equation may remind you of the Chain Rule, and indeed there is a connection.

Assume that f and g are differentiable functions that are inverses of one another, so that $(g \circ f)(x) = x$. Differentiate both sides of this equation with respect to x, using the Chain Rule to express $(g \circ f)'(x)$ as a product of derivatives of g and f. What do you find? (This is not a proof of Theorem 1 because we assume here the theorem's conclusion that $g = f^{-1}$ is differentiable.)

52. Equivalence of the washer and shell methods for finding volume Let f be differentiable and increasing on the interval $a \le x \le b$, with $a > 0$, and suppose that f has a differentiable inverse, f^{-1}. Revolve about the y-axis the region bounded by the graph of f and the lines $x = a$ and $y = f(b)$ to generate a solid. Then the values of the integrals given by the washer and shell methods for the volume have identical values:

$$\int_{f(a)}^{f(b)} \pi((f^{-1}(y))^2 - a^2)\,dy = \int_a^b 2\pi x(f(b) - f(x))\,dx.$$

To prove this equality, define

$$W(t) = \int_{f(a)}^{f(t)} \pi((f^{-1}(y))^2 - a^2)\,dy$$

$$S(t) = \int_a^t 2\pi x(f(t) - f(x))\,dx.$$

Then show that the functions W and S agree at a point of $[a, b]$ and have identical derivatives on $[a, b]$. As you saw in Section 4.8, Exercise 102, this will guarantee $W(t) = S(t)$ for all t in $[a, b]$. In particular, $W(b) = S(b)$. (*Source:* "Disks and Shells Revisited," by Walter Carlip, *American Mathematical Monthly*, Vol. 98, No. 2, Feb. 1991, pp. 154–156.)

COMPUTER EXPLORATIONS

In Exercises 53–60, you will explore some functions and their inverses together with their derivatives and linear approximating functions at specified points. Perform the following steps using your CAS:

a. Plot the function $y = f(x)$ together with its derivative over the given interval. Explain why you know that f is one-to-one over the interval.

b. Solve the equation $y = f(x)$ for x as a function of y, and name the resulting inverse function g.

c. Find the equation for the tangent line to f at the specified point $(x_0, f(x_0))$.

d. Find the equation for the tangent line to g at the point $(f(x_0), x_0)$ located symmetrically across the 45° line $y = x$ (which is the graph of the identity function). Use Theorem 1 to find the slope of this tangent line.

e. Plot the functions f and g, the identity, the two tangent lines, and the line segment joining the points $(x_0, f(x_0))$ and $(f(x_0), x_0)$. Discuss the symmetries you see across the main diagonal.

53. $y = \sqrt{3x - 2}, \quad \frac{2}{3} \le x \le 4, \quad x_0 = 3$

54. $y = \frac{3x + 2}{2x - 11}, \quad -2 \le x \le 2, \quad x_0 = 1/2$

55. $y = \frac{4x}{x^2 + 1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

56. $y = \frac{x^3}{x^2 + 1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

57. $y = x^3 - 3x^2 - 1, \quad 2 \le x \le 5, \quad x_0 = \frac{27}{10}$

58. $y = 2 - x - x^3, \quad -2 \le x \le 2, \quad x_0 = \frac{3}{2}$

59. $y = e^x, \quad -3 \le x \le 5, \quad x_0 = 1$

60. $y = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \quad x_0 = 1$

In Exercises 61 and 62, repeat the steps above to solve for the functions $y = f(x)$ and $x = f^{-1}(y)$ defined implicitly by the given equations over the interval.

61. $y^{1/3} - 1 = (x + 2)^3, \quad -5 \le x \le 5, \quad x_0 = -3/2$

62. $\cos y = x^{1/5}, \quad 0 \le x \le 1, \quad x_0 = 1/2$

7.2 Natural Logarithms

For any positive number a, the function value $f(x) = a^x$ is easy to define when x is an integer or rational number. When x is irrational, the meaning of a^x is not so clear. Similarly, the definition of the logarithm $\log_a x$, the inverse function of $f(x) = a^x$, is not completely obvious. In this section we use integral calculus to define the *natural logarithm* function, for which the number a is a particularly important value. This function allows us to define and analyze general exponential and logarithmic functions, $y = a^x$ and $y = \log_a x$.

Logarithms originally played important roles in arithmetic computations. Historically, considerable labor went into producing long tables of logarithms, correct to five, eight, or even more, decimal places of accuracy. Prior to the modern age of electronic calculators and computers, every engineer owned slide rules marked with logarithmic scales. Calculations with logarithms made possible the great seventeenth-century advances in offshore navigation and celestial mechanics. Today we know such calculations are done using calculators or computers, but the properties and numerous applications of logarithms are as important as ever.

Definition of the Natural Logarithm Function

One solid approach to defining and understanding logarithms begins with a study of the natural logarithm function defined as an integral through the Fundamental Theorem of Calculus. While this approach may seem indirect, it enables us to derive quickly the familiar properties of logarithmic and exponential functions. The functions we have studied so far were analyzed using the techniques of calculus, but here we do something more fundamental. We use calculus for the very definition of the logarithmic and exponential functions.

The natural logarithm of a positive number x, written as $\ln x$, is the value of an integral.

DEFINITION The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t}\, dt, \qquad x > 0$$

If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Figure 7.9). For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1. The function is not defined for $x \leq 0$. From the Zero Width Interval Rule for definite integrals, we also have

$$\ln 1 = \int_1^1 \frac{1}{t}\, dt = 0.$$

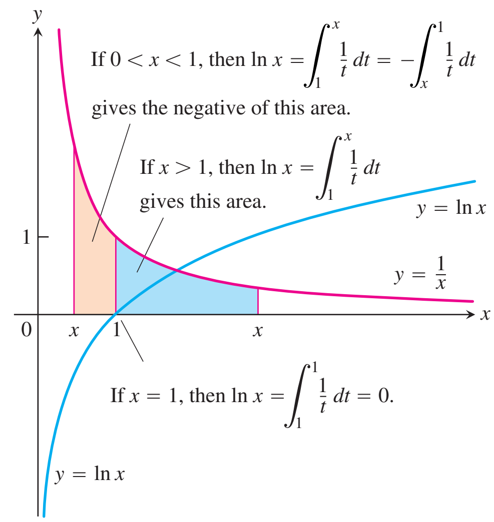


FIGURE 7.9 The graph of $y = \ln x$ and its relation to the function $y = 1/x, x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

TABLE 7.1 Typical 2-place values of $\ln x$

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

Notice that we show the graph of $y = 1/x$ in Figure 7.9 but use $y = 1/t$ in the integral. Using x for everything would have us writing

$$\ln x = \int_1^x \frac{1}{x}\,dx,$$

with x meaning two different things. So we change the variable of integration to t.

By using rectangles to obtain finite approximations of the area under the graph of $y = 1/t$ and over the interval between $t = 1$ and $t = x$, as in Section 5.1, we can approximate the values of the function $\ln x$. Several values are given in Table 7.1. There is an important number whose natural logarithm equals 1.

DEFINITION ***The Number e***

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

Geometrically, the number e corresponds to the point on the x-axis for which the area under the graph of $y = 1/t$ and above the interval $[1, e]$ is the exact area of the unit square. The area of the region shaded blue in Figure 7.9 is 1 sq unit when $x = e$.

The Derivative of $y = \ln x$

By the first part of the Fundamental Theorem of Calculus (Section 5.4),

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

For every positive value of x, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Therefore, the function $y = \ln x$ is a solution to the initial value problem $dy/dx = 1/x$, $x > 0$, with $y(1) = 0$. Notice that the derivative is always positive so the natural logarithm is an increasing function, hence it is one-to-one and invertible. Its inverse is studied in Section 7.3.

If u is a differentiable function of x whose values are positive, so that $\ln u$ is defined, then applying the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to the function $y = \ln u$ gives

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \qquad (1)$$

EXAMPLE 1 Derivatives of Natural Logarithms

(a) $\dfrac{d}{dx}\ln 2x = \dfrac{1}{2x}\dfrac{d}{dx}(2x) = \dfrac{1}{2x}(2) = \dfrac{1}{x}$

(b) Equation (1) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2 + 3) = \frac{1}{x^2 + 3}\cdot\frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3}\cdot 2x = \frac{2x}{x^2 + 3}.$$

Notice the remarkable occurrence in Example 1a. The function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln ax$ for any positive number a:

$$\frac{d}{dx}\ln ax = \frac{1}{ax}\cdot\frac{d}{dx}(ax) = \frac{1}{ax}(a) = \frac{1}{x}. \tag{2}$$

Since they have the same derivative, the functions $y = \ln ax$ and $y = \ln x$ differ by a constant.

HISTORICAL BIOGRAPHY

John Napier
(1550–1617)

Biography

Properties of Logarithms

Logarithms were invented by John Napier and were the single most important improvement in arithmetic calculation before the modern electronic computer. What made them so useful is that the properties of logarithms enable multiplication of positive numbers by addition of their logarithms, division of positive numbers by subtraction of their logarithms, and exponentiation of a number by multiplying its logarithm by the exponent. We summarize these properties as a series of rules in Theorem 2. For the moment, we restrict the exponent r in Rule 4 to be a rational number; you will see why when we prove the rule.

THEOREM 2 Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ — Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ — r rational

We illustrate how these rules apply.

You Try It

EXAMPLE 2 Interpreting the Properties of Logarithms

(a) $\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3$ — Product

(b) $\ln 4 - \ln 5 = \ln\frac{4}{5} = \ln 0.8$ — Quotient

(c) $\ln\frac{1}{8} = -\ln 8$ — Reciprocal

$= -\ln 2^3 = -3\ln 2$ — Power ■

EXAMPLE 3 Applying the Properties to Function Formulas

(a) $\ln 4 + \ln \sin x = \ln(4 \sin x)$ — Product

(b) $\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ — Quotient

(c) $\ln \sec x = \ln \dfrac{1}{\cos x} = -\ln \cos x$ Reciprocal

(d) $\ln \sqrt[3]{x+1} = \ln (x+1)^{1/3} = \dfrac{1}{3} \ln (x+1)$ Power ■

We now give the proof of Theorem 2. The steps in the proof are similar to those used in solving problems involving logarithms.

Proof that $\ln ax = \ln a + \ln x$ The argument is unusual—and elegant. It starts by observing that $\ln ax$ and $\ln x$ have the same derivative (Equation 2). According to Corollary 2 of the Mean Value Theorem, then, the functions must differ by a constant, which means that

$$\ln ax = \ln x + C$$

for some C.

Since this last equation holds for all positive values of x, it must hold for $x = 1$. Hence,

$$\begin{aligned} \ln (a \cdot 1) &= \ln 1 + C \\ \ln a &= 0 + C \qquad \ln 1 = 0 \\ C &= \ln a. \end{aligned}$$

By substituting we conclude,

$$\ln ax = \ln a + \ln x.$$

Proof that $\ln x^r = r \ln x$ (assuming r rational) We use the same-derivative argument again. For all positive values of x,

$$\begin{aligned} \frac{d}{dx} \ln x^r &= \frac{1}{x^r} \frac{d}{dx}(x^r) && \text{Eq. (1) with } u = x^r \\ &= \frac{1}{x^r} r x^{r-1} \\ &= r \cdot \frac{1}{x} = \frac{d}{dx}(r \ln x). \end{aligned}$$

Here is where we need r to be rational, at least for now. We have proved the Power Rule only for rational exponents.

Since $\ln x^r$ and $r \ln x$ have the same derivative,

$$\ln x^r = r \ln x + C$$

for some constant C. Taking x to be 1 identifies C as zero, and we're done.

You are asked to prove Rule 2 in Exercise 84. Rule 3 is a special case of Rule 2, obtained by setting $a = 1$ and noting that $\ln 1 = 0$. So we have established all cases of Theorem 2. ■

We have not yet proved Rule 4 for r irrational; we will return to this case in Section 7.3. The rule does hold for all r, rational or irrational.

The Graph and Range of ln *x*

The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down.

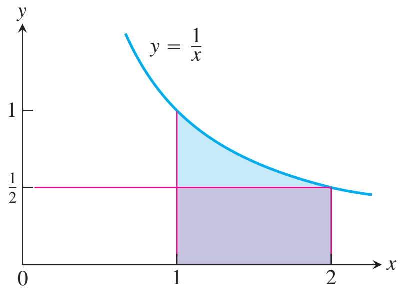


FIGURE 7.10 The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \leq x \leq 2$.

We can estimate the value of ln 2 by considering the area under the graph of $y = 1/x$ and above the interval [1, 2]. In Figure 7.10 a rectangle of height 1/2 over the interval [1, 2] fits under the graph. Therefore the area under the graph, which is ln 2, is greater than the area, 1/2, of the rectangle. So $\ln 2 > 1/2$. Knowing this we have,

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}$$

and

$$\ln 2^{-n} = -n \ln 2 < -n\left(\frac{1}{2}\right) = -\frac{n}{2}.$$

It follows that

$$\lim_{x \to \infty} \ln x = \infty \qquad \text{and} \qquad \lim_{x \to 0^+} \ln x = -\infty.$$

We defined $\ln x$ for $x > 0$, so the domain of $\ln x$ is the set of positive real numbers. The above discussion and the Intermediate Value Theorem show that its range is the entire real line giving the graph of $y = \ln x$ shown in Figure 7.9.

The Integral $\int (1/u)\, du$

Equation (1) leads to the integral formula

$$\int \frac{1}{u}\, du = \ln u + C \tag{3}$$

when u is a positive differentiable function, but what if u is negative? If u is negative, then $-u$ is positive and

$$\begin{aligned} \int \frac{1}{u}\, du &= \int \frac{1}{(-u)}\, d(-u) && \text{Eq. (3) with } u \text{ replaced by } -u \\ &= \ln(-u) + C. && (4) \end{aligned}$$

We can combine Equations (3) and (4) into a single formula by noticing that in each case the expression on the right is $\ln |u| + C$. In Equation (3), $\ln u = \ln |u|$ because $u > 0$; in Equation (4), $\ln(-u) = \ln |u|$ because $u < 0$. Whether u is positive or negative, the integral of $(1/u)\, du$ is $\ln |u| + C$.

> If u is a differentiable function that is never zero,
>
> $$\int \frac{1}{u}\, du = \ln |u| + C. \tag{5}$$

Equation (5) applies anywhere on the domain of $1/u$, the points where $u \neq 0$.

We know that

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1 \text{ and rational}$$

Equation (5) explains what to do when n equals -1. Equation (5) says integrals of a certain *form* lead to logarithms. If $u = f(x)$, then $du = f'(x)\,dx$ and

$$\int \frac{1}{u}\,du = \int \frac{f'(x)}{f(x)}\,dx.$$

So Equation (5) gives

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

whenever $f(x)$ is a differentiable function that maintains a constant sign on the domain given for it.

EXAMPLE 4 Applying Equation (5)

(a)

$$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

$u = x^2 - 5,\quad du = 2x\,dx,$
$u(0) = -5,\quad u(2) = -1$

(b)

$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3 + 2\sin\theta}\,d\theta = \int_1^5 \frac{2}{u}\,du$$

$$= 2\ln|u|\Big]_1^5$$

$$= 2\ln|5| - 2\ln|1| = 2\ln 5$$

$u = 3 + 2\sin\theta,\quad du = 2\cos\theta\,d\theta,$
$u(-\pi/2) = 1,\quad u(\pi/2) = 5$

Note that $u = 3 + 2\sin\theta$ is always positive on $[-\pi/2, \pi/2]$, so Equation (5) applies. ■

The Integrals of tan *x* and cot *x*

Equation (5) tells us at last how to integrate the tangent and cotangent functions. For the tangent function,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$$

$u = \cos x > 0$ on $(-\pi/2, \pi/2)$,
$du = -\sin x\,dx$

$$= -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C$$

Reciprocal Rule

$$= \ln|\sec x| + C.$$

For the cotangent,

$$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u}$$

$u = \sin x,$
$du = \cos x\,dx$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

$$\int \tan u\, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u\, du = \ln|\sin u| + C = -\ln|\csc x| + C$$

EXAMPLE 5

$$\int_0^{\pi/6} \tan 2x\, dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u\, du$$

$$= \frac{1}{2}\ln|\sec u|\Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi/3$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

EXAMPLE 6 Using Logarithmic Differentiation

You Try It

Find dy/dx if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\ln y = \ln\frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

$$= \ln((x^2+1)(x+3)^{1/2}) - \ln(x-1) \qquad \text{Rule 2}$$

$$= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) \qquad \text{Rule 1}$$

$$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). \qquad \text{Rule 3}$$

We then take derivatives of both sides with respect to x, using Equation (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right). \qquad \blacksquare$$

A direct computation in Example 6, using the Quotient and Product Rules, would be much longer.

EXERCISES 7.2

Using the Properties of Logarithms

1. Express the following logarithms in terms of ln 2 and ln 3.
   a. $\ln 0.75$ b. $\ln(4/9)$ c. $\ln(1/2)$
   d. $\ln \sqrt[3]{9}$ e. $\ln 3\sqrt{2}$ f. $\ln \sqrt{13.5}$
2. Express the following logarithms in terms of ln 5 and ln 7.
   a. $\ln(1/125)$ b. $\ln 9.8$ c. $\ln 7\sqrt{7}$
   d. $\ln 1225$ e. $\ln 0.056$
   f. $(\ln 35 + \ln(1/7))/(\ln 25)$

Use the properties of logarithms to simplify the expressions in Exercises 3 and 4.

3. a. $\ln \sin\theta - \ln\left(\frac{\sin\theta}{5}\right)$ b. $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right)$
   c. $\frac{1}{2}\ln(4t^4) - \ln 2$
4. a. $\ln \sec\theta + \ln\cos\theta$ b. $\ln(8x + 4) - 2\ln 2$
   c. $3\ln\sqrt[3]{t^2 - 1} - \ln(t + 1)$

Derivatives of Logarithms

In Exercises 5–36, find the derivative of y with respect to x, t, or θ, as appropriate.

5. $y = \ln 3x$
6. $y = \ln kx$, k constant
7. $y = \ln(t^2)$
8. $y = \ln(t^{3/2})$
9. $y = \ln\frac{3}{x}$
10. $y = \ln\frac{10}{x}$
11. $y = \ln(\theta + 1)$
12. $y = \ln(2\theta + 2)$
13. $y = \ln x^3$
14. $y = (\ln x)^3$
15. $y = t(\ln t)^2$
16. $y = t\sqrt{\ln t}$
17. $y = \frac{x^4}{4}\ln x - \frac{x^4}{16}$
18. $y = \frac{x^3}{3}\ln x - \frac{x^3}{9}$
19. $y = \frac{\ln t}{t}$
20. $y = \frac{1 + \ln t}{t}$
21. $y = \frac{\ln x}{1 + \ln x}$
22. $y = \frac{x\ln x}{1 + \ln x}$
23. $y = \ln(\ln x)$
24. $y = \ln(\ln(\ln x))$
25. $y = \theta(\sin(\ln\theta) + \cos(\ln\theta))$
26. $y = \ln(\sec\theta + \tan\theta)$
27. $y = \ln\frac{1}{x\sqrt{x + 1}}$
28. $y = \frac{1}{2}\ln\frac{1 + x}{1 - x}$
29. $y = \frac{1 + \ln t}{1 - \ln t}$
30. $y = \sqrt{\ln\sqrt{t}}$
31. $y = \ln(\sec(\ln\theta))$
32. $y = \ln\left(\frac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$
33. $y = \ln\left(\frac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$
34. $y = \ln\sqrt{\frac{(x + 1)^5}{(x + 2)^{20}}}$
35. $y = \int_{x^2/2}^{x^2} \ln\sqrt{t}\,dt$
36. $y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\,dt$

Integration

Evaluate the integrals in Exercises 37–54.

37. $\int_{-3}^{-2}\frac{dx}{x}$
38. $\int_{-1}^{0}\frac{3\,dx}{3x - 2}$
39. $\int\frac{2y\,dy}{y^2 - 25}$
40. $\int\frac{8r\,dr}{4r^2 - 5}$
41. $\int_0^{\pi}\frac{\sin t}{2 - \cos t}\,dt$
42. $\int_0^{\pi/3}\frac{4\sin\theta}{1 - 4\cos\theta}\,d\theta$
43. $\int_1^2\frac{2\ln x}{x}\,dx$
44. $\int_2^4\frac{dx}{x\ln x}$
45. $\int_2^4\frac{dx}{x(\ln x)^2}$
46. $\int_2^{16}\frac{dx}{2x\sqrt{\ln x}}$
47. $\int\frac{3\sec^2 t}{6 + 3\tan t}\,dt$
48. $\int\frac{\sec y\tan y}{2 + \sec y}\,dy$
49. $\int_0^{\pi/2}\tan\frac{x}{2}\,dx$
50. $\int_{\pi/4}^{\pi/2}\cot t\,dt$
51. $\int_{\pi/2}^{\pi}2\cot\frac{\theta}{3}\,d\theta$
52. $\int_0^{\pi/12}6\tan 3x\,dx$
53. $\int\frac{dx}{2\sqrt{x} + 2x}$
54. $\int\frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}}$

Logarithmic Differentiation

In Exercises 55–68, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

55. $y = \sqrt{x(x+1)}$

56. $y = \sqrt{(x^2+1)(x-1)^2}$

57. $y = \sqrt{\dfrac{t}{t+1}}$

58. $y = \sqrt{\dfrac{1}{t(t+1)}}$

59. $y = \sqrt{\theta + 3}\sin\theta$

60. $y = (\tan\theta)\sqrt{2\theta + 1}$

61. $y = t(t+1)(t+2)$

62. $y = \dfrac{1}{t(t+1)(t+2)}$

63. $y = \dfrac{\theta + 5}{\theta\cos\theta}$

64. $y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$

65. $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

66. $y = \sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$

67. $y = \sqrt[3]{\dfrac{x(x-2)}{x^2+1}}$

68. $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

Theory and Applications

69. Locate and identify the absolute extreme values of

a. $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,

b. $\cos(\ln x)$ on $[1/2, 2]$.

70. a. Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.

b. Using part (a), show that $\ln x < x$ if $x > 1$.

71. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

72. Find the area between the curve $y = \tan x$ and the x-axis from $x = -\pi/4$ to $x = \pi/3$.

73. The region in the first quadrant bounded by the coordinate axes, the line $y = 3$, and the curve $x = 2/\sqrt{y+1}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

74. The region between the curve $y = \sqrt{\cot x}$ and the x-axis from $x = \pi/6$ to $x = \pi/2$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

75. The region between the curve $y = 1/x^2$ and the x-axis from $x = 1/2$ to $x = 2$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

76. In Section 6.2, Exercise 6, we revolved about the y-axis the region between the curve $y = 9x/\sqrt{x^3+9}$ and the x-axis from $x = 0$ to $x = 3$ to generate a solid of volume 36π. What volume do you get if you revolve the region about the x-axis instead? (See Section 6.2, Exercise 6, for a graph.)

77. Find the lengths of the following curves.

a. $y = (x^2/8) - \ln x, \quad 4 \le x \le 8$

b. $x = (y/4)^2 - 2\ln(y/4), \quad 4 \le y \le 12$

78. Find a curve through the point (1, 0) whose length from $x = 1$ to $x = 2$ is

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\,dx.$$

79. a. Find the centroid of the region between the curve $y = 1/x$ and the x-axis from $x = 1$ to $x = 2$. Give the coordinates to two decimal places.

b. Sketch the region and show the centroid in your sketch.

80. a. Find the center of mass of a thin plate of constant density covering the region between the curve $y = 1/\sqrt{x}$ and the x-axis from $x = 1$ to $x = 16$.

b. Find the center of mass if, instead of being constant, the density function is $\delta(x) = 4/\sqrt{x}$.

Solve the initial value problems in Exercises 81 and 82.

81. $\dfrac{dy}{dx} = 1 + \dfrac{1}{x}, \quad y(1) = 3$

82. $\dfrac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1$

83. The linearization of $\ln(1 + x)$ at $x = 0$ Instead of approximating $\ln x$ near $x = 1$, we approximate $\ln(1 + x)$ near $x = 0$. We get a simpler formula this way.

a. Derive the linearization $\ln(1 + x) \approx x$ at $x = 0$.

b. Estimate to five decimal places the error involved in replacing $\ln(1 + x)$ by x on the interval [0, 0.1].

c. Graph $\ln(1 + x)$ and x together for $0 \le x \le 0.5$. Use different colors, if available. At what points does the approximation of $\ln(1 + x)$ seem best? Least good? By reading coordinates from the graphs, find as good an upper bound for the error as your grapher will allow.

84. Use the same-derivative argument, as was done to prove Rules 1 and 4 of Theorem 2, to prove the Quotient Rule property of logarithms.

Grapher Explorations

85. Graph $\ln x$, $\ln 2x$, $\ln 4x$, $\ln 8x$, and $\ln 16x$ (as many as you can) together for $0 < x \le 10$. What is going on? Explain.

86. Graph $y = \ln|\sin x|$ in the window $0 \le x \le 22$, $-2 \le y \le 0$. Explain what you see. How could you change the formula to turn the arches upside down?

87. a. Graph $y = \sin x$ and the curves $y = \ln(a + \sin x)$ for $a = 2$, 4, 8, 20, and 50 together for $0 \le x \le 23$.

b. Why do the curves flatten as a increases? (*Hint:* Find an a-dependent upper bound for $|y'|$.)

88. Does the graph of $y = \sqrt{x} - \ln x$, $x > 0$, have an inflection point? Try to answer the question **(a)** by graphing, **(b)** by using calculus.

7.3 The Exponential Function

Having developed the theory of the function $\ln x$, we introduce the exponential function $\exp x = e^x$ as the inverse of $\ln x$. We study its properties and compute its derivative and integral. Knowing its derivative, we prove the power rule to differentiate x^n when n is *any* real number, rational or irrational.

The Inverse of ln *x* and the Number *e*

The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. As you can see in Figure 7.11,

$$\lim_{x\to\infty} \ln^{-1} x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} \ln^{-1} x = 0.$$

The function $\ln^{-1} x$ is also denoted by $\exp x$.

