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Functional analysis and its applications

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Introduction

Functional analysis plays an important role in the applied sciences as well as in mathematics itself. These notes are intended to familiarize the student with the basic concepts, principles and methods of functional analysis and its applications, and they are intended for senior undergraduate or beginning graduate students.

The notes are elementary assuming no prerequisites beyond knowledge of linear algebra and ordinary calculus (with ϵ-δ arguments). Measure theory is neither assumed, nor discussed, and no knowledge of topology is required. The notes should hence be accessible to a wide spectrum of students, and may also serve to bridge the gap between linear algebra and advanced functional analysis.

Functional analysis is an abstract branch of mathematics that originated from classical analysis. The impetus came from applications: problems related to ordinary and partial differential equations, numerical analysis, calculus of variations, approximation theory, integral equations, and so on. In ordinary calculus, one dealt with limiting processes in finite-dimensional vector spaces ($\mathbb{R}$ or $\mathbb{R}^n$), but problems arising in the above applications required a calculus in spaces of functions (which are infinite-dimensional vector spaces). For instance, we mention the following optimization problem.

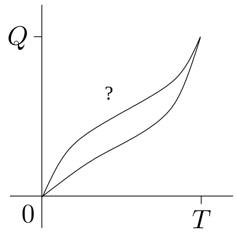
Problem. A copper mining company intends to remove all of the copper ore from a region that contains an estimated Q tons, over a time period of T years. As it is extracted, they will sell it for processing at a net price per ton of

$$p = P - ax(t) - bx'(t)$$

for positive constants P, a, and b, where $x(t)$ denotes the total tonnage sold by time t. If the company wishes to maximize its total profit given by

$$I(x) = \int_0^T [P - ax(t) - bx'(t)]x'(t)dt,$$

where $x(0) = 0$ and $x(T) = Q$, how might it proceed?



The optimal mining operation problem: what curve gives the maximum profit?

We observe that this is an optimization problem: to each curve between the points $(0, 0)$ and (T, Q), we associate a number (the associated profit), and the problem is to find the shape of the curve that minimizes this function

$$I : \{\text{curves between } (0, 0) \text{ and } (T, Q)\} \to \mathbb{R}.$$

This problem does not fit into the usual framework of calculus, where typically one has a function from some subset of the *finite* dimensional vector space $\mathbb{R}^n$ to $\mathbb{R}$, and one wishes to find a vector in $\mathbb{R}^n$ that minimizes/maximizes the function, while in the above problem one has a subset of an *infinite* dimensional function space.

Thus the need arises for developing calculus in more general spaces than $\mathbb{R}^n$. Although we have only considered one example, problems requiring calculus in infinite-dimensional vector spaces arise from many applications and from various disciplines such as economics, engineering, physics, and so on. Mathematicians observed that different problems from varied fields often have related features and properties. This fact was used for an effective unifying approach towards such problems, the unification being obtained by the omission of unessential details. Hence the advantage of an *abstract* approach is that it concentrates on the essential facts, so that these facts become clearly visible and one's attention is not disturbed by unimportant details. Moreover, by developing a box of tools in the abstract framework, one is equipped to solve many different problems (that are really the same problem in disguise!). For example, while fishing for various different species of fish (bass, sardines, perch, and so on), one notices that in each of these different algorithms, the basic steps are the same: all one needs is a fishing rod and some bait. Of course, what bait one uses, where and when one fishes, depends on the particular species one wants to catch, but underlying these minor details, the basic technique is the same. So one can come up with an abstract algorithm for fishing, and applying this general algorithm to the particular species at hand, one gets an algorithm for catching that particular species. Such an abstract approach also has the advantage that it helps us to tackle unseen problems. For instance, if we are faced with a hitherto unknown species of fish, all that one has to do in order to catch it is to find out what it eats, and then by applying the general fishing algorithm, one would also be able to catch this new species.

In the abstract approach, one usually starts from a set of elements satisfying certain axioms. The theory then consists of logical consequences which result from the axioms and are derived as theorems once and for all. These general theorems can then later be applied to various concrete special sets satisfying the axioms.

We will develop such an abstract scheme for doing calculus in function spaces and other infinite-dimensional spaces, and this is what this course is about. Having done this, we will be equipped with a box of tools for solving many problems, and in particular, we will return to the optimal mining operation problem again and solve it.

These notes contain many exercises, which form an integral part of the text, as some results relegated to the exercises are used in proving theorems. Some of the exercises are routine, and the harder ones are marked by an asterisk $(*)$.

Most applications of functional analysis are drawn from the rudiments of the theory, but not all are, and no one can tell what topics will become important. In these notes we have described a few topics from functional analysis which find widespread use, and by no means is the choice of topics 'complete'. However, equipped with this basic knowledge of the elementary facts in functional analysis, the student can undertake a serious study of a more advanced treatise on the subject, and the bibliography gives a few textbooks which might be suitable for further reading.

It is a pleasure to thank Prof. Erik Thomas from the University of Groningen for many useful comments and suggestions.

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Contents

Chapter 1

Normed and Banach spaces

1.1 Vector spaces

In this section we recall the definition of a vector space. Roughly speaking it is a set of elements, called "vectors". Any two vectors can be "added", resulting in a new vector, and any vector can be multiplied by an element from $\mathbb{R}$ (or $\mathbb{C}$, depending on whether we consider a *real* or *complex* vector space), so as to give a new vector. The precise definition is given below.

Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (or more generally[1] a *field*). A *vector space* over $\mathbb{K}$, is a set X together with two functions, $+ : X \times X \to X$, called *vector addition*, and $\cdot : \mathbb{K} \times X \to X$, called *scalar multiplication* that satisfy the following:

V1. For all $x_1, x_2, x_3 \in X$, $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$.

V2. There exists an element, denoted by 0 (called the *zero vector*) such that for all $x \in X$, $x + 0 = 0 + x = x$.

V3. For every $x \in X$, there exists an element, denoted by $-x$, such that $x + (-x) = (-x) + x = 0$.

V4. For all x_1, x_2 in X, $x_1 + x_2 = x_2 + x_1$.

V5. For all $x \in X$, $1 \cdot x = x$.

V6. For all $x \in X$ and all $\alpha, \beta \in \mathbb{K}$, $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$.

V7. For all $x \in X$ and all $\alpha, \beta \in \mathbb{K}$, $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.

V8. For all $x_1, x_2 \in X$ and all $\alpha \in \mathbb{K}$, $\alpha \cdot (x_1 + x_2) = \alpha \cdot x_1 + \alpha \cdot x_2$.

Examples.

1. $\mathbb{R}$ is a vector space over $\mathbb{R}$, with vector addition being the usual addition of real numbers, and scalar multiplication being the usual multiplication of real numbers.

[1] Unless stated otherwise, the underlying field is always assumed to be $\mathbb{R}$ or $\mathbb{C}$ throughout these notes.

2. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, with addition and scalar multiplication defined as follows:

$$\text{if } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n, \text{ then } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix};$$

$$\text{if } \alpha \in \mathbb{R} \text{ and } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \text{ then } \alpha \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}.$$

3. The sequence space ℓ^∞. This example and the next one give a first impression of how surprisingly general the concept of a vector space is.

   Let ℓ^∞ denote the vector space of all bounded sequences with values in $\mathbb{K}$, and with addition and scalar multiplication defined as follows:

$$(x_n)_{n\in\mathbb{N}} + (y_n)_{n\in\mathbb{N}} = (x_n + y_n)_{n\in\mathbb{N}}, \quad (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \ell^\infty; \tag{1.1}$$

$$\alpha(x_n)_{n\in\mathbb{N}} = (\alpha x_n)_{n\in\mathbb{N}}, \quad \alpha \in \mathbb{K}, (x_n)_{n\in\mathbb{N}} \in \ell^\infty. \tag{1.2}$$

4. The function space $C[a,b]$. Let $a, b \in \mathbb{R}$ and $a < b$. Consider the vector space comprising functions $f : [a,b] \to \mathbb{K}$ that are continuous on $[a,b]$, with addition and scalar multiplication defined as follows. If $f, g \in C[a,b]$, then $f + g \in C[a,b]$ is the function given by

$$(f+g)(x) = f(x) + g(x), \quad x \in [a,b]. \tag{1.3}$$

   If $\alpha \in \mathbb{K}$ and $f \in C[a,b]$, then $\alpha f \in C[a,b]$ is the function given by

$$(\alpha f)(x) = \alpha f(x), \quad x \in [a,b]. \tag{1.4}$$

   $C[a,b]$ is referred to as a 'function space', since each vector in $C[a,b]$ is a function (from $[a,b]$ to $\mathbb{K}$). $\Diamond$

Exercises.

1. Let $y_a, y_b \in \mathbb{R}$, and let

$$S(y_a, y_b) = \{x \in C[a,b] \mid x(a) = y_a \text{ and } x(b) = y_b\}.$$

   For what values of y_a, y_b is $S(y_a, y_b)$ a vector space?

2. Show that $C[0,1]$ is not a finite dimensional vector space.

   Hint: One can prove this by contradiction. Let $C[0,1]$ be a finite dimensional vector space with dimension d, say. First show that the set $B = \{x, x^2, \dots, x^d\}$ is linearly independent. Then B is a basis for $C[0,1]$, and so the constant function 1 should be a linear combination of the functions from B. Derive a contradiction.

3. Let V be a vector space, and let $\{V_n \mid n \in \mathbb{N}\}$ be a set of subspaces of V. Prove that $\displaystyle\bigcap_{n=1}^{\infty} V_n$ is a subspace of V.

4. Let λ_1, λ_2 be two distinct real numbers, and let $f_1, f_2 \in C[0,1]$ be

$$f_1(x) = e^{\lambda_1 x} \text{ and } f_2(x) = e^{\lambda_2 x}, \quad x \in [0,1].$$

   Show that the functions f_1 and f_2 are linearly independent in the vector space $C[0,1]$.

1.2 Normed spaces

In order to do ‘calculus’ (that is, speak about limiting processes, convergence, approximation, continuity) in vector spaces, we need a notion of ‘distance’ or ‘closeness’ between the vectors of the vector space. This is provided by the notion of a norm.

Definitions. Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A *norm* on X is a function $\|\cdot\| : X \to [0, +\infty)$ such that:

N1. *(Positive definiteness)* For all $x \in X$, $\|x\| \geq 0$. If $x \in X$, then $\|x\| = 0$ iff $x = 0$.

N2. For all $\alpha \in \mathbb{R}$ (respectively $\mathbb{C}$) and for all $x \in X$, $\|\alpha x\| = |\alpha|\|x\|$.

N3. *(Triangle inequality)* For all $x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$.

A *normed space* is a vector space X equipped with a norm.

If $x, y \in X$, then the number $\|x - y\|$ provides a notion of closeness of points x and y in X, that is, a ‘distance’ between them. Thus $\|x\| = \|x - 0\|$ is the distance of x from the zero vector in X.

We now give a few examples of normed spaces.

Examples.

1. $\mathbb{R}$ is a vector space over $\mathbb{R}$, and if we define $\|\cdot\| : \mathbb{R} \to [0, +\infty)$ by

$$\|x\| = |x|, \quad x \in \mathbb{R},$$

then it becomes a normed space.

2. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, and let

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

Then $\mathbb{R}^n$ is a normed space (see Exercise 5a on page 5).

This is not the only norm that can be defined on $\mathbb{R}^n$. For example,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \text{and} \quad \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n,$$

are also examples of norms (see Exercise 5a on page 5).

Note that $(\mathbb{R}^n, \|\cdot\|_2)$, $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{R}^n, \|\cdot\|_\infty)$ are all *different* normed spaces. This illustrates the important fact that from a given vector space, we can obtain various normed spaces by choosing different norms. What norm is considered depends on the particular application at hand. We illustrate this in the next paragraph.

Suppose that we are interested in comparing the economic performance of a country from year to year, using certain economic indicators. For example, let the ordered 365-tuple

$x = (x_1, \dots, x_{365})$ be the record of the daily industrial averages. A measure of differences in yearly performance is given by

$$\|x - y\| = \sum_{i=1}^{365} |x_i - y_i|.$$

Thus the space $(\mathbb{R}^{365}, \|\cdot\|_1)$ arises naturally. We might also be interested in the monthly cost of living index. Let the record of this index for a year be given by 12-tuples $x = (x_1, \dots, x_{12})$. A measure of differences in yearly performance of the cost of living index is given by

$$\|x - y\| = \max\{|x_1 - y_1|, \dots, |x_{12} - y_{12}|\},$$

which is the distance between x and y in the normed space $(\mathbb{R}^{12}, \|\cdot\|_\infty)$.

3. The sequence space ℓ^∞. This example and the next one give a first impression of how surprisingly general the concept of a normed space is.

   Let ℓ^∞ denote the vector space of all bounded sequences, with the addition and scalar multiplication defined earlier in (1.1)-(1.2).

   Define
$$\|(x_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|, \quad (x_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

   Then it is easy to check that $\|\cdot\|_\infty$ is a norm, and so $(\ell^\infty, \|\cdot\|_\infty)$ is a normed space.

4. The function space $C[a,b]$. Let $a, b \in \mathbb{R}$ and $a < b$. Consider the vector space comprising functions that are continuous on $[a,b]$, with addition and scalar multiplication defined earlier by (1.3)-(1.4).

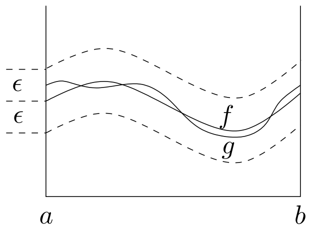

Figure 1.1: The set of all continuous functions g whose graph lies between the two dotted lines is the 'ball' $B(f,\epsilon) = \{g \in C[a,b] \mid \|g - f\|_\infty < \epsilon\}$.

Define
$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|, \quad f \in C[a,b]. \tag{1.5}$$

Then $\|\cdot\|_\infty$ is a norm on $C[a,b]$. Another norm is given by
$$\|f\|_1 = \int_a^b |f(x)|dx, \quad f \in C[a,b]. \tag{1.6}$$

$\Diamond$

Exercises.

1. Let $(X, \|\cdot\|)$ be a normed space. Prove that for all $x, y \in X$, $|\|x\| - \|y\|| \le \|x - y\|$.
2. If $x \in \mathbb{R}$, then let $\|x\| = |x|^2$. Is $\|\cdot\|$ a norm on $\mathbb{R}$?

3. Let $(X, \|\cdot\|)$ be a normed space and $r > 0$. Show that the function $x \mapsto r\|x\|$ defines a norm on X.

   Thus there are infinitely many other norms on any normed space.

4. Let X be a normed space $\|\cdot\|_X$ and Y be a subspace of X. Prove that Y is also a normed space with the norm $\|\cdot\|_Y$ defined simply as the restriction of the norm $\|\cdot\|_X$ to Y. This norm on Y is called the *induced norm*.

5. Let $1 < p < +\infty$ and q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Then *Hölder's inequality*[2] says that if $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are any real or complex numbers, then

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}.$$

   If $1 \leq p \leq +\infty$, and $n \in \mathbb{N}$, then for

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n,$$

   define

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \text{ if } 1 \leq p < +\infty, \quad \text{and} \quad \|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}. \tag{1.7}$$

   (a) Show that the function $x \mapsto \|x\|_p$ is a norm on $\mathbb{R}^n$.
   Hint: Use Hölder's inequality to obtain

$$\sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} \leq \|x\|_p \|x + y\|_p^{\frac{p}{q}} \quad \text{and} \quad \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1} \leq \|y\|_p \|x + y\|_p^{\frac{p}{q}}.$$

   Adding these, we obtain the triangle inequality:

$$\|x + y\|_p^p = \sum_{i=1}^{n} |x_i + y_i||x_i + y_i|^{p-1} \leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1}.$$

   (b) Let $n = 2$. Depict the following sets pictorially:

$$\begin{aligned} B_2(0,1) &= \{x \in \mathbb{R}^2 \mid \|x\|_2 < 1\}, \\ B_1(0,1) &= \{x \in \mathbb{R}^2 \mid \|x\|_1 < 1\}, \\ B_\infty(0,1) &= \{x \in \mathbb{R}^2 \mid \|x\|_\infty < 1\}. \end{aligned}$$

   (c) Let $x \in \mathbb{R}^n$. Prove that $(\|x\|_p)_{p \in \mathbb{N}}$ is a convergent sequence in $\mathbb{R}$ and $\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$.
   Describe what happens to the sets $B_p(0,1) = \{x \in \mathbb{R}^2 \mid \|x\|_p < 1\}$ as p tends to ∞.

