Introduction to Complex Analysis
by Hilary Priestly
Unofficial Solutions Manual

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Dedicated to my Parents

Preface

This is an ongoing Solutions Manual for *Introduction to Complex Analysis* by Hilary Priestly [1]. The main reason for taking up such a project is to have an electronic backup of my own handwritten solutions.

Mathematics cannot be done without actually *doing* it. However at the undergraduate level many students are put off attempting problems unless they have access to written solutions. Thus I am making my work publicly available in the hope that it will encourage undergraduates (or even dedicated high school students) to attempt the exercises and gain confidence in their own problem-solving ability.

I am aware that questions from textbooks are often set as assessed homework for students. Thus in making available these solutions there arises the danger of plagiarism. In order to address this issue I have attempted to write the solutions in a manner which conveys the general idea, but leaves it to the reader to fill in the details.

At the time of writing this work is far from complete. While I will do my best to add additional solutions whenever possible, I can not guarantee that any one solution will be available at a given time. Updates will be made whenever I am free to do so.

I should point out that my solutions are not the only ways to tackle the questions. It is possible that many 'better' solutions exist for any given problem. Additionally my work has not been peer reviewed, so it is not guaranteed to be free of errors. Anyone using these solutions does so at their own risk.

I also wish to emphasize that this is an *unofficial* work, in that it has nothing to do with the original author or publisher. However, in respect of their copyright, I have chosen to omit statements of all the questions. Indeed it should be quite impossible for one to read this work without having a copy of the book [1] present.

I hope that the reader will find this work useful and wish him the best of luck in his Mathematical studies.

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The end of a solution is indicated by ■. Any reference such as 'Theorem 13.1', 'Question 23.10' refers to the relevant numbered item in Priestly's book [1]. This work has been prepared using LaTeX.

The latest version of this file can be found at: `http://akhtarmath.wordpress.com/`

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Chapter 1

The Complex Plane

1.12) Given $z, w \in \mathbb{C}$ we have that:

$$\begin{aligned}|1-\bar{z}w|^2 - |z-w|^2 &= (1-\bar{z}w)(1-z\bar{w}) - (z-w)(\bar{z}-\bar{w}) \\ &= 1+|z|^2|w|^2-|z|^2-|w|^2 = \left(1-|z|^2\right)\left(1-|w|^2\right)\end{aligned}$$

as required. Now suppose $|z|<1$ and $|w|<1$. We observe that:

$$\left|\frac{z-w}{1-\bar{z}w}\right| < 1 \iff |z-w| < |1-\bar{z}w| \iff |z-w|^2 < |1-\bar{z}w|^2$$

where the last equivalence follows because both $|z-w| \geq 0$ and $|1-\bar{z}w| \geq 0$. Furthermore $|z-w|^2 < |1-\bar{z}w|^2$ if and only if $\left(1-|z|^2\right)\left(1-|w|^2\right) > 0$ by the previous part. This last inequality is true because both $|z|<1$ and $|w|<1$ by assumption. The result follows. ■

1.13) (i) Recall that for a complex number a, we have $2\mathrm{Re}(a) = a+\bar{a}$, and if $a \neq 0$ then $\overline{(a^{-1})} = (\bar{a})^{-1}$. With these ideas in mind, select $z, w \in \mathbb{C}$ with $z \neq w$. Then:

$$\begin{aligned}\mathrm{Re}\left(\frac{w+z}{w-z}\right) &= \frac{1}{2}\left[\left(\frac{w+z}{w-z}\right) + \overline{\left(\frac{w+z}{w-z}\right)}\right] = \frac{1}{2}\left[\left(\frac{w+z}{w-z}\right) + \frac{(\bar{w}+\bar{z})}{\overline{(w-z)}}\right] \\ &= \frac{1}{2}\left[\left(\frac{w+z}{w-z}\right) + \frac{(\bar{w}+\bar{z})(w-z)}{\overline{(w-z)}(w-z)}\right] \\ &= \frac{1}{2}\left[\frac{(w+z)(\bar{w}-\bar{z})+(\bar{w}+\bar{z})(w-z)}{|w-z|^2}\right] \\ &= \frac{1}{2}\left[\frac{2\left(|w|^2-|z|^2\right)}{|w-z|^2}\right] = \frac{|w|^2-|z|^2}{|w-z|^2}\end{aligned}$$