In Section 7.2 we defined the number e by the equation $\ln(e) = 1$, so $e = \ln^{-1}(1) = \exp(1)$. Although e is not a rational number, later in this section we see one way to express it as a limit. In Chapter 11, we will calculate its value with a computer to as many places of accuracy as we want with a different formula (Section 11.9, Example 6). To 15 places,

$$e = 2.718281828459045.$$

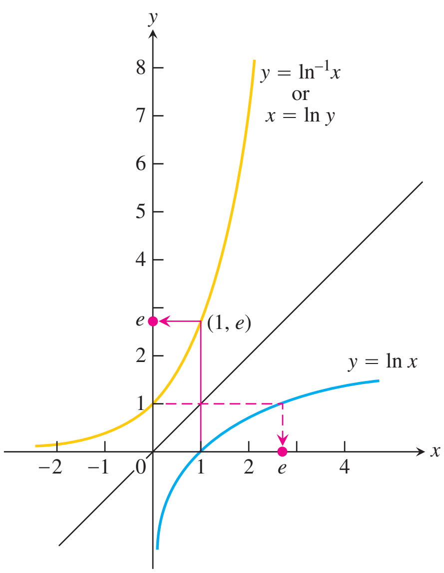


FIGURE 7.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

The Function $y = e^x$

We can raise the number e to a rational power r in the usual way:

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e},$$

and so on. Since e is positive, e^r is positive too. Thus, e^r has a logarithm. When we take the logarithm, we find that

$$\ln e^r = r \ln e = r \cdot 1 = r.$$

Since $\ln x$ is one-to-one and $\ln(\ln^{-1} r) = r$, this equation tells us that

$$e^r = \ln^{-1} r = \exp r \qquad \text{for } r \text{ rational}. \tag{1}$$

We have not yet found a way to give an obvious meaning to e^x for x irrational. But $\ln^{-1} x$ has meaning for any x, rational or irrational. So Equation (1) provides a way to extend the definition of e^x to irrational values of x. The function $\ln^{-1} x$ is defined for all x, so we use it to assign a value to e^x at every point where e^x had no previous definition.

Typical values of e^x

x	e^x **(rounded)**
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

DEFINITION **The Natural Exponential Function**

For every real number x, $e^x = \ln^{-1} x = \exp x$.

For the first time we have a precise meaning for an irrational exponent. Usually the exponential function is denoted by e^x rather than $\exp x$. Since $\ln x$ and e^x are inverses of one another, we have

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \qquad (2)$$

$$\ln(e^x) = x \qquad (\text{all } x) \qquad (3)$$

The domain of $\ln x$ is $(0, \infty)$ and its range is $(-\infty, \infty)$. So the domain of e^x is $(-\infty, \infty)$ and its range is $(0, \infty)$.

Transcendental Numbers and Transcendental Functions

Numbers that are solutions of polynomial equations with rational coefficients are called **algebraic**: -2 is algebraic because it satisfies the equation $x + 2 = 0$, and $\sqrt{3}$ is algebraic because it satisfies the equation $x^2 - 3 = 0$. Numbers that are not algebraic are called **transcendental**, like e and π. In 1873, Charles Hermite proved the transcendence of e in the sense that we describe. In 1882, C.L.F. Lindemann proved the transcendence of π.

Today, we call a function $y = f(x)$ algebraic if it satisfies an equation of the form

$$P_n y^n + \cdots + P_1 y + P_0 = 0$$

in which the P's are polynomials in x with rational coefficients. The function $y = 1/\sqrt{x+1}$ is algebraic because it satisfies the equation $(x + 1)y^2 - 1 = 0$. Here the polynomials are $P_2 = x + 1$, $P_1 = 0$, and $P_0 = -1$. Functions that are not algebraic are called transcendental.

EXAMPLE 1 Using the Inverse Equations

(a) $\ln e^2 = 2$

(b) $\ln e^{-1} = -1$

(c) $\ln \sqrt{e} = \frac{1}{2}$

(d) $\ln e^{\sin x} = \sin x$

(e) $e^{\ln 2} = 2$

(f) $e^{\ln(x^2+1)} = x^2 + 1$

(g) $e^{3 \ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$ — One way

(h) $e^{3 \ln 2} = (e^{\ln 2})^3 = 2^3 = 8$ — Another way ■

EXAMPLE 2 Solving for an Exponent

Find k if $e^{2k} = 10$.

Solution Take the natural logarithm of both sides:

$$e^{2k} = 10$$

$$\ln e^{2k} = \ln 10$$

$$2k = \ln 10 \qquad \text{Eq. (3)}$$

$$k = \frac{1}{2} \ln 10. \qquad ■$$

The General Exponential Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can think of a^x as $(e^{\ln a})^x = e^{x \ln a}$. We therefore make the following definition.

DEFINITION **General Exponential Functions**

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

When $a = e$, the definition gives $a^x = e^{x \ln a} = e^{x \ln e} = e^{x \cdot 1} = e^x$.

HISTORICAL BIOGRAPHY

Siméon Denis Poisson
(1781–1840)

Biography

EXAMPLE 3 Evaluating Exponential Functions

(a) $2^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$

(b) $2^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$ ■

We study the calculus of general exponential functions and their inverses in the next section. Here we need the definition in order to discuss the laws of exponents for e^x.

Laws of Exponents

Even though e^x is defined in a seemingly roundabout way as $\ln^{-1} x$, it obeys the familiar laws of exponents from algebra. Theorem 3 shows us that these laws are consequences of the definitions of $\ln x$ and e^x.

THEOREM 3 **Laws of Exponents for e^x**

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Proof of Law 1 Let

$$y_1 = e^{x_1} \qquad \text{and} \qquad y_2 = e^{x_2}. \tag{4}$$

Then

$$\begin{aligned}
x_1 &= \ln y_1 \quad \text{and} \quad x_2 = \ln y_2 && \text{Take logs of both sides of Eqs. (4).}\\
x_1 + x_2 &= \ln y_1 + \ln y_2 \\
&= \ln y_1 y_2 && \text{Product Rule for logarithms}\\
e^{x_1 + x_2} &= e^{\ln y_1 y_2} && \text{Exponentiate.}\\
&= y_1 y_2 && e^{\ln u} = u\\
&= e^{x_1} e^{x_2}.
\end{aligned}$$
■

The proof of Law 4 is similar. Laws 2 and 3 follow from Law 1 (Exercise 78).

EXAMPLE 4 Applying the Exponent Laws

(a) $e^{x + \ln 2} = e^x \cdot e^{\ln 2} = 2e^x$ Law 1

(b) $e^{-\ln x} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$ Law 2

(c) $\dfrac{e^{2x}}{e} = e^{2x-1}$ Law 3

(d) $(e^3)^x = e^{3x} = (e^x)^3$ Law 4 ■

Theorem 3 is also valid for a^x, the exponential function with base a. For example,

$$\begin{aligned} a^{x_1} \cdot a^{x_2} &= e^{x_1 \ln a} \cdot e^{x_2 \ln a} && \text{Definition of } a^x \\ &= e^{x_1 \ln a + x_2 \ln a} && \text{Law 1} \\ &= e^{(x_1+x_2)\ln a} && \text{Factor } \ln a \\ &= a^{x_1+x_2}. && \text{Definition of } a^x \end{aligned}$$

The Derivative and Integral of e^x

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero (Theorem 1). We calculate its derivative using Theorem 1 and our knowledge of the derivative of $\ln x$. Let

$$f(x) = \ln x \quad \text{and} \quad y = e^x = \ln^{-1} x = f^{-1}(x).$$

Then,

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}(e^x) &= \frac{d}{dx} \ln^{-1} x \\ &= \frac{d}{dx} f^{-1}(x) \\ &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\ &= \frac{1}{f'(e^x)} && f^{-1}(x) = e^x \\ &= \frac{1}{\left(\frac{1}{e^x}\right)} && f'(z) = \frac{1}{z} \text{ with } z = e^x \\ &= e^x. \end{aligned}$$

That is, for $y = e^x$, we find that $dy/dx = e^x$ so the natural exponential function e^x is its own derivative. We will see in Section 7.5 that the only functions that behave this way are constant multiples of e^x. In summary,

$$\frac{d}{dx} e^x = e^x \tag{5}$$

EXAMPLE 5 Differentiating an Exponential

You Try It

$$\begin{aligned} \frac{d}{dx}(5e^x) &= 5\frac{d}{dx} e^x \\ &= 5e^x \end{aligned}$$

■

The Chain Rule extends Equation (5) in the usual way to a more general form.

If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}. \tag{6}$$

EXAMPLE 6 Applying the Chain Rule with Exponentials

(a) $\frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (6) with $u = -x$

(b) $\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cdot\cos x$ Eq. (6) with $u = \sin x$ ■

The integral equivalent of Equation (6) is

$$\int e^u\,du = e^u + C.$$

EXAMPLE 7 Integrating Exponentials

(a)

$$\begin{aligned}\int_0^{\ln 2} e^{3x}\,dx &= \int_0^{\ln 8} e^u\cdot\frac{1}{3}\,du \\ &= \frac{1}{3}\int_0^{\ln 8} e^u\,du \\ &= \frac{1}{3}e^u\Big]_0^{\ln 8} \\ &= \frac{1}{3}(8-1) = \frac{7}{3}\end{aligned}$$

$u = 3x$, $\frac{1}{3}du = dx$, $u(0) = 0$, $u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

(b)

$$\begin{aligned}\int_0^{\pi/2} e^{\sin x}\cos x\,dx &= e^{\sin x}\Big]_0^{\pi/2} \\ &= e^1 - e^0 = e - 1\end{aligned}$$

Antiderivative from Example 6 ■

EXAMPLE 8 Solving an Initial Value Problem

Solve the initial value problem

$$e^y\frac{dy}{dx} = 2x, \qquad x > \sqrt{3}; \qquad y(2) = 0.$$

Solution We integrate both sides of the differential equation with respect to x to obtain

$$e^y = x^2 + C.$$

We use the initial condition $y(2) = 0$ to determine C:

$$C = e^0 - (2)^2$$
$$= 1 - 4 = -3.$$

This completes the formula for e^y:

$$e^y = x^2 - 3.$$

To find y, we take logarithms of both sides:

$$\ln e^y = \ln(x^2 - 3)$$
$$y = \ln(x^2 - 3).$$

Notice that the solution is valid for $x > \sqrt{3}$.

Let's check the solution in the original equation.

$$e^y \frac{dy}{dx} = e^y \frac{d}{dx} \ln(x^2 - 3)$$
$$= e^y \frac{2x}{x^2 - 3} \qquad \text{Derivative of } \ln(x^2 - 3)$$
$$= e^{\ln(x^2-3)} \frac{2x}{x^2 - 3} \qquad y = \ln(x^2 - 3)$$
$$= (x^2 - 3)\frac{2x}{x^2 - 3} \qquad e^{\ln y} = y$$
$$= 2x.$$

The solution checks. ■

The Number e Expressed as a Limit

We have defined the number e as the number for which $\ln e = 1$, or the value $\exp(1)$. We see that e is an important constant for the logarithmic and exponential functions, but what is its numerical value? The next theorem shows one way to calculate e as a limit.

THEOREM 4 The Number e as a Limit

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1 + x) - f(1)}{x}$$
$$= \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1 + x) \qquad \ln 1 = 0$$
$$= \lim_{x \to 0} \ln(1 + x)^{1/x} = \ln\left[\lim_{x \to 0}(1 + x)^{1/x}\right] \qquad \text{ln is continuous.}$$

Because $f'(1) = 1$, we have

$$\ln\left[\lim_{x\to 0}(1 + x)^{1/x}\right] = 1$$

Therefore,

$$\lim_{x\to 0}(1 + x)^{1/x} = e \qquad \text{ln } e = 1 \text{ and ln is one-to-one}$$ ■

By substituting $y = 1/x$, we can also express the limit in Theorem 4 as

$$e = \lim_{y\to\infty}\left(1 + \frac{1}{y}\right)^{y}. \tag{7}$$

At the beginning of the section we noted that $e = 2.718281828459045$ to 15 decimal places.

The Power Rule (General Form)

We can now define x^n for any $x > 0$ and any real number n as $x^n = e^{n \ln x}$. Therefore, the n in the equation $\ln x^n = n \ln x$ no longer needs to be rational—it can be any number as long as $x > 0$:

$$\ln x^n = \ln\left(e^{n \ln x}\right) = n \ln x \qquad \ln e^u = u, \text{ any } u$$

Together, the law $a^x/a^y = a^{x-y}$ and the definition $x^n = e^{n \ln x}$ enable us to establish the Power Rule for differentiation in its final form. Differentiating x^n with respect to x gives

$$\begin{aligned}
\frac{d}{dx}x^n &= \frac{d}{dx}e^{n \ln x} && \text{Definition of } x^n,\ x > 0\\
&= e^{n \ln x}\cdot\frac{d}{dx}(n \ln x) && \text{Chain Rule for } e^u\\
&= x^n \cdot \frac{n}{x} && \text{The definition again}\\
&= nx^{n-1}.
\end{aligned}$$

In short, as long as $x > 0$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

The Chain Rule extends this equation to the Power Rule's general form.

Power Rule (General Form)

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

EXAMPLE 9 Using the Power Rule with Irrational Powers

(a) $\dfrac{d}{dx} x^{\sqrt{2}} = \sqrt{2} x^{\sqrt{2}-1} \qquad (x > 0)$

(b) $$\begin{aligned} \frac{d}{dx}(2 + \sin 3x)^{\pi} &= \pi(2 + \sin 3x)^{\pi-1}(\cos 3x) \cdot 3 \\ &= 3\pi(2 + \sin 3x)^{\pi-1}(\cos 3x). \end{aligned}$$

■

EXERCISES 7.3

Algebraic Calculations with the Exponential and Logarithm

Find simpler expressions for the quantities in Exercises 1–4.

1. a. $e^{\ln 7.2}$ **b.** $e^{-\ln x^2}$ **c.** $e^{\ln x - \ln y}$

2. a. $e^{\ln(x^2+y^2)}$ **b.** $e^{-\ln 0.3}$ **c.** $e^{\ln \pi x - \ln 2}$

3. a. $2\ln\sqrt{e}$ **b.** $\ln(\ln e^e)$ **c.** $\ln(e^{-x^2-y^2})$

4. a. $\ln(e^{\sec\theta})$ **b.** $\ln(e^{(e^x)})$ **c.** $\ln(e^{2\ln x})$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 5–10, solve for y in terms of t or x, as appropriate.

5. $\ln y = 2t + 4$

6. $\ln y = -t + 5$

7. $\ln(y - 40) = 5t$

8. $\ln(1 - 2y) = t$

9. $\ln(y - 1) - \ln 2 = x + \ln x$

10. $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

In Exercises 11 and 12, solve for k.

11. a. $e^{2k} = 4$ **b.** $100e^{10k} = 200$ **c.** $e^{k/1000} = a$

12. a. $e^{5k} = \frac{1}{4}$ **b.** $80e^k = 1$ **c.** $e^{(\ln 0.8)k} = 0.8$

In Exercises 13–16, solve for t.

13. a. $e^{-0.3t} = 27$ **b.** $e^{kt} = \frac{1}{2}$ **c.** $e^{(\ln 0.2)t} = 0.4$

14. a. $e^{-0.01t} = 1000$ **b.** $e^{kt} = \frac{1}{10}$ **c.** $e^{(\ln 2)t} = \frac{1}{2}$

15. $e^{\sqrt{t}} = x^2$

16. $e^{(x^2)}e^{(2x+1)} = e^t$

Derivatives

In Exercises 17–36, find the derivative of y with respect to x, t, or θ, as appropriate.

17. $y = e^{-5x}$

18. $y = e^{2x/3}$

19. $y = e^{5-7x}$

20. $y = e^{(4\sqrt{x}+x^2)}$

21. $y = xe^x - e^x$

22. $y = (1 + 2x)e^{-2x}$

23. $y = (x^2 - 2x + 2)e^x$

24. $y = (9x^2 - 6x + 2)e^{3x}$

25. $y = e^{\theta}(\sin\theta + \cos\theta)$

26. $y = \ln(3\theta e^{-\theta})$

27. $y = \cos(e^{-\theta^2})$

28. $y = \theta^3 e^{-2\theta}\cos 5\theta$

29. $y = \ln(3te^{-t})$

30. $y = \ln(2e^{-t}\sin t)$

31. $y = \ln\left(\dfrac{e^{\theta}}{1 + e^{\theta}}\right)$

32. $y = \ln\left(\dfrac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right)$

33. $y = e^{(\cos t + \ln t)}$

34. $y = e^{\sin t}(\ln t^2 + 1)$

35. $y = \displaystyle\int_0^{\ln x} \sin e^t \, dt$

36. $y = \displaystyle\int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

In Exercises 37–40, find dy/dx.

37. $\ln y = e^y \sin x$

38. $\ln xy = e^{x+y}$

39. $e^{2x} = \sin(x + 3y)$

40. $\tan y = e^x + \ln x$

Integrals

Evaluate the integrals in Exercises 41–62.

41. $\displaystyle\int (e^{3x} + 5e^{-x})\, dx$

42. $\displaystyle\int (2e^x - 3e^{-2x})\, dx$

43. $\displaystyle\int_{\ln 2}^{\ln 3} e^x \, dx$

44. $\displaystyle\int_{-\ln 2}^{0} e^{-x} \, dx$

45. $\displaystyle\int 8e^{(x+1)}\, dx$

46. $\displaystyle\int 2e^{(2x-1)}\, dx$

47. $\displaystyle\int_{\ln 4}^{\ln 9} e^{x/2}\, dx$

48. $\displaystyle\int_0^{\ln 16} e^{x/4}\, dx$

49. $\displaystyle\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\, dr$

50. $\displaystyle\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\, dr$

51. $\displaystyle\int 2t\, e^{-t^2}\, dt$

52. $\displaystyle\int t^3 e^{(t^4)}\, dt$

53. $\displaystyle\int \frac{e^{1/x}}{x^2}\, dx$

54. $\displaystyle\int \frac{e^{-1/x^2}}{x^3}\, dx$

55. $\displaystyle\int_0^{\pi/4} (1 + e^{\tan\theta})\sec^2\theta \, d\theta$

56. $\displaystyle\int_{\pi/4}^{\pi/2} (1 + e^{\cot\theta})\csc^2\theta \, d\theta$

57. $\displaystyle\int e^{\sec \pi t} \sec \pi t \tan \pi t \, dt$

58. $\displaystyle\int e^{\csc(\pi+t)} \csc(\pi + t)\cot(\pi + t)\, dt$

59. $\displaystyle\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^{v}\cos e^{v}\,dv$

60. $\displaystyle\int_{0}^{\sqrt{\ln \pi}} 2x\,e^{x^2}\cos\left(e^{x^2}\right)dx$

61. $\displaystyle\int \frac{e^r}{1+e^r}\,dr$

62. $\displaystyle\int \frac{dx}{1+e^x}$

Initial Value Problems

Solve the initial value problems in Exercises 63–66.

63. $\dfrac{dy}{dt} = e^t\sin(e^t - 2),\quad y(\ln 2) = 0$

64. $\dfrac{dy}{dt} = e^{-t}\sec^2(\pi e^{-t}),\quad y(\ln 4) = 2/\pi$

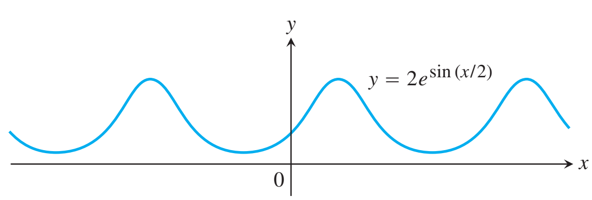
65. $\dfrac{d^2y}{dx^2} = 2e^{-x},\quad y(0) = 1 \quad\text{and}\quad y'(0) = 0$

66. $\dfrac{d^2y}{dt^2} = 1 - e^{2t},\quad y(1) = -1 \quad\text{and}\quad y'(1) = 0$

Theory and Applications

67. Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

68. Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?



69. Find the absolute maximum value of $f(x) = x^2\ln(1/x)$ and say where it is assumed.

T 70. Graph $f(x) = (x-3)^2e^x$ and its first derivative together. Comment on the behavior of f in relation to the signs and values of f'. Identify significant points on the graphs with calculus, as necessary.

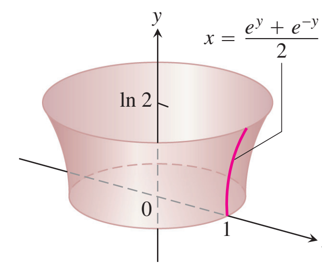
71. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.

72. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2\ln 2$.

73. Find a curve through the origin in the xy-plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\,dx.$$

74. Find the area of the surface generated by revolving the curve $x = (e^y + e^{-y})/2,\ 0 \le y \le \ln 2$, about the y-axis.



75. a. Show that $\int \ln x\,dx = x\ln x - x + C$.

b. Find the average value of $\ln x$ over $[1, e]$.

76. Find the average value of $f(x) = 1/x$ on $[1, 2]$.

77. The linearization of e^x at $x = 0$

a. Derive the linear approximation $e^x \approx 1 + x$ at $x = 0$.

T b. Estimate to five decimal places the magnitude of the error involved in replacing e^x by $1 + x$ on the interval $[0, 0.2]$.

T c. Graph e^x and $1 + x$ together for $-2 \le x \le 2$. Use different colors, if available. On what intervals does the approximation appear to overestimate e^x? Underestimate e^x?

78. Laws of Exponents

a. Starting with the equation $e^{x_1}e^{x_2} = e^{x_1+x_2}$, derived in the text, show that $e^{-x} = 1/e^x$ for any real number x. Then show that $e^{x_1}/e^{x_2} = e^{x_1-x_2}$ for any numbers x_1 and x_2.

b. Show that $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$ for any numbers x_1 and x_2.

T 79. A decimal representation of e Find e to as many decimal places as your calculator allows by solving the equation $\ln x = 1$.

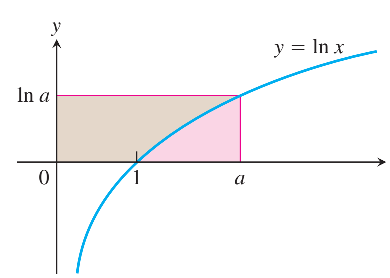
T 80. The inverse relation between e^x and $\ln x$ Find out how good your calculator is at evaluating the composites

$$e^{\ln x} \quad\text{and}\quad \ln(e^x).$$

81. Show that for any number $a > 1$

$$\int_1^a \ln x\,dx + \int_0^{\ln a} e^y\,dy = a\ln a.$$

(See accompanying figure.)



82. The geometric, logarithmic, and arithmetic mean inequality

a. Show that the graph of e^x is concave up over every interval of x-values.

b. Show, by reference to the accompanying figure, that if $0 < a < b$ then

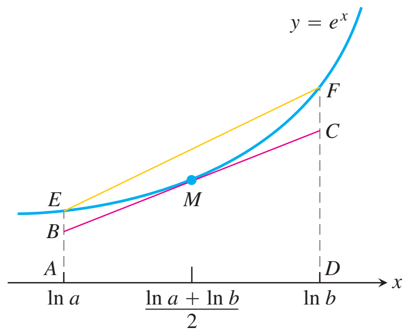
$$e^{(\ln a+\ln b)/2}\cdot(\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x\,dx < \frac{e^{\ln a}+e^{\ln b}}{2}\cdot(\ln b - \ln a).$$

c. Use the inequality in part (b) to conclude that

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}.$$

This inequality says that the geometric mean of two positive numbers is less than their logarithmic mean, which in turn is less than their arithmetic mean.

(For more about this inequality, see "The Geometric, Logarithmic, and Arithmetic Mean Inequality" by Frank Burk, *American Mathematical Monthly*, Vol. 94, No. 6, June–July 1987, pp. 527–528.)



NOT TO SCALE

7.4 a^x and $\log_a x$

We have defined general exponential functions such as 2^x, 10^x, and π^x. In this section we compute their derivatives and integrals. We also define the general logarithmic functions such as $\log_2 x$, $\log_{10} x$, and $\log_\pi x$, and find their derivatives and integrals as well.

The Derivative of a^u

We start with the definition $a^x = e^{x \ln a}$:

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \frac{d}{dx} e^u = e^u \frac{du}{dx}$$
$$= a^x \ln a.$$

If $a > 0$, then

$$\frac{d}{dx} a^x = a^x \ln a.$$

With the Chain Rule, we get a more general form.

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}. \qquad (1)$$

These equations show why e^x is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx} e^x = e^x \ln e = e^x.$$

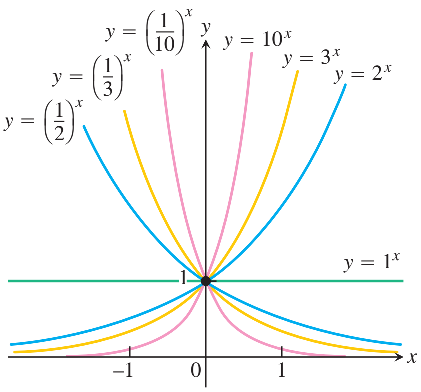


FIGURE 7.12 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

EXAMPLE 1 Differentiating General Exponential Functions

(a) $\frac{d}{dx} 3^x = 3^x \ln 3$

(b) $\frac{d}{dx} 3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x} \ln 3$

(c) $\frac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$ ■

From Equation (1), we see that the derivative of a^x is positive if $\ln a > 0$, or $a > 1$, and negative if $\ln a < 0$, or $0 < a < 1$. Thus, a^x is an increasing function of x if $a > 1$ and a decreasing function of x if $0 < a < 1$. In each case, a^x is one-to-one. The second derivative

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}(a^x \ln a) = (\ln a)^2 a^x$$

is positive for all x, so the graph of a^x is concave up on every interval of the real line (Figure 7.12).

Other Power Functions

The ability to raise positive numbers to arbitrary real powers makes it possible to define functions like x^x and $x^{\ln x}$ for $x > 0$. We find the derivatives of such functions by rewriting the functions as powers of e.

EXAMPLE 2 Differentiating a General Power Function

Find dy/dx if $y = x^x, \quad x > 0$.

Solution Write x^x as a power of e:

$$y = x^x = e^{x \ln x}. \qquad a^x \text{ with } a = x.$$

Then differentiate as usual:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} e^{x \ln x} \\ &= e^{x \ln x} \frac{d}{dx}(x \ln x) \\ &= x^x \left(x \cdot \frac{1}{x} + \ln x\right) \\ &= x^x (1 + \ln x). \end{aligned}$$ ■

The Integral of a^u

If $a \neq 1$, so that $\ln a \neq 0$, we can divide both sides of Equation (1) by $\ln a$ to obtain

$$a^u \frac{du}{dx} = \frac{1}{\ln a}\frac{d}{dx}(a^u).$$

Integrating with respect to x then gives

$$\int a^u \frac{du}{dx}\,dx = \int \frac{1}{\ln a}\frac{d}{dx}(a^u)\,dx = \frac{1}{\ln a}\int \frac{d}{dx}(a^u)\,dx = \frac{1}{\ln a}a^u + C.$$

Writing the first integral in differential form gives

$$\int a^u\,du = \frac{a^u}{\ln a} + C. \qquad (2)$$

EXAMPLE 3 Integrating General Exponential Functions

You Try It

(a) $\int 2^x\,dx = \frac{2^x}{\ln 2} + C$ — Eq. (2) with $a = 2$, $u = x$

(b) $\int 2^{\sin x}\cos x\,dx$

$= \int 2^u\,du = \frac{2^u}{\ln 2} + C$ — $u = \sin x$, $du = \cos x\,dx$, and Eq. (2)

$= \frac{2^{\sin x}}{\ln 2} + C$ — u replaced by $\sin x$ ■

Logarithms with Base *a*

As we saw earlier, if a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the **logarithm of *x* with base *a*** and denote it by $\boldsymbol{\log_a x}$.

DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$\log_a x$ is the inverse function of a^x.