6. A subset C of a vector space X is said to be *convex* if for all $x, y \in C$, and all $\alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in C$; see Figure 1.2.

   (a) Show that the unit ball $B(0,1) = \{x \in X \mid \|x\| < 1\}$ is convex in any normed space $(X, \|\cdot\|)$.
   (b) Sketch the curve $\{(x_1, x_2) \in \mathbb{R}^2 \mid \sqrt{|x_1|} + \sqrt{|x_2|} = 1\}$.

[2] A proof of this inequality can be obtained by elementary calculus, and we refer the interested student to §1.4 at the end of this chapter.

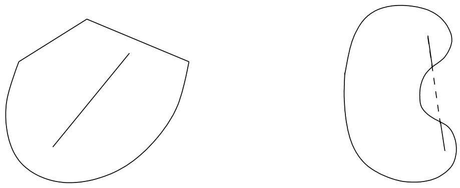

Figure 1.2: Examples of convex and nonconvex sets in $\mathbb{R}^2$.

(c) Prove that

$$\|x\|_{\frac{1}{2}} := \left(\sqrt{|x_1|} + \sqrt{|x_2|}\right)^2, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2,$$

does not define a norm on $\mathbb{R}^2$.

7. (a) Show that the *polyhedron*

$$P_n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \;\middle|\; \forall i \in \{1, \dots, n\},\ x_i > 0 \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}$$

is convex in $\mathbb{R}^n$. Sketch P_2.

(b) Prove that

$$\text{if } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in P_n, \text{ then } \sum_{i=1}^{n} \frac{1}{x_i} \geq n^2. \tag{1.8}$$

HINT: Use Hölder's inequality with $p = 2$.

(c) In the financial world, there is a method of investment called *dollar cost averaging*. Roughly speaking, this means that one invests a fixed amount of money regularly instead of a lumpsum. It is claimed that a person using dollar cost averaging should be better off than one who invests all the amount at one time. Suppose a fixed amount A is used to buy shares at prices $p_1, \dots, p_n$. Then the total number of shares is then $\frac{A}{p_1} + \dots + \frac{A}{p_n}$. If one invests the amount nA at a time when the share price is the average of $p_1, \dots, p_n$, then the number of shares which one can purchase is $\frac{n^2 A}{p_1 + \dots + p_n}$. Using the inequality (1.8), conclude that dollar cost averaging is at least as good as purchasing at the average share price.

8. $(*)$ *(p-adic norm)* Consider the vector space of the rational numbers $\mathbb{Q}$ over the field $\mathbb{Q}$. Let p be a prime number. Define the *p-adic norm* $|\cdot|_p$ on the set of rational numbers as follows: if $r \in \mathbb{Q}$, then

$$|r|_p = \begin{cases} \frac{1}{p^k} \text{ where } r = p^k \frac{m}{n}, \quad k, m, n \in \mathbb{Z} \text{ and } p \not| m, n, & \text{if } r \neq 0, \\ 0 & \text{if } r = 0. \end{cases}$$

So in this context, a rational number is close to 0 precisely when it is highly divisible by p.

(a) Show that $|\cdot|_p$ is well-defined on $\mathbb{Q}$.

(b) If $r \in \mathbb{Q}$, then prove that $|r|_p \geq 0$, and that $|r|_p = 0$ iff $r = 0$.

(c) For all $r_1, r_2 \in \mathbb{Q}$, show that $|r_1 r_2|_p = |r_1|_p |r_2|_p$.

(d) For all $r_1, r_2 \in \mathbb{Q}$, prove that $|r_1 + r_2|_p \leq \max\{|r_1|_p, |r_2|_p\}$. In particular, for all $r_1, r_2 \in Q$, $|r_1 + r_2|_p \leq |r_1|_p + |r_2|_p$.

9. Show that (1.6) defines a norm on $C[a, b]$.

1.3 Banach spaces

In a normed space, we have a notion of 'distance' between vectors, and we can say when two vectors are close by and when they are far away. So we can talk about convergent sequences. In the same way as in $\mathbb{R}$ or $\mathbb{C}$, we can define convergent sequences and Cauchy sequences in a normed space:

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $x \in X$. The sequence $(x_n)_{n\in\mathbb{N}}$ *converges to* x if

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \text{ such that for all } n \in \mathbb{N} \text{ satisfying } n \geq N, \ \|x_n - x\| < \epsilon. \tag{1.9}$$

Note that (1.9) says that the real sequence $(\|x_n - x\|)_{n\in\mathbb{N}}$ converges to 0: $\lim\limits_{n\to\infty} \|x_n - x\| = 0$, that is the distance of the vector x_n to the limit x tends to zero, and this matches our geometric intuition. One can show in the same way as with $\mathbb{R}$, that the limit is unique: a convergent sequence has only one limit. We write

$$\lim_{n\to\infty} x_n = x.$$

Example. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ in the normed space $(C[0,1], \|\cdot\|_\infty)$, where

$$f_n = \frac{\sin(2\pi n x)}{n^2}.$$

The first few terms of the sequence are shown in Figure 1.3.

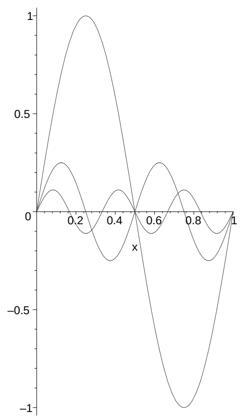

Figure 1.3: The first three terms of the sequence $(f_n)_{n\in\mathbb{N}}$.

From the figure, we see that the terms seem to converge to the zero function. Indeed we have $\|f_n - 0\|_\infty = \dfrac{1}{n^2}\|\sin(2\pi n x)\|_\infty = \dfrac{1}{n^2} < \epsilon$ for all $n > N > \frac{1}{\sqrt{\epsilon}}$. $\Diamond$

Definition. The sequence $(x_n)_{n\in\mathbb{N}}$ is a called a *Cauchy sequence* if

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \text{ such that for all } m, n \in \mathbb{N} \text{ satisfying } m, n \geq N, \ \|x_m - x_n\| < \epsilon. \tag{1.10}$$

Every convergent sequence is a Cauchy sequence, since $\|x_m - x_n\| \leq \|x_m - x\| + \|x - x_n\|$.

Definition. A normed space $(X, \|\cdot\|)$ is called *complete* if every Cauchy sequence is convergent.

Complete normed spaces are called *Banach spaces* after the Polish mathematician Stephan Banach (1892-1945) who was the first to set up the general theory (in his Ph.D. thesis in 1920).

Thus in a complete normed space, or Banach space, the Cauchy condition is sufficient for convergence: the sequence $(x_n)_{n\in\mathbb{N}}$ converges iff it is a Cauchy sequence, that is if (1.10) holds. So we can determine convergence a priori without the knowledge of the limit. Just as it was possible to introduce new numbers in $\mathbb{R}$, in the same way in a Banach space it is possible to show the existence of elements with some property of interest, by making use of the Cauchy criterion. In this manner, one can sometimes show that certain equations have a unique solution. In many cases, one cannot write them explicitly. After existence and uniqueness of the solution is demonstrated, then one can do numerical approximations.

The following theorem is an instance where one uses the Cauchy criterion:

Theorem 1.3.1 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach space and let $s_n = x_1 + \cdots + x_n$. If*

$$\sum_{n=1}^{\infty} \|x_n\| < +\infty, \tag{1.11}$$

then the series $\sum_{n=1}^{\infty} x_n$ converges, that is, the sequence $(s_n)_{n\in\mathbb{N}}$ converges.

If we denote $\lim_{n\to\infty} s_n$ by $\sum_{n=1}^{\infty} x_n$, then we have $\left\|\sum_{n=1}^{\infty} x_n\right\| \le \sum_{n=1}^{\infty} \|x_n\|$.

Proof For $k > n$, we have $s_k - s_n = \sum_{i=n+1}^{k} x_i$ so that:

$$\|s_k - s_n\| \le \sum_{i=n+1}^{k} \|x_i\| \le \sum_{i=n+1}^{\infty} \|x_i\| < \epsilon$$

for $k > n \ge N$ sufficiently large. It follows that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence is convergent. If $s = \lim_{n\to\infty} s_n$, then using the triangle inequality (see Exercise 1 on page 5), we have $|\|s\| - \|s_n\|| \le \|s - s_n\|$ so that $\|s\| = \lim_{n\to\infty} \|s_n\|$. Since

$$\|s_n\| \le \sum_{i=1}^{n} \|x_i\| \le \sum_{i=1}^{\infty} \|x_i\|,$$

we obtain $\|\sum_{n=1}^{\infty} x_n\| \le \sum_{n=1}^{\infty} \|x_n\|$ by taking the limit. ∎

We will use this theorem later to show that e^A converges, where A is a square matrix. This matrix-valued function plays an important role in the theory of ordinary differential equations.

Examples.

1. The space $\mathbb{R}^n$ equipped with the norm $\|\cdot\|_p$, given by (1.7) is a Banach space. We must show that these spaces are complete. Let $(x^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^n$ (respectively $\mathbb{C}^n$). Then we have
$$\|x^{(k)} - x^{(m)}\| < \epsilon \quad \text{for all } m, k \ge N$$

that is

$$\sum_{i=1}^{n} |x_i^{(k)} - x_i^{(m)}|^2 < \epsilon^2 \quad \text{for all } m, k \geq N. \tag{1.12}$$

Thus it follows that for every $i \in \{1, \dots, n\}$, $|x_i^{(k)} - x_i^{(m)}| < \epsilon$ for all $m, k \geq N$, that is the sequence $(x_i^{(m)})_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ (respectively $\mathbb{C}$) and consequently it is convergent. Let $x_i = \lim\limits_{m\to\infty} x_i^{(m)}$. Then $x = (x_1, \dots, x_n)$ belongs to $\mathbb{R}^n$ (respectively $\mathbb{C}^n$). Now let k go to infinity in (1.12), and we obtain:

$$\sum_{i=1}^{n} |x_i - x_i^{(m)}|^2 \leq \epsilon^2 \quad \text{for all } m \geq N,$$

that is

$$\|x - x^{(m)}\|_2 \leq \epsilon \quad \text{for all } m \geq N$$

and so $x = \lim\limits_{m\to\infty} x^{(m)}$ in the normed space. This completes the proof.

2. The spaces ℓ^p.

   Let $1 \leq p < +\infty$. Then one defines the space ℓ^p as follows:

$$\ell^p = \left\{ x = (x_i)_{i\in\mathbb{N}} \;\middle|\; \sum_{i=1}^{\infty} |x_i|^p < +\infty \right\}$$

with the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}. \tag{1.13}$$

For $p = +\infty$, we define the space ℓ^∞ by

$$\ell^\infty = \left\{ x = (x_i)_{i\in\mathbb{N}} \;\middle|\; \sup_{i\in\mathbb{N}} |x_i| < +\infty \right\}$$

with the norm

$$\|x\|_\infty = \sup_{i\in\mathbb{N}} |x_i|.$$

(See Exercise 1 on page 12.) The most important of these spaces are ℓ^1, ℓ^∞ and ℓ^2.

Theorem 1.3.2 *The spaces ℓ^p are Banach spaces for $1 \leq p \leq +\infty$.*

Proof We prove this for instance in the case of the space ℓ^2. From the inequality $|x_i+y_i|^2 \leq 2|x_i|^2 + 2|y_i|^2$, we see that ℓ^2, equipped with the operations

$$(x_n)_{n\in\mathbb{N}} + (y_n)_{n\in\mathbb{N}} = (x_n + y_n)_{n\in\mathbb{N}}, \quad (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \ell^2,$$
$$\alpha(x_n)_{n\in\mathbb{N}} = (\alpha x_n)_{n\in\mathbb{N}}, \quad \alpha \in \mathbb{K}, (x_n)_{n\in\mathbb{N}} \in \ell^2,$$

is a vector space.

We must now show that ℓ^2 is complete. Let $(x^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ^2. The proof of the completeness will be carried out in three steps:

Step 1. We seek a candidate limit x for the sequence $(x^{(n)})_{n\in\mathbb{N}}$.

We have

$$\|x^{(k)} - x^{(n)}\|_2 < \epsilon \quad \text{for all } n, k \geq N,$$

that is

$$\sum_{i=1}^{\infty} |x_i^{(k)} - x_i^{(n)}|^2 < \epsilon^2 \quad \text{for all } n, k \geq N. \tag{1.14}$$

Thus for every $i \in \mathbb{N}$, $|x_i^{(k)} - x_i^{(n)}| < \epsilon$ for all $n, k \geq N$, that is, the sequence $(x_i^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ (respectively $\mathbb{C}$) and consequently, it is convergent. Let $x_i = \lim\limits_{n\to\infty} x_i^{(n)}$.

Step 2. We show that indeed x belongs to the desired space (here ℓ^2).

The sequence $x = (x_i)_{i\in\mathbb{N}}$ belongs to ℓ^2. Let $m \in \mathbb{N}$. Then from (1.14) it follows that

$$\sum_{i=1}^{m} |x_i^{(k)} - x_i^{(n)}|^2 < \epsilon^2 \quad \text{for all } n, k \geq N.$$

Now we let k go to ∞. Then we see that

$$\sum_{i=1}^{m} |x_i - x_i^{(n)}|^2 \leq \epsilon^2 \quad \text{for all } n \geq N.$$

Since this is true for all $m \in \mathbb{N}$, we have

$$\sum_{i=1}^{\infty} |x_i - x_i^{(n)}|^2 \leq \epsilon^2 \quad \text{for all } n \geq N. \tag{1.15}$$

This means that for $n \geq N$, the sequence $x - x^{(n)}$, and thus also the sequence $x = x - x^{(n)} + x^{(n)}$, belongs to ℓ^2.

Step 3. We show that indeed $\|x - x^{(n)}\|$ goes to 0, that is, $x^{(n)}$ converges to x in the given normed space (here ℓ^2).

The equation (1.15) is equivalent with

$$\|x - x^{(n)}\|_2 \leq \epsilon \quad \text{for all } n \geq N,$$

and so it follows that $x = \lim\limits_{n\to\infty} x^{(n)}$ in the normed space ℓ^2. This completes the proof. ∎

3. Spaces of continuous functions.

Theorem 1.3.3 *Let $a, b \in \mathbb{R}$ and $a < b$. The space $(C[a,b], \|\cdot\|_\infty)$ is a Banach space.*

Proof It is clear that linear combinations of continuous functions are continuous, so that $C[a,b]$ is a vector space. The equation (1.5) defines a norm, and the space $C[a,b]$ is a normed space.