as required. (ii) We observe that $|w-z|^2 = (w-z)(\bar{w}-\bar{z}) = |w|^2 - 2\mathrm{Re}(z\bar{w}) + |z|^2$. Writing $z = re^{\mathrm{i}\theta}$ and $w = Re^{\mathrm{i}\varphi}$ we find that $|w-z|^2 = R^2 - 2\mathrm{Re}\left(Rre^{\mathrm{i}(\theta-\varphi)}\right) + r^2$ and since $e^{\mathrm{i}\alpha} = \cos(\alpha) + \mathrm{i}\sin(\alpha)$ for $\alpha \in \mathbb{R}$ we conclude that $|w-z|^2 = R^2 - 2Rr\cos(\theta-\varphi) + r^2$. By substituting this result, along with $|z| = r$ and $|w| = R$ into the expression obtained Part (i) we deduce that

$$\mathrm{Re}\left(\frac{w+z}{w-z}\right) = \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}$$

which was what we wanted to show. ■

1.14) Suppose there exists a relation $>$ on $\mathbb{C}$ satisfying both (a) and (b). Since $\mathrm{i} \neq 0$, condition (a) implies that exactly one of $\mathrm{i} > 0$ or $-\mathrm{i} > 0$ holds. If $\mathrm{i} > 0$ then condition (b) implies that

$i^2 = -1 > 0$. Now[1] since both $i > 0$ and $-1 > 0$, it follows once again from condition (b)that $(-1)(i) = -i > 0$. But now both $i > 0$ and $-i > 0$, which contradicts condition (a). On the other hand, if $-i > 0$ then we once again find that $(-i)^2 = -1 > 0$ which, using condition (b) implies $(-1)(-i) = i > 0$. This once again contradicts condition (a) as before. ■

[1]Note that the fact that $-1 > 0$ does not in itself yield a contradiction. This is because $>$ may not be the usual order relation on $\mathbb{R}$.

Chapter 5

Holomorphic functions

5.1) (i) For any $z = x + iy \in \mathbb{C}$, $\operatorname{Im} z = u(x, y) + iv(x, y)$, where $u(x, y) = 0$ and $v(x, y) = y$ for all $(x, y) \in \mathbb{R}^2$. In particular $u_x = 0 \neq 1 = v_y$ at any $(x, y) \in \mathbb{R}^2$. So the Cauchy-Riemann equations fail at every $z = x + iy \in \mathbb{C}$. We conclude $\operatorname{Im} z$ is not differentiable anywhere.
(ii) For any $z = x + iy \in \mathbb{C}$, $\bar{z} = u(x, y) + iv(x, y)$, where $u(x, y) = x$ and $v(x, y) = -y$ for all $(x, y) \in \mathbb{R}^2$. In particular, $u_x = 1 \neq -1 = v_y$. So the Cauchy-Riemann equations fail at every $z = x + iy \in \mathbb{C}$. We conclude $\bar{z}$ is not differentiable anywhere. ■

5.3) If $f = u+iv$ is differentiable at $z = a+ib \in \mathbb{C}$, then all the partial derivatives u_x, u_y, v_x, v_y exist at z, and in particular, $v_x = -u_y$ there. Now $f'(z) = u_x + iv_x$ and also $f'(z) = v_y - iu_y$. Substituting $v_x = -u_y$ into both of these yields $f'(z) = u_x - iu_y$ and $f'(z) = v_y + iv_x$. ■

5.7) (a)(i) This is a polynomial and so is holomorphic at all $z \in \mathbb{C}$. (a)(ii) Holomorphic at all $z \in \mathbb{C}$ except at $z = 0, 1$ and 2. (a)(iii) Holomorphic at all $z \in \mathbb{C}$ except at $z = 1, \omega, \omega^2, \omega^3$ and ω^4, where $\omega = \exp(2\pi i/5)$. (b)(i) First select any $a \in \mathbb{C}\backslash\{0\}$. If $1/|z|$ is holomorphic at a then in particular $1/|z|$ is differentiable at a. Also $1/|a| \neq 0$ so $(1/|z|)^{-1} = |z|$ is also differentiable at a. This is a contradiction since $|z|$ does not satisfy the Cauchy-Riemann equations at a $(\neq 0)$. So $1/|z|$ is not holomorphic at any $a \in \mathbb{C}\backslash\{0\}$. It follows that $1/|z|$ can not be holomorphic in any disc centered at 0. Thus $1/|z|$ is not holomorphic at any $a \in \mathbb{C}$. (b)(ii) Suppose that $z|z|$ is homomorphic at $a \in \mathbb{C}\backslash\{0\}$. The function $1/z$ is also homomorphic at a $(\neq 0)$. Therefore $(1/z)(z|z|) = |z|$ is also holomorphic at a. This is a contradiction since $|z|$ is not even differentiable at a. The argument now follows as in (b)(i). ■

Bibliography

[1] Priestly, H.A. *Introduction to Complex Analysis, 2nd Ed.*, 2003. Oxford University Press.