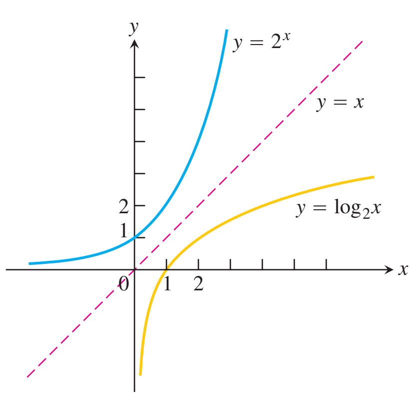


FIGURE 7.13 The graph of 2^x and its inverse, $\log_2 x$.

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the 45° line $y = x$ (Figure 7.13). When $a = e$, we have $\log_e x =$ inverse of $e^x = \ln x$. Since $\log_a x$ and a^x are inverses of one another, composing them in either order gives the identity function.

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \qquad (3)$$

$$\log_a(a^x) = x \qquad (\text{all } x) \qquad (4)$$

EXAMPLE 4 Applying the Inverse Equations

(a) $\log_2(2^5) = 5$ **(b)** $\log_{10}(10^{-7}) = -7$

(c) $2^{\log_2(3)} = 3$ **(d)** $10^{\log_{10}(4)} = 4$ ■

Evaluation of $\log_a x$

The evaluation of $\log_a x$ is simplified by the observation that $\log_a x$ is a numerical multiple of $\ln x$.

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \tag{5}$$

We can derive this equation from Equation (3):

$$\begin{aligned} a^{\log_a(x)} &= x && \text{Eq. (3)} \\ \ln a^{\log_a(x)} &= \ln x && \text{Take the natural logarithm of both sides.} \\ \log_a(x) \cdot \ln a &= \ln x && \text{The Power Rule in Theorem 2} \\ \log_a x &= \frac{\ln x}{\ln a} && \text{Solve for } \log_a x. \end{aligned}$$

For example,

$$\log_{10} 2 = \frac{\ln 2}{\ln 10} \approx \frac{0.69315}{2.30259} \approx 0.30103$$

The arithmetic rules satisfied by $\log_a x$ are the same as the ones for $\ln x$ (Theorem 2). These rules, given in Table 7.2, can be proved by dividing the corresponding rules for the natural logarithm function by $\ln a$. For example,

$$\begin{aligned} \ln xy &= \ln x + \ln y && \text{Rule 1 for natural logarithms } \ldots \\ \frac{\ln xy}{\ln a} &= \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} && \ldots \text{ divided by } \ln a \ldots \\ \log_a xy &= \log_a x + \log_a y. && \ldots \text{ gives Rule 1 for base } a \text{ logarithms.} \end{aligned}$$

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:* $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:* $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:* $\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:* $\log_a x^y = y \log_a x$

Derivatives and Integrals Involving $\log_a x$

To find derivatives or integrals involving base a logarithms, we convert them to natural logarithms.

If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$$

EXAMPLE 5

(a) $$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$$

(b) $$\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x, \quad du = \frac{1}{x}dx$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$ ■

Base 10 Logarithms

Base 10 logarithms, often called **common logarithms**, appear in many scientific formulas. For example, earthquake intensity is often reported on the logarithmic **Richter scale**. Here the formula is

$$\text{Magnitude } R = \log_{10}\left(\frac{a}{T}\right) + B,$$

where a is the amplitude of the ground motion in microns at the receiving station, T is the period of the seismic wave in seconds, and B is an empirical factor that accounts for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.

EXAMPLE 6 Earthquake Intensity

For an earthquake 10,000 km from the receiving station, $B = 6.8$. If the recorded vertical ground motion is $a = 10$ microns and the period is $T = 1$ sec, the earthquake's magnitude is

$$R = \log_{10}\left(\frac{10}{1}\right) + 6.8 = 1 + 6.8 = 7.8.$$

An earthquake of this magnitude can do great damage near its epicenter. ■

The **pH scale** for measuring the acidity of a solution is a base 10 logarithmic scale. The pH value (hydrogen potential) of the solution is the common logarithm of the reciprocal of the solution's hydronium ion concentration, $[H_3O^+]$:

$$\text{pH} = \log_{10}\frac{1}{[H_3O^+]} = -\log_{10}[H_3O^+].$$

Most foods are acidic ($\text{pH} < 7$).

Food	pH Value
Bananas	4.5–4.7
Grapefruit	3.0–3.3
Oranges	3.0–4.0
Limes	1.8–2.0
Milk	6.3–6.6
Soft drinks	2.0–4.0
Spinach	5.1–5.7

The hydronium ion concentration is measured in moles per liter. Vinegar has a pH of three, distilled water a pH of 7, seawater a pH of 8.15, and household ammonia a pH of 12. The total scale ranges from about 0.1 for normal hydrochloric acid to 14 for a normal solution of sodium hydroxide.

Another example of the use of common logarithms is the **decibel** or dB ("dee bee") **scale** for measuring loudness. If I is the **intensity** of sound in watts per square meter, the decibel level of the sound is

$$\textbf{Sound level} = 10\log_{10}(I \times 10^{12})\text{ dB}. \qquad (6)$$

Typical sound levels

Threshold of hearing	0 dB
Rustle of leaves	10 dB
Average whisper	20 dB
Quiet automobile	50 dB
Ordinary conversation	65 dB
Pneumatic drill 10 feet away	90 dB
Threshold of pain	120 dB

If you ever wondered why doubling the power of your audio amplifier increases the sound level by only a few decibels, Equation (6) provides the answer. As the following example shows, doubling I adds only about 3 dB.

EXAMPLE 7 Sound Intensity

Doubling I in Equation (6) adds about 3 dB. Writing log for $\log_{10}$ (a common practice), we have

$$\begin{aligned}
\text{Sound level with } I \text{ doubled} &= 10 \log (2I \times 10^{12}) && \text{Eq. (6) with } 2I \text{ for } I \\
&= 10 \log (2 \cdot I \times 10^{12}) \\
&= 10 \log 2 + 10 \log (I \times 10^{12}) \\
&= \text{original sound level} + 10 \log 2 \\
&\approx \text{original sound level} + 3. && \log_{10} 2 \approx 0.30
\end{aligned}$$

■

EXERCISES 7.4

Algebraic Calculations With a^x and $\log_a x$

Simplify the expressions in Exercises 1–4.

1. a. $5^{\log_5 7}$ b. $8^{\log_8 \sqrt{2}}$ c. $1.3^{\log_{1.3} 75}$
   d. $\log_4 16$ e. $\log_3 \sqrt{3}$ f. $\log_4 \left(\frac{1}{4}\right)$
2. a. $2^{\log_2 3}$ b. $10^{\log_{10}(1/2)}$ c. $\pi^{\log_\pi 7}$
   d. $\log_{11} 121$ e. $\log_{121} 11$ f. $\log_3 \left(\frac{1}{9}\right)$
3. a. $2^{\log_4 x}$ b. $9^{\log_3 x}$ c. $\log_2 \left(e^{(\ln 2)(\sin x)}\right)$
4. a. $25^{\log_5 (3x^2)}$ b. $\log_e (e^x)$ c. $\log_4 \left(2^{e^x \sin x}\right)$

Express the ratios in Exercises 5 and 6 as ratios of natural logarithms and simplify.

5. a. $\dfrac{\log_2 x}{\log_3 x}$ b. $\dfrac{\log_2 x}{\log_8 x}$ c. $\dfrac{\log_x a}{\log_{x^2} a}$
6. a. $\dfrac{\log_9 x}{\log_3 x}$ b. $\dfrac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$ c. $\dfrac{\log_a b}{\log_b a}$

Solve the equations in Exercises 7–10 for x.

7. $3^{\log_3 (7)} + 2^{\log_2 (5)} = 5^{\log_5 (x)}$
8. $8^{\log_8 (3)} - e^{\ln 5} = x^2 - 7^{\log_7 (3x)}$
9. $3^{\log_3 (x^2)} = 5e^{\ln x} - 3 \cdot 10^{\log_{10} (2)}$
10. $\ln e + 4^{-2 \log_4 (x)} = \dfrac{1}{x} \log_{10} (100)$

Derivatives

In Exercises 11–38, find the derivative of y with respect to the given independent variable.

11. $y = 2^x$
12. $y = 3^{-x}$
13. $y = 5^{\sqrt{s}}$
14. $y = 2^{(s^2)}$
15. $y = x^\pi$
16. $y = t^{1-e}$
17. $y = (\cos \theta)^{\sqrt{2}}$
18. $y = (\ln \theta)^\pi$
19. $y = 7^{\sec \theta} \ln 7$
20. $y = 3^{\tan \theta} \ln 3$
21. $y = 2^{\sin 3t}$
22. $y = 5^{-\cos 2t}$
23. $y = \log_2 5\theta$
24. $y = \log_3 (1 + \theta \ln 3)$
25. $y = \log_4 x + \log_4 x^2$
26. $y = \log_{25} e^x - \log_5 \sqrt{x}$
27. $y = \log_2 r \cdot \log_4 r$
28. $y = \log_3 r \cdot \log_9 r$
29. $y = \log_3 \left(\left(\dfrac{x+1}{x-1} \right)^{\ln 3} \right)$
30. $y = \log_5 \sqrt{\left(\dfrac{7x}{3x+2} \right)^{\ln 5}}$
31. $y = \theta \sin (\log_7 \theta)$
32. $y = \log_7 \left(\dfrac{\sin \theta \cos \theta}{e^\theta 2^\theta} \right)$
33. $y = \log_5 e^x$
34. $y = \log_2 \left(\dfrac{x^2 e^2}{2\sqrt{x+1}} \right)$
35. $y = 3^{\log_2 t}$
36. $y = 3 \log_8 (\log_2 t)$
37. $y = \log_2 (8t^{\ln 2})$
38. $y = t \log_3 \left(e^{(\sin t)(\ln 3)} \right)$

Logarithmic Differentiation

In Exercises 39–46, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

39. $y = (x+1)^x$
40. $y = x^{(x+1)}$
41. $y = (\sqrt{t})^t$
42. $y = t^{\sqrt{t}}$
43. $y = (\sin x)^x$
44. $y = x^{\sin x}$
45. $y = x^{\ln x}$
46. $y = (\ln x)^{\ln x}$

Integration

Evaluate the integrals in Exercises 47–56.

47. $\displaystyle\int 5^x \, dx$
48. $\displaystyle\int (1.3)^x \, dx$

49. $\int_0^1 2^{-\theta}\, d\theta$

50. $\int_{-2}^0 5^{-\theta}\, d\theta$

51. $\int_1^{\sqrt{2}} x2^{(x^2)}\, dx$

52. $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\, dx$

53. $\int_0^{\pi/2} 7^{\cos t} \sin t\, dt$

54. $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t\, dt$

55. $\int_2^4 x^{2x}(1 + \ln x)\, dx$

56. $\int_1^2 \frac{2^{\ln x}}{x}\, dx$

Evaluate the integrals in Exercises 57–60.

57. $\int 3x^{\sqrt{3}}\, dx$

58. $\int x^{\sqrt{2}-1}\, dx$

59. $\int_0^3 (\sqrt{2} + 1)x^{\sqrt{2}}\, dx$

60. $\int_1^e x^{(\ln 2)-1}\, dx$

Evaluate the integrals in Exercises 61–70.

61. $\int \frac{\log_{10} x}{x}\, dx$

62. $\int_1^4 \frac{\log_2 x}{x}\, dx$

63. $\int_1^4 \frac{\ln 2 \log_2 x}{x}\, dx$

64. $\int_1^e \frac{2 \ln 10 \log_{10} x}{x}\, dx$

65. $\int_0^2 \frac{\log_2 (x + 2)}{x + 2}\, dx$

66. $\int_{1/10}^{10} \frac{\log_{10} (10x)}{x}\, dx$

67. $\int_0^9 \frac{2 \log_{10} (x + 1)}{x + 1}\, dx$

68. $\int_2^3 \frac{2 \log_2 (x - 1)}{x - 1}\, dx$

69. $\int \frac{dx}{x \log_{10} x}$

70. $\int \frac{dx}{x(\log_8 x)^2}$

Evaluate the integrals in Exercises 71–74.

71. $\int_1^{\ln x} \frac{1}{t}\, dt, \quad x > 1$

72. $\int_1^{e^x} \frac{1}{t}\, dt$

73. $\int_1^{1/x} \frac{1}{t}\, dt, \quad x > 0$

74. $\frac{1}{\ln a} \int_1^x \frac{1}{t}\, dt, \quad x > 0$

Theory and Applications

75. Find the area of the region between the curve $y = 2x/(1 + x^2)$ and the interval $-2 \le x \le 2$ of the x-axis.

76. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \le x \le 1$ of the x-axis.

77. **Blood pH** The pH of human blood normally falls between 7.37 and 7.44. Find the corresponding bounds for $[H_3O^+]$.

78. **Brain fluid pH** The cerebrospinal fluid in the brain has a hydronium ion concentration of about $[H_3O^+] = 4.8 \times 10^{-8}$ moles per liter. What is the pH?

79. **Audio amplifiers** By what factor k do you have to multiply the intensity of I of the sound from your audio amplifier to add 10 dB to the sound level?

80. **Audio amplifiers** You multiplied the intensity of the sound of your audio system by a factor of 10. By how many decibels did this increase the sound level?

81. In any solution, the product of the hydronium ion concentration $[H_3O^+]$ (moles/L) and the hydroxyl ion concentration $[OH^-]$ (moles/L) is about 10^{-14}.
   a. What value of $[H_3O^+]$ minimizes the sum of the concentrations, $S = [H_3O^+] + [OH^-]$? (*Hint:* Change notation. Let $x = [H_3O^+]$.)
   b. What is the pH of a solution in which S has this minimum value?
   c. What ratio of $[H_3O^+]$ to $[OH^-]$ minimizes S?

82. Could $\log_a b$ possibly equal $1/\log_b a$? Give reasons for your answer.

T 83. The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

T 84. Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for $x > 0$? Graph the two functions and explain what you see.

85. **The linearization of 2^x**
   a. Find the linearization of $f(x) = 2^x$ at $x = 0$. Then round its coefficients to two decimal places.
   T b. Graph the linearization and function together for $-3 \le x \le 3$ and $-1 \le x \le 1$.

86. **The linearization of $\log_3 x$**
   a. Find the linearization of $f(x) = \log_3 x$ at $x = 3$. Then round its coefficients to two decimal places.
   T b. Graph the linearization and function together in the window $0 \le x \le 8$ and $2 \le x \le 4$.

Calculations with Other Bases

T 87. Most scientific calculators have keys for $\log_{10} x$ and $\ln x$. To find logarithms to other bases, we use the Equation (5), $\log_a x = (\ln x)/(\ln a)$.

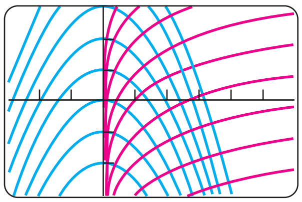
   Find the following logarithms to five decimal places.
   a. $\log_3 8$
   b. $\log_7 0.5$
   c. $\log_{20} 17$
   d. $\log_{0.5} 7$
   e. $\ln x$, given that $\log_{10} x = 2.3$
   f. $\ln x$, given that $\log_2 x = 1.4$
   g. $\ln x$, given that $\log_2 x = -1.5$
   h. $\ln x$, given that $\log_{10} x = -0.7$

88. **Conversion factors**
   a. Show that the equation for converting base 10 logarithms to base 2 logarithms is
   $$\log_2 x = \frac{\ln 10}{\ln 2} \log_{10} x.$$
   b. Show that the equation for converting base a logarithms to base b logarithms is
   $$\log_b x = \frac{\ln a}{\ln b} \log_a x.$$

89. Orthogonal families of curves Prove that all curves in the family

$$y = -\frac{1}{2}x^2 + k$$

(k any constant) are perpendicular to all curves in the family $y = \ln x + c$ (c any constant) at their points of intersection. (See the accompanying figure.)



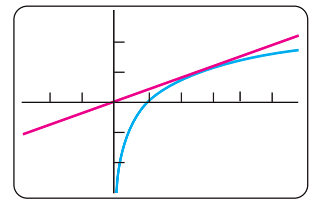
T **90. The inverse relation between e^x and $\ln x$** Find out how good your calculator is at evaluating the composites

$$e^{\ln x} \quad \text{and} \quad \ln(e^x).$$

T **91. A decimal representation of e** Find e to as many decimal places as your calculator allows by solving the equation $\ln x = 1$.

T **92. Which is bigger, π^e or e^π?** Calculators have taken some of the mystery out of this once-challenging question. (Go ahead and check; you will see that it is a surprisingly close call.) You can answer the question without a calculator, though.

a. Find an equation for the line through the origin tangent to the graph of $y = \ln x$.



[–3, 6] by [–3, 3]

b. Give an argument based on the graphs of $y = \ln x$ and the tangent line to explain why $\ln x < x/e$ for all positive $x \neq e$.

c. Show that $\ln(x^e) < x$ for all positive $x \neq e$.

d. Conclude that $x^e < e^x$ for all positive $x \neq e$.

e. So which is bigger, π^e or e^π?

7.5 Exponential Growth and Decay

Exponential functions increase or decrease very rapidly with changes in the independent variable. They describe growth or decay in a wide variety of natural and industrial situations. The variety of models based on these functions partly accounts for their importance.

The Law of Exponential Change

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t. Examples of such quantities include the amount of a decaying radioactive material, funds earning interest in a bank account, the size of a population, and the temperature difference between a hot cup of coffee and the room in which it sits. Such quantities change according to the *law of exponential change*, which we derive in this section.

If the amount present at time $t = 0$ is called y_0, then we can find y as a function of t by solving the following initial value problem:

$$\begin{aligned} &\text{Differential equation:} \quad && \frac{dy}{dt} = ky \\ &\text{Initial condition:} \quad && y = y_0 \quad \text{when} \quad t = 0. \end{aligned} \tag{1}$$

If y is positive and increasing, then k is positive, and we use Equation (1) to say that the rate of growth is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative, and we use Equation (1) to say that the rate of decay is proportional to the amount still left.

We see right away that the constant function $y = 0$ is a solution of Equation (1) if $y_0 = 0$. To find the nonzero solutions, we divide Equation (1) by y:

$$
\begin{aligned}
\frac{1}{y} \cdot \frac{dy}{dt} &= k && \\
\int \frac{1}{y}\frac{dy}{dt}\,dt &= \int k\,dt && \text{Integrate with respect to } t; \\
\ln|y| &= kt + C && \textstyle\int (1/u)\,du = \ln|u| + C. \\
|y| &= e^{kt+C} && \text{Exponentiate.} \\
|y| &= e^{C} \cdot e^{kt} && e^{a+b} = e^{a} \cdot e^{b} \\
y &= \pm e^{C} e^{kt} && \text{If } |y| = r, \text{ then } y = \pm r. \\
y &= A e^{kt}. && A \text{ is a shorter name for } \pm e^{C}.
\end{aligned}
$$

By allowing A to take on the value 0 in addition to all possible values $\pm e^{C}$, we can include the solution $y = 0$ in the formula.

We find the value of A for the initial value problem by solving for A when $y = y_0$ and $t = 0$:

$$y_0 = A e^{k \cdot 0} = A.$$

The solution of the initial value problem is therefore $y = y_0 e^{kt}$.

Quantities changing in this way are said to undergo **exponential growth** if $k > 0$, and **exponential decay** if $k < 0$.

Animation

The Law of Exponential Change

$$y = y_0 e^{kt} \qquad (2)$$

Growth: $k > 0$ Decay: $k < 0$

The number k is the **rate constant** of the equation.

The derivation of Equation (2) shows that the only functions that are their own derivatives are constant multiples of the exponential function.

Unlimited Population Growth

Strictly speaking, the number of individuals in a population (of people, plants, foxes, or bacteria, for example) is a discontinuous function of time because it takes on discrete values. However, when the number of individuals becomes large enough, the population can be approximated by a continuous function. Differentiability of the approximating function is another reasonable hypothesis in many settings, allowing for the use of calculus to model and predict population sizes.

If we assume that the proportion of reproducing individuals remains constant and assume a constant fertility, then at any instant t the birth rate is proportional to the number $y(t)$ of individuals present. Let's assume, too, that the death rate of the population is stable and proportional to $y(t)$. If, further, we neglect departures and arrivals, the growth rate

dy/dt is the birth rate minus the death rate, which is the difference of the two proportionalities under our assumptions. In other words, $dy/dt = ky$, so that $y = y_0 e^{kt}$, where y_0 is the size of the population at time $t = 0$. As with all kinds of growth, there may be limitations imposed by the surrounding environment, but we will not go into these here. (This situation is analyzed in Section 9.5.)

In the following example we assume this population model to look at how the number of individuals infected by a disease within a given population decreases as the disease is appropriately treated.

EXAMPLE 1 Reducing the Cases of an Infectious Disease

One model for the way diseases die out when properly treated assumes that the rate dy/dt at which the number of infected people changes is proportional to the number y. The number of people cured is proportional to the number that have the disease. Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?

Solution We use the equation $y = y_0 e^{kt}$. There are three things to find: the value of y_0, the value of k, and the time t when $y = 1000$.

The value of y_0. We are free to count time beginning anywhere we want. If we count from today, then $y = 10{,}000$ when $t = 0$, so $y_0 = 10{,}000$. Our equation is now

$$y = 10{,}000 e^{kt}. \tag{3}$$

The value of k. When $t = 1$ year, the number of cases will be 80% of its present value, or 8000. Hence,

$$\begin{aligned} 8000 &= 10{,}000 e^{k(1)} && \text{Eq. (3) with } t = 1 \text{ and } y = 8000 \\ e^k &= 0.8 \\ \ln(e^k) &= \ln 0.8 && \text{Logs of both sides} \\ k &= \ln 0.8 < 0. \end{aligned}$$

At any given time t,

$$y = 10{,}000 e^{(\ln 0.8)t}. \tag{4}$$

The value of t *that makes* $y = 1000$. We set y equal to 1000 in Equation (4) and solve for t:

$$\begin{aligned} 1000 &= 10{,}000 e^{(\ln 0.8)t} \\ e^{(\ln 0.8)t} &= 0.1 \\ (\ln 0.8)t &= \ln 0.1 && \text{Logs of both sides} \\ t &= \frac{\ln 0.1}{\ln 0.8} \approx 10.32 \text{ years}. \end{aligned}$$

It will take a little more than 10 years to reduce the number of cases to 1000. ■

Continuously Compounded Interest

If you invest an amount A_0 of money at a fixed annual interest rate r (expressed as a decimal) and if interest is added to your account k times a year, the formula for the amount of money you will have at the end of t years is

$$A_t = A_0 \left(1 + \frac{r}{k}\right)^{kt}. \tag{5}$$

The interest might be added ("compounded," bankers say) monthly ($k = 12$), weekly ($k = 52$), daily ($k = 365$), or even more frequently, say by the hour or by the minute. By taking the limit as interest is compounded more and more often, we arrive at the following formula for the amount after t years,

$$\begin{aligned}
\lim_{k\to\infty} A_t &= \lim_{k\to\infty} A_0\left(1 + \frac{r}{k}\right)^{kt} \\
&= A_0 \lim_{k\to\infty}\left(1 + \frac{r}{k}\right)^{\frac{k}{r}\cdot rt} \\
&= A_0\left[\lim_{\frac{r}{k}\to 0}\left(1 + \frac{r}{k}\right)^{\frac{k}{r}}\right]^{rt} && \text{As } k \to \infty, \frac{r}{k} \to 0 \\
&= A_0\left[\lim_{x\to 0}(1 + x)^{1/x}\right]^{rt} && \text{Substitute } x = \frac{r}{k} \\
&= A_0 e^{rt} && \text{Theorem 4}
\end{aligned}$$

The resulting formula for the amount of money in your account after t years is

$$A(t) = A_0 e^{rt}. \tag{6}$$

Interest paid according to this formula is said to be **compounded continuously**. The number r is called the **continuous interest rate.** The amount of money after t years is calculated with the law of exponential change given in Equation (6).

EXAMPLE 2 A Savings Account

Suppose you deposit \$621 in a bank account that pays 6% compounded continuously. How much money will you have 8 years later?

Solution We use Equation (6) with $A_0 = 621$, $r = 0.06$, and $t = 8$:

$$A(8) = 621e^{(0.06)(8)} = 621e^{0.48} = 1003.58 \qquad \text{Nearest cent}$$

Had the bank paid interest quarterly ($k = 4$ in Equation 5), the amount in your account would have been \$1000.01. Thus the effect of continuous compounding, as compared with quarterly compounding, has been an addition of \$3.57. A bank might decide it would be worth this additional amount to be able to advertise, "We compound interest every second, night and day—better yet, we compound the interest continuously." ■

Radioactivity

Some atoms are unstable and can spontaneously emit mass or radiation. This process is called **radioactive decay**, and an element whose atoms go spontaneously through this process is called **radioactive**. Sometimes when an atom emits some of its mass through this process of radioactivity, the remainder of the atom re-forms to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium, through a number of intermediate radioactive steps, decays into lead.

For radon-222 gas, t is measured in days and $k = 0.18$. For radium-226, which used to be painted on watch dials to make them glow at night (a dangerous practice), t is measured in years and $k = 4.3 \times 10^{-4}$.

Experiments have shown that at any given time the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. It is conventional to use

$-k(k > 0)$ here instead of $k(k < 0)$ to emphasize that y is decreasing. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0 e^{-kt}, \qquad k > 0.$$

EXAMPLE 3 Half-Life of a Radioactive Element

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. It is an interesting fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

To see why, let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0 e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$\begin{aligned} y_0 e^{-kt} &= \frac{1}{2} y_0 \\ e^{-kt} &= \frac{1}{2} \\ -kt &= \ln \frac{1}{2} = -\ln 2 && \text{Reciprocal Rule for logarithms} \\ t &= \frac{\ln 2}{k} \end{aligned}$$

This value of t is the half-life of the element. It depends only on the value of k; the number y_0 does not enter in.

$$\text{Half-life} = \frac{\ln 2}{k} \tag{7}$$

■

EXAMPLE 4 Half-Life of Polonium-210

You Try It

The effective radioactive lifetime of polonium-210 is so short we measure it in days rather than years. The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0 e^{-5 \times 10^{-3} t}.$$

Find the element's half-life.

Solution

$$\begin{aligned} \text{Half-life} &= \frac{\ln 2}{k} && \text{Eq. (7)} \\ &= \frac{\ln 2}{5 \times 10^{-3}} && \text{The } k \text{ from polonium's decay equation} \\ &\approx 139 \text{ days} \end{aligned}$$

■

EXAMPLE 5 Carbon-14 Dating

You Try It

The decay of radioactive elements can sometimes be used to date events from the Earth's past. In a living organism, the ratio of radioactive carbon, carbon-14, to ordinary carbon stays fairly constant during the lifetime of the organism, being approximately equal to the ratio in the organism's surroundings at the time. After the organism's death, however, no new carbon is ingested, and the proportion of carbon-14 in the organism's remains decreases as the carbon-14 decays.