We must now show the completeness. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C[a,b]$. Let $\epsilon > 0$ be given. Then there exists a $N \in \mathbb{N}$ such that for all $x \in [a,b]$, we have

$$|f_k(x) - f_n(x)| \leq \|f_k - f_n\|_\infty < \epsilon \quad \text{for all } k, n \geq N. \tag{1.16}$$

Thus it follows that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) is complete, the limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ exists. We must now show that the limit is continuous. If we let k go to ∞ in (1.16), then we see that for all $x \in [a,b]$,

$$|f(x) - f_n(x)| \leq \epsilon \quad \text{for all } n \geq N. \tag{1.17}$$

Using the continuity of the f_n's and (1.17) above, we now show that f is continuous on $[a, b]$. Let $x_0 \in [a, b]$. Given any $\zeta > 0$, let $\epsilon = \frac{\zeta}{3}$. Choose $N \in \mathbb{N}$ large enough so that (1.17) holds. As f_N is continuous on $[a, b]$, it follows that there exists a $\delta > 0$ such that

$$\text{for all } x \in [a, b] \text{ such that } |x - x_0| < \delta, \quad |f_N(x) - f_N(x_0)| < \epsilon.$$

Consequently, for all $x \in [a, b]$ such that $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \epsilon + \epsilon + \epsilon = \zeta. \end{aligned}$$

So f must be continuous, and hence it belongs to $C[a, b]$.

Finally, from (1.17), we have

$$\|f - f_n\|_\infty \leq \epsilon \quad \text{for all } n \geq N,$$

and so f_n converges to f in the normed space $C[a, b]$. ■

It can be shown that $C[a, b]$ is not complete when it is equipped with the norm

$$\|f\|_1 = \int_a^b |f(x)| dx, \quad f \in C[a, b];$$

see Exercise 6 below.

Using the fact that $(C[0, 1], \|\cdot\|_\infty)$ is a Banach space, and using Theorem 1.3.1, let us show that

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} \tag{1.18}$$

converges in $(C[0, 1], \|\cdot\|_\infty)$. Indeed, we have

$$\left\| \frac{\sin(2\pi nx)}{n^2} \right\|_\infty \leq \frac{1}{n^2},$$

and as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, it follows that (1.18) converges in the $\|\cdot\|_\infty$-norm to a continuous function.

In fact, we can get a pretty good idea of the limit by computing the first N terms (with a large enough N) and plotting the resulting function–the error can then be bounded as follows:

$$\left\| \sum_{n=N+1}^{\infty} \frac{\sin(2\pi nx)}{n^2} \right\|_\infty \leq \sum_{n=N+1}^{\infty} \left\| \frac{\sin(2\pi nx)}{n^2} \right\|_\infty \leq \sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

For example, if $N = 10$, then the error is bounded above by

$$\sum_{n=11}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} - \left(1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100}\right) \approx 0.09516637.$$

Using Maple, we have plotted the partial sum of (1.18) with $N = 10$ in Figure 1.4. Thus the sum converges to a continuous function that lies in the strip of width 0.96 around the graph shown in the figure. ♢

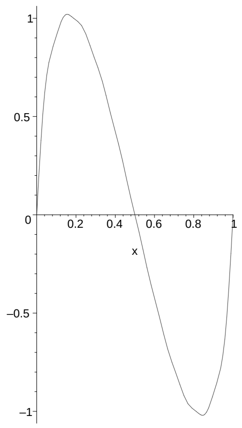

Figure 1.4: Partial sum of (1.18).

Exercises.

1. Show that if $1 \leq p \leq +\infty$, then ℓ^p is a normed space. (That is, ℓ^p is a vector space and that $\|\cdot\|_p$ defined by (1.13) gives a norm on ℓ^p.)

   HINT: Use Exercise 5 on page 5.

2. Let X be a normed space, and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X with limit x. Prove that $(\|x_n\|)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{R}$ and that

$$\lim_{n\to\infty} \|x_n\| = \|x\|.$$

3. Let c_{00} denote the set of all sequences that have only finitely many nonzero terms.

   (a) Show that c_{00} is a subspace of ℓ^p for all $1 \leq p \leq +\infty$.

   (b) Prove that for all $1 \leq p \leq +\infty$, c_{00} is not complete with the induced norm from ℓ^p.

4. Show that $\ell^1 \subsetneq \ell^2 \subsetneq \ell^\infty$.

5. $(*)$ Let $C^1[a,b]$ denote the space of continuously differentiable[3] functions on $[a,b]$:

$$C^1[a,b] = \{f : [a,b] \to \mathbb{K} \mid f \text{ is continuously differentiable}\},$$

   equipped with the norm

$$\|f\|_{1,\infty} = \|f\|_\infty + \|f'\|_\infty, \quad f \in C^1[a,b]. \tag{1.19}$$

   Show that $(C^1[a,b], \|\cdot\|_{1,\infty})$ is a Banach space.

6. $(*)$ Prove that $C[0,1]$ is not complete if it is equipped with the norm

$$\|f\|_1 = \int_0^1 |f(x)|dx, \quad f \in C[0,1].$$

   HINT: See Exercise 5 on page 54.

7. Show that a convergent sequence $(x_n)_{n\in\mathbb{N}}$ in a normed space X has a unique limit.

[3] A function $f : [a,b] \to \mathbb{K}$ is *continuously differentiable* if for every $c \in [a,b]$, the derivative of f at c, namely $f'(c)$, exists, and the map $c \mapsto f'(c) : [a,b] \to \mathbb{K}$ is a continuous function.

8. Show that if $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in a normed space X, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

9. Prove that a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in a normed space X is *bounded*, that is, there exists a $M > 0$ such that for all $n \in \mathbb{N}$, $\|x_n\| \le M$.

   In particular, every convergent sequence in a normed space is bounded.

10. Let X be a normed space.

    (a) If a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X has a convergent subsequence, then show that $(x_n)_{n\in\mathbb{N}}$ is convergent.

    (b) $(*)$ If every series in X with the property (1.11) is convergent, then prove that X is complete.

    HINT: Construct a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of a given Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ possessing the property that if $n > n_k$, then $\|x_n - x_{n_k}\| < \frac{1}{2^k}$. Define $u_1 = x_{n_1}$, $u_{k+1} = x_{n_{k+1}} - x_{n_k}$, $k \in \mathbb{N}$, and consider $\displaystyle\sum_{k=1}^{\infty} \|u_k\|$.

11. Let X be a normed space and S be a subset of X. A point $x \in X$ is said to be a *limit point of* S if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $S\setminus\{x\}$ with limit x. The set of all points and limit points of S is denoted by $\overline{S}$. Prove that if Y is a subspace of X, then $\overline{Y}$ is also a subspace of X. This subspace is called the *closure of* Y.

1.4 Appendix: proof of Hölder's inequality

Let $p \in (1 + \infty)$ and q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $a, b \in \mathbb{R}$ and $a, b \ge 0$. We begin by showing that

$$\frac{a}{p} + \frac{b}{q} \ge a^{\frac{1}{p}} b^{\frac{1}{q}}. \tag{1.20}$$

If $a = 0$ or $b = 0$, then the conclusion is clear, and so we assume that both a and b are positive. We will use the following result:

CLAIM: If $\alpha \in (0, 1)$, then for all $x \in [1, \infty)$, $\alpha(x - 1) + 1 \ge x^\alpha$.

Proof Given $\alpha \in (0, 1)$, define $f : [1, \infty) \to \mathbb{R}$ by

$$f_\alpha(x) = \alpha(x - 1) - x^\alpha + 1, \quad x \in [1, \infty).$$

Note that

$$f_\alpha(1) = \alpha \cdot 0 - 1^\alpha + 1 = 0,$$

and for all $x \in [1, \infty)$,

$$f'_\alpha(x) = \alpha - \alpha \cdot x^{\alpha-1} = \alpha\left(1 - \frac{1}{x^{1-\alpha}}\right) \ge 0.$$

Hence using the fundamental theorem of calculus, we have for any $x > 1$,

$$f_\alpha(x) - f_\alpha(1) = \int_0^x f'_\alpha(y)dy \ge 0,$$

and so we obtain $f_\alpha(x) \ge 0$ for all $x \in [1, \infty)$. ■

As $p \in (1, \infty)$, it follows that $\frac{1}{p} \in (0, 1)$. Applying the above with $\alpha = \frac{1}{p}$ and

$$x = \begin{cases} \frac{a}{b} & \text{if } a \geq b \\ \frac{b}{a} & \text{if } a \leq b \end{cases}$$

we obtain inequality (1.20).

Hölder's inequality is obvious if

$$\sum_{i=1}^{n} |x_i|^p = 0 \quad \text{or} \quad \sum_{i=1}^{n} |y_i|^q = 0.$$

So we assume that neither is 0, and proceed as follows. Define

$$a_i = \frac{|x_i|^p}{\displaystyle\sum_{i=1}^{n} |x_i|^p} \quad \text{and } b_i = \frac{|y_i|^q}{\displaystyle\sum_{i=1}^{n} |y_i|^q}, \quad i \in \{1, \dots, n\}.$$

Applying the inequality (1.20) to a_i, b_i, we obtain for each $i \in \{1, \dots, n\}$:

$$\frac{|x_i y_i|}{\left(\displaystyle\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\displaystyle\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}} \leq \frac{|x_i|^p}{p\displaystyle\sum_{i=1}^{n} |x_i|^p} + \frac{|y_i|^p}{q\displaystyle\sum_{i=1}^{n} |y_i|^q}.$$

Adding these n inequalities, we obtain Hölder's inequality.

Chapter 2

Continuous maps

In this chapter, we consider continuous maps from a normed space X to a normed space Y. The spaces X and Y have a notion of distance between vectors (namely the norm of the difference between the two vectors). Hence we can talk about continuity of maps between these normed spaces, just as in the case of ordinary calculus.

Since the normed spaces are also vector spaces, linear maps play an important role. Recall that linear maps are those maps that preserve the vector space operations of addition and scalar multiplication. These are already familiar to the reader from elementary linear algebra, and they are called *linear transformations*.

In the context of normed spaces, it is then natural to focus attention on those linear transformations that are also continuous. These are important from the point of view of applications, and they are called *bounded linear operators*. The reason for this terminology will become clear in Theorem 2.3.3.

The set of all bounded linear operators is itself a vector space, with obvious operations of addition and scalar multiplication, and as we shall see, it also has a natural notion of a norm, called the *operator norm*. Equipped with the operator norm, the vector space of bounded linear operators is a Banach space, provided that the co-domain is a Banach space. This is a useful result, which we will use in order to prove the existence of solutions to integral and differential equations.

2.1 Linear transformations

We recall the definition of linear transformations below. Roughly speaking, linear transformations are maps that respect vector space operations.

Definition. Let X and Y be vector spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A map $T : X \to Y$ is called a *linear transformation* if it satisfies the following:

L1. For all $x_1, x_2 \in X$, $T(x_1 + x_2) = T(x_1) + T(x_2)$.

L2. For all $x \in X$ and all $\alpha \in \mathbb{K}$, $T(\alpha \cdot x) = \alpha \cdot T(x)$.

Examples.

1. Let $m, n \in \mathbb{N}$ and $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. If

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n},$$

then the function $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{k=1}^{n} a_{1k}x_k \\ \vdots \\ \displaystyle\sum_{k=1}^{n} a_{mk}x_k \end{bmatrix} \text{ for all } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \tag{2.1}$$

is a linear transformation from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$. Indeed,

$$T_A \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right) = T_A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + T_A \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \text{ for all } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n,$$

and so L1 holds. Moreover,

$$T_A \left(\alpha \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) = \alpha \cdot T_A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ for all } \alpha \in \mathbb{R} \text{ and all } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n,$$

and so L2 holds as well. Hence T_A is a linear transformation.

2. Let $X = Y = \ell^2$. Consider maps R, L from ℓ^2 to ℓ^2, defined as follows: if $(x_n)_{n\in\mathbb{N}} \in \ell^2$, then

$$R((x_1, x_2, x_3, \dots)) = (x_2, x_3, a_4, \dots) \quad \text{and} \quad L((x_1, x_2, x_3, \dots)) = (0, x_1, x_2, x_3, \dots).$$

Then it is easy to see that R and L are linear transformations.

3. The map $T : C[a, b] \to \mathbb{K}$ given by

$$Tf = f\left(\frac{a+b}{2}\right) \quad \text{for all } f \in C[a, b],$$

is a linear transformation from the vector space $C[a, b]$ to the vector space $\mathbb{K}$. Indeed, we have

$$T(f+g) = (f+g)\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right) = T(f) + T(g), \text{ for all } f, g \in C[a, b],$$

and so L1 holds. Furthermore

$$T(\alpha \cdot f) = (\alpha \cdot f)\left(\frac{a+b}{2}\right) = \alpha f\left(\frac{a+b}{2}\right) = \alpha T(f), \text{ for all } \alpha \in \mathbb{K} \text{ and all } f \in C[a, b],$$

and so L2 holds too. Thus T is a linear transformation.

Similarly, the map $I : C[a,b] \to \mathbb{K}$ given by

$$I(f) = \int_a^b f(x)dx \quad \text{for all } f \in C[a,b],$$

is a linear transformation.

Another example of a linear transformation is the operation of differentiation: let $X = C^1[a,b]$ and $Y = C[a,b]$. Define $D : C^1[a,b] \to C[a,b]$ as follows: if $f \in C^1[a,b]$, then

$$(D(f))(x) = \frac{df}{dx}(x), \quad x \in [a,b].$$

It is easy to check that D is a linear transformation from the space of continuously differentiable functions to the space of continuous functions. $\Diamond$

Exercises.

1. Let $a, b \in \mathbb{R}$, not both zeros, and consider the two real-valued functions f_1, f_2 defined on $\mathbb{R}$ by
$$f_1(x) = e^{ax}\cos(bx) \quad \text{and} \quad f_2(x) = e^{ax}\sin(bx), \quad x \in \mathbb{R}.$$
f_1 and f_2 are vectors belonging to the infinite-dimensional vector space over $\mathbb{R}$ (denoted by $C^1(\mathbb{R},\mathbb{R})$), comprising all continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Denote by $\mathscr{V}$ the span of the two functions f_1 and f_2.
   (a) Prove that f_1 and f_2 are linearly independent in $C^1(\mathbb{R},\mathbb{R})$.
   (b) Show that the differentiation map D, $f \mapsto \frac{df}{dx}$, is a linear transformation from $\mathscr{V}$ to $\mathscr{V}$.
   (c) What is the matrix $[D]_{\mathscr{B}}$ of D with respect to the basis $\mathscr{B} = \{f_1, f_2\}$?
   (d) Prove that D is invertible, and write down the matrix corresponding to the inverse of D.
   (e) Using the result above, compute the indefinite integrals
$$\int e^{ax}\cos(bx)dx \quad \text{and} \quad \int e^{ax}\sin(bx)dx.$$

2. *(Delay line)* Consider a system whose output is a delayed version of the input, that is, if u is the input, then the output y is given by
$$y(t) = u(t - \Delta), \quad t \in \mathbb{R}, \tag{2.2}$$
where Δ (≥ 0) is the delay.
Let $D : C(\mathbb{R}) \to C(\mathbb{R})$ denote the map modelling the system operation (2.2) corresponding to delay Δ:
$$(Df)(t) = f(t - \Delta), \quad t \in \mathbb{R}, \quad f \in C(\mathbb{R}).$$
Show that D is a linear transformation.

3. Consider the squaring map $S : C[a,b] \to C[a,b]$ defined as follows:
$$(S(u))(t) = (u(t))^2, \quad t \in [a,b], \quad u \in C[a,b].$$
Show that S is not a linear transformation.

2.2 Continuous maps

Let X and Y be normed spaces. As there is a notion of distance between pairs of vectors in either space (provided by the norm of the difference of the pair of vectors in each respective space), one can talk about continuity of maps. Within the huge collection of all maps, the class of continuous maps form an important subset. Continuous maps play a prominent role in functional analysis since they possess some useful properties.

Before discussing the case of a function between normed spaces, let us first of all recall the notion of continuity of a function $f : \mathbb{R} \to \mathbb{R}$.

2.2.1 Continuity of functions from $\mathbb{R}$ to $\mathbb{R}$

In everyday speech, a 'continuous' process is one that proceeds without gaps of interruptions or sudden changes. What does it mean for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous? The common informal definition of this concept states that a function f is continuous if one can sketch its graph without lifting the pencil. In other words, the graph of f has no breaks in it. If a break does occur in the graph, then this break will occur at some point. Thus (based on this visual view of continuity), we first give the formal definition of the continuity of a function *at a point* below. Next, if a function is continuous at *each* point, then it will be called continuous.

If a function has a break at a point, say x_0, then even if points x are close to x_0, the points $f(x)$ do not get close to $f(x_0)$. See Figure 2.1.

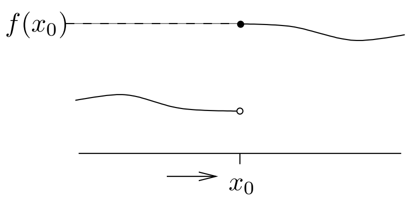

Figure 2.1: A function with a break at x_0. If x lies to the left of x_0, then $f(x)$ is not close to $f(x_0)$, no matter how close x comes to x_0.

This motivates the definition of continuity in calculus, which guarantees that if a function is continuous at a point x_0, then we can make $f(x)$ as close as we like to $f(x_0)$, by choosing x sufficiently close to x_0. See Figure 2.2.

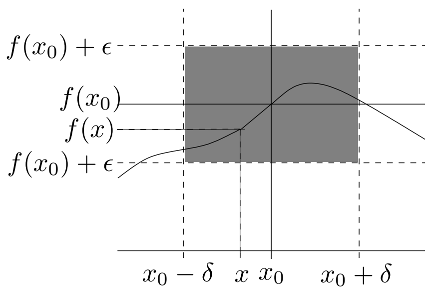

Figure 2.2: The definition of the continuity of a function at point x_0. If the function is continuous at x_0, then given any $\epsilon > 0$ (which determines a strip around the line $y = f(x_0)$ of width 2ϵ), there exists a $\delta > 0$ (which determines an interval of width 2δ around the point x_0) such that whenever x lies in this width (so that x satisfies $|x - x_0| < \delta$) and then $f(x)$ satisfies $|f(x) - f(x_0)| < \epsilon$.

Definitions. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at* x_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ satisfying $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$.

A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if for every $x_0 \in \mathbb{R}$, f is continuous at x_0.

For instance, if $\alpha \in \mathbb{R}$, then the linear map $x \mapsto x$ is continuous. It can be seen that sums and products of continuous functions are also continuous, and so it follows that all polynomial functions belong to the class of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

2.2.2 Continuity of functions between normed spaces

We now define the set of continuous maps from a normed space X to a normed space Y.

We observe that in the definition of continuity in ordinary calculus, if x, y are real numbers, then $|x - y|$ is a measure of the distance between them, and that the absolute value $|\cdot|$ is a norm in the finite (1-)dimensional normed space $\mathbb{R}$.

So it is natural to define continuity in arbitrary normed spaces by simply replacing the absolute values by the corresponding norms, since the norm provides a notion of distance between vectors.

Definitions. Let X and Y be normed spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Let $x_0 \in X$. A map $f : X \to Y$ is said to be *continuous at* x_0 if

$$\forall \epsilon > 0, \quad \exists \delta > 0 \text{ such that } \forall x \in X \text{ satisfying } \|x - x_0\| < \delta, \quad \|f(x) - f(x_0)\| < \epsilon. \tag{2.3}$$

The map $f : X \to Y$ is called *continuous* if for all $x_0 \in X$, f is continuous at x_0.

We will see in the next section that the examples of the linear transformations given in the previous section are all continuous maps, if the vector spaces are equipped with their usual norms. Here we give an example of a *nonlinear* map which is continuous.

Example. Consider the squaring map $S : C[a, b] \to C[a, b]$ defined as follows:

$$(S(u))(t) = (u(t))^2, \quad t \in [a, b], \quad u \in C[a, b]. \tag{2.4}$$

The map is not linear, but it is continuous. Indeed, let $u_0 \in C[a, b]$. Let

$$M = \max\{|u(t)| \mid t \in [a, b]\}$$

(extreme value theorem). Given any $\epsilon > 0$, let

$$\delta = \min\left\{1, \frac{\epsilon}{2M + 1}\right\}.$$

Then for any $u \in C[a, b]$, such that $\|u - u_0\| < \delta$, we have for all $t \in [a, b]$

$$\begin{aligned}
|(u(t))^2 - (u_0(t))^2| &= |u(t) - u_0(t)||u(t) + u_0(t)| \\
&< \delta(|u(t) - u_0(t) + 2u_0(t)|) \\
&\leq \delta(|u(t) - u_0(t)| + 2|u_0(t)|) \\
&\leq \delta(\|u - u_0\| + 2M) \\
&< \delta(\delta + 2M) \\
&\leq \delta(1 + 2M) \leq \epsilon.
\end{aligned}$$

Hence for all $u \in C[a,b]$ satisfying $\|u - u_0\| < \delta$, we have

$$\|S(u) - S(u_0)\| = \sup_{t\in[a,b]} |(u(t))^2 - (u_0(t))^2| \le \epsilon.$$

So S is continuous at u_0. As the choice of $u_0 \in C[a,b]$ was arbitrary, it follows that S is continuous on $C[a,b]$. $\Diamond$

Exercises.

1. Let $(X, \|\cdot\|)$ be a normed space. Show that the norm $\|\cdot\| : X \to \mathbb{R}$ is a continuous map.
2. $(*)$ Let X, Y be normed spaces and suppose that $f : X \to Y$ is a map. Prove that f is continuous at $x_0 \in X$ iff

$$\boxed{\begin{array}{l}\text{for every convergent sequence } (x_n)_{n\in\mathbb{N}} \text{ contained in } X \text{ with limit } x_0,\\ (f(x_n))_{n\in\mathbb{N}} \text{ is convergent and } \lim\limits_{n\to\infty} f(x_n) = f(x_0).\end{array}} \tag{2.5}$$

   In the above claim, can "and $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$" be dropped from (2.5)?
3. $(*)$ This exercise concerns the norm $\|\cdot\|_{1,\infty}$ on $C^1[a,b]$ considered in Exercise 5 on page 12.

   Since we want to be able to use ordinary calculus in the setting when we have a map with domain as a function space, then, given a function $F : C^1[a,b] \to \mathbb{R}$, it is reasonable to choose a norm on $C^1[a,b]$ such that F is continuous.

   (a) It might seem that induced norm on $C^1[a,b]$ from the space $C[a,b]$ (of which $C^1[a,b]$ as a subspace) would be adequate. However, this is not true in some instances. For example, prove that the arc length function $L : C^1[0,1] \to \mathbb{R}$ given by

   $$L(f) = \int_0^1 \sqrt{1 + (f'(x))^2}dx \tag{2.6}$$

   is not continuous if we equip $C^1[0,1]$ with the norm induced from $C[0,1]$.

   HINT: For every curve, we can find another curve arbitrarily close to the first in the sense of the norm of $C[0,1]$, whose length differs from that of the first curve by a factor of 10, say.

   (b) Show that the arc length function L given by (2.6) is continuous if we equip $C^1[0,1]$ with the norm given by (1.19).

2.3 The normed space $\mathscr{L}(X,Y)$

In this section we study those linear transformations from a normed space X to a normed space Y that are also continuous, and we denote this set by $\mathscr{L}(X,Y)$:

$$\mathscr{L}(X,Y) = \{F : X \to Y \mid F \text{ is a linear transformation}\} \bigcap \{F : X \to Y \mid F \text{ is continuous}\}.$$

We begin by giving a characterization of continuous linear transformations.

Theorem 2.3.1 *Let X and Y be normed spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Let $T : X \to Y$ be a linear transformation. Then the following properties of T are equivalent:*

1. *T is continuous.*

2. *T is continuous at* 0.

3. *There exists a number M such that for all $x \in X$, $\|Tx\| \leq M\|x\|$.*

Proof

$1 \Rightarrow 2$. Evident.

$2 \Rightarrow 3$. For every $\epsilon > 0$, for example $\epsilon = 1$, there exists a $\delta > 0$ such that $\|x\| \leq \delta$ implies $\|Tx\| \leq 1$. This yields:

$$\|Tx\| \leq \frac{1}{\delta}\|x\| \quad \text{for all } x \in X. \tag{2.7}$$

This is true if $\|x\| = \delta$. But if (2.7) holds for some x, then owing to the homogeneity of T and of the norm, it also holds for αx, for any arbitrary $\alpha \in \mathbb{K}$. Since every x can be written in the form $x = \alpha y$ with $\|y\| = \delta$ (take $\alpha = \frac{\|x\|}{\delta}$), (2.7) is valid for all x. Thus we have that for all $x \in X$, $\|Tx\| \leq M\|x\|$ with $M = \frac{1}{\delta}$.

$3 \Rightarrow 1$. From linearity, we have: $\|Tx - Ty\| = \|T(x-y)\| \leq M\|x-y\|$ for all $x, y \in X$. The continuity follows immediately. ■

Owing to the characterization of continuous linear transformations by the existence of a bound as in item 3 above, they are called *bounded* linear operators.

Theorem 2.3.2 *Let X and Y be normed spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).*

1. *Let $T : X \to Y$ be a linear operator. Of all the constants M possible in 3 of Theorem 2.3.3, there is a smallest one, and this is given by:*

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|. \tag{2.8}$$

2. *The set $\mathscr{L}(X,Y)$ of bounded linear operators from X to Y with addition and scalar multiplication defined by:*

$$(T+S)x = Tx + Sx, \quad x \in X, \tag{2.9}$$
$$(\alpha T)x = \alpha Tx, \quad x \in X, \quad \alpha \in \mathbb{K}, \tag{2.10}$$

is a vector space. The map $T \mapsto \|T\|$ is a norm on this space.

Proof 1. From item 3 of Theorem 2.3.3, it follows immediately that $\|T\| \leq M$. Conversely we have, by the definition of $\|T\|$, that $\|x\| \leq 1 \;\Rightarrow\; \|Tx\| \leq \|T\|$. Owing to the homogeneity of T and of the norm, it again follows from this that:

$$\|Tx\| \leq \|T\|\|x\| \quad \text{for all } x \in X \tag{2.11}$$

which means that $\|T\|$ is the smallest constant M that can occur in item 3 of Theorem 2.3.3.

2. We already know from linear algebra that the space of *all* linear transformations from a vector space X to a vector space Y, equipped with the operations of addition and scalar multiplication given by (2.9) and (2.10), forms a vector space. We now prove that the subset $\mathscr{L}(X,Y)$ comprising *bounded* linear transformations is a subspace of this vector space, and consequently it is itself a vector space.

We first prove that if T, S are in bounded linear transformations, then so are $T+S$ and αT. It is clear that $T+S$ and αT are linear transformations. Moreover, there holds that

$$\|(T+S)x\| \leq \|Tx\| + \|Sx\| \leq (\|T\| + \|S\|)\|x\|, \quad x \in X, \tag{2.12}$$

from which it follows that $T+S$ is bounded. Also there holds:

$$\|\alpha T\| = \sup_{\|x\|\leq 1} \|\alpha Tx\| = \sup_{\|x\|\leq 1} |\alpha|\|Tx\| = |\alpha| \sup_{\|x\|\leq 1} \|Tx\| = |\alpha|\|T\|. \tag{2.13}$$

Finally, the 0 operator, is bounded and so it belongs to $\mathscr{L}(X, Y)$.

Furthermore, $\mathscr{L}(X, Y)$ is a normed space. Indeed, from (2.12), it follows that $\|T+S\| \leq \|T\| + \|S\|$, and so N3 holds. Also, from (2.13) we see that N2 holds. We have $\|T\| \geq 0$; from (2.11) it follows that if $\|T\| = 0$, then $Tx = 0$ for all $x \in X$, that is, $T = 0$, the operator 0, which is the zero vector of the space $\mathscr{L}(X, Y)$. This shows that N1 holds. ∎

So far we have shown that the space of all continuous linear transformations (which we also call the space of bounded linear operators), $\mathscr{L}(X, Y)$, can be equipped with the operator norm given by (2.8), so that $\mathscr{L}(X, Y)$ becomes a normed space. We will now prove that in fact $\mathscr{L}(X, Y)$ with the operator norm is in fact a Banach space provided that the co-domain Y is a Banach space.

Theorem 2.3.3 *If Y is complete, then $\mathscr{L}(X, Y)$ is also complete.*

Proof Let $(T_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{L}(X, Y)$. Then given a $\epsilon > 0$ there exists a number N such that: $\|T_n - T_m\| \leq \epsilon$ for all $n, m \geq N$, and so, if $x \in X$, then

$$\forall n, m \geq N, \quad \|T_n x - T_m x\| \leq \|T_n - T_m\|\|x\| \leq \epsilon\|x\|. \tag{2.14}$$

This implies that the sequence $(T_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Since Y is complete, $\lim\limits_{n\to\infty} T_n x$ exists:

$$Tx = \lim_{n\to\infty} T_n x.$$

This holds for every point $x \in X$. It is clear that the map $T : x \mapsto T(x)$ is linear. Moreover, the map T is continuous: we see this by observing that a Cauchy sequence in a normed space is bounded. Thus there exists a number M such that $\|T_n\| \leq M$ for all n (take $M = \max(\|T_1\|, \dots, \|T_{N-1}\|, \epsilon + \|T_N\|)$). Since

$$\forall n \in \mathbb{N}, \quad \forall x \in X, \quad \|T_n x\| \leq M\|x\|,$$

by passing the limit, we obtain:

$$\forall x \in X, \quad \|Tx\| \leq M\|x\|,$$

and so T is bounded.

Finally we show that $\lim\limits_{n\to\infty} \|T_n - T\| = 0$. By letting m go to ∞ in (2.14), we see that

$$\forall n \geq N, \quad \forall x \in X, \quad \|T_n x - Tx\| \leq \epsilon\|x\|.$$

This means that for all $n \geq N$, $\|T_n - T\| \leq \epsilon$, which gives the desired result. ∎

Corollary 2.3.4 *Let X be a normed space over $\mathbb{R}$ or $\mathbb{C}$. Then the normed space $\mathscr{L}(X, \mathbb{R})$ ($\mathscr{L}(X, \mathbb{C})$ respectively) is a Banach space.*

Remark. The space $\mathscr{L}(X,\mathbb{R})$ ($\mathscr{L}(X,\mathbb{C})$ respectively) is denoted by X' (sometimes X^*) and is called the *dual space*. Elements of the dual space are called *bounded linear functionals*.

Examples.

1. Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

We equip X and Y with the Euclidean norm. From Hölder's inequality with $p = 2$, it follows that

$$\left(\sum_{j=1}^n a_{ij}x_j\right)^2 \le \left(\sum_{j=1}^n a_{ij}^2\right)\|x\|^2,$$

for each $i \in \{1,\dots m\}$. This yields $\|T_A x\| \le \|A\|_2\|x\|$ where

$$\|A\|_2 = \left(\sum_{i=1}^m\sum_{j=1}^n a_{ij}^2\right)^{\frac{1}{2}}. \tag{2.15}$$

Thus we see that *all* linear transformations in finite dimensional spaces are continuous, and that if X and Y are equipped with the Euclidean norm, then the operator norm is majorized by the Euclidean norm of the matrix:

$$\|A\| \le \|A\|_2.$$

Remark. There does not exist any 'formula' for $\|T_A\|$ in terms of the matrix coefficients except in the special cases $n = 1$ or $m = 1$). The map $A \mapsto \|A\|_2$ given by (2.15) is also a norm on $\mathbb{R}^{m\times n}$, and is called the *Hilbert-Schmidt norm* of A.