Scientists who do carbon-14 dating use a figure of 5700 years for its half-life (more about carbon-14 dating in the exercises). Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Solution We use the decay equation $y = y_0 e^{-kt}$. There are two things to find: the value of k and the value of t when y is $0.9y_0$ (90% of the radioactive nuclei are still present). That is, find t when $y_0 e^{-kt} = 0.9y_0$, or $e^{-kt} = 0.9$.

The value of k. We use the half-life Equation (7):

$$k = \frac{\ln 2}{\text{half-life}} = \frac{\ln 2}{5700} \qquad (\text{about } 1.2 \times 10^{-4})$$

The value of t that makes $e^{-kt} = 0.9$:

$$e^{-kt} = 0.9$$

$$e^{-(\ln 2/5700)t} = 0.9$$

$$-\frac{\ln 2}{5700} t = \ln 0.9 \qquad \text{Logs of both sides}$$

$$t = -\frac{5700 \ln 0.9}{\ln 2} \approx 866 \text{ years}.$$

The sample is about 866 years old. ■

Heat Transfer: Newton's Law of Cooling

Hot soup left in a tin cup cools to the temperature of the surrounding air. A hot silver ingot immersed in a large tub of water cools to the temperature of the surrounding water. In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. This observation is called *Newton's law of cooling*, although it applies to warming as well, and there is an equation for it.

If H is the temperature of the object at time t and H_S is the constant surrounding temperature, then the differential equation is

$$\frac{dH}{dt} = -k(H - H_S). \tag{8}$$

If we substitute y for $(H - H_S)$, then

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{dt}(H - H_S) = \frac{dH}{dt} - \frac{d}{dt}(H_S) \\
&= \frac{dH}{dt} - 0 && H_S \text{ is a constant.} \\
&= \frac{dH}{dt} \\
&= -k(H - H_S) && \text{Eq. (8)} \\
&= -ky. && H - H_S = y.
\end{aligned}$$

Now we know that the solution of $dy/dt = -ky$ is $y = y_0 e^{-kt}$, where $y(0) = y_0$. Substituting $(H - H_S)$ for y, this says that

$$H - H_S = (H_0 - H_S)e^{-kt}, \tag{9}$$

where H_0 is the temperature at $t = 0$. This is the equation for Newton's Law of Cooling.

EXAMPLE 6 Cooling a Hard-Boiled Egg

You Try It

A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 min, the egg's temperature is 38°C. Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach 20°C?

Solution We find how long it would take the egg to cool from 98°C to 20°C and subtract the 5 min that have already elapsed. Using Equation (9) with $H_S = 18$ and $H_0 = 98$, the egg's temperature t min after it is put in the sink is

$$H = 18 + (98 - 18)e^{-kt} = 18 + 80e^{-kt}.$$

To find k, we use the information that $H = 38$ when $t = 5$:

$$\begin{aligned}
38 &= 18 + 80e^{-5k} \\
e^{-5k} &= \frac{1}{4} \\
-5k &= \ln\frac{1}{4} = -\ln 4 \\
k &= \frac{1}{5}\ln 4 = 0.2 \ln 4 \qquad (\text{about } 0.28).
\end{aligned}$$

The egg's temperature at time t is $H = 18 + 80e^{-(0.2 \ln 4)t}$. Now find the time t when $H = 20$:

$$\begin{aligned}
20 &= 18 + 80e^{-(0.2 \ln 4)t} \\
80e^{-(0.2 \ln 4)t} &= 2 \\
e^{-(0.2 \ln 4)t} &= \frac{1}{40} \\
-(0.2 \ln 4)t &= \ln\frac{1}{40} = -\ln 40 \\
t &= \frac{\ln 40}{0.2 \ln 4} \approx 13 \text{ min}.
\end{aligned}$$

The egg's temperature will reach 20°C about 13 min after it is put in the water to cool. Since it took 5 min to reach 38°C, it will take about 8 min more to reach 20°C. ■

EXERCISES 7.5

The answers to most of the following exercises are in terms of logarithms and exponentials. A calculator can be helpful, enabling you to express the answers in decimal form.

1. **Human evolution continues** The analysis of tooth shrinkage by C. Loring Brace and colleagues at the University of Michigan's Museum of Anthropology indicates that human tooth size is continuing to decrease and that the evolutionary process did not come to a halt some 30,000 years ago as many scientists contend. In northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

a. If t represents time in years and y represents tooth size, use the condition that $y = 0.99y_0$ when $t = 1000$ to find the value of k in the equation $y = y_0 e^{kt}$. Then use this value of k to answer the following questions.

b. In about how many years will human teeth be 90% of their present size?

c. What will be our descendants' tooth size 20,000 years from now (as a percentage of our present tooth size)?

(*Source: LSA Magazine*, Spring 1989, Vol. 12, No. 2, p. 19, Ann Arbor, MI.)

2. Atmospheric pressure The earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1013 millibars (about 14.7 pounds per square inch) and that the pressure at an altitude of 20 km is 90 millibars.

a. Solve the initial value problem

Differential equation: $dp/dh = kp$ (k a constant)
Initial condition: $p = p_0$ when $h = 0$

to express p in terms of h. Determine the values of p_0 and k from the given altitude-pressure data.

b. What is the atmospheric pressure at $h = 50$ km?

c. At what altitude does the pressure equal 900 millibars?

3. First-order chemical reactions In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

when t is measured in hours. If there are 100 grams of δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

4. The inversion of sugar The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 hours, how much raw sugar will remain after another 14 hours?

5. Working underwater The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL.$$

As a diver, you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can you expect to work without artificial light?

6. Voltage in a discharging capacitor Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage V across its terminals and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

Solve this equation for V, using V_0 to denote the value of V when $t = 0$. How long will it take the voltage to drop to 10% of its original value?

7. Cholera bacteria Suppose that the bacteria in a colony can grow unchecked, by the law of exponential change. The colony starts with 1 bacterium and doubles every half-hour. How many bacteria will the colony contain at the end of 24 hours? (Under favorable laboratory conditions, the number of cholera bacteria can double every 30 min. In an infected person, many bacteria are destroyed, but this example helps explain why a person who feels well in the morning may be dangerously ill by evening.)

8. Growth of bacteria A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000. How many bacteria were present initially?

9. The incidence of a disease (*Continuation of Example 1.*) Suppose that in any given year the number of cases can be reduced by 25% instead of 20%.

a. How long will it take to reduce the number of cases to 1000?

b. How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1?

10. The U.S. population The Museum of Science in Boston displays a running total of the U.S. population. On May 11, 1993, the total was increasing at the rate of 1 person every 14 sec. The displayed population figure for 3:45 P.M. that day was 257,313,431.

a. Assuming exponential growth at a constant rate, find the rate constant for the population's growth (people per 365-day year).

b. At this rate, what will the U.S. population be at 3:45 P.M. Boston time on May 11, 2008?

11. Oil depletion Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

12. Continuous price discounting To encourage buyers to place 100-unit orders, your firm's sales department applies a continuous discount that makes the unit price a function $p(x)$ of the number of units x ordered. The discount decreases the price at the rate of \$0.01 per unit ordered. The price per unit for a 100-unit order is $p(100) = \$20.09$.

a. Find $p(x)$ by solving the following initial value problem:

Differential equation: $\dfrac{dp}{dx} = -\dfrac{1}{100}p$
Initial condition: $p(100) = 20.09.$

b. Find the unit price $p(10)$ for a 10-unit order and the unit price $p(90)$ for a 90-unit order.

c. The sales department has asked you to find out if it is discounting so much that the firm's revenue, $r(x) = x \cdot p(x)$, will actually be less for a 100-unit order than, say, for a 90-unit order. Reassure them by showing that r has its maximum value at $x = 100$.

T **d.** Graph the revenue function $r(x) = xp(x)$ for $0 \le x \le 200$.

13. Continuously compounded interest You have just placed A_0 dollars in a bank account that pays 4% interest, compounded continuously.

a. How much money will you have in the account in 5 years?

b. How long will it take your money to double? To triple?

14. John Napier's question John Napier (1550–1617), the Scottish laird who invented logarithms, was the first person to answer the question, What happens if you invest an amount of money at 100% interest, compounded continuously?

a. What does happen?

b. How long does it take to triple your money?

c. How much can you earn in a year?

Give reasons for your answers.

15. Benjamin Franklin's will The Franklin Technical Institute of Boston owes its existence to a provision in a codicil to Benjamin Franklin's will. In part the codicil reads:

> I wish to be useful even after my Death, if possible, in forming and advancing other young men that may be serviceable to their Country in both Boston and Philadelphia. To this end I devote Two thousand Pounds Sterling, which I give, one thousand thereof to the Inhabitants of the Town of Boston in Massachusetts, and the other thousand to the inhabitants of the City of Philadelphia, in Trust and for the Uses, Interests and Purposes hereinafter mentioned and declared.

Franklin's plan was to lend money to young apprentices at 5% interest with the provision that each borrower should pay each year along

> . . . with the yearly Interest, one tenth part of the Principal, which sums of Principal and Interest shall be again let to fresh Borrowers. . . . If this plan is executed and succeeds as projected without interruption for one hundred Years, the Sum will then be one hundred and thirty-one thousand Pounds of which I would have the Managers of the Donation to the Inhabitants of the Town of Boston, then lay out at their discretion one hundred thousand Pounds in Public Works. . . . The remaining thirty-one thousand Pounds, I would have continued to be let out on Interest in the manner above directed for another hundred Years. . . . At the end of this second term if no unfortunate accident has prevented the operation the sum will be Four Millions and Sixty-one Thousand Pounds.

It was not always possible to find as many borrowers as Franklin had planned, but the managers of the trust did the best they could. At the end of 100 years from the reception of the Franklin gift, in January 1894, the fund had grown from 1000 pounds to almost exactly 90,000 pounds. In 100 years the original capital had multiplied about 90 times instead of the 131 times Franklin had imagined.

What rate of interest, compounded continuously for 100 years, would have multiplied Benjamin Franklin's original capital by 90?

16. (*Continuation of Exercise 15.*) In Benjamin Franklin's estimate that the original 1000 pounds would grow to 131,000 in 100 years, he was using an annual rate of 5% and compounding once each year. What rate of interest per year when compounded continuously for 100 years would multiply the original amount by 131?

17. Radon-222 The decay equation for radon-222 gas is known to be $y = y_0 e^{-0.18t}$, with t in days. About how long will it take the radon in a sealed sample of air to fall to 90% of its original value?

18. Polonium-210 The half-life of polonium is 139 days, but your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives has disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

19. The mean life of a radioactive nucleus Physicists using the radioactivity equation $y = y_0 e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon nucleus is about $1/0.18 = 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in a sample will disintegrate within three mean lifetimes, i.e., by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.

20. Californium-252 What costs \$27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope so rare that only 8 g of it have been made in the western world since its discovery by Glenn Seaborg in 1950. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per second.

a. What is the value of k in the decay equation for this isotope?

b. What is the isotope's mean life? (See Exercise 19.)

c. How long will it take 95% of a sample's radioactive nuclei to disintegrate?

21. Cooling soup Suppose that a cup of soup cooled from 90°C to 60°C after 10 min in a room whose temperature was 20°C. Use Newton's law of cooling to answer the following questions.

a. How much longer would it take the soup to cool to 35°C?

b. Instead of being left to stand in the room, the cup of 90°C soup is put in a freezer whose temperature is -15°C. How long will it take the soup to cool from 90°C to 35°C?

22. A beam of unknown temperature An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°F. After 10 min, the beam warmed to 35°F and after another 10 min it was 50°F. Use Newton's law of cooling to estimate the beam's initial temperature.

23. Surrounding medium of unknown temperature A pan of warm water (46°C) was put in a refrigerator. Ten minutes later, the water's temperature was 39°C; 10 min after that, it was 33°C. Use Newton's law of cooling to estimate how cold the refrigerator was.

24. Silver cooling in air The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

a. 15 min from now?

b. 2 hours from now?

c. When will the silver be 10°C above room temperature?

25. The age of Crater Lake The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

26. The sensitivity of carbon-14 dating to measurement To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, consider this hypothetical situation:

a. A fossilized bone found in central Illinois in the year A.D. 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.

b. Repeat part (a) assuming 18% instead of 17%.

c. Repeat part (a) assuming 16% instead of 17%.

27. Art forgery A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

7.6 Relative Rates of Growth

It is often important in mathematics, computer science, and engineering to compare the rates at which functions of x grow as x becomes large. Exponential functions are important in these comparisons because of their very fast growth, and logarithmic functions because of their very slow growth. In this section we introduce the *little-oh* and *big-oh* notation used to describe the results of these comparisons. We restrict our attention to functions whose values eventually become and remain positive as $x \to \infty$.

Growth Rates of Functions

You may have noticed that exponential functions like 2^x and e^x seem to grow more rapidly as x gets large than do polynomials and rational functions. These exponentials certainly grow more rapidly than x itself, and you can see 2^x outgrowing x^2 as x increases in Figure 7.14. In fact, as $x \to \infty$, the functions 2^x and e^x grow faster than any power of x, even $x^{1{,}000{,}000}$ (Exercise 19).

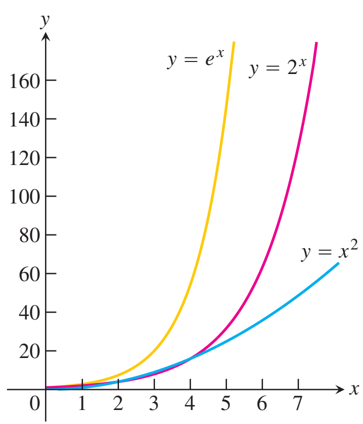


FIGURE 7.14 The graphs of e^x, 2^x, and x^2.

To get a feeling for how rapidly the values of $y = e^x$ grow with increasing x, think of graphing the function on a large blackboard, with the axes scaled in centimeters. At $x = 1$ cm, the graph is $e^1 \approx 3$ cm above the x-axis. At $x = 6$ cm, the graph is $e^6 \approx 403$ cm ≈ 4 m high (it is about to go through the ceiling if it hasn't done so already). At $x = 10$ cm, the graph is $e^{10} \approx 22{,}026$ cm ≈ 220 m high, higher than most buildings. At $x = 24$ cm, the graph is more than halfway to the moon, and at $x = 43$ cm from the origin, the graph is high enough to reach past the sun's closest stellar neighbor, the red dwarf star Proxima Centauri:

$$
\begin{aligned}
e^{43} &\approx 4.73 \times 10^{18} \text{ cm} \\
&= 4.73 \times 10^{13} \text{ km} \\
&\approx 1.58 \times 10^{8} \text{ light-seconds} \\
&\approx 5.0 \text{ light-years}
\end{aligned}
$$

In a vacuum, light travels at 300,000 km/sec.

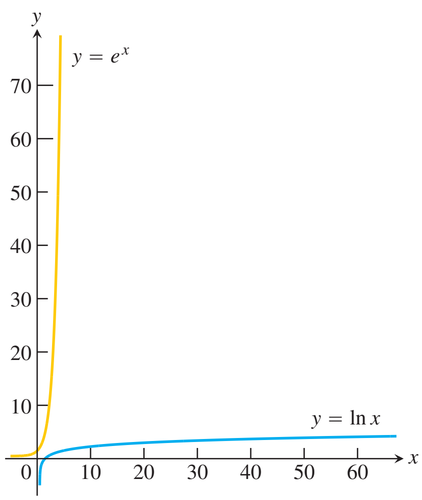


FIGURE 7.15 Scale drawings of the graphs of e^x and $\ln x$.

The distance to Proxima Centauri is about 4.22 light-years. Yet with $x = 43$ cm from the origin, the graph is still less than 2 feet to the right of the y-axis.

In contrast, logarithmic functions like $y = \log_2 x$ and $y = \ln x$ grow more slowly as $x \to \infty$ than any positive power of x (Exercise 21). With axes scaled in centimeters, you have to go nearly 5 light-years out on the x-axis to find a point where the graph of $y = \ln x$ is even $y = 43$ cm high. See Figure 7.15.

These important comparisons of exponential, polynomial, and logarithmic functions can be made precise by defining what it means for a function $f(x)$ to grow faster than another function $g(x)$ as $x \to \infty$.

DEFINITION Rates of Growth as $x \to \infty$

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

2. f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

According to these definitions, $y = 2x$ does not grow faster than $y = x$. The two functions grow at the same rate because

$$\lim_{x\to\infty} \frac{2x}{x} = \lim_{x\to\infty} 2 = 2,$$

which is a finite, nonzero limit. The reason for this apparent disregard of common sense is that we want "f grows faster than g" to mean that for large x-values g is negligible when compared with f.

EXAMPLE 1 Several Useful Comparisons of Growth Rates

(a) e^x grows faster than x^2 as $x \to \infty$ because

$$\underbrace{\lim_{x\to\infty} \frac{e^x}{x^2}}_{\infty/\infty} = \underbrace{\lim_{x\to\infty} \frac{e^x}{2x}}_{\infty/\infty} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Using l'Hôpital's Rule twice

(b) 3^x grows faster than 2^x as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{3^x}{2^x} = \lim_{x\to\infty} \left(\frac{3}{2}\right)^x = \infty.$$

(c) x^2 grows faster than $\ln x$ as $x \to \infty$, because

$$\lim_{x\to\infty} \frac{x^2}{\ln x} = \lim_{x\to\infty} \frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty. \qquad \text{l'Hôpital's Rule}$$

(d) $\ln x$ grows slower than x as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} \qquad \text{l'Hôpital's Rule}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0.$$

■

EXAMPLE 2 Exponential and Logarithmic Functions with Different Bases

You Try It

(a) As Example 1b suggests, exponential functions with different bases never grow at the same rate as $x \to \infty$. If $a > b > 0$, then a^x grows faster than b^x. Since $(a/b) > 1$,

$$\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = \infty.$$

(b) In contrast to exponential functions, logarithmic functions with different bases a and b always grow at the same rate as $x \to \infty$:

$$\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x/\ln a}{\ln x/\ln b} = \frac{\ln b}{\ln a}.$$

The limiting ratio is always finite and never zero. ■

If f grows at the same rate as g as $x \to \infty$, and g grows at the same rate as h as $x \to \infty$, then f grows at the same rate as h as $x \to \infty$. The reason is that

$$\lim_{x\to\infty} \frac{f}{g} = L_1 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{g}{h} = L_2$$

together imply

$$\lim_{x\to\infty} \frac{f}{h} = \lim_{x\to\infty} \frac{f}{g} \cdot \frac{g}{h} = L_1 L_2.$$

If L_1 and L_2 are finite and nonzero, then so is $L_1 L_2$.

EXAMPLE 3 Functions Growing at the Same Rate

You Try It

Show that $\sqrt{x^2 + 5}$ and $(2\sqrt{x} - 1)^2$ grow at the same rate as $x \to \infty$.

Solution We show that the functions grow at the same rate by showing that they both grow at the same rate as the function $g(x) = x$:

$$\lim_{x\to\infty} \frac{\sqrt{x^2 + 5}}{x} = \lim_{x\to\infty} \sqrt{1 + \frac{5}{x^2}} = 1,$$

$$\lim_{x\to\infty} \frac{(2\sqrt{x} - 1)^2}{x} = \lim_{x\to\infty} \left(\frac{2\sqrt{x} - 1}{\sqrt{x}}\right)^2 = \lim_{x\to\infty} \left(2 - \frac{1}{\sqrt{x}}\right)^2 = 4.$$

■

Order and Oh-Notation

Here we introduce the "little-oh" and "big-oh" notation invented by number theorists a hundred years ago and now commonplace in mathematical analysis and computer science.

DEFINITION Little-oh

A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\boldsymbol{f = o(g)}$ ("f is little-oh of g").

Notice that saying $f = o(g)$ as $x \to \infty$ is another way to say that f grows slower than g as $x \to \infty$.

EXAMPLE 4 Using Little-oh Notation

(a) $\ln x = o(x)$ as $x \to \infty$ because $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

(b) $x^2 = o(x^3 + 1)$ as $x \to \infty$ because $\lim_{x\to\infty} \frac{x^2}{x^3 + 1} = 0$ ■

DEFINITION Big-oh

Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \le M,$$

for x sufficiently large. We indicate this by writing $\boldsymbol{f = O(g)}$ ("f is big-oh of g").

EXAMPLE 5 Using Big-oh Notation

(a) $x + \sin x = O(x)$ as $x \to \infty$ because $\frac{x + \sin x}{x} \le 2$ for x sufficiently large.

(b) $e^x + x^2 = O(e^x)$ as $x \to \infty$ because $\frac{e^x + x^2}{e^x} \to 1$ as $x \to \infty$.

(c) $x = O(e^x)$ as $x \to \infty$ because $\frac{x}{e^x} \to 0$ as $x \to \infty$. ■

If you look at the definitions again, you will see that $f = o(g)$ implies $f = O(g)$ for functions that are positive for x sufficiently large. Also, if f and g grow at the same rate, then $f = O(g)$ and $g = O(f)$ (Exercise 11).

Sequential vs. Binary Search

Computer scientists often measure the efficiency of an algorithm by counting the number of steps a computer must take to execute the algorithm. There can be significant differences in how efficiently algorithms perform, even if they are designed to accomplish the same task. These differences are often described in big-oh notation. Here is an example.

Webster's Third New International Dictionary lists about 26,000 words that begin with the letter a. One way to look up a word, or to learn if it is not there, is to read through the list one word at a time until you either find the word or determine that it is not there. This method, called sequential search, makes no particular use of the words' alphabetical arrangement. You are sure to get an answer, but it might take 26,000 steps.

Another way to find the word or to learn it is not there is to go straight to the middle of the list (give or take a few words). If you do not find the word, then go to the middle of the half that contains it and forget about the half that does not. (You know which half contains it because you know the list is ordered alphabetically.) This method eliminates roughly 13,000 words in a single step. If you do not find the word on the second try, then jump to the middle of the half that contains it. Continue this way until you have either found the word or divided the list in half so many times there are no words left. How many times do you have to divide the list to find the word or learn that it is not there? At most 15, because

$$(26{,}000/2^{15}) < 1.$$

That certainly beats a possible 26,000 steps.

For a list of length n, a sequential search algorithm takes on the order of n steps to find a word or determine that it is not in the list. A binary search, as the second algorithm is called, takes on the order of $\log_2 n$ steps. The reason is that if $2^{m-1} < n \leq 2^m$, then $m - 1 < \log_2 n \leq m$, and the number of bisections required to narrow the list to one word will be at most $m = \lceil \log_2 n \rceil$, the integer ceiling for $\log_2 n$.

Big-oh notation provides a compact way to say all this. The number of steps in a sequential search of an ordered list is $O(n)$; the number of steps in a binary search is $O(\log_2 n)$. In our example, there is a big difference between the two (26,000 vs. 15), and the difference can only increase with n because n grows faster than $\log_2 n$ as $n \to \infty$ (as in Example 1d).

EXERCISES 7.6

Comparisons with the Exponential e^x

Exercises

1. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate as e^x? Which grow slower?

   a. $x + 3$ **b.** $x^3 + \sin^2 x$
   c. $\sqrt{x}$ **d.** 4^x
   e. $(3/2)^x$ **f.** $e^{x/2}$
   g. $e^x/2$ **h.** $\log_{10} x$

2. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate as e^x? Which grow slower?

   a. $10x^4 + 30x + 1$ **b.** $x \ln x - x$
   c. $\sqrt{1 + x^4}$ **d.** $(5/2)^x$
   e. e^{-x} **f.** xe^x
   g. $e^{\cos x}$ **h.** e^{x-1}

Comparisons with the Power x^2

Exercises

3. Which of the following functions grow faster than x^2 as $x \to \infty$? Which grow at the same rate as x^2? Which grow slower?

   a. $x^2 + 4x$ **b.** $x^5 - x^2$
   c. $\sqrt{x^4 + x^3}$ **d.** $(x + 3)^2$
   e. $x \ln x$ **f.** 2^x
   g. $x^3 e^{-x}$ **h.** $8x^2$

4. Which of the following functions grow faster than x^2 as $x \to \infty$? Which grow at the same rate as x^2? Which grow slower?

   a. $x^2 + \sqrt{x}$ **b.** $10x^2$
   c. $x^2 e^{-x}$ **d.** $\log_{10}(x^2)$
   e. $x^3 - x^2$ **f.** $(1/10)^x$
   g. $(1.1)^x$ **h.** $x^2 + 100x$

Comparisons with the Logarithm ln *x*

5. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

a. $\log_3 x$ **b.** $\ln 2x$
c. $\ln \sqrt{x}$ **d.** $\sqrt{x}$
e. x **f.** $5 \ln x$
g. $1/x$ **h.** e^x

6. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

a. $\log_2 (x^2)$ **b.** $\log_{10} 10x$
c. $1/\sqrt{x}$ **d.** $1/x^2$
e. $x - 2 \ln x$ **f.** e^{-x}
g. $\ln (\ln x)$ **h.** $\ln (2x + 5)$

Ordering Functions by Growth Rates

7. Order the following functions from slowest growing to fastest growing as $x \to \infty$.

a. e^x **b.** x^x
c. $(\ln x)^x$ **d.** $e^{x/2}$

8. Order the following functions from slowest growing to fastest growing as $x \to \infty$.

a. 2^x **b.** x^2
c. $(\ln 2)^x$ **d.** e^x

Big-oh and Little-oh; Order

9. True, or false? As $x \to \infty$,

a. $x = o(x)$ **b.** $x = o(x + 5)$
c. $x = O(x + 5)$ **d.** $x = O(2x)$
e. $e^x = o(e^{2x})$ **f.** $x + \ln x = O(x)$
g. $\ln x = o(\ln 2x)$ **h.** $\sqrt{x^2 + 5} = O(x)$

10. True, or false? As $x \to \infty$,

a. $\dfrac{1}{x + 3} = O\left(\dfrac{1}{x}\right)$ **b.** $\dfrac{1}{x} + \dfrac{1}{x^2} = O\left(\dfrac{1}{x}\right)$
c. $\dfrac{1}{x} - \dfrac{1}{x^2} = o\left(\dfrac{1}{x}\right)$ **d.** $2 + \cos x = O(2)$
e. $e^x + x = O(e^x)$ **f.** $x \ln x = o(x^2)$
g. $\ln (\ln x) = O(\ln x)$ **h.** $\ln (x) = o(\ln (x^2 + 1))$

11. Show that if positive functions $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$, then $f = O(g)$ and $g = O(f)$.

12. When is a polynomial $f(x)$ of smaller order than a polynomial $g(x)$ as $x \to \infty$? Give reasons for your answer.

13. When is a polynomial $f(x)$ of at most the order of a polynomial $g(x)$ as $x \to \infty$? Give reasons for your answer.

14. What do the conclusions we drew in Section 2.4 about the limits of rational functions tell us about the relative growth of polynomials as $x \to \infty$?

Other Comparisons

T **15.** Investigate

$$\lim_{x\to\infty} \frac{\ln (x + 1)}{\ln x} \quad \text{and} \quad \lim_{x\to\infty} \frac{\ln (x + 999)}{\ln x}.$$

Then use l'Hôpital's Rule to explain what you find.