2. *Integral operators.* We take $X = Y = C[a,b]$. Let $k : [a,b]\times[a,b] \to \mathbb{K}$ be a *uniformly continuous* function, that is,

$$\begin{array}{c}\forall\epsilon > 0,\ \ \exists\delta > 0 \text{ such that } \forall(x_1,y_1),(x_2,y_2) \in [a,b]\times[a,b] \text{ satisfying} \\ |x_1 - x_2| < \delta \text{ and } |y_1 - y_2| < \delta, \quad |k(x_1,y_1) - k(x_2,y_2)| < \epsilon.\end{array} \tag{2.16}$$

Such a k defines an operator

$$K : C[a,b] \to C[a,b]$$

via the formula

$$(Kf)(x) = \int_a^b k(x,y)f(y)dy. \tag{2.17}$$

We first show that $Kf \in C[a,b]$. Let $x_0 \in X$, and suppose that $\epsilon > 0$. Then choose a $\delta > 0$ such that

$$\text{if } |x_1 - x_2| < \delta \text{ and } |y_1 - y_2| < \delta, \text{ then } |k(x_1,y_1) - k(x_2,y_2)| < \frac{\epsilon}{\|f\|_\infty(b-a)}.$$

Then we obtain

$$\begin{aligned}
|(Kf)(x) - (Kf)(x_0)| &= \left| \int_a^b k(x,y)f(y)dy - \int_a^b k(x_0,y)f(y)dy \right| \\
&= \left| \int_a^b (k(x,y) - k(x_0,y))f(y)dy \right| \\
&\leq \int_a^b |k(x,y) - k(x_0,y)||f(y)|dy \\
&\leq (b-a)\frac{\epsilon}{\|f\|_\infty (b-a)} \|f\|_\infty = \epsilon.
\end{aligned}$$

This proves that Kf is a continuous function, that is, it belongs to $X = C[a,b]$. The map $f \mapsto Kf$ is clearly linear. Moreover,

$$|(Kf)(t)| \leq \int_a^b |k(t,s)|\ |f(s)|ds \leq \int_a^b |k(t,s)|ds\ \|f\|_\infty$$

so that if $\|k\|_\infty$ denotes the supremum[1] of $|k|$ on $[a,b]^2$, we have:

$$\|Kf\|_\infty \leq (b-a)\|k\|_\infty \|f\|_\infty \text{ for all } f \in C[a,b].$$

Thus it follows that K is bounded, and that

$$\|K\| \leq (b-a)\|k\|_\infty.$$

Remark. Note that the formula (2.17) is analogous to the matrix product

$$(Kf)_i = \sum_{j=1}^n k_{ij} f_j.$$

Operators of the type (2.17) are called *integral operators*. It used to be common to call the function k that plays the role of the matrix (k_{ij}), as the 'kernel' of the integral operator. However, this has nothing to do with the null space: $\{f \mid Kf = 0\}$, which is also called the kernel. Many variations of the integral operator are possible. $\Diamond$

Exercises.

1. Let X, Y be normed spaces with $X \neq 0$, and $T \in \mathscr{L}(X,Y)$. Show that

$$\|T\| = \sup\{\|Tx\| \mid x \in X \text{ and } \|x\| = 1\} = \sup\left\{ \frac{\|Tx\|}{\|x\|} \;\middle|\; x \in X \text{ and } x \neq 0 \right\}.$$

So one can think of $\|T\|$ as the maximum possible 'amplification factor' of the norm as a vector x is taken by T to the vector Tx.

2. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a bounded sequence of scalars, and consider the *diagonal operator* $D : \ell^2 \to \ell^2$ defined as follows:

$$D(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots), \quad (x_n)_{n\in\mathbb{N}} \in \ell^2. \tag{2.18}$$

Prove that $D \in \mathscr{L}(\ell^2)$ and that

$$\|D\| = \sup_{n\in\mathbb{N}} |\lambda_n|.$$

[1] This is finite, and can be seen from (2.16), since $[a,b]^2$ can be covered by *finitely* many boxes of width 2δ.

3. An analogue of the diagonal operator in the context of function spaces is the multiplication operator. Let l be a continuous function on $[a, b]$. Define the *multiplication operator* $M : C[a, b] \to C[a, b]$ as follows:

$$(Mf)(x) = l(x)f(x), \quad x \in [a, b], \quad f \in C[a, b].$$

Is M a bounded linear operator?

4. A linear transformation on a vector space X may be continuous with respect to some norm on X, but discontinuous with respect to another norm on X. To illustrate this, let X be the space c_{00} of all sequences with only finitely many nonzero terms. This is a subspace of $\ell^1 \cap \ell^2$. Consider the linear transformation $T : c_{00} \to \mathbb{R}$ given by

$$T((x_n)_{n\in\mathbb{N}}) = x_1 + x_2 + x_3 + \dots, \quad (x_n)_{n\in\mathbb{N}} \in c_{00}.$$

(a) Let c_{00} be equipped with the induced norm from ℓ^1. Prove that T is a bounded linear operator from $(c_{00}, \|\cdot\|_1)$ to $\mathbb{R}$.

(b) Let c_{00} be equipped with the induced norm from ℓ^2. Prove that T is not a bounded linear operator from $(c_{00}, \|\cdot\|_2)$ to $\mathbb{R}$.
Hint: Consider the sequences $\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{m}, 0, \dots\right)$, $m \in \mathbb{N}$.

5. Prove that the *averaging operator* $A : \ell^\infty \to \ell^\infty$, defined by

$$A(x_1, x_2, x_3, \dots) = \left(x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \dots\right), \tag{2.19}$$

is a bounded linear operator. What is the norm of A?

6. $(*)$ A subspace $\mathscr{V}$ of a normed space X is said to be an *invariant subspace* with respect to a linear transformation $T : X \to X$ if $T\mathscr{V} \subset \mathscr{V}$.

Let $A : \ell^\infty \to \ell^\infty$ be the averaging operator given by (2.19). Show that the subspace (of ℓ^∞) c comprising convergent sequences is an invariant subspace of the averaging operator A.

Hint: Prove that if $x \in c$ has limit L, then Ax has limit L as well.

Remark. Invariant subspaces are useful since they are helpful in studying complicated operators by breaking down them into smaller operators acting on invariant subspaces. This is already familiar to the student from the diagonalization procedure in linear algebra, where one decomposes the vector space into eigenspaces, and in these eigenspaces the linear transformation acts trivially. One of the open problems in modern functional analysis is the *invariant subspace problem*:

Does every bounded linear operator on a separable Hilbert space X over $\mathbb{C}$ have a non-trivial invariant subspace?

Hilbert spaces are just special types of Banach spaces, and we will learn about Hilbert spaces in Chapter 4. We will also learn about separability. Non-trivial means that the invariant subspace must be different from 0 or X. In the case of Banach spaces, the answer to the above question is 'no': during the annual meeting of the American Mathematical Society in Toronto in 1976, the young Swedish mathematician Per Enflo announced the existence of a Banach space and a bounded linear operator on it without any non-trivial invariant subspace.

7. $(*)$ *(Dual of $C[a, b]$)* In this exercise we will learn a representation of bounded linear functionals on $C[a, b]$.

A function $w : [a, b] \to \mathbb{R}$ is said to be of *bounded variation on* $[a, b]$ if its *total variation* $\mathrm{Var}(w)$ on $[a, b]$ is finite, where

$$\mathrm{Var}(w) = \sup_{\mathscr{P}} \sum_{j=1}^{n} |w(x_j) - w(x_{j-1})|,$$

the supremum being taken over the set $\mathscr{P}$ of all *partitions*

$$a = x_0 < x_1 < \cdots < x_n = b \tag{2.20}$$

of the interval $[a, b]$; here, $n \in \mathbb{N}$ is arbitrary and so is the choice of the values $x_1, \ldots, x_{n-1}$ in $[a, b]$, which, however, must satisfy (2.20).

Show that the set of all functions of bounded variations on $[a, b]$, with the usual operations forms a vector space, denoted by $\mathrm{BV}[a, b]$.

Define $\|\cdot\| : \mathrm{BV}[a, b] \to [0, +\infty)$ as follows: if $w \in \mathrm{BV}[a, b]$, then

$$\|w\| = |w(a)| + \mathrm{Var}(w). \tag{2.21}$$

Prove that $\|\cdot\|$ given by (2.21) is a norm on $\mathrm{BV}[a, b]$.

We now obtain the concept of a Riemann-Stieltjes integral as follows. Let $f \in C[a, b]$ and $w \in \mathrm{BV}[a, b]$. Let P_n be any partition of $[a, b]$ given by (2.20) and denote by $\Delta(P_n)$ the length of a largest interval $[x_{j-1}, x_j]$, that is,

$$\Delta(P_n) = \max\{x_1 - x_0, \ldots, x_n - x_{n-1}\}.$$

For every partition P_n of $[a, b]$, we consider the sum

$$S(P_n) = \sum_{j=1}^{n} f(x_j)(w(x_j) - w(x_{j-1})).$$

Then the following can be shown:

Fact: There exists a unique number $\mathscr{S}$ with the property that for every $\epsilon > 0$ there is a $\delta > 0$ such that if P_n is a partition satisfying $\Delta(P_n) < \delta$, then $|\mathscr{S} - S(P_n)| < \epsilon$.

$\mathscr{S}$ is called the *Riemann-Stieltjes integral* of f over $[a, b]$ with respect to w, and is denoted by

$$\int_a^b f(x)dw(x).$$

It can be seen that

$$\int_a^b f_1(x) + f_2(x)dw(x) = \int_a^b f_1(x)dw(x) + \int_a^b f_2(x)dw(x) \text{ for all } f_1, f_2 \in C[a, b], \tag{2.22}$$

$$\int_a^b \alpha f(x)dw(x) = \alpha \int_a^b f(x)dw(x) \text{ for all } f \in C[a, b] \text{ and all } \alpha \in \mathbb{K}. \tag{2.23}$$

Prove the following inequality:

$$\left| \int_a^b f(x)dw(x) \right| \le \|f\|_\infty \mathrm{Var}(w),$$

where $f \in C[a, b]$ and $w \in \mathrm{BV}[a, b]$. Conclude that every $w \in \mathrm{BV}[a, b]$ gives rise to a bounded linear functional $T_w \in \mathscr{L}(C[a, b], \mathbb{K})$ as follows:

$$f \mapsto \int_a^b f(x)dw(x),$$

and that $\|T_w\| \le \mathrm{Var}(w)$.

The following converse result was proved by F. Riesz:

Theorem 2.3.5 (Riesz's theorem about functionals on $C[a,b]$) *If $T \in \mathscr{L}(C[a,b],\mathbb{K})$, then there exists a $w \in BV[a,b]$ such that*

$$\forall f \in C[a,b], \quad T(f) = \int_a^b f(x)dw(x),$$

and $\|T\| = \mathrm{Var}(w)$.

In other words, every bounded linear functional on $C[a,b]$ can be represented by a Riemann-Stieltjes integral.

Now consider the bounded linear functional on $C[a,b]$ given by $f \mapsto f(b)$. Find a corresponding $w \in \mathrm{BV}[a,b]$.

8. (a) Consider the subspace c of ℓ^∞ comprising convergent sequences. Prove that the limit map $l : c \to \mathbb{K}$ given by

$$l(x_n)_{n\in\mathbb{N}} = \lim_{n\to\infty} x_n, \quad (x_n)_{n\in\mathbb{N}} \in c, \tag{2.24}$$

is an element in the dual space $\mathscr{L}(c,\mathbb{K})$ of c, when c is equipped with the induced norm from ℓ^∞.

(b) $(*)$ The Hahn-Banach theorem is a deep result in functional analysis, which says the following:

Theorem 2.3.6 (Hahn-Banach) *Let X be a normed space and Y be a subspace of X. If $l \in \mathscr{L}(Y,\mathbb{K})$, then there exists a $L \in \mathscr{L}(X,\mathbb{K})$ such that $L|_Y = l$ and $\|L\| = \|l\|$.*

Thus the theorem says that bounded linear functionals can be extended from subspaces to the whole space, while preserving the norm.

Consider the following set in ℓ^∞:

$$Y = \left\{ (x_n)_{n\in\mathbb{N}} \in \ell^\infty \;\middle|\; \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \text{ exists} \right\}.$$

Show that Y is a subspace of ℓ^∞, and that for all $x \in \ell^\infty$, $x - Sx \in Y$, where $S : \ell^\infty \to \ell^\infty$ denotes the shift operator:

$$S(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots), \quad (x_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

Furthermore, prove that $c \subset Y$.

Consider the limit functional l on c given by (2.24). Prove that there exists an $L \in \mathscr{L}(\ell^\infty,\mathbb{K})$ such that $L|_c = l$ and moreover, $LS = L$.

This gives a generalization of the concept of a limit, and Lx is called a *Banach limit* of a (possibly divergent!) sequence $x \in \ell^\infty$.

Hint: First observe that $L_0 : Y \to \mathbb{K}$ defined by

$$L_0(x_n)_{n\in\mathbb{N}} = \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n}, \quad (x_n)_{n\in\mathbb{N}} \in Y,$$

is an extension of the functional l from c to Y. Now use the Hahn-Banach theorem to extend L_0 from Y to ℓ^∞.

(c) Find the Banach limit of the divergent sequence $((-1)^n)_{n\in\mathbb{N}}$.

2.4 The Banach algebra $\mathscr{L}(X)$. The Neumann series

In this section we study $\mathscr{L}(X,Y)$ when $X = Y$, and X is a Banach space.

Let X, Y, Z be normed spaces over $\mathbb{K}$.

Theorem 2.4.1 *If $B : X \to Y$ and $A : Y \to Z$ are bounded linear operators, then the composition $AB : X \to Z$ is a bounded linear operator, and there holds:*

$$\|AB\| \leq \|A\|\|B\|. \tag{2.25}$$

Proof For all $x \in X$, we have

$$\|ABx\| \leq \|A\|\|Bx\| \leq \|A\|\|B\|\|x\|$$

and so AB is a bounded linear operator, and $\|AB\| \leq \|A\|\|B\|$. ■

We shall use (2.25) mostly in the situations when $X = Y = Z$. The space $\mathscr{L}(X,X)$ is denoted in short by $\mathscr{L}(X)$, and it is an algebra.

Definitions. An *algebra* is a vector space X in which an associative and distributive multiplication is defined, that is,

$$x(yz) = (xy)z, \quad (x+y)z = xz + yz, \quad x(y+z) = xy + xz$$

for $x, y, z \in X$, and which is related to scalar multiplication so that

$$\alpha(xy) = x(\alpha y) = (\alpha x)y \tag{2.26}$$

for $x, y \in X$ and $\alpha \in \mathbb{K}$. An element $e \in X$ is called an *identity element* if

$$\forall x \in X, \quad ex = x = xe.$$

From the previous proposition, we have that if $A, B \in \mathscr{L}(X)$, then $AB \in \mathscr{L}(X)$. We see that $\mathscr{L}(X)$ is an algebra with the product $(A, B) \mapsto AB$. Moreover $\mathscr{L}(X)$ has an identity element, namely the identity operator I.

Definitions. A *normed algebra* is an algebra equipped with a norm that satisfies (2.25). A *Banach algebra* is a normed algebra which is complete.

Thus we have the following theorem:

Theorem 2.4.2 *If X is a Banach space, then $\mathscr{L}(X)$ is a Banach algebra with identity. Moreover, $\|I\| = 1$.*

Remark. *(To be read after a Hilbert spaces are introduced.)* If instead of Banach spaces, we are interested only in Hilbert spaces, then still the notion of a Banach space is indispensable, since $\mathscr{L}(X)$ is a Banach space, but not a Hilbert space in general.