16. (*Continuation of Exercise 15.*) Show that the value of

$$\lim_{x\to\infty} \frac{\ln (x + a)}{\ln x}$$

is the same no matter what value you assign to the constant a. What does this say about the relative rates at which the functions $f(x) = \ln (x + a)$ and $g(x) = \ln x$ grow?

17. Show that $\sqrt{10x + 1}$ and $\sqrt{x + 1}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as $\sqrt{x}$ as $x \to \infty$.

18. Show that $\sqrt{x^4 + x}$ and $\sqrt{x^4 - x^3}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as x^2 as $x \to \infty$.

19. Show that e^x grows faster as $x \to \infty$ than x^n for any positive integer n, even $x^{1{,}000{,}000}$. (*Hint:* What is the nth derivative of x^n?)

20. The function e^x outgrows any polynomial Show that e^x grows faster as $x \to \infty$ than any polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

21. a. Show that $\ln x$ grows slower as $x \to \infty$ than $x^{1/n}$ for any positive integer n, even $x^{1/1{,}000{,}000}$.

T **b.** Although the values of $x^{1/1{,}000{,}000}$ eventually overtake the values of $\ln x$, you have to go way out on the x-axis before this happens. Find a value of x greater than 1 for which $x^{1/1{,}000{,}000} > \ln x$. You might start by observing that when $x > 1$ the equation $\ln x = x^{1/1{,}000{,}000}$ is equivalent to the equation $\ln (\ln x) = (\ln x)/1{,}000{,}000$.

T **c.** Even $x^{1/10}$ takes a long time to overtake $\ln x$. Experiment with a calculator to find the value of x at which the graphs of $x^{1/10}$ and $\ln x$ cross, or, equivalently, at which $\ln x = 10 \ln (\ln x)$. Bracket the crossing point between powers of 10 and then close in by successive halving.

T **d.** (*Continuation of part (c).*) The value of x at which $\ln x = 10 \ln (\ln x)$ is too far out for some graphers and root finders to identify. Try it on the equipment available to you and see what happens.

22. The function ln *x* grows slower than any polynomial Show that $\ln x$ grows slower as $x \to \infty$ than any nonconstant polynomial.

Algorithms and Searches

23. a. Suppose you have three different algorithms for solving the same problem and each algorithm takes a number of steps that is of the order of one of the functions listed here:

$$n \log_2 n, \quad n^{3/2}, \quad n(\log_2 n)^2.$$

Which of the algorithms is the most efficient in the long run? Give reasons for your answer.

T **b.** Graph the functions in part (a) together to get a sense of how rapidly each one grows.

24. Repeat Exercise 23 for the functions

$$n, \quad \sqrt{n} \log_2 n, \quad (\log_2 n)^2.$$

T **25.** Suppose you are looking for an item in an ordered list one million items long. How many steps might it take to find that item with a sequential search? A binary search?

T **26.** You are looking for an item in an ordered list 450,000 items long (the length of *Webster's Third New International Dictionary*). How many steps might it take to find the item with a sequential search? A binary search?

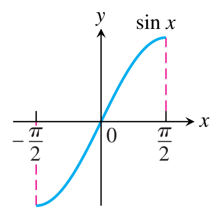
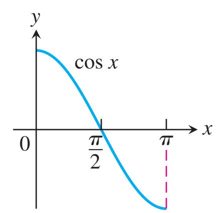
7.7 Inverse Trigonometric Functions

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations. This section shows how these functions are defined, graphed, and evaluated, how their derivatives are computed, and why they appear as important antiderivatives.

Defining the Inverses

The six basic trigonometric functions are not one-to-one (their values repeat periodically). However we can restrict their domains to intervals on which they are one-to-one. The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse $\sin^{-1} x$ (Figure 7.16). Similar domain restrictions can be applied to all six trigonometric functions.

Domain restrictions that make the trigonometric functions one-to-one

Function	Domain	Range	
$\sin x$	$[-\pi/2, \pi/2]$	$[-1, 1]$	
$\cos x$	$[0, \pi]$	$[-1, 1]$	

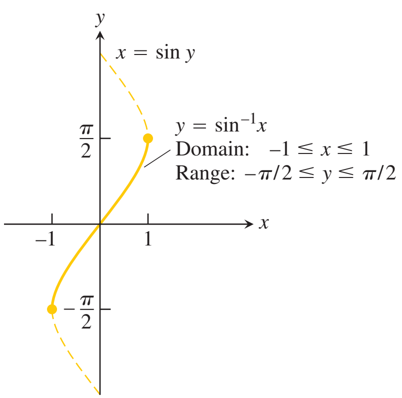
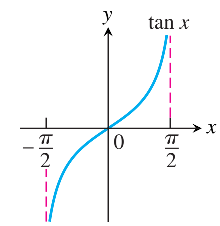
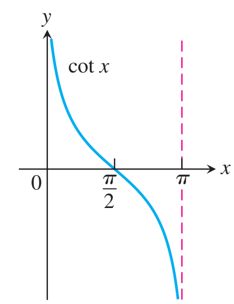
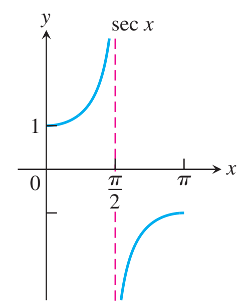
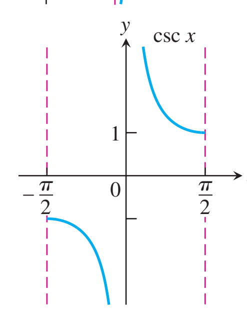


FIGURE 7.16 The graph of $y = \sin^{-1} x$.

$\tan x$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$	
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$	
$\sec x$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$	
$\csc x$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$	

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x$$
$$y = \cos^{-1} x \quad \text{or} \quad y = \arccos x$$
$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x$$
$$y = \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x$$
$$y = \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x$$
$$y = \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x$$

These equations are read "y equals the arcsine of x" or "y equals arcsin x" and so on.

CAUTION The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The graphs of the six inverse trigonometric functions are shown in Figure 7.17. We can obtain these graphs by reflecting the graphs of the restricted trigonometric functions through the line $y = x$, as in Section 7.1. We now take a closer look at these functions and their derivatives.

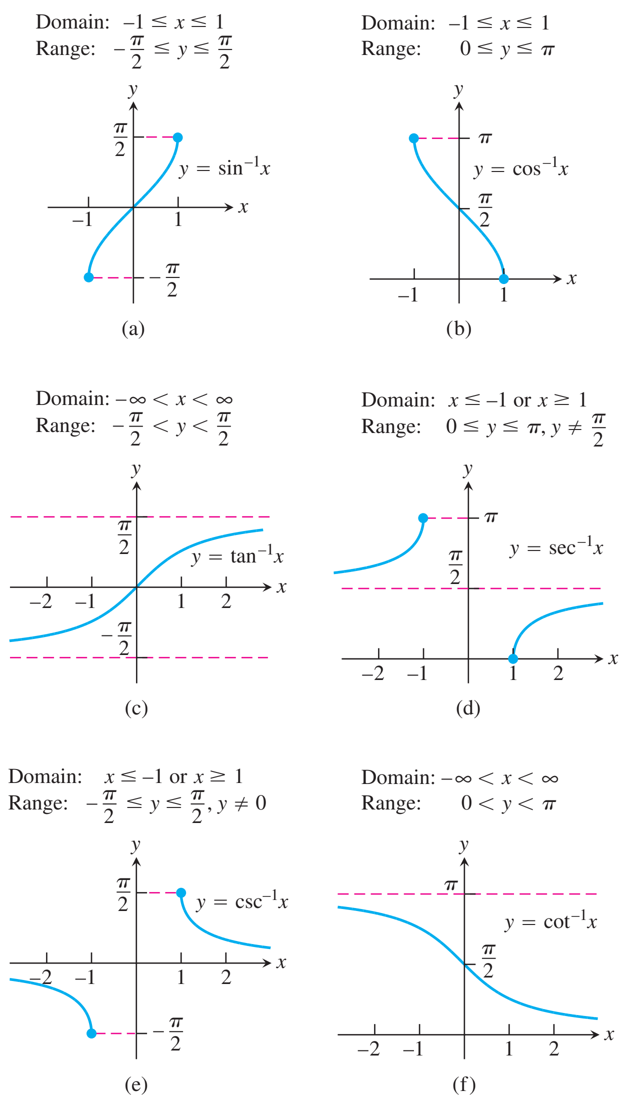


FIGURE 7.17 Graphs of the six basic inverse trigonometric functions.

The Arcsine and Arccosine Functions

The arcsine of x is the angle in $[-\pi/2, \pi/2]$ whose sine is x. The arccosine is an angle in $[0, \pi]$ whose cosine is x.

DEFINITION Arcsine and Arccosine Functions

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

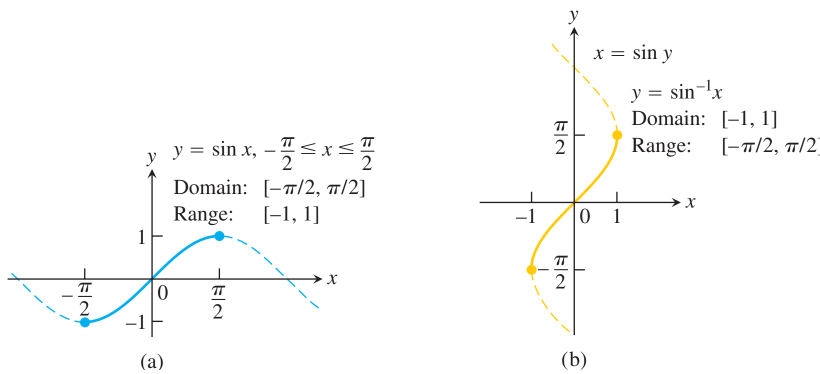
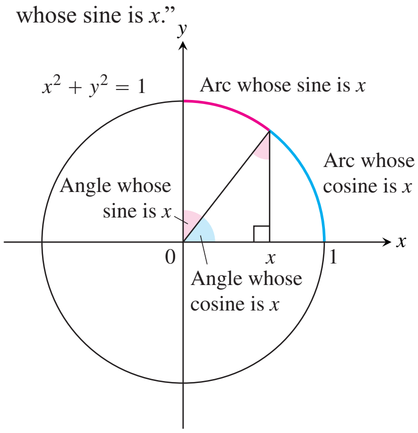


FIGURE 7.18 The graphs of (a) $y = \sin x$, $-\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

The "Arc" in Arc Sine and Arc Cosine

The accompanying figure gives a geometric interpretation of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for radian angles in the first quadrant. For a unit circle, the equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



The graph of $y = \sin^{-1} x$ (Figure 7.18) is symmetric about the origin (it lies along the graph of $x = \sin y$). The arcsine is therefore an odd function:

$$\sin^{-1}(-x) = -\sin^{-1} x. \tag{1}$$

The graph of $y = \cos^{-1} x$ (Figure 7.19) has no such symmetry.

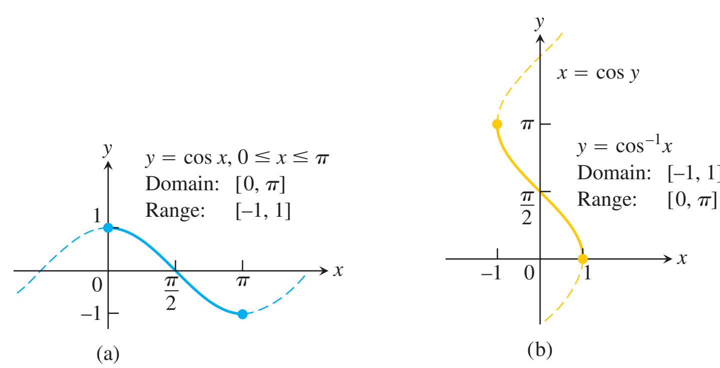
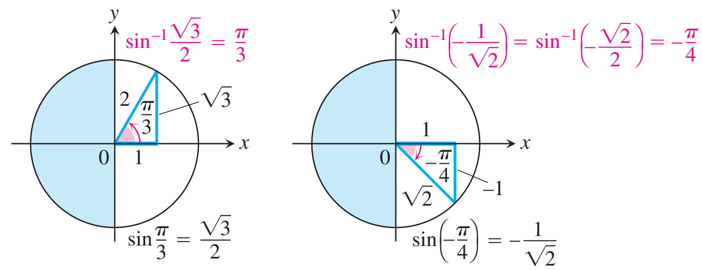


FIGURE 7.19 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Known values of $\sin x$ and $\cos x$ can be inverted to find values of $\sin^{-1} x$ and $\cos^{-1} x$.

EXAMPLE 1 Common Values of $\sin^{-1} x$

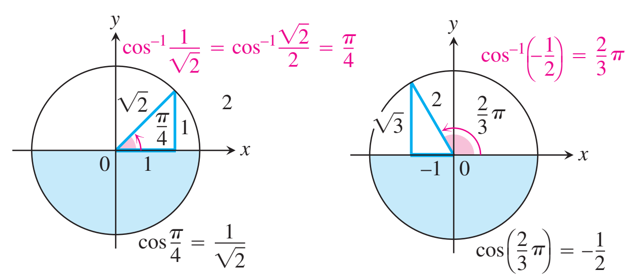
x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$



The angles come from the first and fourth quadrants because the range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$. ■

EXAMPLE 2 Common Values of $\cos^{-1} x$

x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$



The angles come from the first and second quadrants because the range of $\cos^{-1} x$ is $[0, \pi]$. ■

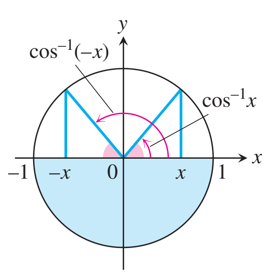


FIGURE 7.20 $\cos^{-1} x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

Identities Involving Arcsine and Arccosine

As we can see from Figure 7.20, the arccosine of x satisfies the identity

$$\cos^{-1} x + \cos^{-1}(-x) = \pi, \tag{2}$$

or

$$\cos^{-1}(-x) = \pi - \cos^{-1} x. \tag{3}$$

Also, we can see from the triangle in Figure 7.21 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2. \tag{4}$$

Equation (4) holds for the other values of x in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Figure 7.21. It is, however, a consequence of Equations (1) and (3) (Exercise 131).

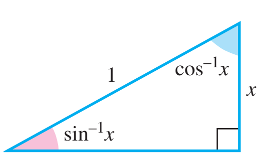


FIGURE 7.21 $\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

Inverses of tan *x*, cot *x*, sec *x*, and csc *x*

The arctangent of x is an angle whose tangent is x. The arccotangent of x is an angle whose cotangent is x.

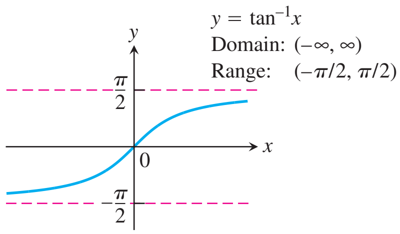


FIGURE 7.22 The graph of $y = \tan^{-1} x$.

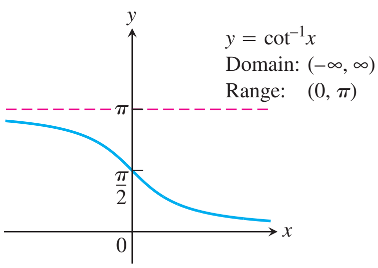


FIGURE 7.23 The graph of $y = \cot^{-1} x$.

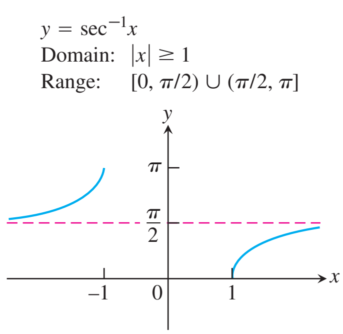


FIGURE 7.24 The graph of $y = \sec^{-1} x$.

> **DEFINITION Arctangent and Arccotangent Functions**
>
> $y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.
>
> $y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

We use open intervals to avoid values where the tangent and cotangent are undefined.

The graph of $y = \tan^{-1} x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Figure 7.22). Algebraically this means that

$$\tan^{-1}(-x) = -\tan^{-1} x;$$

the arctangent is an odd function. The graph of $y = \cot^{-1} x$ has no such symmetry (Figure 7.23).

The inverses of the restricted forms of $\sec x$ and $\csc x$ are chosen to be the functions graphed in Figures 7.24 and 7.25.

CAUTION There is no general agreement about how to define $\sec^{-1} x$ for negative values of x. We chose angles in the second quadrant between $\pi/2$ and π. This choice makes $\sec^{-1} x = \cos^{-1}(1/x)$. It also makes $\sec^{-1} x$ an increasing function on each interval of its domain. Some tables choose $\sec^{-1} x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Figure 7.26). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1} x = \cos^{-1}(1/x)$. From this, we can derive the identity

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right) \tag{5}$$

by applying Equation (4).

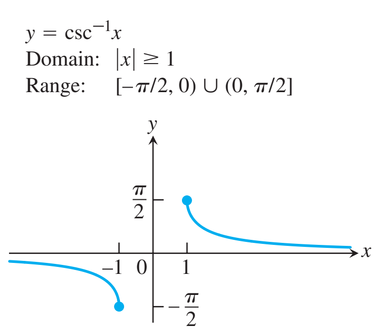


FIGURE 7.25 The graph of $y = \csc^{-1} x$.

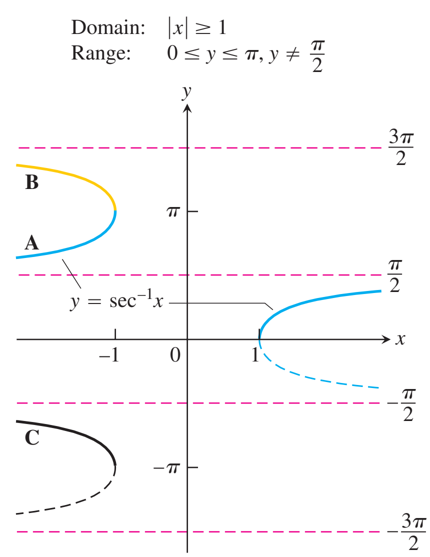
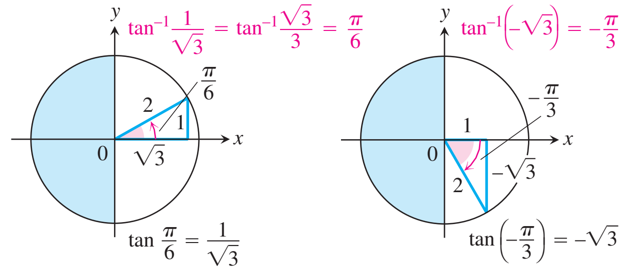


FIGURE 7.26 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, $\sec^{-1} x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

EXAMPLE 3 Common Values of $\tan^{-1} x$



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$. ■

EXAMPLE 4 Find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$ if

$$\alpha = \sin^{-1}\frac{2}{3}.$$

Solution This equation says that $\sin\alpha = 2/3$. We picture α as an angle in a right triangle with opposite side 2 and hypotenuse 3 (Figure 7.27). The length of the remaining side is

$$\sqrt{(3)^2 - (2)^2} = \sqrt{9-4} = \sqrt{5}. \qquad \text{Pythagorean theorem}$$

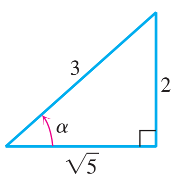


FIGURE 7.27 If $\alpha = \sin^{-1}(2/3)$, then the values of the other basic trigonometric functions of α can be read from this triangle (Example 4).

We add this information to the figure and then read the values we want from the completed triangle:

$$\cos\alpha = \frac{\sqrt{5}}{3}, \quad \tan\alpha = \frac{2}{\sqrt{5}}, \quad \sec\alpha = \frac{3}{\sqrt{5}}, \quad \csc\alpha = \frac{3}{2}, \quad \cot\alpha = \frac{\sqrt{5}}{2}.$$ ■

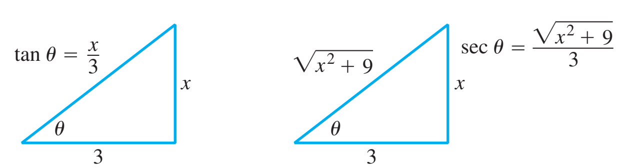
EXAMPLE 5 Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite}/\text{adjacent} = x/3.$$

The length of the triangle's hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$



Thus,

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta$$
$$= \frac{\sqrt{x^2+9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$ ■

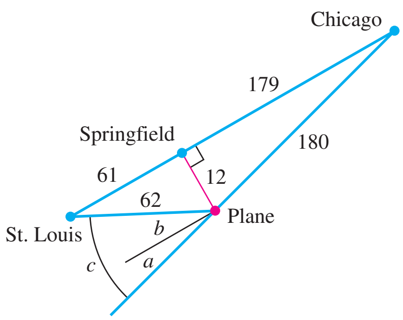


FIGURE 7.28 Diagram for drift correction (Example 6), with distances rounded to the nearest mile (drawing not to scale).

EXAMPLE 6 Drift Correction

During an airplane flight from Chicago to St. Louis the navigator determines that the plane is 12 mi off course, as shown in Figure 7.28. Find the angle a for a course parallel to the original, correct course, the angle b, and the correction angle $c = a + b$.

Solution

$$a = \sin^{-1}\frac{12}{180} \approx 0.067 \text{ radian} \approx 3.8°$$

$$b = \sin^{-1}\frac{12}{62} \approx 0.195 \text{ radian} \approx 11.2°$$

$$c = a + b \approx 15°.$$

The Derivative of $y = \sin^{-1} u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 1 in Section 7.1 therefore assures us that the inverse function $y = \sin^{-1} x$ is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points (see Figure 7.29).

We find the derivative of $y = \sin^{-1} x$ by applying Theorem 1 with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$.

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1}\\
&= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u\\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u}\\
&= \frac{1}{\sqrt{1 - x^2}} && \sin(\sin^{-1} x) = x
\end{aligned}
$$

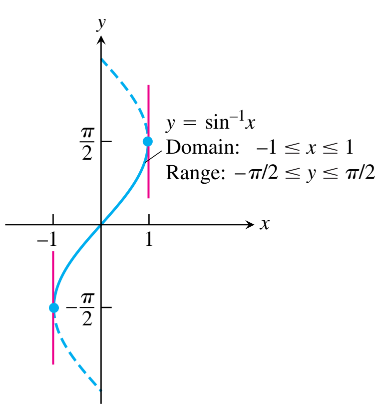


FIGURE 7.29 The graph of $y = \sin^{-1} x$.

Alternate Derivation: Instead of applying Theorem 1 directly, we can find the derivative of $y = \sin^{-1} x$ using implicit differentiation as follows:

$$
\begin{aligned}
\sin y &= x && y = \sin^{-1} x \Leftrightarrow \sin y = x\\
\frac{d}{dx}(\sin y) &= 1 && \text{Derivative of both sides with respect to } x\\
\cos y \frac{dy}{dx} &= 1 && \text{Chain Rule}\\
\frac{dy}{dx} &= \frac{1}{\cos y} && \text{We can divide because } \cos y > 0 \text{ for } -\pi/2 < y < \pi/2.\\
&= \frac{1}{\sqrt{1 - x^2}} && \cos y = \sqrt{1 - \sin^2 y}
\end{aligned}
$$

No matter which derivation we use, we have that the derivative of $y = \sin^{-1} x$ with respect to x is

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \qquad |u| < 1.$$

EXAMPLE 7 Applying the Derivative Formula

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1 - x^4}}$$ ■

The Derivative of $y = \tan^{-1} u$

We find the derivative of $y = \tan^{-1} x$ by applying Theorem 1 with $f(x) = \tan x$ and $f^{-1}(x) = \tan^{-1} x$. Theorem 1 can be applied because the derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$.

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u \\
&= \frac{1}{1 + \tan^2(\tan^{-1} x)} && \sec^2 u = 1 + \tan^2 u \\
&= \frac{1}{1 + x^2} && \tan(\tan^{-1} x) = x
\end{aligned}$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2} \frac{du}{dx}.$$

EXAMPLE 8 A Moving Particle

A particle moves along the x-axis so that its position at any time $t \geq 0$ is $x(t) = \tan^{-1}\sqrt{t}$. What is the velocity of the particle when $t = 16$?

Solution

$$v(t) = \frac{d}{dt}\tan^{-1}\sqrt{t} = \frac{1}{1 + (\sqrt{t})^2} \cdot \frac{d}{dt}\sqrt{t} = \frac{1}{1 + t} \cdot \frac{1}{2\sqrt{t}}$$

When $t = 16$, the velocity is

$$v(16) = \frac{1}{1 + 16} \cdot \frac{1}{2\sqrt{16}} = \frac{1}{136}.$$

The Derivative of $y = \sec^{-1} u$

Since the derivative of $\sec x$ is positive for $0 < x < \pi/2$ and $\pi/2 < x < \pi$, Theorem 1 says that the inverse function $y = \sec^{-1} x$ is differentiable. Instead of applying the formula in Theorem 1 directly, we find the derivative of $y = \sec^{-1} x$, $|x| > 1$, using implicit differentiation and the Chain Rule as follows:

$$y = \sec^{-1} x$$

$$\sec y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\sec y) = \frac{d}{dx} x \qquad \text{Differentiate both sides.}$$

$$\sec y \tan y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \qquad \text{Since } |x| > 1,\ y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \neq 0.$$

To express the result in terms of x, we use the relationships

$$\sec y = x \qquad \text{and} \qquad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

Can we do anything about the $\pm$ sign? A glance at Figure 7.30 shows that the slope of the graph $y = \sec^{-1} x$ is always positive. Thus,

$$\frac{d}{dx} \sec^{-1} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

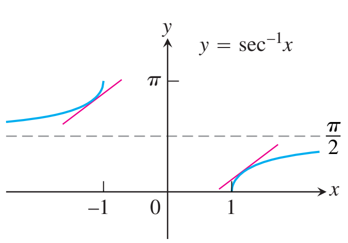


FIGURE 7.30 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

With the absolute value symbol, we can write a single expression that eliminates the "±" ambiguity:

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \qquad |u| > 1.$$

EXAMPLE 9 Using the Formula

$$\begin{aligned}\frac{d}{dx}\sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2-1}}\frac{d}{dx}(5x^4)\\ &= \frac{1}{5x^4\sqrt{25x^8-1}}(20x^3) \qquad 5x^4>0\\ &= \frac{4}{x\sqrt{25x^8-1}}\end{aligned}$$

Derivatives of the Other Three

We could use the same techniques to find the derivatives of the other three inverse trigonometric functions—arccosine, arccotangent, and arccosecant—but there is a much easier way, thanks to the following identities.

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1}x = \pi/2 - \sin^{-1}x$$
$$\cot^{-1}x = \pi/2 - \tan^{-1}x$$
$$\csc^{-1}x = \pi/2 - \sec^{-1}x$$

We saw the first of these identities in Equation (4). The others are derived in a similar way. It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions. For example, the derivative of $\cos^{-1}x$ is calculated as follows:

$$\begin{aligned}\frac{d}{dx}(\cos^{-1}x) &= \frac{d}{dx}\left(\frac{\pi}{2}-\sin^{-1}x\right) \qquad \text{Identity}\\ &= -\frac{d}{dx}(\sin^{-1}x)\\ &= -\frac{1}{\sqrt{1-x^2}} \qquad \text{Derivative of arcsine}\end{aligned}$$

EXAMPLE 10 A Tangent Line to the Arccotangent Curve

Find an equation for the line tangent to the graph of $y = \cot^{-1}x$ at $x = -1$.

Solution First we note that

$$\cot^{-1}(-1) = \pi/2 - \tan^{-1}(-1) = \pi/2 - (-\pi/4) = 3\pi/4.$$

The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\left.\frac{1}{1+x^2}\right|_{x=-1} = -\frac{1}{1+(-1)^2} = -\frac{1}{2},$$

so the tangent line has equation $y - 3\pi/4 = (-1/2)(x+1)$.

The derivatives of the inverse trigonometric functions are summarized in Table 7.3.

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1+u^2}$

4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1+u^2}$

5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

Integration Formulas

The derivative formulas in Table 7.3 yield three useful integration formulas in Table 7.4. The formulas are readily verified by differentiating the functions on the right-hand sides.

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad$ (Valid for $u^2 < a^2$)

2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad$ (Valid for $|u| > a > 0$)

The derivative formulas in Table 7.3 have $a = 1$, but in most integrations $a \neq 1$, and the formulas in Table 7.4 are more useful.

EXAMPLE 11 Using the Integral Formulas

(a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big]_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(b) $$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x \Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

(c) $$\int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x \Big]_{2/\sqrt{3}}^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$ ■

EXAMPLE 12 Using Substitution and Table 7.4

(a) $$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{(3)^2 - x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$ Table 7.4 Formula 1, with $a = 3$, $u = x$

(b) $$\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}}$$ $a = \sqrt{3}$, $u = 2x$, and $du/2 = dx$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$ Formula 1

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$ ■

EXAMPLE 13 Completing the Square

Evaluate

$$\int \frac{dx}{\sqrt{4x-x^2}}.$$

Solution The expression $\sqrt{4x-x^2}$ does not match any of the formulas in Table 7.4, so we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x-2)^2.$$

Then we substitute $a = 2$, $u = x - 2$, and $du = dx$ to get

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= \int \frac{du}{\sqrt{a^2-u^2}}$$ $a = 2$, $u = x - 2$, and $du = dx$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$ Table 7.4, Formula 1

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$ ■

EXAMPLE 14 Completing the Square

Evaluate

$$\int \frac{dx}{4x^2 + 4x + 2}.$$

Solution We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.$$

Then,

$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x + 1)^2 + 1} = \frac{1}{2}\int \frac{du}{u^2 + a^2} \qquad a = 1, u = 2x + 1, \text{ and } du/2 = dx$$

$$= \frac{1}{2} \cdot \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 7.4, Formula 2}$$

$$= \frac{1}{2}\tan^{-1}(2x + 1) + C \qquad a = 1, u = 2x + 1$$

■

EXAMPLE 15 Using Substitution

You Try It

Evaluate

$$\int \frac{dx}{\sqrt{e^{2x} - 6}}.$$

Solution

$$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{du/u}{\sqrt{u^2 - a^2}} \qquad u = e^x, du = e^x\,dx, \; dx = du/e^x = du/u, \; a = \sqrt{6}$$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad \text{Table 7.4, Formula 3}$$

$$= \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

■

EXERCISES 7.7

Common Values of Inverse Trignonometric Functions

Use reference triangles like those in Examples 1–3 to find the angles in Exercises 1–12.

1. **a.** $\tan^{-1} 1$ **b.** $\tan^{-1}(-\sqrt{3})$ **c.** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

2. **a.** $\tan^{-1}(-1)$ **b.** $\tan^{-1}\sqrt{3}$ **c.** $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

3. **a.** $\sin^{-1}\left(\frac{-1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

4. **a.** $\sin^{-1}\left(\frac{1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

5. **a.** $\cos^{-1}\left(\frac{1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

6. **a.** $\cos^{-1}\left(\frac{-1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

7. **a.** $\sec^{-1}(-\sqrt{2})$ **b.** $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **c.** $\sec^{-1}(-2)$

8. **a.** $\sec^{-1}\sqrt{2}$ **b.** $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ **c.** $\sec^{-1} 2$

9. **a.** $\csc^{-1}\sqrt{2}$ **b.** $\csc^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ **c.** $\csc^{-1} 2$

10. **a.** $\csc^{-1}(-\sqrt{2})$ **b.** $\csc^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **c.** $\csc^{-1}(-2)$

11. **a.** $\cot^{-1}(-1)$ **b.** $\cot^{-1}(\sqrt{3})$ **c.** $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

12. **a.** $\cot^{-1}(1)$ **b.** $\cot^{-1}(-\sqrt{3})$ **c.** $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Trigonometric Function Values

13. Given that $\alpha = \sin^{-1}(5/13)$, find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

14. Given that $\alpha = \tan^{-1}(4/3)$, find $\sin\alpha$, $\cos\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

15. Given that $\alpha = \sec^{-1}(-\sqrt{5})$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

16. Given that $\alpha = \sec^{-1}(-\sqrt{13}/2)$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

Evaluating Trigonometric and Inverse Trigonometric Terms

Find the values in Exercises 17–28.

17. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ **18.** $\sec\left(\cos^{-1}\frac{1}{2}\right)$

19. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ **20.** $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

21. $\csc(\sec^{-1} 2) + \cos(\tan^{-1}(-\sqrt{3}))$

22. $\tan(\sec^{-1} 1) + \sin(\csc^{-1}(-2))$

23. $\sin\left(\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)\right)$

24. $\cot\left(\sin^{-1}\left(-\frac{1}{2}\right) - \sec^{-1} 2\right)$

25. $\sec(\tan^{-1} 1 + \csc^{-1} 1)$ **26.** $\sec(\cot^{-1}\sqrt{3} + \csc^{-1}(-1))$

27. $\sec^{-1}\left(\sec\left(-\frac{\pi}{6}\right)\right)$ (The answer is *not* $-\pi/6$.)

28. $\cot^{-1}\left(\cot\left(-\frac{\pi}{4}\right)\right)$ (The answer is *not* $-\pi/4$.)

Finding Trigonometric Expressions

Evaluate the expressions in Exercises 29–40.

29. $\sec\left(\tan^{-1}\frac{x}{2}\right)$ **30.** $\sec(\tan^{-1} 2x)$

31. $\tan(\sec^{-1} 3y)$ **32.** $\tan\left(\sec^{-1}\frac{y}{5}\right)$

33. $\cos(\sin^{-1} x)$ **34.** $\tan(\cos^{-1} x)$

35. $\sin(\tan^{-1}\sqrt{x^2 - 2x}), \quad x \geq 2$

36. $\sin\left(\tan^{-1}\frac{x}{\sqrt{x^2+1}}\right)$ **37.** $\cos\left(\sin^{-1}\frac{2y}{3}\right)$

38. $\cos\left(\sin^{-1}\frac{y}{5}\right)$ **39.** $\sin\left(\sec^{-1}\frac{x}{4}\right)$

40. $\sin\sec^{-1}\left(\frac{\sqrt{x^2+4}}{x}\right)$

Limits

Find the limits in Exercises 41–48. (If in doubt, look at the function's graph.)

41. $\lim_{x\to 1^-}\sin^{-1} x$ **42.** $\lim_{x\to -1^+}\cos^{-1} x$

43. $\lim_{x\to\infty}\tan^{-1} x$ **44.** $\lim_{x\to -\infty}\tan^{-1} x$

45. $\lim_{x\to\infty}\sec^{-1} x$ **46.** $\lim_{x\to -\infty}\sec^{-1} x$

47. $\lim_{x\to\infty}\csc^{-1} x$ **48.** $\lim_{x\to -\infty}\csc^{-1} x$

Finding Derivatives

In Exercises 49–70, find the derivative of y with respect to the appropriate variable.

49. $y = \cos^{-1}(x^2)$ **50.** $y = \cos^{-1}(1/x)$

51. $y = \sin^{-1}\sqrt{2}\,t$ **52.** $y = \sin^{-1}(1 - t)$

53. $y = \sec^{-1}(2s + 1)$ **54.** $y = \sec^{-1} 5s$

55. $y = \csc^{-1}(x^2 + 1), \quad x > 0$

56. $y = \csc^{-1}\frac{x}{2}$

57. $y = \sec^{-1}\frac{1}{t}, \quad 0 < t < 1$ **58.** $y = \sin^{-1}\frac{3}{t^2}$

59. $y = \cot^{-1}\sqrt{t}$ **60.** $y = \cot^{-1}\sqrt{t - 1}$

61. $y = \ln(\tan^{-1} x)$ **62.** $y = \tan^{-1}(\ln x)$

63. $y = \csc^{-1}(e^t)$ **64.** $y = \cos^{-1}(e^{-t})$

65. $y = s\sqrt{1 - s^2} + \cos^{-1} s$ **66.** $y = \sqrt{s^2 - 1} - \sec^{-1} s$

67. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x, \quad x > 1$

68. $y = \cot^{-1}\frac{1}{x} - \tan^{-1} x$ **69.** $y = x\sin^{-1} x + \sqrt{1 - x^2}$

70. $y = \ln(x^2 + 4) - x\tan^{-1}\left(\frac{x}{2}\right)$

Evaluating Integrals

Evaluate the integrals in Exercises 71–94.

71. $\int \frac{dx}{\sqrt{9 - x^2}}$ **72.** $\int \frac{dx}{\sqrt{1 - 4x^2}}$

73. $\int \frac{dx}{17 + x^2}$

74. $\int \frac{dx}{9 + 3x^2}$

75. $\int \frac{dx}{x\sqrt{25x^2 - 2}}$

76. $\int \frac{dx}{x\sqrt{5x^2 - 4}}$

77. $\int_0^1 \frac{4\,ds}{\sqrt{4 - s^2}}$

78. $\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9 - 4s^2}}$

79. $\int_0^2 \frac{dt}{8 + 2t^2}$

80. $\int_{-2}^2 \frac{dt}{4 + 3t^2}$

81. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$

82. $\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2 - 1}}$

83. $\int \frac{3\,dr}{\sqrt{1 - 4(r - 1)^2}}$

84. $\int \frac{6\,dr}{\sqrt{4 - (r + 1)^2}}$

85. $\int \frac{dx}{2 + (x - 1)^2}$

86. $\int \frac{dx}{1 + (3x + 1)^2}$

87. $\int \frac{dx}{(2x - 1)\sqrt{(2x - 1)^2 - 4}}$

88. $\int \frac{dx}{(x + 3)\sqrt{(x + 3)^2 - 25}}$

89. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{1 + (\sin\theta)^2}$

90. $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\,dx}{1 + (\cot x)^2}$

91. $\int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{1 + e^{2x}}$

92. $\int_1^{e^{\pi/4}} \frac{4\,dt}{t(1 + \ln^2 t)}$

93. $\int \frac{y\,dy}{\sqrt{1 - y^4}}$

94. $\int \frac{\sec^2 y\,dy}{\sqrt{1 - \tan^2 y}}$

Evaluate the integrals in Exercises 95–104.

95. $\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$

96. $\int \frac{dx}{\sqrt{2x - x^2}}$

97. $\int_{-1}^0 \frac{6\,dt}{\sqrt{3 - 2t - t^2}}$

98. $\int_{1/2}^1 \frac{6\,dt}{\sqrt{3 + 4t - 4t^2}}$

99. $\int \frac{dy}{y^2 - 2y + 5}$

100. $\int \frac{dy}{y^2 + 6y + 10}$

101. $\int_1^2 \frac{8\,dx}{x^2 - 2x + 2}$

102. $\int_2^4 \frac{2\,dx}{x^2 - 6x + 10}$

103. $\int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}$

104. $\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$

Evaluate the integrals in Exercises 105–112.

105. $\int \frac{e^{\sin^{-1} x}\,dx}{\sqrt{1 - x^2}}$

106. $\int \frac{e^{\cos^{-1} x}\,dx}{\sqrt{1 - x^2}}$

107. $\int \frac{(\sin^{-1} x)^2\,dx}{\sqrt{1 - x^2}}$

108. $\int \frac{\sqrt{\tan^{-1} x}\,dx}{1 + x^2}$

109. $\int \frac{dy}{(\tan^{-1} y)(1 + y^2)}$

110. $\int \frac{dy}{(\sin^{-1} y)\sqrt{1 - y^2}}$

111. $\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1} x)\,dx}{x\sqrt{x^2 - 1}}$

112. $\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x)\,dx}{x\sqrt{x^2 - 1}}$

Limits

Find the limits in Exercises 113–116.

113. $\lim_{x \to 0} \frac{\sin^{-1} 5x}{x}$

114. $\lim_{x \to 1^+} \frac{\sqrt{x^2 - 1}}{\sec^{-1} x}$

115. $\lim_{x \to \infty} x \tan^{-1} \frac{2}{x}$

116. $\lim_{x \to 0} \frac{2\tan^{-1} 3x^2}{7x^2}$

Integration Formulas

Verify the integration formulas in Exercises 117–120.

117. $\int \frac{\tan^{-1} x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1} x}{x} + C$

118. $\int x^3 \cos^{-1} 5x\,dx = \frac{x^4}{4}\cos^{-1} 5x + \frac{5}{4}\int \frac{x^4\,dx}{\sqrt{1 - 25x^2}}$

119. $\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1 - x^2}\sin^{-1} x + C$

120. $\int \ln(a^2 + x^2)\,dx = x\ln(a^2 + x^2) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

Initial Value Problems

Solve the initial value problems in Exercises 121–124.

121. $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}, \quad y(0) = 0$

122. $\frac{dy}{dx} = \frac{1}{x^2 + 1} - 1, \quad y(0) = 1$

123. $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}, \quad x > 1; \quad y(2) = \pi$

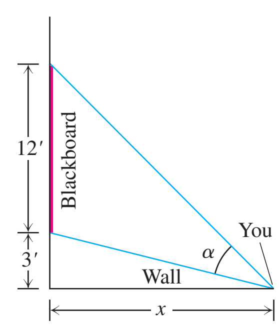
124. $\frac{dy}{dx} = \frac{1}{1 + x^2} - \frac{2}{\sqrt{1 - x^2}}, \quad y(0) = 2$

Applications and Theory

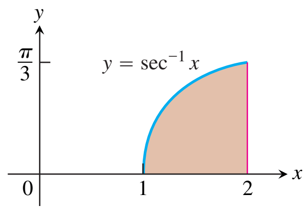
125. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$$

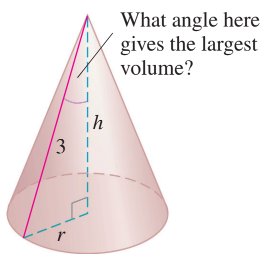
if you are x ft from the front wall.



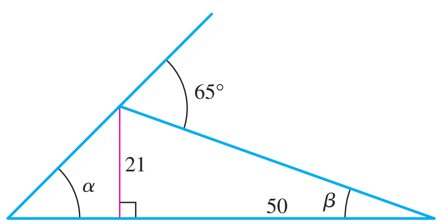
126. The region between the curve $y = \sec^{-1} x$ and the x-axis from $x = 1$ to $x = 2$ (shown here) is revolved about the y-axis to generate a solid. Find the volume of the solid.



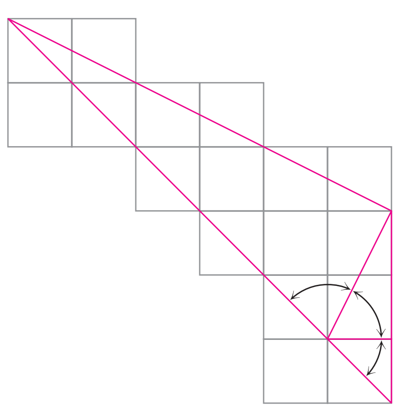
127. The slant height of the cone shown here is 3 m. How large should the indicated angle be to maximize the cone's volume?



128. Find the angle α.

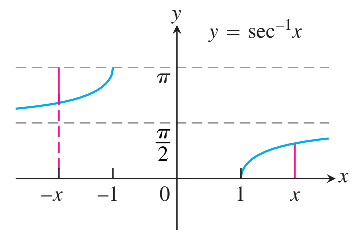


129. Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



130. Two derivations of the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$

a. (*Geometric*) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1} x$. See if you can tell what is going on.



b. (*Algebraic*) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad \text{Eq. (3)}$$

$$\sec^{-1} x = \cos^{-1}(1/x) \qquad \text{Eq. (5)}$$

131. The identity $\sin^{-1} x + \cos^{-1} x = \pi/2$ Figure 7.21 establishes the identity for $0 < x < 1$. To establish it for the rest of $[-1, 1]$, verify by direct calculation that it holds for $x = 1$, 0, and -1. Then, for values of x in $(-1, 0)$, let $x = -a$, $a > 0$, and apply Eqs. (1) and (3) to the sum $\sin^{-1}(-a) + \cos^{-1}(-a)$.

132. Show that the sum $\tan^{-1} x + \tan^{-1}(1/x)$ is constant.

Which of the expressions in Exercises 133–136 are defined, and which are not? Give reasons for your answers.

133. **a.** $\tan^{-1} 2$ **b.** $\cos^{-1} 2$

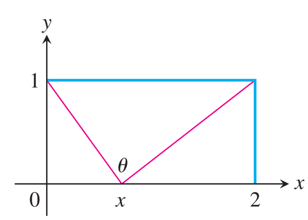
134. **a.** $\csc^{-1}(1/2)$ **b.** $\csc^{-1} 2$

135. **a.** $\sec^{-1} 0$ **b.** $\sin^{-1}\sqrt{2}$

136. **a.** $\cot^{-1}(-1/2)$ **b.** $\cos^{-1}(-5)$

137. (*Continuation of Exercise 125.*) You want to position your chair along the wall to maximize your viewing angle α. How far from the front of the room should you sit?

138. What value of x maximizes the angle θ shown here? How large is θ at that point? Begin by showing that $\theta = \pi - \cot^{-1} x - \cot^{-1}(2 - x)$.



139. Can the integrations in (a) and (b) both be correct? Explain.

a. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$

b. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = -\int -\frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1} x + C$

140. Can the integrations in (a) and (b) both be correct? Explain.

a. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = -\int -\frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1} x + C$

b. $$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{-du}{\sqrt{1-(-u)^2}} \qquad \begin{aligned} x &= -u, \\ dx &= -du \end{aligned}$$
$$= \int \frac{-du}{\sqrt{1-u^2}}$$
$$= \cos^{-1} u + C$$
$$= \cos^{-1}(-x) + C \qquad u = -x$$

141. Use the identity
$$\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u$$
to derive the formula for the derivative of $\csc^{-1} u$ in Table 7.3 from the formula for the derivative of $\sec^{-1} u$.

142. Derive the formula
$$\frac{dy}{dx} = \frac{1}{1+x^2}$$
for the derivative of $y = \tan^{-1} x$ by differentiating both sides of the equivalent equation $\tan y = x$.

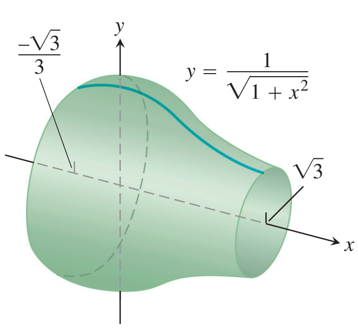
143. Use the Derivative Rule in Section 7.1, Theorem 1, to derive
$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

144. Use the identity
$$\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u$$
to derive the formula for the derivative of $\cot^{-1} u$ in Table 7.3 from the formula for the derivative of $\tan^{-1} u$.

145. What is special about the functions
$$f(x) = \sin^{-1} \frac{x-1}{x+1}, \quad x \geq 0, \quad \text{and} \quad g(x) = 2\tan^{-1}\sqrt{x}?$$
Explain.

146. What is special about the functions
$$f(x) = \sin^{-1} \frac{1}{\sqrt{x^2+1}} \quad \text{and} \quad g(x) = \tan^{-1} \frac{1}{x}?$$
Explain.

147. Find the volume of the solid of revolution shown here.



148. Arc length Find the length of the curve $y = \sqrt{1-x^2}$, $-1/2 \leq x \leq 1/2$.

Volumes by Slicing

Find the volumes of the solids in Exercises 149 and 150.

149. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are

a. circles whose diameters stretch from the curve $y = -1/\sqrt{1+x^2}$ to the curve $y = 1/\sqrt{1+x^2}$.

b. vertical squares whose base edges run from the curve $y = -1/\sqrt{1+x^2}$ to the curve $y = 1/\sqrt{1+x^2}$.

150. The solid lies between planes perpendicular to the x-axis at $x = -\sqrt{2}/2$ and $x = \sqrt{2}/2$. The cross-sections perpendicular to the x-axis are

a. circles whose diameters stretch from the x-axis to the curve $y = 2/\sqrt[4]{1-x^2}$.

b. squares whose diagonals stretch from the x-axis to the curve $y = 2/\sqrt[4]{1-x^2}$.

T Calculator and Grapher Explorations

151. Find the values of

a. $\sec^{-1} 1.5$ **b.** $\csc^{-1}(-1.5)$ **c.** $\cot^{-1} 2$

152. Find the values of

a. $\sec^{-1}(-3)$ **b.** $\csc^{-1} 1.7$ **c.** $\cot^{-1}(-2)$

In Exercises 153–155, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

153. a. $y = \tan^{-1}(\tan x)$ **b.** $y = \tan(\tan^{-1} x)$

154. a. $y = \sin^{-1}(\sin x)$ **b.** $y = \sin(\sin^{-1} x)$

155. a. $y = \cos^{-1}(\cos x)$ **b.** $y = \cos(\cos^{-1} x)$

156. Graph $y = \sec(\sec^{-1} x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

157. Newton's serpentine Graph Newton's serpentine, $y = 4x/(x^2+1)$. Then graph $y = 2\sin(2\tan^{-1} x)$ in the same graphing window. What do you see? Explain.

158. Graph the rational function $y = (2-x^2)/x^2$. Then graph $y = \cos(2\sec^{-1} x)$ in the same graphing window. What do you see? Explain.

159. Graph $f(x) = \sin^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

160. Graph $f(x) = \tan^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

7.8 Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and they are important in applications. For instance, they are used in problems such as computing the tension in a cable suspended by its two ends, as in an electric transmission line. They also play an important role in finding solutions to differential equations. In this section, we give a brief introduction to hyperbolic functions, their graphs, how their derivatives are calculated, and why they appear as important antiderivatives.

Even and Odd Parts of the Exponential Function

Recall the definitions of even and odd functions from Section 1.4, and the symmetries of their graphs. An even function f satisfies $f(-x) = f(x)$, while an odd function satisfies $f(-x) = -f(x)$. Every function f that is defined on an interval centered at the origin can be written in a unique way as the sum of one even function and one odd function. The decomposition is

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd part}}.$$

If we write e^x this way, we get

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even part}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd part}}.$$

The even and odd parts of e^x, called the hyperbolic cosine and hyperbolic sine of x, respectively, are useful in their own right. They describe the motions of waves in elastic solids and the temperature distributions in metal cooling fins. The centerline of the Gateway Arch to the West in St. Louis is a weighted hyperbolic cosine curve.

Definitions and Identities

The hyperbolic cosine and hyperbolic sine functions are defined by the first two equations in Table 7.5. The table also lists the definitions of the hyperbolic tangent, cotangent, secant, and cosecant. As we will see, the hyperbolic functions bear a number of similarities to the trigonometric functions after which they are named. (See Exercise 84 as well.)

The notation cosh x is often read "kosh x," rhyming with "gosh x," and sinh x is pronounced as if spelled "cinch x," rhyming with "pinch x."

Hyperbolic functions satisfy the identities in Table 7.6. Except for differences in sign, these resemble identities we already know for trigonometric functions.

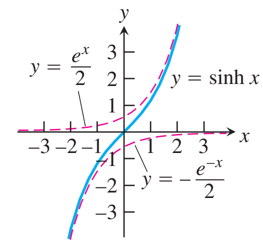
The second equation is obtained as follows:

$$\begin{aligned} 2 \sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{e^{2x} - e^{-2x}}{2} \\ &= \sinh 2x. \end{aligned}$$

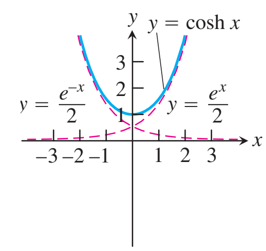
TABLE 7.5 The six basic hyperbolic functions

Hyperbolic sine of x:	$\sinh x = \dfrac{e^x - e^{-x}}{2}$
Hyperbolic cosine of x:	$\cosh x = \dfrac{e^x + e^{-x}}{2}$
Hyperbolic tangent:	$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
Hyperbolic cotangent:	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
Hyperbolic secant:	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
Hyperbolic cosecant:	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

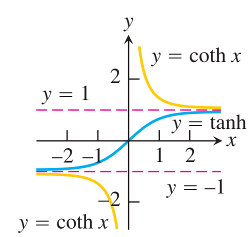
FIGURE 7.31



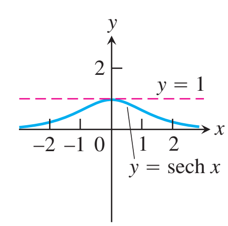
(a)



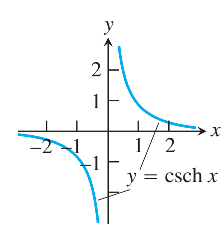
(b)



(c)



(d)



(e)

TABLE 7.6 Identities for hyperbolic functions

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

$\coth^2 x = 1 + \operatorname{csch}^2 x$

The other identities are obtained similarly, by substituting in the definitions of the hyperbolic functions and using algebra. Like many standard functions, hyperbolic functions and their inverses are easily evaluated with calculators, which have special keys or keystroke sequences for that purpose.

Derivatives and Integrals

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined (Table 7.7). Again, there are similarities with trigonometric functions. The derivative formulas in Table 7.7 lead to the integral formulas in Table 7.8.

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u\, du = \cosh u + C$$

$$\int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

The derivative formulas are derived from the derivative of e^u:

$$\begin{aligned}
\frac{d}{dx}(\sinh u) &= \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) && \text{Definition of } \sinh u \\
&= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2} && \text{Derivative of } e^u \\
&= \cosh u \frac{du}{dx} && \text{Definition of } \cosh u
\end{aligned}$$

This gives the first derivative formula. The calculation

$$\begin{aligned}
\frac{d}{dx}(\operatorname{csch} u) &= \frac{d}{dx}\left(\frac{1}{\sinh u}\right) && \text{Definition of } \operatorname{csch} u \\
&= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} && \text{Quotient Rule} \\
&= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} && \text{Rearrange terms.} \\
&= -\operatorname{csch} u \coth u \frac{du}{dx} && \text{Definitions of } \operatorname{csch} u \text{ and } \coth u
\end{aligned}$$

gives the last formula. The others are obtained similarly.