Definition. Let X be a normed space. An element $A \in \mathscr{L}(X)$ is *invertible* if there exists an element $B \in \mathscr{L}(X)$ such that:

$$AB = BA = I.$$

Such an element B is then uniquely defined: Indeed, if $AB' = B'A = I$, then thanks to the associativity, we have:

$$B' = B'I = B'(AB) = (B'A)B = IB = B.$$

The element B, the inverse of A, is denoted by A^{-1}. Thus we have:

$$AA^{-1} = A^{-1}A = I.$$

In particular,

$$AA^{-1}x = A^{-1}Ax = x \quad \text{for all } x \in X$$

so that $A : X \to X$ is bijective[2].

Theorem 2.4.3 *Let X be a Banach space.*

1. *Let $A \in \mathscr{L}(X)$ be a linear operator with $\|A\| < 1$. Then the operator $I - A$ is invertible and*

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^n + \cdots = \sum_{n=0}^{\infty} A^n. \tag{2.27}$$

2. *In particular, $I - A : X \to X$ is bijective: for all $y \in X$, there exists a unique solution $x \in X$ of the equation*

$$x - Ax = y,$$

and moreover, there holds that:

$$\|x\| \leq \frac{1}{1 - \|A\|}\|y\|.$$

The geometric series in (2.27) is called the *Neumann series* after Carl Neumann, who used this in connection with the solution of the Dirichlet problem for a convex domain.

In order to prove Theorem 2.4.3, we we will need the following result.

Lemma 2.4.4 *There holds*

$$\|A^n\| \leq \|A\|^n \text{ for all } n \in \mathbb{N}. \tag{2.28}$$

Proof This follows by using induction on n from (2.25): if (2.28) holds for n, then from (2.25) we have $\|A^{n+1}\| \leq \|A^n\| \, \|A\| \leq \|A\|^n \|A\| = \|A\|^{n+1}$. ■

Proof *(of Theorem 2.4.3.)* Since $\|A\| < 1$, we have

$$\sum_{n=0}^{\infty} \|A^n\| \leq \sum_{n=0}^{\infty} \|A\|^n = \frac{1}{1 - \|A\|} < +\infty,$$

so that the Neumann series converges in the Banach space $\mathscr{L}(X)$ (see Theorem 1.3.1). Let

$$S_n = I + A + \cdots + A^n \text{ and } S = I + A + \cdots + A^n + \cdots = \sum_{n=0}^{\infty} A^n = \lim_{n\to\infty} S_n.$$

From the inequality (2.25), it follows that for a fixed $A \in \mathscr{L}(X)$, the maps $B \mapsto AB$ and $B \mapsto BA$ are continuous from $\mathscr{L}(X)$ to itself. We have: $AS_n = S_nA = A + A^2 + \cdots + A^{n+1} = S_{n+1} - I$. In

[2] In fact if X is a Banach space, then it can be shown that every bijective linear operator is invertible, and this is a consequence of a deep theorem, known as the *open mapping theorem*.

the limit, this yields: $AS = SA = S - I$, and so $(I - A)S = S(I - A) = I$. Thus $I - A$ is invertible and $(I - A)^{-1} = S$. Again, using Theorem 1.3.1, we have $\|S\| \leq \frac{1}{1-\|A\|}$.

The second claim is a direct consequence of the above. ■

Example. Let k be a uniformly continuous function on $[a, b]^2$. Assume that $(b - a)\|k\|_\infty < 1$. Then for every $g \in C[a, b]$, there exists a unique solution $f \in C[a, b]$ of the integral equation:

$$f(x) - \int_a^b k(x, y)f(y)dy = g(x) \quad \text{for all } x \in [a, b]. \tag{2.29}$$

Integral equations of the type (2.29) are called *Fredholm integral equations of the second kind.*

Fredholm integral equations of the first kind, that is: $\displaystyle\int_a^b k(x, y)g(y)dy = g(x)$ for all $x \in [a, b]$ are much more difficult to handle.

What can we say about the solution $f \in C[a, b]$ except that it is a continuous function? In general nothing. Indeed the operator $I - K$ is bijective, and so every $f \in C[a, b]$ is of the form $(I - K)^{-1}g$ for $g = (I - K)f$. ♢

Exercises.

1. Consider the system

$$\left.\begin{array}{rcl} x_1 &=& \frac{1}{2}x_1 + \frac{1}{3}x_2 + 1, \\ x_2 &=& \frac{1}{3}x_1 + \frac{1}{4}x_2 + 2, \end{array}\right\} \tag{2.30}$$

in the unknown variables $(x_1, x_2) \in \mathbb{R}^2$. This system can be written as $(I - K)x = y$, where I denotes the identity matrix,

$$K = \begin{bmatrix} \frac{1}{2} & \frac{-1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) Show that if $\mathbb{R}^2$ is equipped with the norm $\|\cdot\|_2$, then $\|K\| < 1$. Conclude that the system (2.30) has a unique solution (denoted by x in the sequel).

(b) Find out the unique solution x by computing $(I - K)^{-1}$.

n	approximate solution $x_n = (I + K + K^2 + \cdots + K^n)y$	relative error (%) $\frac{\|x-x_n\|_2}{\|x\|_2}$
2	$(3.0278, 3.4306)$	38.63
3	$(3.6574, 3.8669)$	28.24
5	$(4.4541, 4.4190)$	15.09
10	$(5.1776, 4.9204)$	3.15
15	$(5.3286, 5.0250)$	0.66
20	$(5.3601, 5.0469)$	0.14
25	$(5.3667, 5.0514)$	0.03
30	$(5.3681, 5.0524)$	0.01

Table 2.1: Convergence of the Neumann series to the solution $x \approx (5.3684, 5.0526)$.

(c) Write a computer program to compute $x_n = (I + K + K^2 + K^3 + \cdots + K^n)y$ and the relative error $\frac{\|x-x_0\|}{\|x\|_2}$ for various values of n (say, until the relative error is less than

1%). See Table 2.1. We see that the convergence of the Neumann series converges very slowly.

2. (a) Let X be a normed space, and let $(T_n)_{n\in\mathbb{N}}$ be a convergent sequence with limit T in $\mathscr{L}(X)$. If $S \in \mathscr{L}(X)$, then show that $(ST_n)_{n\in\mathbb{N}}$ is convergent in $\mathscr{L}(X)$, with limit ST.

   (b) Let X be a Banach space, and let $A \in \mathscr{L}(X)$ be such that $\|A\| < 1$. Consider the sequence $(P_n)_{n\in\mathbb{N}}$ defined as follows:
$$P_n = (I + A)(I + A^2)(I + A^4)\dots(I + A^{2^n}), \quad n \in \mathbb{N}.$$
      i. Using induction, show that $(I - A)P_n = I - A^{2^{n+1}}$ for all $n \in \mathbb{N}$.
      ii. Prove that $(P_n)_{n\in\mathbb{N}}$ is convergent in $\mathscr{L}(X)$ in the operator norm. What is the limit of $(P_n)_{n\in\mathbb{N}}$?
      HINT: Use $\|A^m\| \le \|A\|^m$ $(m \in \mathbb{N})$. Also use part 2a with $S = (I - A)^{-1}$.

2.5 The exponential of an operator

Let X be a Banach space and $A \in \mathscr{L}(X)$ be a bounded linear operator. In this section, we will study the exponential e^A, where A is an operator.

Theorem 2.5.1 *Let X be a Banach space. If $A \in \mathscr{L}(X)$, then the series*
$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{2.31}$$
converges in $\mathscr{L}(X)$.

Proof That the series (2.31) converges absolutely is an immediate consequence of the inequality:
$$\left\| \frac{1}{n!} A^n \right\| \le \frac{\|A\|^n}{n!},$$
and the fact that the real series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges for all real $x \in \mathbb{R}$. Using Theorem 1.3.1, we obtain the desired result. ∎

The exponential of an operator plays an important role in the theory of differential equations. Let X be a Banach space, for example, $\mathbb{R}^n$, and let $x_0 \in X$. It can be shown that there exists precisely one continuously differentiable function $t \mapsto x(t) \in X$, namely $x(t) = e^{tA}x_0$ such that:
$$\frac{dx}{dt}(t) = Ax(t), \quad t \in \mathbb{R} \tag{2.32}$$
$$x(0) = x_0. \tag{2.33}$$
Briefly: The Cauchy initial value problem (2.32)-(2.33) has a unique solution.

Exercises.

1. A matrix $A \in \mathbb{B}^{n\times n}$ is said to be *nilpotent* if there exists a $n \ge 0$ such that $A^n = 0$. The series for e^A is a finite sum if A is nilpotent. Compute e^A, where
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{2.34}$$

2. Let $D \in \mathbb{C}^{n\times n}$ be a *diagonal* matrix
$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$
for some $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Find e^D.

3. If P is an invertible $n\times n$ matrix, then show that for any $n\times n$ matrix Q, $e^{PQP^{-1}} = Pe^Q P^{-1}$.

   A matrix $A \in \mathbb{C}^{n\times n}$ is said to be *diagonalizable* if there exists a matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

   Diagonalize
$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$
   and show that
$$e^A = e^a \begin{bmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{bmatrix}.$$

4. It can be shown that if X is a Banach space and $A, B \in \mathscr{L}(X)$ commute (that is, $AB = BA$), then $e^{A+B} = e^A e^B$.
   (a) Let X be a Banach space. Prove that for all $A \in \mathscr{L}(X)$, e^A is invertible.
   (b) Let X be a Banach space and $A \in \mathscr{L}(X)$. Show that if $s, t \in \mathbb{R}$, then $e^{(s+t)A} = e^{sA}e^{tA}$.
   (c) Give an example of 2×2 matrices A and B such that $e^{A+B} \neq e^A e^B$.
   Hint: Take for instance A given by (2.34) and $B = -A^\top$.

2.6 Left and right inverses

We have already remarked that the product in $\mathscr{L}(X)$ is not commutative, that is, in general, $AB \neq BA$ (except when $X = \mathbb{K}$ and so $\mathscr{L}(X) = \mathbb{K}$). This can be already seen in the case of operators in $\mathbb{R}^2$ or $\mathbb{C}^2$. For example a rotation followed by a reflection is, in general, not the same as this reflection followed by the same rotation.

Definitions. Let X be a normed space and suppose that $A \in \mathscr{L}(X)$. If there exists a $B \in \mathscr{L}(X)$ such that $AB = I$, then one says that B is a *right inverse* of A. If there exists a $C \in \mathscr{L}(X)$ such that $CA = I$, then C is called the *left inverse* of A.

If A has both a right inverse (say B) and a left inverse (say C), then we have
$$B = IB = (CA)B = C(AB) = CI = C.$$
Thus if the operator A has a left and a right inverse, then they must be equal, and A is then invertible.

If X is finite dimensional, then one can show that $A \in \mathscr{L}(X)$ is invertible iff A has a left (or right) inverse. For example, $AB = I$ shows that A is surjective, which implies that A is bijective, and hence invertible.

However, in an infinite dimensional space X, this is no longer the case in general. There exist injective operators in X that are not surjective, and there exist surjective operators that are not injective.

Example. Consider the right shift and left shift operators $R : \ell^2 \to \ell^2$, $L : \ell^2 \to \ell^2$, respectively, given by

$$R(x_1, x_2, \dots) = (0, x_1, x_2, \dots) \quad \text{and} \quad L(x_1, x_2, \dots) = (x_2, x_3, x_4, \dots), \quad (x_n)_{n\in\mathbb{N}} \in \ell^2.$$

Then R is not surjective, and L is not injective, but there holds: $LR = I$. Hence we see that L has a right inverse, but it is not bijective, and a fortiori not invertible, and that R has a left inverse, but is not bijective and a fortiori not invertible. $\Diamond$

Remark. The term 'invertible' is not always used in the same manner. Sometimes the operator $A \in \mathscr{L}(X)$ is called invertible if it is injective. The inverse is then an operator which is defined on the image of A. However in these notes, invertible always means that $A \in \mathscr{L}(X)$ has an inverse in the algebra $\mathscr{L}(X)$.

Exercises.

1. Verify that R and L are bounded linear operators on ℓ^2. (R is in fact an *isometry*, that is, it satisfies $\|Rx\| = \|x\|$ for all $x \in \ell^2$).

2. The *trace* of a square matrix

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \in \mathbb{C}^{n\times n}$$

   is the sum of its diagonal entries:

$$\operatorname{tr}(A) = a_{11} + \dots + a_{nn}.$$

   Show that $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$ and that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

   Prove that there cannot exist matrices $A, B \in \mathbb{C}^{n\times n}$ such that $AB - BA = I$, where I denotes the $n \times n$ identity matrix.

   Let $C^\infty(\mathbb{R}, \mathbb{R})$ denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N}$, $f^{(n)}$ exists and is continuous. It is easy to see that this forms a subspace of the vector space $C(\mathbb{R}, \mathbb{R})$ with the usual operations, and it is called the *space of infinitely differentiable functions.*

   Consider the operators $A, B : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ given as follows: if $f \in C^\infty(\mathbb{R}, \mathbb{R})$, then

$$(Af)(x) = \frac{df}{dx}(x) \quad \text{and} \quad (Bf)(x) = xf(x), \quad x \in \mathbb{R}.$$

   Show that $AB - BA = I$, where I denotes the identity operator on $C^\infty(\mathbb{R}, \mathbb{R})$.

3. Let X be a normed space, and suppose that $A, B \in \mathscr{L}(X)$. Show that if $I + AB$ is invertible, then $I + BA$ is also invertible, with inverse $I - B(I + AB)^{-1}A$.

4. Consider the diagonal operator considered in Exercise 2 on page 24. Under what condition on the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is D invertible?

Chapter 3

Differentiation

In the last chapter we studied continuity of operators from a normed space X to a normed space Y. In this chapter, we will study differentiation: we will define the (Frechet) derivative of a map $F : X \to Y$ at a point $x_0 \in X$. Roughly speaking, the derivative of a nonlinear map at a point is a local approximation by means of a continuous linear transformation. Thus the derivative at a point will be a bounded linear operator. This theme of local approximation is the basis of several computational methods in nonlinear functional analysis.

As an application of the notion of differentiation, we will indicate the use of the derivative in solving optimization problems in normed spaces. We outline this below by reviewing the situation in the case of functions from $\mathbb{R}$ to $\mathbb{R}$.

Consider the quadratic function $f(x) = ax^2 + bx + c$. Suppose that one wants to know the points x_0 at which f assumes a maximum or a minimum. We know that if f has a maximum or a minimum at the point x_0, then the derivative of the function must be zero at that point: $f'(x_0) = 0$. See Figure 3.1.

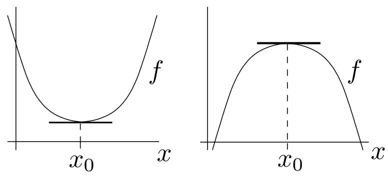

Figure 3.1: Necessary condition for x_0 to be an extremal point for f is that $f'(x_0) = 0$.

So one can then one can proceed as follows. First find the expression for the derivative: $f'(x) = 2ax + b$. Next solve for the unknown x_0 in the equation $f'(x_0) = 0$, that is,

$$2ax_0 + b = 0 \tag{3.1}$$

and so we find that a candidate for the point x_0 which minimizes or maximizes f is $x_0 = -\frac{b}{2a}$, which is obtained by solving the algebraic equation (3.1) above.

We wish to do the above with maps living on function spaces, such as $C[a, b]$, and taking values in $\mathbb{R}$. In order to do this we need a notion of derivative of a map from a function space to $\mathbb{R}$, and an analogue of the fact above concerning the necessity of the vanishing derivative at extremal points. We define the derivative of a map $I : X \to Y$ (where X, Y are normed spaces) in §3.1, and in the case that $Y = \mathbb{R}$, we prove Theorem 3.2.1, which says that this derivative must vanish at local maximum/minimum of the map I.