EXAMPLE 1 Finding Derivatives and Integrals

(a) $$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1 = t^2}\cdot\frac{d}{dt}\left(\sqrt{1+t^2}\right)$$
$$= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

(b) $$\int \coth 5x\,dx = \int\frac{\cosh 5x}{\sinh 5x}\,dx = \frac{1}{5}\int\frac{du}{u}$$ $u = \sinh 5x$, $du = 5\cosh 5x\,dx$
$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

(c) $$\int_0^1 \sinh^2 x\,dx = \int_0^1 \frac{\cosh 2x - 1}{2}\,dx$$ Table 7.6
$$= \frac{1}{2}\int_0^1(\cosh 2x - 1)\,dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$
$$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$ Evaluate with a calculator

(d) $$\int_0^{\ln 2} 4e^x\sinh x\,dx = \int_0^{\ln 2} 4e^x\,\frac{e^x - e^{-x}}{2}\,dx = \int_0^{\ln 2}(2e^{2x} - 2)\,dx$$
$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0)$$
$$= 4 - 2\ln 2 - 1$$
$$\approx 1.6137$$

Inverse Hyperbolic Functions

The inverses of the six basic hyperbolic functions are very useful in integration. Since $d(\sinh x)/dx = \cosh x > 0$, the hyperbolic sine is an increasing function of x. We denote its inverse by

$$y = \sinh^{-1} x.$$

For every value of x in the interval $-\infty < x < \infty$, the value of $y = \sinh^{-1} x$ is the number whose hyperbolic sine is x. The graphs of $y = \sinh x$ and $y = \sinh^{-1} x$ are shown in Figure 7.32a.

The function $y = \cosh x$ is not one-to-one, as we can see from the graph in Figure 7.31b. The restricted function $y = \cosh x$, $x \geq 0$, however, is one-to-one and therefore has an inverse, denoted by

$$y = \cosh^{-1} x.$$

For every value of $x \geq 1$, $y = \cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x. The graphs of $y = \cosh x$, $x \geq 0$, and $y = \cosh^{-1} x$ are shown in Figure 7.32b.

Like $y = \cosh x$, the function $y = \operatorname{sech} x = 1/\cosh x$ fails to be one-to-one, but its restriction to nonnegative values of x does have an inverse, denoted by

$$y = \operatorname{sech}^{-1} x.$$

For every value of x in the interval $(0, 1]$, $y = \operatorname{sech}^{-1} x$ is the nonnegative number whose hyperbolic secant is x. The graphs of $y = \operatorname{sech} x$, $x \geq 0$, and $y = \operatorname{sech}^{-1} x$ are shown in Figure 7.32c.

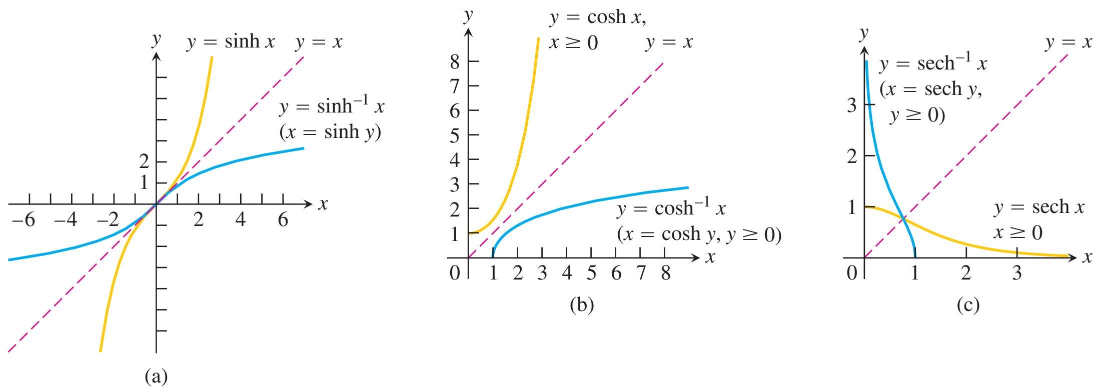


FIGURE 7.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \qquad y = \coth^{-1} x, \qquad y = \operatorname{csch}^{-1} x.$$

These functions are graphed in Figure 7.33.

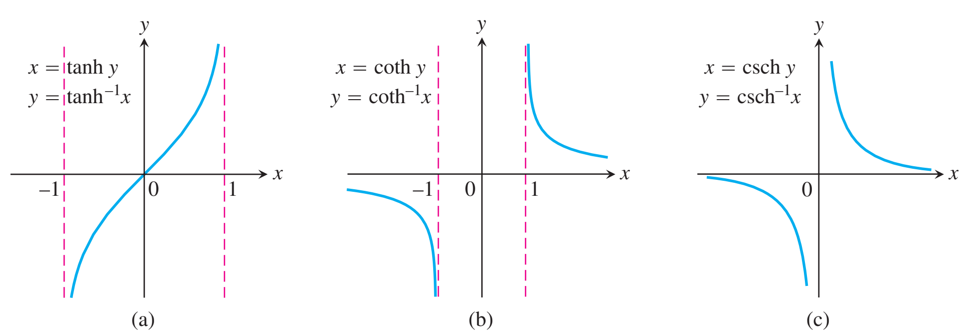


FIGURE 7.33 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

TABLE 7.9 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

Useful Identities

We use the identities in Table 7.9 to calculate the values of $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$, and $\coth^{-1} x$ on calculators that give only $\cosh^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$. These identities are direct consequences of the definitions. For example, if $0 < x \leq 1$, then

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x$$

so

$$\cosh^{-1}\left(\frac{1}{x}\right) = \operatorname{sech}^{-1} x$$

since the hyperbolic secant is one-to-one on (0, 1].

Derivatives and Integrals

The chief use of inverse hyperbolic functions lies in integrations that reverse the derivative formulas in Table 7.10.

TABLE 7.10 Derivatives of inverse hyperbolic functions

$\dfrac{d(\sinh^{-1} u)}{dx} = \dfrac{1}{\sqrt{1 + u^2}} \dfrac{du}{dx}$	
$\dfrac{d(\cosh^{-1} u)}{dx} = \dfrac{1}{\sqrt{u^2 - 1}} \dfrac{du}{dx},$	$u > 1$
$\dfrac{d(\tanh^{-1} u)}{dx} = \dfrac{1}{1 - u^2} \dfrac{du}{dx},$	$\lvert u \rvert < 1$
$\dfrac{d(\coth^{-1} u)}{dx} = \dfrac{1}{1 - u^2} \dfrac{du}{dx},$	$\lvert u \rvert > 1$
$\dfrac{d(\operatorname{sech}^{-1} u)}{dx} = \dfrac{-du/dx}{u\sqrt{1 - u^2}},$	$0 < u < 1$
$\dfrac{d(\operatorname{csch}^{-1} u)}{dx} = \dfrac{-du/dx}{\lvert u \rvert\sqrt{1 + u^2}},$	$u \neq 0$

The restrictions $|u| < 1$ and $|u| > 1$ on the derivative formulas for $\tanh^{-1} u$ and $\coth^{-1} u$ come from the natural restrictions on the values of these functions. (See Figure 7.33a and b.) The distinction between $|u| < 1$ and $|u| > 1$ becomes important when we convert the derivative formulas into integral formulas. If $|u| < 1$, the integral of $1/(1 - u^2)$ is $\tanh^{-1} u + C$. If $|u| > 1$, the integral is $\coth^{-1} u + C$.

We illustrate how the derivatives of the inverse hyperbolic functions are found in Example 2, where we calculate $d(\cosh^{-1} u)/dx$. The other derivatives are obtained by similar calculations.

HISTORICAL BIOGRAPHY

Sonya Kovalevsky
(1850–1891)

EXAMPLE 2 Derivative of the Inverse Hyperbolic Cosine

Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

Solution First we find the derivative of $y = \cosh^{-1} x$ for $x > 1$ by applying Theorem 1 with $f(x) = \cosh x$ and $f^{-1}(x) = \cosh^{-1} x$. Theorem 1 can be applied because the derivative of $\cosh x$ is positive for $0 < x$.

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\sinh(\cosh^{-1} x)} && f'(u) = \sinh u \\
&= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} && \cosh^2 u - \sinh^2 u = 1, \ \sinh u = \sqrt{\cosh^2 u - 1} \\
&= \frac{1}{\sqrt{x^2 - 1}} && \cosh(\cosh^{-1} x) = x
\end{aligned}$$

In short,

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}.$$

The Chain Rule gives the final result:

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

■

Instead of applying Theorem 1 directly, as in Example 2, we could also find the derivative of $y = \cosh^{-1} x, x > 1$, using implicit differentiation and the Chain Rule:

$$\begin{aligned}
y &= \cosh^{-1} x \\
x &= \cosh y && \text{Equivalent equation} \\
1 &= \sinh y \frac{dy}{dx} && \text{Implicit differentiation with respect to } x \text{, and the Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} && \text{Since } x > 1, y > 0 \text{ and } \sinh y > 0 \\
&= \frac{1}{\sqrt{x^2 - 1}}. && \cosh y = x
\end{aligned}$$

With appropriate substitutions, the derivative formulas in Table 7.10 lead to the integration formulas in Table 7.11. Each of the formulas in Table 7.11 can be verified by differentiating the expression on the right-hand side.

EXAMPLE 3 Using Table 7.11

Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}.$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1.	$\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C,$	$a > 0$
2.	$\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C,$	$u > a > 0$
3.	$\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$	
4.	$\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C,$	$0 < u < a$
5.	$\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left\lvert\frac{u}{a}\right\rvert + C,$	$u \neq 0$ and $a > 0$

Solution The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} \qquad u = 2x,\quad du = 2\,dx,\quad a = \sqrt{3}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.11}$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Big]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

■

EXERCISES 7.8

Hyperbolic Function Values and Identities

Each of Exercises 1–4 gives a value of sinh x or cosh x. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\frac{3}{4}$

2. $\sinh x = \frac{4}{3}$

3. $\cosh x = \frac{17}{15}, \quad x > 0$

4. $\cosh x = \frac{13}{5}, \quad x > 0$

Rewrite the expressions in Exercises 5–10 in terms of exponentials and simplify the results as much as you can.

5. $2 \cosh (\ln x)$

6. $\sinh (2 \ln x)$

7. $\cosh 5x + \sinh 5x$

8. $\cosh 3x - \sinh 3x$

9. $(\sinh x + \cosh x)^4$

10. $\ln (\cosh x + \sinh x) + \ln (\cosh x - \sinh x)$

11. Use the identities

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

to show that

a. $\sinh 2x = 2 \sinh x \cosh x$

b. $\cosh 2x = \cosh^2 x + \sinh^2 x$.

12. Use the definitions of $\cosh x$ and $\sinh x$ to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

Derivatives

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

13. $y = 6 \sinh \frac{x}{3}$

14. $y = \frac{1}{2} \sinh(2x+1)$

15. $y = 2\sqrt{t} \tanh \sqrt{t}$

16. $y = t^2 \tanh \frac{1}{t}$

17. $y = \ln(\sinh z)$

18. $y = \ln(\cosh z)$

19. $y = \operatorname{sech} \theta(1 - \ln \operatorname{sech} \theta)$

20. $y = \operatorname{csch} \theta(1 - \ln \operatorname{csch} \theta)$

21. $y = \ln \cosh v - \frac{1}{2} \tanh^2 v$

22. $y = \ln \sinh v - \frac{1}{2} \coth^2 v$

23. $y = (x^2+1) \operatorname{sech}(\ln x)$

(*Hint:* Before differentiating, express in terms of exponentials and simplify.)

24. $y = (4x^2-1) \operatorname{csch}(\ln 2x)$

In Exercises 25–36, find the derivative of y with respect to the appropriate variable.

25. $y = \sinh^{-1} \sqrt{x}$

26. $y = \cosh^{-1} 2\sqrt{x+1}$

27. $y = (1-\theta) \tanh^{-1} \theta$

28. $y = (\theta^2 + 2\theta) \tanh^{-1}(\theta+1)$

29. $y = (1-t) \coth^{-1} \sqrt{t}$

30. $y = (1-t^2) \coth^{-1} t$

31. $y = \cos^{-1} x - x \operatorname{sech}^{-1} x$

32. $y = \ln x + \sqrt{1-x^2} \operatorname{sech}^{-1} x$

33. $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$

34. $y = \operatorname{csch}^{-1} 2^{\theta}$

35. $y = \sinh^{-1}(\tan x)$

36. $y = \cosh^{-1}(\sec x), \quad 0 < x < \pi/2$

Integration Formulas

Verify the integration formulas in Exercises 37–40.

37. a. $\int \operatorname{sech} x \, dx = \tan^{-1}(\sinh x) + C$

b. $\int \operatorname{sech} x \, dx = \sin^{-1}(\tanh x) + C$

38. $\int x \operatorname{sech}^{-1} x \, dx = \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$

39. $\int x \coth^{-1} x \, dx = \frac{x^2-1}{2} \coth^{-1} x + \frac{x}{2} + C$

40. $\int \tanh^{-1} x \, dx = x \tanh^{-1} x + \frac{1}{2} \ln(1-x^2) + C$

Indefinite Integrals

Evaluate the integrals in Exercises 41–50.

41. $\int \sinh 2x \, dx$

42. $\int \sinh \frac{x}{5} \, dx$

43. $\int 6 \cosh\left(\frac{x}{2} - \ln 3\right) dx$

44. $\int 4 \cosh(3x - \ln 2) \, dx$

45. $\int \tanh \frac{x}{7} \, dx$

46. $\int \coth \frac{\theta}{\sqrt{3}} \, d\theta$

47. $\int \operatorname{sech}^2\left(x - \frac{1}{2}\right) dx$

48. $\int \operatorname{csch}^2(5-x) \, dx$

49. $\int \frac{\operatorname{sech} \sqrt{t} \tanh \sqrt{t} \, dt}{\sqrt{t}}$

50. $\int \frac{\operatorname{csch}(\ln t) \coth(\ln t) \, dt}{t}$

Definite Integrals

Evaluate the integrals in Exercises 51–60.

51. $\int_{\ln 2}^{\ln 4} \coth x \, dx$

52. $\int_0^{\ln 2} \tanh 2x \, dx$

53. $\int_{-\ln 4}^{-\ln 2} 2e^{\theta} \cosh \theta \, d\theta$

54. $\int_0^{\ln 2} 4e^{-\theta} \sinh \theta \, d\theta$

55. $\int_{-\pi/4}^{\pi/4} \cosh(\tan \theta) \sec^2 \theta \, d\theta$

56. $\int_0^{\pi/2} 2 \sinh(\sin \theta) \cos \theta \, d\theta$

57. $\int_1^2 \frac{\cosh(\ln t)}{t} \, dt$

58. $\int_1^4 \frac{8 \cosh \sqrt{x}}{\sqrt{x}} \, dx$

59. $\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right) dx$

60. $\int_0^{\ln 10} 4 \sinh^2\left(\frac{x}{2}\right) dx$

Evaluating Inverse Hyperbolic Functions and Related Integrals

When hyperbolic function keys are not available on a calculator, it is still possible to evaluate the inverse hyperbolic functions by expressing them as logarithms, as shown here.

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right),$	$-\infty < x < \infty$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right),$	$x \geq 1$
$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x},$	$\lvert x \rvert < 1$
$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),$	$0 < x \leq 1$
$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x \rvert}\right),$	$x \neq 0$
$\coth^{-1} x = \frac{1}{2} \ln \frac{x+1}{x-1},$	$\lvert x \rvert > 1$

Use the formulas in the box here to express the numbers in Exercises 61–66 in terms of natural logarithms.

61. $\sinh^{-1}(-5/12)$ **62.** $\cosh^{-1}(5/3)$

63. $\tanh^{-1}(-1/2)$ **64.** $\coth^{-1}(5/4)$

65. $\operatorname{sech}^{-1}(3/5)$ **66.** $\operatorname{csch}^{-1}(-1/\sqrt{3})$

Evaluate the integrals in Exercises 67–74 in terms of

a. inverse hyperbolic functions.

b. natural logarithms.

67. $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}}$ **68.** $\displaystyle\int_0^{1/3} \frac{6\,dx}{\sqrt{1+9x^2}}$

69. $\displaystyle\int_{5/4}^{2} \frac{dx}{1-x^2}$ **70.** $\displaystyle\int_0^{1/2} \frac{dx}{1-x^2}$

71. $\displaystyle\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}}$ **72.** $\displaystyle\int_1^{2} \frac{dx}{x\sqrt{4+x^2}}$

73. $\displaystyle\int_0^{\pi} \frac{\cos x\,dx}{\sqrt{1+\sin^2 x}}$ **74.** $\displaystyle\int_1^{e} \frac{dx}{x\sqrt{1+(\ln x)^2}}$

Applications and Theory

75. a. Show that if a function f is defined on an interval symmetric about the origin (so that f is defined at $-x$ whenever it is defined at x), then

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}. \qquad (1)$$

Then show that $(f(x)+f(-x))/2$ is even and that $(f(x)-f(-x))/2$ is odd.

b. Equation (1) simplifies considerably if f itself is (i) even or (ii) odd. What are the new equations? Give reasons for your answers.

76. Derive the formula $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$, $-\infty < x < \infty$. Explain in your derivation why the plus sign is used with the square root instead of the minus sign.

77. Skydiving If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity t sec into the fall satisfies the differential equation

$$m\frac{dv}{dt} = mg - kv^2,$$

where k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is short enough so that the variation in the air's density will not affect the outcome significantly.)

a. Show that

$$v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}\,t\right)$$

satisfies the differential equation and the initial condition that $v = 0$ when $t = 0$.

b. Find the body's *limiting velocity*, $\lim_{t\to\infty} v$.

c. For a 160-lb skydiver $(mg = 160)$, with time in seconds and distance in feet, a typical value for k is 0.005. What is the diver's limiting velocity?

78. Accelerations whose magnitudes are proportional to displacement Suppose that the position of a body moving along a coordinate line at time t is

a. $s = a\cos kt + b\sin kt$

b. $s = a\cosh kt + b\sinh kt$.

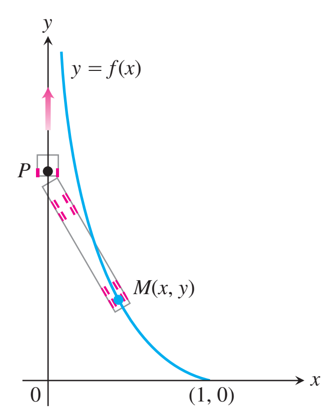
Show in both cases that the acceleration d^2s/dt^2 is proportional to s but that in the first case it is directed toward the origin, whereas in the second case it is directed away from the origin.

79. Tractor trailers and the tractrix When a tractor trailer turns into a cross street or driveway, its rear wheels follow a curve like the one shown here. (This is why the rear wheels sometimes ride up over the curb.) We can find an equation for the curve if we picture the rear wheels as a mass M at the point (1, 0) on the x-axis attached by a rod of unit length to a point P representing the cab at the origin. As the point P moves up the y-axis, it drags M along behind it. The curve traced by M—called a *tractrix* from the Latin word *tractum*, for "drag"—can be shown to be the graph of the function $y = f(x)$ that solves the initial value problem

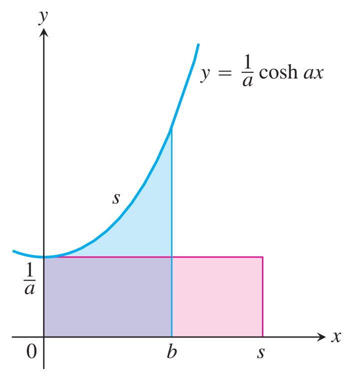
Differential equation: $\displaystyle\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$

Initial condition: $y = 0$ when $x = 1$.

Solve the initial value problem to find an equation for the curve. (You need an inverse hyperbolic function.)



80. Area Show that the area of the region in the first quadrant enclosed by the curve $y = (1/a)\cosh ax$, the coordinate axes, and the line $x = b$ is the same as the area of a rectangle of height $1/a$ and length s, where s is the length of the curve from $x = 0$ to $x = b$. (See accompanying figure.)

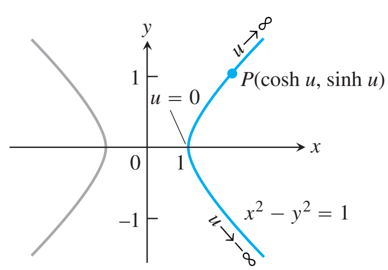


81. Volume A region in the first quadrant is bounded above by the curve $y = \cosh x$, below by the curve $y = \sinh x$, and on the left and right by the y-axis and the line $x = 2$, respectively. Find the volume of the solid generated by revolving the region about the x-axis.

82. Volume The region enclosed by the curve $y = \operatorname{sech} x$, the x-axis, and the lines $x = \pm \ln \sqrt{3}$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

83. Arc length Find the length of the segment of the curve $y = (1/2)\cosh 2x$ from $x = 0$ to $x = \ln \sqrt{5}$.

84. The hyperbolic in hyperbolic functions In case you are wondering where the name *hyperbolic* comes from, here is the answer: Just as $x = \cos u$ and $y = \sin u$ are identified with points (x, y) on the unit circle, the functions $x = \cosh u$ and $y = \sinh u$ are identified with points (x, y) on the right-hand branch of the unit hyperbola, $x^2 - y^2 = 1$.



Since $\cosh^2 u - \sinh^2 u = 1$, the point $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$ for every value of u (Exercise 84).

Another analogy between hyperbolic and circular functions is that the variable u in the coordinates $(\cosh u, \sinh u)$ for the points of the right-hand branch of the hyperbola $x^2 - y^2 = 1$ is twice the area of the sector AOP pictured in the accompanying figure. To see why this is so, carry out the following steps.

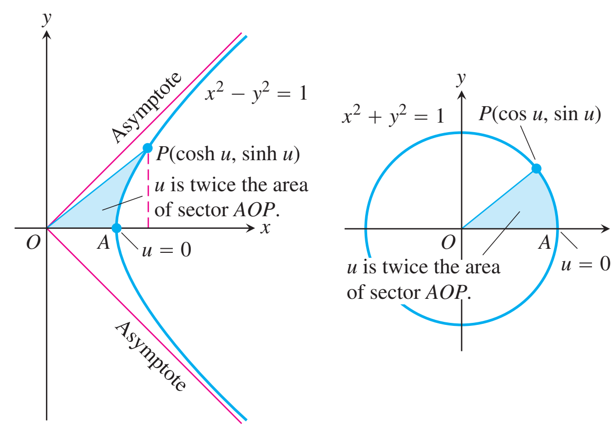
a. Show that the area $A(u)$ of sector AOP is

$$A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\, dx.$$

b. Differentiate both sides of the equation in part (a) with respect to u to show that

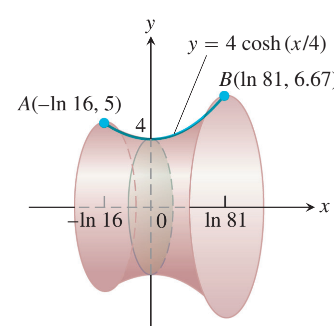
$$A'(u) = \frac{1}{2}.$$

c. Solve this last equation for $A(u)$. What is the value of $A(0)$? What is the value of the constant of integration C in your solution? With C determined, what does your solution say about the relationship of u to $A(u)$?



One of the analogies between hyperbolic and circular functions is revealed by these two diagrams (Exercise 84).

85. A minimal surface Find the area of the surface swept out by revolving about the x-axis the curve $y = 4\cosh(x/4)$, $-\ln 16 \le x \le \ln 81$.



It can be shown that, of all continuously differentiable curves joining points A and B in the figure, the curve $y = 4\cosh(x/4)$ generates the surface of least area. If you made a rigid wire frame of the end-circles through A and B and dipped them in a soap-film solution, the surface spanning the circles would be the one generated by the curve.

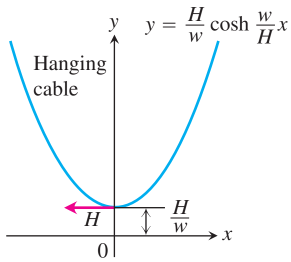
T **86. a.** Find the centroid of the curve $y = \cosh x$, $-\ln 2 \le x \le \ln 2$.

b. Evaluate the coordinates to two decimal places. Then sketch the curve and plot the centroid to show its relation to the curve.

Hanging Cables

87. Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is w and the horizontal tension at its lowest point is a vector of length H. If we choose a coordinate system for the plane of the cable in which the x-axis is horizontal, the force of gravity is straight down, the positive y-axis points straight up, and the lowest point of the cable lies at the point $y = H/w$ on the y-axis (see accompanying figure), then it can be shown that the cable lies along the graph of the hyperbolic cosine

$$y = \frac{H}{w}\cosh\frac{w}{H}x.$$

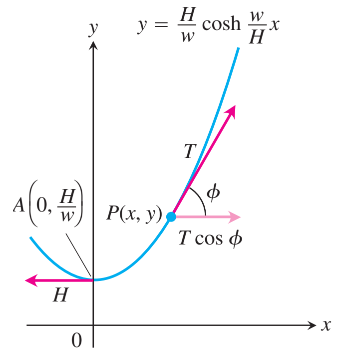


Such a curve is sometimes called a **chain curve** or a **catenary**, the latter deriving from the Latin *catena*, meaning "chain."

a. Let $P(x, y)$ denote an arbitrary point on the cable. The next accompanying figure displays the tension at P as a vector of length (magnitude) T, as well as the tension H at the lowest point A. Show that the cable's slope at P is

$$\tan\phi = \frac{dy}{dx} = \sinh\frac{w}{H}x.$$

b. Using the result from part (a) and the fact that the horizontal tension at P must equal H (the cable is not moving), show that $T = wy$. Hence, the magnitude of the tension at $P(x, y)$ is exactly equal to the weight of y units of cable.



88. (*Continuation of Exercise 87.*) The length of arc AP in the Exercise 87 figure is $s = (1/a)\sinh ax$, where $a = w/H$. Show that the coordinates of P may be expressed in terms of s as

$$x = \frac{1}{a}\sinh^{-1} as, \qquad y = \sqrt{s^2 + \frac{1}{a^2}}.$$

89. The sag and horizontal tension in a cable The ends of a cable 32 ft long and weighing 2 lb/ft are fastened at the same level to posts 30 ft apart.

a. Model the cable with the equation

$$y = \frac{1}{a}\cosh ax, \quad -15 \le x \le 15.$$

Use information from Exercise 88 to show that a satisfies the equation

$$16a = \sinh 15a. \qquad (2)$$

T **b.** Solve Equation (2) graphically by estimating the coordinates of the points where the graphs of the equations $y = 16a$ and $y = \sinh 15a$ intersect in the ay-plane.

T **c.** Solve Equation (2) for a numerically. Compare your solution with the value you found in part (b).

d. Estimate the horizontal tension in the cable at the cable's lowest point.

T **e.** Using the value found for a in part (c), graph the catenary

$$y = \frac{1}{a}\cosh ax$$

over the interval $-15 \le x \le 15$. Estimate the sag in the cable at its center.

Chapter 7 Additional and Advanced Exercises

Limits

Find the limits in Exercises 1–6.

1. $\lim_{b \to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$

2. $\lim_{x \to \infty} \frac{1}{x} \int_0^x \tan^{-1} t \, dt$

3. $\lim_{x \to 0^+} (\cos \sqrt{x})^{1/x}$

4. $\lim_{x \to \infty} (x + e^x)^{2/x}$

5. $\lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$

6. $\lim_{n \to \infty} \frac{1}{n} \left(e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n} + e^{n/n} \right)$

7. Let $A(t)$ be the area of the region in the first quadrant enclosed by the coordinate axes, the curve $y = e^{-x}$, and the vertical line $x = t, t > 0$. Let $V(t)$ be the volume of the solid generated by revolving the region about the x-axis. Find the following limits.

a. $\lim_{t \to \infty} A(t)$ **b.** $\lim_{t \to \infty} V(t)/A(t)$ **c.** $\lim_{t \to 0^+} V(t)/A(t)$

8. Varying a logarithm's base

a. Find $\lim \log_a 2$ as $a \to 0^+, 1^-, 1^+$, and ∞.

T **b.** Graph $y = \log_a 2$ as a function of a over the interval $0 < a \le 4$.

Theory and Examples

9. Find the areas between the curves $y = 2(\log_2 x)/x$ and $y = 2(\log_4 x)/x$ and the x-axis from $x = 1$ to $x = e$. What is the ratio of the larger area to the smaller?

T **10.** Graph $f(x) = \tan^{-1} x + \tan^{-1}(1/x)$ for $-5 \le x \le 5$. Then use calculus to explain what you see. How would you expect f to behave beyond the interval $[-5, 5]$? Give reasons for your answer.

11. For what $x > 0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.

T **12.** Graph $f(x) = (\sin x)^{\sin x}$ over $[0, 3\pi]$. Explain what you see.

13. Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \int_2^x \frac{t}{1+t^4} dt$.

14. a. Find df/dx if

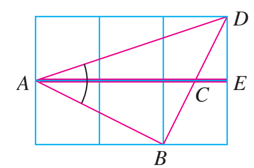
$$f(x) = \int_1^{e^x} \frac{2 \ln t}{t} dt.$$

b. Find $f(0)$.

c. What can you conclude about the graph of f? Give reasons for your answer.

15. The figure here shows an informal proof that

$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{4}.$$

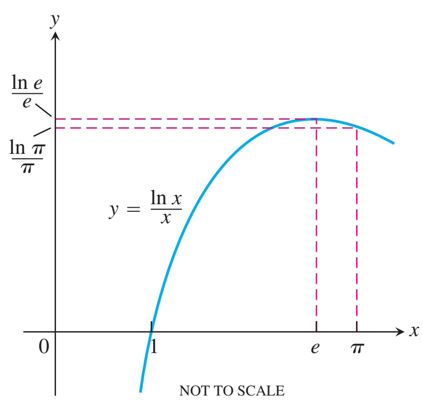


How does the argument go? (*Source*: "Behold! Sums of Arctan," by Edward M. Harris, *College Mathematics Journal*, Vol. 18, No. 2, Mar. 1987, p. 141.)

16. $\pi^e < e^\pi$

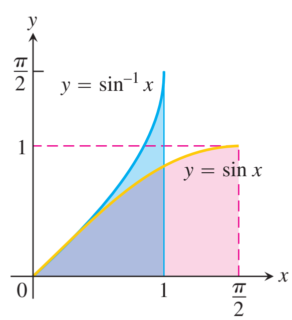
a. Why does the accompanying figure "prove" that $\pi^e < e^\pi$? (*Source:* "Proof Without Words," by Fouad Nakhil, *Mathematics Magazine*, Vol. 60, No. 3, June 1987, p. 165.)

b. The accompanying figure assumes that $f(x) = (\ln x)/x$ has an absolute maximum value at $x = e$. How do you know it does?



17. Use the accompanying figure to show that

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x \, dx.$$

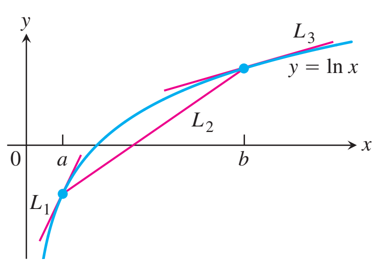


18. Napier's inequality Here are two pictorial proofs that

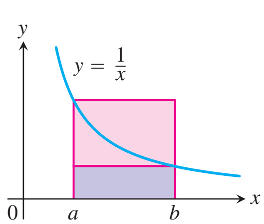
$$b > a > 0 \quad \Rightarrow \quad \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}.$$

Explain what is going on in each case.

a.



b.



(*Source:* Roger B. Nelson, *College Mathematics Journal*, Vol. 24, No. 2, March 1993, p. 165.)

19. Even-odd decompositions

a. Suppose that g is an even function of x and h is an odd function of x. Show that if $g(x) + h(x) = 0$ for all x then $g(x) = 0$ for all x and $h(x) = 0$ for all x.

b. Use the result in part (a) to show that if $f(x) = f_E(x) + f_O(x)$ is the sum of an even function $f_E(x)$ and an odd function $f_O(x)$, then

$$f_E(x) = (f(x) + f(-x))/2 \quad \text{and} \quad f_O(x) = (f(x) - f(-x))/2.$$

c. What is the significance of the result in part (b)?

20. Let g be a function that is differentiable throughout an open interval containing the origin. Suppose g has the following properties:

i. $g(x + y) = \dfrac{g(x) + g(y)}{1 - g(x)g(y)}$ for all real numbers x, y, and $x + y$ in the domain of g.

ii. $\lim_{h \to 0} g(h) = 0$

iii. $\lim_{h \to 0} \dfrac{g(h)}{h} = 1$

a. Show that $g(0) = 0$.

b. Show that $g'(x) = 1 + [g(x)]^2$.

c. Find $g(x)$ by solving the differential equation in part (b).

Applications

21. Center of mass Find the center of mass of a thin plate of constant density covering the region in the first and fourth quadrants enclosed by the curves $y = 1/(1 + x^2)$ and $y = -1/(1 + x^2)$ and by the lines $x = 0$ and $x = 1$.

22. Solid of revolution The region between the curve $y = 1/(2\sqrt{x})$ and the x-axis from $x = 1/4$ to $x = 4$ is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

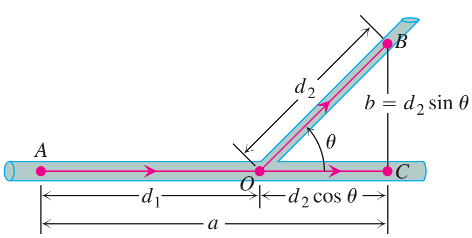
b. Find the centroid of the region.

23. The Rule of 70 If you use the approximation $\ln 2 \approx 0.70$ (in place of $0.69314\ldots$), you can derive a rule of thumb that says, "To estimate how many years it will take an amount of money to double when invested at r percent compounded continuously, divide r into 70." For instance, an amount of money invested at 5% will double in about $70/5 = 14$ years. If you want it to double in 10 years instead, you have to invest it at $70/10 = 7\%$. Show how the Rule of 70 is derived. (A similar "Rule of 72" uses 72 instead of 70, because 72 has more integer factors.)

24. Free fall in the fourteenth century In the middle of the fourteenth century, Albert of Saxony (1316–1390) proposed a model of free fall that assumed that the velocity of a falling body was proportional to the distance fallen. It seemed reasonable to think that a body that had fallen 20 ft might be moving twice as fast as a body that had fallen 10 ft. And besides, none of the instruments in use at the time were accurate enough to prove otherwise. Today we can see just how far off Albert of Saxony's model was by solving the initial value problem implicit in his model. Solve the problem and compare your solution graphically with the equation $s = 16t^2$. You will see that it describes a motion that starts too slowly at first and then becomes too fast too soon to be realistic.

25. The best branching angles for blood vessels and pipes When a smaller pipe branches off from a larger one in a flow system, we may want it to run off at an angle that is best from some energy-saving point of view. We might require, for instance, that energy loss due to friction be minimized along the section *AOB* shown in the accompanying figure. In this diagram, *B* is a given point to be reached by the smaller pipe, *A* is a point in the larger pipe upstream from *B*, and *O* is the point where the branching occurs. A law due to Poiseuille states that the loss of energy due to friction in nonturbulent flow is proportional to the length of the path and inversely proportional to the fourth power of the radius. Thus, the loss along *AO* is $(kd_1)/R^4$ and along *OB* is $(kd_2)/r^4$, where k is a constant, d_1 is the length of *AO*, d_2 is the length of *OB*, R is the radius of the larger pipe, and r is the radius of the smaller pipe. The angle θ is to be chosen to minimize the sum of these two losses:

$$L = k\frac{d_1}{R^4} + k\frac{d_2}{r^4}.$$



In our model, we assume that $AC = a$ and $BC = b$ are fixed. Thus we have the relations

$$d_1 + d_2\cos\theta = a \qquad d_2\sin\theta = b,$$

so that

$$d_2 = b\csc\theta,$$
$$d_1 = a - d_2\cos\theta = a - b\cot\theta.$$

We can express the total loss L as a function of θ:

$$L = k\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right).$$

a. Show that the critical value of θ for which $dL/d\theta$ equals zero is

$$\theta_c = \cos^{-1}\frac{r^4}{R^4}.$$

b. If the ratio of the pipe radii is $r/R = 5/6$, estimate to the nearest degree the optimal branching angle given in part (a).

The mathematical analysis described here is also used to explain the angles at which arteries branch in an animal's body. (See *Introduction to Mathematics for Life Scientists*, Second Edition, by E. Batschelet [New York: Springer-Verlag, 1976].)

T **26. Group blood testing** During World War II it was necessary to administer blood tests to large numbers of recruits. There are two standard ways to administer a blood test to N people. In method 1, each person is tested separately. In method 2, the blood samples of x people are pooled and tested as one large sample. If the test is negative, this one test is enough for all x people. If the test is positive, then each of the x people is tested separately, requiring a total of $x + 1$ tests. Using the second method and some probability theory it can be shown that, on the average, the total number of tests y will be

$$y = N\left(1 - q^x + \frac{1}{x}\right).$$

With $q = 0.99$ and $N = 1000$, find the integer value of x that minimizes y. Also find the integer value of x that maximizes y. (This second result is not important to the real-life situation.) The group testing method was used in World War II with a savings of 80% over the individual testing method, but not with the given value of q.

Chapter 7 Practice Exercises

Differentiation

In Exercises 1–24, find the derivative of y with respect to the appropriate variable.

1. $y = 10e^{-x/5}$
2. $y = \sqrt{2}e^{\sqrt{2}x}$
3. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$
4. $y = x^2e^{-2/x}$
5. $y = \ln(\sin^2\theta)$
6. $y = \ln(\sec^2\theta)$
7. $y = \log_2(x^2/2)$
8. $y = \log_5(3x - 7)$
9. $y = 8^{-t}$
10. $y = 9^{2t}$
11. $y = 5x^{3.6}$
12. $y = \sqrt{2}x^{-\sqrt{2}}$
13. $y = (x + 2)^{x+2}$
14. $y = 2(\ln x)^{x/2}$
15. $y = \sin^{-1}\sqrt{1 - u^2}, \quad 0 < u < 1$
16. $y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right), \quad v > 1$
17. $y = \ln\cos^{-1}x$
18. $y = z\cos^{-1}z - \sqrt{1 - z^2}$
19. $y = t\tan^{-1}t - \frac{1}{2}\ln t$
20. $y = (1 + t^2)\cot^{-1}2t$
21. $y = z\sec^{-1}z - \sqrt{z^2 - 1}, \quad z > 1$
22. $y = 2\sqrt{x - 1}\sec^{-1}\sqrt{x}$
23. $y = \csc^{-1}(\sec\theta), \quad 0 < \theta < \pi/2$
24. $y = (1 + x^2)e^{\tan^{-1}x}$

Logarithmic Differentiation

In Exercises 25–30, use logarithmic differentiation to find the derivative of y with respect to the appropriate variable.

25. $y = \frac{2(x^2 + 1)}{\sqrt{\cos 2x}}$
26. $y = \sqrt[10]{\frac{3x + 4}{2x - 4}}$
27. $y = \left(\frac{(t + 1)(t - 1)}{(t - 2)(t + 3)}\right)^5, \quad t > 2$
28. $y = \frac{2u2^u}{\sqrt{u^2 + 1}}$
29. $y = (\sin\theta)^{\sqrt{\theta}}$
30. $y = (\ln x)^{1/(\ln x)}$

Integration

Evaluate the integrals in Exercises 31–78.

31. $\int e^x\sin(e^x)\,dx$
32. $\int e^t\cos(3e^t - 2)\,dt$

33. $\int e^x \sec^2(e^x - 7)\,dx$

34. $\int e^y \csc(e^y + 1)\cot(e^y + 1)\,dy$

35. $\int \sec^2(x)e^{\tan x}\,dx$

36. $\int \csc^2 x\, e^{\cot x}\,dx$

37. $\int_{-1}^{1} \frac{dx}{3x - 4}$

38. $\int_{1}^{e} \frac{\sqrt{\ln x}}{x}\,dx$

39. $\int_{0}^{\pi} \tan\frac{x}{3}\,dx$

40. $\int_{1/6}^{1/4} 2\cot \pi x\,dx$

41. $\int_{0}^{4} \frac{2t}{t^2 - 25}\,dt$

42. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1 - \sin t}\,dt$

43. $\int \frac{\tan(\ln v)}{v}\,dv$

44. $\int \frac{dv}{v \ln v}$

45. $\int \frac{(\ln x)^{-3}}{x}\,dx$

46. $\int \frac{\ln(x - 5)}{x - 5}\,dx$

47. $\int \frac{1}{r}\csc^2(1 + \ln r)\,dr$

48. $\int \frac{\cos(1 - \ln v)}{v}\,dv$

49. $\int x3^{x^2}\,dx$

50. $\int 2^{\tan x}\sec^2 x\,dx$

51. $\int_{1}^{7} \frac{3}{x}\,dx$

52. $\int_{1}^{32} \frac{1}{5x}\,dx$

53. $\int_{1}^{4} \left(\frac{x}{8} + \frac{1}{2x}\right)dx$

54. $\int_{1}^{8} \left(\frac{2}{3x} - \frac{8}{x^2}\right)dx$

55. $\int_{-2}^{-1} e^{-(x+1)}\,dx$

56. $\int_{-\ln 2}^{0} e^{2w}\,dw$

57. $\int_{0}^{\ln 5} e^r(3e^r + 1)^{-3/2}\,dr$

58. $\int_{0}^{\ln 9} e^\theta(e^\theta - 1)^{1/2}\,d\theta$

59. $\int_{1}^{e} \frac{1}{x}(1 + 7\ln x)^{-1/3}\,dx$

60. $\int_{e}^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx$

61. $\int_{1}^{3} \frac{(\ln(v + 1))^2}{v + 1}\,dv$

62. $\int_{2}^{4} (1 + \ln t)t \ln t\,dt$

63. $\int_{1}^{8} \frac{\log_4 \theta}{\theta}\,d\theta$

64. $\int_{1}^{e} \frac{8\ln 3 \log_3 \theta}{\theta}\,d\theta$

65. $\int_{-3/4}^{3/4} \frac{6\,dx}{\sqrt{9 - 4x^2}}$

66. $\int_{-1/5}^{1/5} \frac{6\,dx}{\sqrt{4 - 25x^2}}$

67. $\int_{-2}^{2} \frac{3\,dt}{4 + 3t^2}$

68. $\int_{\sqrt{3}}^{3} \frac{dt}{3 + t^2}$

69. $\int \frac{dy}{y\sqrt{4y^2 - 1}}$

70. $\int \frac{24\,dy}{y\sqrt{y^2 - 16}}$

71. $\int_{\sqrt{2}/3}^{2/3} \frac{dy}{|y|\sqrt{9y^2 - 1}}$

72. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{dy}{|y|\sqrt{5y^2 - 3}}$

73. $\int \frac{dx}{\sqrt{-2x - x^2}}$

74. $\int \frac{dx}{\sqrt{-x^2 + 4x - 1}}$

75. $\int_{-2}^{-1} \frac{2\,dv}{v^2 + 4v + 5}$

76. $\int_{-1}^{1} \frac{3\,dv}{4v^2 + 4v + 4}$

77. $\int \frac{dt}{(t + 1)\sqrt{t^2 + 2t - 8}}$

78. $\int \frac{dt}{(3t + 1)\sqrt{9t^2 + 6t}}$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 79–84, solve for y.

79. $3^y = 2^{y+1}$

80. $4^{-y} = 3^{y+2}$

81. $9e^{2y} = x^2$

82. $3^y = 3\ln x$

83. $\ln(y - 1) = x + \ln y$

84. $\ln(10 \ln y) = \ln 5x$

Evaluating Limits

Find the limits in Exercises 85–96.

85. $\lim_{x\to 0} \frac{10^x - 1}{x}$

86. $\lim_{\theta\to 0} \frac{3^\theta - 1}{\theta}$

87. $\lim_{x\to 0} \frac{2^{\sin x} - 1}{e^x - 1}$

88. $\lim_{x\to 0} \frac{2^{-\sin x} - 1}{e^x - 1}$

89. $\lim_{x\to 0} \frac{5 - 5\cos x}{e^x - x - 1}$

90. $\lim_{x\to 0} \frac{4 - 4e^x}{xe^x}$

91. $\lim_{t\to 0^+} \frac{t - \ln(1 + 2t)}{t^2}$

92. $\lim_{x\to 4} \frac{\sin^2(\pi x)}{e^{x-4} + 3 - x}$

93. $\lim_{t\to 0^+} \left(\frac{e^t}{t} - \frac{1}{t}\right)$

94. $\lim_{y\to 0^+} e^{-1/y}\ln y$

95. $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^x$

96. $\lim_{x\to 0^+} \left(1 + \frac{3}{x}\right)^x$

Comparing Growth Rates of Functions

97. Does f grow faster, slower, or at the same rate as g as $x \to \infty$? Give reasons for your answers.

a. $f(x) = \log_2 x$, $g(x) = \log_3 x$

b. $f(x) = x$, $g(x) = x + \frac{1}{x}$

c. $f(x) = x/100$, $g(x) = xe^{-x}$

d. $f(x) = x$, $g(x) = \tan^{-1} x$

e. $f(x) = \csc^{-1} x$, $g(x) = 1/x$

f. $f(x) = \sinh x$, $g(x) = e^x$

98. Does f grow faster, slower, or at the same rate as g as $x \to \infty$? Give reasons for your answers.

a. $f(x) = 3^{-x}$, $g(x) = 2^{-x}$

b. $f(x) = \ln 2x$, $g(x) = \ln x^2$

c. $f(x) = 10x^3 + 2x^2$, $g(x) = e^x$

d. $f(x) = \tan^{-1}(1/x)$, $g(x) = 1/x$

e. $f(x) = \sin^{-1}(1/x)$, $g(x) = 1/x^2$

f. $f(x) = \operatorname{sech} x$, $g(x) = e^{-x}$

99. True, or false? Give reasons for your answers.

a. $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^2}\right)$ **b.** $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^4}\right)$

c. $x = o(x + \ln x)$ **d.** $\ln(\ln x) = o(\ln x)$

e. $\tan^{-1} x = O(1)$ **f.** $\cosh x = O(e^x)$

100. True, or false? Give reasons for your answers.

a. $\frac{1}{x^4} = O\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$ **b.** $\frac{1}{x^4} = o\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$

c. $\ln x = o(x + 1)$ **d.** $\ln 2x = O(\ln x)$

e. $\sec^{-1} x = O(1)$ **f.** $\sinh x = O(e^x)$

Theory and Applications

101. The function $f(x) = e^x + x$, being differentiable and one-to-one, has a differentiable inverse $f^{-1}(x)$. Find the value of df^{-1}/dx at the point $f(\ln 2)$.

102. Find the inverse of the function $f(x) = 1 + (1/x), x \neq 0$. Then show that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$ and that

$$\left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{1}{f'(x)}.$$

In Exercises 103 and 104, find the absolute maximum and minimum values of each function on the given interval.

103. $y = x \ln 2x - x, \quad \left[\frac{1}{2e}, \frac{e}{2}\right]$

104. $y = 10x(2 - \ln x), \quad (0, e^2]$

105. Area Find the area between the curve $y = 2(\ln x)/x$ and the x-axis from $x = 1$ to $x = e$.

106. Area

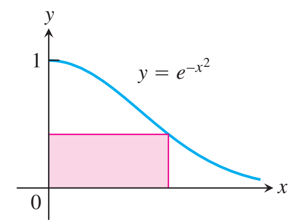
a. Show that the area between the curve $y = 1/x$ and the x-axis from $x = 10$ to $x = 20$ is the same as the area between the curve and the x-axis from $x = 1$ to $x = 2$.

b. Show that the area between the curve $y = 1/x$ and the x-axis from ka to kb is the same as the area between the curve and the x-axis from $x = a$ to $x = b$ $(0 < a < b, k > 0)$.

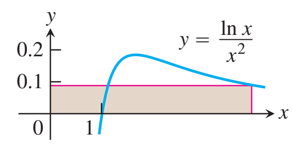
107. A particle is traveling upward and to the right along the curve $y = \ln x$. Its x-coordinate is increasing at the rate $(dx/dt) = \sqrt{x}$ m/sec. At what rate is the y-coordinate changing at the point $(e^2, 2)$?

108. A girl is sliding down a slide shaped like the curve $y = 9e^{-x/3}$. Her y-coordinate is changing at the rate $dy/dt = (-1/4)\sqrt{9 - y}$ ft/sec. At approximately what rate is her x-coordinate changing when she reaches the bottom of the slide at $x = 9$ ft? (Take e^3 to be 20 and round your answer to the nearest ft/sec.)

109. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What dimensions give the rectangle its largest area, and what is that area?



110. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = (\ln x)/x^2$. What dimensions give the rectangle its largest area, and what is that area?



111. The functions $f(x) = \ln 5x$ and $g(x) = \ln 3x$ differ by a constant. What constant? Give reasons for your answer.

112. a. If $(\ln x)/x = (\ln 2)/2$, must $x = 2$?

b. If $(\ln x)/x = -2 \ln 2$, must $x = 1/2$?

Give reasons for your answers.

113. The quotient $(\log_4 x)/(\log_2 x)$ has a constant value. What value? Give reasons for your answer.

114. $\mathbf{\log_x (2)}$ **vs.** $\mathbf{\log_2 (x)}$ How does $f(x) = \log_x(2)$ compare with $g(x) = \log_2(x)$? Here is one way to find out.

a. Use the equation $\log_a b = (\ln b)/(\ln a)$ to express $f(x)$ and $g(x)$ in terms of natural logarithms.

b. Graph f and g together. Comment on the behavior of f in relation to the signs and values of g.

115. Graph the following functions and use what you see to locate and estimate the extreme values, identify the coordinates of the inflection points, and identify the intervals on which the graphs are concave up and concave down. Then confirm your estimates by working with the functions' derivatives.

a. $y = (\ln x)/\sqrt{x}$ **b.** $y = e^{-x^2}$ **c.** $y = (1 + x)e^{-x}$

116. Graph $f(x) = x \ln x$. Does the function appear to have an absolute minimum value? Confirm your answer with calculus.

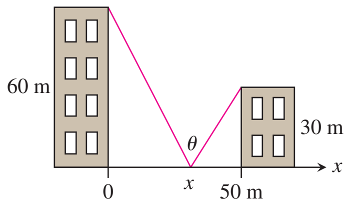
117. What is the age of a sample of charcoal in which 90% of the carbon-14 originally present has decayed?

118. Cooling a pie A deep-dish apple pie, whose internal temperature was 220°F when removed from the oven, was set out on a breezy 40°F porch to cool. Fifteen minutes later, the pie's internal temperature was 180°F. How long did it take the pie to cool from there to 70°F?

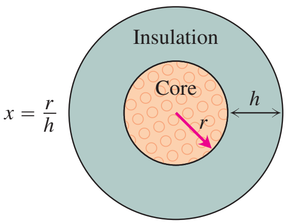
119. Locating a solar station You are under contract to build a solar station at ground level on the east–west line between the two buildings shown here. How far from the taller building should you place the station to maximize the number of hours it will be in the sun on a day when the sun passes directly overhead? Begin by observing that

$$\theta = \pi - \cot^{-1}\frac{x}{60} - \cot^{-1}\frac{50 - x}{30}.$$

Then find the value of x that maximizes θ.



120. A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If x denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $v = x^2 \ln(1/x)$. If the radius of the core is 1 cm, what insulation thickness h will allow the greatest transmission speed?



Chapter 7 Questions to Guide Your Review

1. What functions have inverses? How do you know if two functions f and g are inverses of one another? Give examples of functions that are (are not) inverses of one another.
2. How are the domains, ranges, and graphs of functions and their inverses related? Give an example.
3. How can you sometimes express the inverse of a function of x as a function of x?
4. Under what circumstances can you be sure that the inverse of a function f is differentiable? How are the derivatives of f and f^{-1} related?

5. What is the natural logarithm function? What are its domain, range, and derivative? What arithmetic properties does it have? Comment on its graph.
6. What is logarithmic differentiation? Give an example.
7. What integrals lead to logarithms? Give examples. What are the integrals of $\tan x$ and $\cot x$?
8. How is the exponential function e^x defined? What are its domain, range, and derivative? What laws of exponents does it obey? Comment on its graph.
9. How are the functions a^x and $\log_a x$ defined? Are there any restrictions on a? How is the graph of $\log_a x$ related to the graph of $\ln x$? What truth is there in the statement that there is really only one exponential function and one logarithmic function?
10. Describe some of the applications of base 10 logarithms.
11. What is the law of exponential change? How can it be derived from an initial value problem? What are some of the applications of the law?
12. How do you compare the growth rates of positive functions as $x \to \infty$?
13. What roles do the functions e^x and $\ln x$ play in growth comparisons?
14. Describe big-oh and little-oh notation. Give examples.
15. Which is more efficient—a sequential search or a binary search? Explain.
16. How are the inverse trigonometric functions defined? How can you sometimes use right triangles to find values of these functions? Give examples.
17. What are the derivatives of the inverse trigonometric functions? How do the domains of the derivatives compare with the domains of the functions?
18. What integrals lead to inverse trigonometric functions? How do substitution and completing the square broaden the application of these integrals?
19. What are the six basic hyperbolic functions? Comment on their domains, ranges, and graphs. What are some of the identities relating them?
20. What are the derivatives of the six basic hyperbolic functions? What are the corresponding integral formulas? What similarities do you see here with the six basic trigonometric functions?
21. How are the inverse hyperbolic functions defined? Comment on their domains, ranges, and graphs. How can you find values of $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$, and $\coth^{-1} x$ using a calculator's keys for $\cosh^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$?
22. What integrals lead naturally to inverse hyperbolic functions?