In the last section of the chapter, we apply Theorem 3.2.1 to the concrete case where X comprises continuously differentiable functions, and I is the map

$$I(x) = \int_0^T (P - ax(t) - bx'(t))x'(t)dt. \tag{3.2}$$

Setting the derivative of such a functional to zero, a necessary condition for an extremal curve can be obtained. Instead of an *algebraic* equation (3.1), a *differential* equation (3.12) is obtained. The solution x_0 of this differential equation is the candidate which maximizes/minimizes the function I.

3.1 The derivative

Recall that for a function $f : \mathbb{R} \to \mathbb{R}$, the derivative at a point x_0 is the approximation of f around x_0 by a straight line. See Figure 3.2.

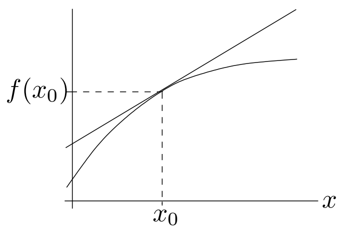

Figure 3.2: The derivative of f at x_0.

In other words, the derivative $f'(x_0)$ gives the slope of the line which is tangent to the function f at the point x_0:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

In other words,

$$\lim_{x \to x_0} \left| \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} \right| = 0,$$

that is,

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in \mathbb{R} \backslash \{x_0\} \text{ satisfying } |x - x_0| < \delta,\ \frac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|} < \epsilon.$$

Observe that every real number α gives rise to a linear transformation from $\mathbb{R}$ to $\mathbb{R}$: the operator in question is simply multiplication by α, that is the map $x \mapsto \alpha x$. We can therefore think of $(f'(x_0))(x - x_0)$ as the action of the linear transformation $L : \mathbb{R} \to \mathbb{R}$ on the vector $x - x_0$, where L is given by

$$L(h) = f'(x_0)h, \quad h \in \mathbb{R}.$$

Hence the derivative $f'(x_0)$ is simply a linear map from $\mathbb{R}$ to $\mathbb{R}$. In the same manner, in the definition of a derivative of a map $F : X \to Y$ between normed spaces X and Y, the derivative of F at a point x_0 will be defined to be a linear transformation from X to Y.

A linear map $L : \mathbb{R} \to \mathbb{R}$ is automatically continuous[1] But this is not true in general if $\mathbb{R}$ is replaced by infinite dimensional normed linear spaces! And we would expect that the derivative

[1]Indeed, every linear map $L : \mathbb{R} \to \mathbb{R}$ is simply given by multiplication, since $L(x) = L(x \cdot 1) = xL(1)$. Consequently $|L(x) - L(y)| = |L(1)||x - y|$, and so L is continuous!

(being the approximation of the map at a point) to have the same property as the function itself at that point. Of course, a differentiable function should first of all be continuous (so that this situation matches with the case of functions from $\mathbb{R}$ to $\mathbb{R}$ from ordinary calculus), and so we expect the derivative to be a continuous linear transformation, that is, it should be a bounded linear operator. So while generalizing the notion of the derivative from ordinary calculus to the case of a map $F : X \to Y$ between normed spaces X and Y, we now specify *continuity* of the derivative as well. Thus, in the definition of a derivative of a map F, the derivative of F at a point x_0 will be defined to be a *bounded* linear transformation from X to Y, that is, an element of $\mathscr{L}(X, Y)$.

This motivates the following definition.

Definition. Let X, Y be normed spaces. If $F : X \to Y$ be a map and $x_0 \in X$, then F is said to be *differentiable at* x_0 if there exists a bounded linear operator $L \in \mathscr{L}(X, Y)$ such that

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in X \setminus \{x_0\} \text{ satisfying } \|x - x_0\| < \delta,\ \frac{\|F(x) - F(x_0) - L(x - x_0)\|}{\|x - x_0\|} < \epsilon. \tag{3.3}$$

The operator L is called a *derivative of* F *at* x_0. If F is differentiable at every point $x \in X$, then it is simply said to be *differentiable.*

We now prove that if F is differentiable at x_0, then its derivative is unique.

Theorem 3.1.1 *Let X, Y be normed spaces. If $F : X \to Y$ is differentiable at $x_0 \in X$, then the derivative of F at x_0 is unique.*

Proof Suppose that $L_1, L_2 \in \mathscr{L}(X, Y)$ are derivatives of F at x_0. Given $\epsilon > 0$, choose a δ such that (3.3) holds with L_1 and L_2 instead of L. Consequently

$$\forall x \in X \setminus \{x_0\} \text{ satisfying } \|x - x_0\| < \delta,\ \frac{\|L_2(x - x_0) - L_1(x - x_0)\|}{\|x - x_0\|} < 2\epsilon. \tag{3.4}$$

Given any $h \in X$ such that $h \neq 0$, define

$$x = x_0 + \frac{\delta}{2\|h\|} h.$$

Then $\|x - x_0\| = \frac{\delta}{2} < \delta$ and so (3.4) yields

$$\|(L_2 - L_1)h\| \leq 2\epsilon \|h\|. \tag{3.5}$$

Hence $\|L_2 - L_1\| \leq 2\epsilon$, and since the choice of $\epsilon > 0$ was arbitrary, we obtain $\|L_2 - L_1\| = 0$. So $L_2 = L_1$, and this completes the proof. ■

Notation. We denote the derivative of F at x_0 by $DF(x_0)$.

Examples.

1. Consider the nonlinear squaring map S from the example on page 19. We had seen that S is continuous. We now show that $S : C[a, b] \to C[a, b]$ is in fact differentiable. We note that

$$(Su - Su_0)(t) = u(t)^2 - u_0(t)^2 = \underbrace{(u(t) + u_0(t))}(u(t) - u_0(t)). \tag{3.6}$$

As u approaches u_0 in $C[a,b]$, the term $u(t)+u_0(t)$ above approaches $2u_0(t)$. So from (3.6), we suspect that $(DS)(u_0)$ would be the multiplication map M by $2u_0$:

$$(Mu)(t) = 2u_0(t)u(t), \quad t \in [a,b].$$

Let us prove this. Let $\epsilon > 0$. We have

$$\begin{aligned} |(Su - Su_0 - M(u-u_0))(t)| &= |u(t)^2 - u_0(t)^2 - 2u_0(t)(u(t)-u_0(t))| \\ &= |u(t)^2 + u_0(t)^2 - 2u_0(t)u(t)| \\ &= |u(t) - u_0(t)|^2 \\ &\leq \|u - u_0\|^2. \end{aligned}$$

Hence if $\delta := \epsilon > 0$, then for all $u \in C[a,b]$ satisfying $\|u - u_0\| < \delta$, we have

$$\|Su - Su_0 - M(u-u_0)\| \leq \|u-u_0\|^2,$$

and so for all $u \in C[a,b] \setminus \{u_0\}$ satisfying $\|u-u_0\| < \delta$, we obtain

$$\frac{\|Su - Su_0 - M(u-u_0)\|}{\|u-u_0\|} \leq \|u-u_0\| < \delta = \epsilon.$$

Thus $DS(u_0) = M$.

2. Let X, Y be normed spaces and let $T \in \mathscr{L}(X,Y)$. Is T differentiable, and if so, what is its derivative?

   Recall that the derivative at a point is the linear transformation that approximates the map at that point. If the map is itself linear, then we expect the derivative to equal the given linear map! We claim that $(DT)(x_0) = T$, and we prove this below.

   Given $\epsilon > 0$, choose any $\delta > 0$. Then for all $x \in X$ satisfying $\|x - x_0\| < \delta$, we have

   $$\|Tx - Tx_0 - T(x-x_0)\| = \|Tx - Tx_0 - Tx + Tx_0\| = 0 < \epsilon.$$

   Consequently $(DT)(x_0) = T$.

   In particular, if $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and $T = T_A$, where $A \in \mathbb{R}^{m\times n}$, then $(DT_A)(x_0) = T_A$. $\Diamond$

Exercises.

1. Let X, Y be normed spaces. Prove that if $F : X \to Y$ is differentiable at x_0, then F is continuous at x_0.

2. Consider the functional $I : C[a,b] \to \mathbb{R}$ given by

   $$I(x) = \int_a^b x(t)dt.$$

   Prove that I is differentiable, and find its derivative at $x_0 \in C[a,b]$.

3. $(*)$ Prove that the square of a differentiable functional $I : X \to \mathbb{R}$ is differentiable, and find an expression for its derivative at $x \in X$.

   Hint: $(I(x))^2 - (I(x_0))^2 = (I(x) + I(x_0))(I(x) - I(x_0)) \approx 2I(x_0)DI(x_0)(x - x_0)$ if $x \approx x_0$.

4. (a) Given x_1, x_2 in a normed space X, define

   $$\varphi(t) = tx_1 + (1-t)x_2.$$

   Prove that if $I : X \to \mathbb{R}$ is differentiable, then $I \circ \varphi : [0,1] \to \mathbb{R}$ is differentiable and

   $$\frac{d}{dt}(I\circ\varphi)(t) = [DI(\varphi(t))](x_1 - x_2).$$

   (b) Prove that if $I_1, I_2 : X \to \mathbb{R}$ are differentiable and their derivatives are equal at every $x \in X$, then I_1 and I_2 differ by a constant.

3.2 Optimization: necessity of vanishing derivative

In this section we take the normed space $Y = \mathbb{R}$, and consider maps maps $I : X \to \mathbb{R}$. We wish to find points $x_0 \in X$ that maximize/minimize I.

In elementary analysis, a necessary condition for a differentiable function $f : \mathbb{R} \to \mathbb{R}$ to have a local extremum (local maximum or local minimum) at $x_0 \in \mathbb{R}$ is that $f'(x_0) = 0$. We will prove a similar necessary condition for a differentiable function $I : X \to \mathbb{R}$.

First we specify what exactly we mean by a 'local maximum/minimum' (collectively termed 'local extremum'). Roughly speaking, a point $x_0 \in X$ is a local maximum/minimum for I if for all points x in some neighbourhood of that point, the values $I(x)$ are all less (respectively greater) than $I(x_0)$.

Definition. Let X be a normed space. A function $I : X \to \mathbb{R}$ is said to have a *local extremum* at x_0 ($\in X$) if there exists a $\delta > 0$ such that

$$\forall x \in X \setminus \{x_0\} \text{ satisfying } \|x - x_0\| < \delta, \quad I(x) \geq I(x_0) \quad \text{(local minimum)}$$
$$\text{or}$$
$$\forall x \in X \text{ satisfying } \|x - x_0\| < \delta, \quad I(x) \leq I(x_0) \quad \text{(local maximum).}$$

Theorem 3.2.1 *Let X be a normed space, and let $I : X \to \mathbb{R}$ be a function that is differentiable at $x_0 \in X$. If I has a local extremum at x_0, then $(DI)(x_0) = 0$.*

Proof We prove the statement in the case that I has a local minimum at x_0. (If instead I has a local maximum at x_0, then the function $-I$ has a local minimum at x_0, and so $(DI)(x_0) = -(D(-I))(x_0) = 0$.)

For notational simplicity, we denote $(DI)(x_0)$ by L. Suppose that $Lh \neq 0$ for some $h \in X$. Let $\epsilon > 0$ be given. Choose a δ such that for all $x \in X$ satisfying $\|x - x_0\| < \delta$, $I(x) \geq I(x_0)$, and moreover if $x \neq x_0$, then

$$\frac{|I(x) - I(x_0) - L(x - x_0)|}{\|x - x_0\|} < \epsilon.$$

Define the sequence

$$x_n = x_0 - \frac{1}{n}\frac{Lh}{|Lh|}h, \quad n \in \mathbb{N}.$$

We note that $\|x_n - x_0\| = \frac{\|h\|}{n}$, and so with N chosen large enough, we have $\|x_n - x_0\| < \delta$ for all $n > N$. It follows that for all $n > N$,

$$0 \leq \frac{I(x_n) - I(x_0)}{\|x_n - x_0\|} < \frac{L(x_n - x_0)}{\|x_n - x_0\|} + \epsilon = -\frac{|Lh|}{\|h\|} + \epsilon.$$

Since the choice of $\epsilon > 0$ was arbitrary, we obtain $|Lh| \leq 0$, and so $Lh = 0$, a contradiction. ■

Remark. Note that this is a *necessary* condition for the existence of a local extremum. Thus a the vanishing of a derivative at some point x_0 doesn't imply local extremality of x_0! This is analogous to the case of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$, for which $f'(0) = 0$, although f clearly does not have a local minimum or maximum at 0. In the next section we study an important class of functions $I : X \to \mathbb{R}$, called *convex* functions, for which a vanishing derivative implies the function has a global minimum at that point!

3.3 Optimization: sufficiency in the convex case

In this section, we will show that if $I : X \to \mathbb{R}$ is a convex function, then a vanishing derivative is enough to conclude that the function has a global minimum at that point. We begin by giving the definition of a convex function.

Definition. Let X be a normed space. A function $F : X \to \mathbb{R}$ is *convex* if for all $x_1, x_2 \in X$ and all $\alpha \in [0, 1]$,

$$F(\alpha x_1 + (1-\alpha)x_2) \le \alpha F(x_1) + (1-\alpha)F(x_2). \tag{3.7}$$

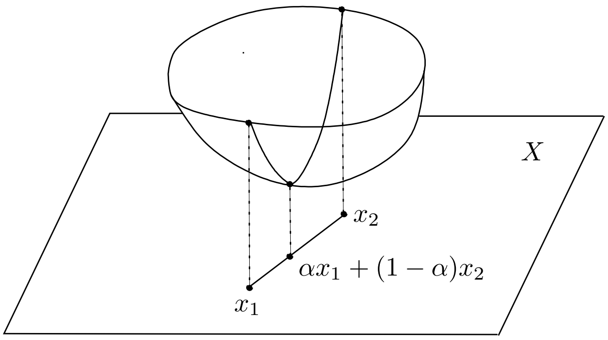

Figure 3.3: Convex function.

Examples.

1. If $X = \mathbb{R}$, then the function $f(x) = x^2$, $x \in \mathbb{R}$, is convex. This is visually obvious from Figure 3.4, since we see that the point B lies above the point A:

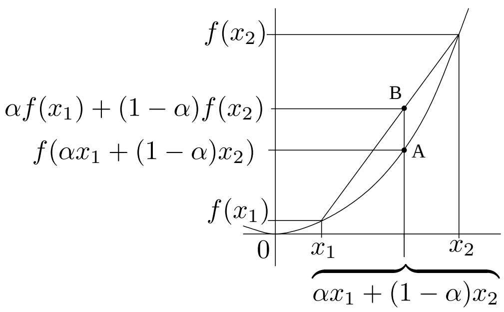

Figure 3.4: The convex function $x \mapsto x^2$.

But one can prove this as follows: for all $x_1, x_2 \in \mathbb{R}$ and all $\alpha \in [0, 1]$, we have

$$\begin{aligned} f(\alpha x_1 + (1-\alpha)x_2) &= (\alpha x_1 + (1-\alpha)x_2)^2 = \alpha^2 x_1^2 + 2\alpha(1-\alpha)x_1x_2 + (1-\alpha)^2 x_2^2 \\ &= \alpha x_1^2 + (1-\alpha)x_2^2 + (\alpha^2-\alpha)x_1^2 + (\alpha^2-\alpha)x_2^2 + 2\alpha(1-\alpha)x_1x_2 \\ &= \alpha x_1^2 + (1-\alpha)x_2^2 - \alpha(1-\alpha)(x_1^2 + x_2^2 - 2x_1x_2) \\ &= \alpha x_1^2 + (1-\alpha)x_2^2 - \alpha(1-\alpha)(x_1 - x_2)^2 \\ &\le \alpha x_1^2 + (1-\alpha)x_2^2 = \alpha f(x_1) + (1-\alpha)f(x_2). \end{aligned}$$

A slick way of proving convexity of smooth functions from $\mathbb{R}$ to $\mathbb{R}$ is to check if f'' is nonnegative; see Exercise 1 below.

2. Consider $I : C[0,1] \to \mathbb{R}$ given by

$$I(f) = \int_0^1 (f(x))^2 dx, \quad f \in C[0,1].$$

Then I is convex, since for all $f_1, f_2 \in C[0,1]$ and all $\alpha \in [0,1]$, we see that

$$\begin{aligned} I(\alpha f_1 + (1-\alpha) f_2) &= \int_0^1 (\alpha f_1(x) + (1-\alpha) f_2(x))^2 dx \\ &\leq \int_0^1 \alpha (f_1(x))^2 + (1-\alpha)(f_2(x))^2 dx \text{ (using the convexity of } y \mapsto y^2) \\ &= \alpha \int_0^1 (f_1(x))^2 dx + (1-\alpha) \int_0^1 (f_2(x))^2 dx \\ &= \alpha I(f_1) + (1-\alpha) I(f_2). \end{aligned}$$

Thus I is convex. ◊

In order to prove the theorem on the sufficiency of the vanishing derivative in the case of a convex function, we will need the following result, which says that if a differentiable function f is convex, then its derivative f' is an increasing function, that is, if $x \leq y$, then $f'(x) \leq f'(y)$. (In Exercise 1 below, we will also prove a converse.)

Lemma 3.3.1 *If $f : \mathbb{R} \to \mathbb{R}$ is convex and differentiable, then f' is an increasing function.*

Proof Let $x < u < y$. If $\alpha = \frac{u-x}{y-1}$, then $\alpha \in (0,1)$, and $1 - \alpha = \frac{y-u}{y-x}$. From the convexity of f, we obtain

$$\frac{u-x}{y-x} f(y) + \frac{y-u}{y-x} f(x) \geq f\left(\frac{u-x}{y-x} y + \frac{y-u}{y-x} x\right) = f(u)$$

that is,

$$(y-x) f(u) \leq (u-x) f(y) + (y-u) f(x). \tag{3.8}$$

From (3.8), we obtain $(y-x)f(u) \leq (u-x)f(y) + (y-x+x-u)f(x)$, that is,

$$(y-x)f(u) - (y-x)f(x) \leq (u-x)f(y) - (u-x)f(x),$$

and so

$$\frac{f(u) - f(x)}{u - x} \leq \frac{f(y) - f(x)}{y - x}. \tag{3.9}$$

From (3.8), we also have $(y-x)f(u) \leq (u-y+y-x)f(y) + (y-u)f(x)$, that is,

$$(y-x)f(u) - (y-x)f(y) \leq (u-y)f(y) - (u-y)f(x),$$

and so

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(y) - f(u)}{y - u}. \tag{3.10}$$

Combining (3.9) and (3.10),

$$\frac{f(u) - f(x)}{u - x} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(y) - f(u)}{y - u}.$$

Passing the limit as $u \searrow x$ and $u \nearrow y$, we obtain $f'(x) \leq \dfrac{f(y) - f(x)}{y - x} \leq f'(y)$, and so f' is increasing. ■

We are now ready to prove the result on the existence of global minima. First of all, we mention that if I is a function from a normed space X to $\mathbb{R}$, then I is said to have a *global minimum* at the point $x_0 \in X$ if for all $x \in X$, $I(x) \geq I(x_0)$. Similarly if $I(x) \leq I(x_0)$ for all x, then I is said to have a *global maximum* at x_0. We also note that the problem of finding a maximizer for a map I can always be converted to a minimization problem by considering $-I$ instead of I. We now prove the following.

Theorem 3.3.2 *Let X be a normed space and $I : X \to \mathbb{R}$ be differentiable. Suppose that I is convex. If $x_0 \in X$ is such that $(DI)(x_0) = 0$, then I has a global minimum at x_0.*

Proof Suppose that $x_1 \in X$ and $I(x_1) < I(x_0)$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(\alpha) = I(\alpha x_1 + (1 - \alpha)x_0), \quad \alpha \in \mathbb{R}.$$

The function f is convex, since if $r \in [0, 1]$ and $\alpha, \beta \in \mathbb{R}$, then we have

$$\begin{aligned} f(r\alpha + (1-r)\beta) &= I((r\alpha + (1-r)\beta)x_1 + (1 - r\alpha - (1-r)\beta)x_0) \\ &= I(r(\alpha x_1 + (1-\alpha)x_0) + (1-r)(\beta x_1 + (1-\beta)x_0)) \\ &\leq rI(\alpha x_1 + (1-\alpha)x_0) + (1-r)I(\beta x_1 + (1-\beta)x_0) \\ &= rf(\alpha) + (1-r)f(\beta). \end{aligned}$$

From Exercise 4a on page 38, it follows that f is differentiable on $[0, 1]$, and

$$f'(0) = ((DI)(x_0))(x_1 - x_0) = 0.$$

Since $f(1) = I(x_1) < I(x_0) = f(0)$, by the mean value theorem[2], there exists a $c \in (0, 1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} < 0 = f'(0).$$

This contradicts the convexity of f (see Lemma 3.3.1 above), and so $I(x_1) \geq I(x_0)$. Hence I has a global minimum at x_0. ■

Exercises.

1. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and $f''(x) > 0$ for all $x \in \mathbb{R}$, then f is convex.
2. Let X be a normed space, and $f \in \mathscr{L}(X, \mathbb{R})$. Show that f is convex.
3. If X is a normed space, then prove that the norm function, $x \mapsto \|x\| : X \to \mathbb{R}$, is a convex.
4. Let X be a normed space, and let $f : X \to \mathbb{R}$ be a function. Define the *epigraph* of f by
$$U(f) = \bigcup_{x \in X} \{x\} \times (f(x), +\infty) \subset X \times \mathbb{R}.$$
This is the 'region above the graph of f'. Show that if f is convex, then $U(f)$ is a convex subset of $X \times \mathbb{R}$. (See Exercise 6 on page 5 for the definition of a convex set).
5. $(*)$ Show that if $f : \mathbb{R} \to \mathbb{R}$ is convex, then for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in \mathbb{R}$, there holds that
$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

[2]The mean value theorem says that if $f : [a, b] \to \mathbb{R}$ is a continuous function that is differentiable in (a, b), then there exists a $c \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

3.4 An example of optimization in a function space

Example. A copper mining company intends to remove all of the copper ore from a region that contains an estimated Q tons, over a time period of T years. As it is extracted, they will sell it for processing at a net price per ton of

$$p(x(t), x'(t)) = P - ax(t) - bx'(t)$$

for positive constants P, a, and b, where $x(t)$ denotes the total tonnage sold by time t. (This pricing model allows the cost of mining to increase with the extent of the mined region and speed of production.)

If the company wishes to maximize its total profit given by

$$I(x) = \int_0^T p(x(t), x'(t))x'(t)dt, \tag{3.11}$$

where $x(0) = 0$ and $x(T) = Q$, how might it proceed?

Step 1. First of all we note that the set of curves in $C^1[0, T]$ satisfying $x(a) = 0$ and $x(T) = Q$ do not form a linear space! So Theorem 3.2.1 is not applicable directly. Hence we introduce a new linear space X, and consider a new function $\tilde{I} : X \to \mathbb{R}$ which is defined in terms of the old function I.

Introduce the linear space

$$X = \{x \in C^1[0, T] \mid x(0) = x(T) = 0\},$$

with the $C^1[0, T]$-norm:

$$\|x\| = \sup_{t\in[0,T]} |x(t)| + \sup_{t\in[0,T]} |x'(t)|.$$

Then for all $h \in X$, $x_0 + h$ satisfies $(x_0 + h)(0) = 0$ and $(x_0 + h)(T) = Q$. Defining $\tilde{I}(h) = I(x_0 + h)$, we note that $\tilde{I} : X \to \mathbb{R}$ has an extremum at 0. It follows from Theorem 3.2.1 that $(D\tilde{I})(0) = 0$. Note that by the 0 in the right hand side of the equality, we mean the zero functional, namely the continuous linear map from X to $\mathbb{R}$, which is defined by $h \mapsto 0$ for all $h \in X$.

Step 2. We now calculate $\tilde{I}'(0)$. We have

$$\begin{aligned}
\tilde{I}(h) - \tilde{I}(0) &= \int_0^T P - a(x_0(t) + h(t)) - b(x_0'(t) + h'(t))dt - \int_0^T P - ax_0(t) - bx_0(t)dt \\
&= \int_0^T P - ax_0(t) - 2bx_0'(t))h'(t) - ax_0'(t)h(t)dt + \int_0^T -ah(t)h'(t) - bh'(t)h'(t)dt.
\end{aligned}$$

Since the map

$$h \mapsto \int_0^T (P - ax_0(t) - 2bx_0'(t))h'(t) - ax_0'(t)h(t)dt$$

is a functional from X to $\mathbb{R}$ and since

$$\left| \int_0^T -ah(t)h'(t) - bh'(t)h'(t)dt \right| \leq T(a + b)\|h\|^2,$$

it follows that

$$[(D\tilde{I})(0)](h) = \int_0^T (P - ax_0(t) - 2bx_0'(t))h'(t) - ax_0'(t)h(t)dt = \int_0^T (P - 2bx_0'(t))h'(t)dt,$$

where the last equality follows using partial integration:

$$\int_0^T ax_0'(t)h(t)dt = -\int_0^T ax_0(t)h'(t)dt + ax_0(t)h(t)|_{t=0}^T = -\int_0^T ax_0(t)h'(t)dt.$$

STEP 3. Since $(D\tilde{I})(0) = 0$, it follows that

$$\int_0^T \left(P - ax_0(t) - 2bx_0'(t) - a\int_0^t x_0'(\tau)d\tau\right) h'(t)dt = 0$$

for all $h \in C^1[0,T]$ with $h(0) = h(T) = 0$. We now prove the following.

CLAIM: If $k \in C[a,b]$ and

$$\int_a^b k(t)h'(t)dt = 0$$

for all $h \in C^1[a,b]$ with $h(a) = h(b) = 0$, then there exists a constant c such that $k(t) = c$ for all $t \in [a,b]$.

Proof Define the constant c and the function h via

$$\int_a^b (k(t) - c)dt = 0 \quad \text{and} \quad h(t) = \int_a^t (k(\tau) - c)d\tau.$$

Then $h \in C^1[a,b]$ and it satisfies $h(a) = h(b) = 0$. Furthermore,

$$\int_a^b (k(t) - c)^2 dt = \int_a^b (k(t) - c)h'(t)dt = \int_a^b k(t)h'(t)dt - c(h(b) - h(a)) = 0.$$

Thus $k(t) - c = 0$ for all $t \in [a,b]$. ■

STEP 4. The above result implies in our case that

$$\forall t \in [0,T], \quad P - 2bx_0'(t) = c. \tag{3.12}$$

Integrating, we obtain $x_0(t) = At + B$, $t \in [0,T]$, for some constants A and B. Using $x_0(0) = 0$ and $x_0(T) = Q$, we obtain

$$x_0(t) = \frac{t}{T}Q, \quad t \in [0,T]. \tag{3.13}$$

STEP 5. Finally we show that this is the optimal mining operation, that is $I(x_0) \geq I(x)$ for all x such that $x(0) = 0$ and $x(T) = Q$. We prove this by showing $-\tilde{I}$ is convex, and so by Theorem 3.3.2, $-\tilde{I}$ in fact has a global minimum at 0.

Let $h_1, h_2 \in X$, and $\alpha \in [0,1]$, and define $x_1 = x_0 + h_1$, $x_2 = x_0 + h_2$. Then we have

$$\int_0^T (\alpha x_1'(t) + (1-\alpha)x_2'(t))^2 dt \leq \int_0^T \alpha(x_1'(t))^2 + (1-\alpha)(x_2'(t))^2 dt, \tag{3.14}$$

using the convexity of $y \mapsto y^2$. Furthermore, $x_1(0) = 0 = x_2(0)$ and $x_1(T) = Q = x_2(T)$, and so

$$\begin{aligned}
& \int_0^T (\alpha x_1'(t) + (1-\alpha)x_2'(t))(\alpha x_1(t) + (1-\alpha)x_2(t))dt \\
= & \frac{1}{2}\int_0^T \frac{d}{dt}(\alpha x_1(t) + (1-\alpha)x_2(t))^2 dt \\
= & \frac{1}{2}Q^2 = \alpha\frac{1}{2}Q^2 + (1-\alpha)\frac{1}{2}Q^2 \\
= & \alpha\int_0^T x_1'(t)x_1(t)dt + (1-\alpha)\int_0^T x_2'(t)x_2(t)dt.
\end{aligned}$$

Hence

$$\begin{aligned}
-\tilde{I}(\alpha h_1 + (1-\alpha)h_2) &= -I(x_0 + \alpha h_1 + (1-\alpha)h_2) \\
&= -I(\alpha x_0 + (1-\alpha)x_0 + \alpha h_1 + (1-\alpha)h_2) \\
&= -I(\alpha x_1 + (1-\alpha)x_2) \\
&= b\int_0^T (\alpha x_1'(t) + (1-\alpha)x_2'(t))^2 dt \\
&\quad +a\int_0^T (\alpha x_1'(t) + (1-\alpha)x_2'(t))(\alpha x_1(t) + (1-\alpha)x_2(t))dt \\
&\quad -P\int_0^T (\alpha x_1'(t) + (1-\alpha)x_2'(t))dt \\
&\leq \alpha\int_0^T (x_1'(t))^2 dt + (1-\alpha)\int_0^T (x_2'(t))^2 dt \\
&\quad +\alpha\int_0^T x_1'(t)x_1(t)dt + (1-\alpha)\int_0^T x_2'(t)x_2(t)dt \\
&\quad -\alpha P\int_0^T x_1'(t)dt - (1-\alpha)P\int_0^T x_2'(t)dt \\
&= \alpha\left(\int_0^T x_1'(t)(bx_1'(t) + ax_1(t) - P)dt\right) \\
&\quad +(1-\alpha)\left(\int_0^T x_2'(t)(bx_2'(t) + ax_2(t) - P)dt\right) \\
&= \alpha(-I(x_1)) + (1-\alpha)(-I(x_2)) = \alpha(-\tilde{I}(h_1)) + (1-\alpha)(-\tilde{I}(h_2)).
\end{aligned}$$

Hence $-\tilde{I}$ is convex. $\Diamond$

Remark. Such optimization problems arising from applications fall under the subject of 'calculus of variations', and we will not delve into this vast area, but we mention the following result. Let I be a function of the form

$$I(x) = \int_a^b F\left(x(t), \frac{dx}{dt}(t), t\right) dt,$$

where $F(\alpha, \beta, \gamma)$ is a 'nice' function and $x \in C^1[a,b]$ is such that $x(a) = y_a$ and $x(b) = y_b$. Then proceeding in a similar manner as above, it can be shown that if I has an extremum at x_0, then x_0 satisfies the *Euler-Lagrange equation*:

$$\frac{\partial F}{\partial \alpha}\left(x_0(t), \frac{dx_0}{dt}(t), t\right) - \frac{d}{dt}\left(\frac{\partial F}{\partial \beta}\left(x_0(t), \frac{dx_0}{dt}(t), t\right)\right) = 0, \quad t \in [a,b]. \tag{3.15}$$

(This equation is abbreviated by $F_x - \frac{d}{dt}F_{x'} = 0$.)

Chapter 4

Geometry of inner product spaces

In a vector space we can add vectors and multiply vectors by scalars. In a normed space, the vector space is also equipped with a norm, so that we can measure the distance between vectors. The plane $\mathbb{R}^2$ or the space $\mathbb{R}^3$ are examples of normed spaces.

However, in the familiar geometry of the plane or of space, we can also measure the angle between lines, provided by the notion of 'dot' product of two vectors. We wish to generalize this notion to abstract spaces, so that we can talk about perpendicularity or orthogonality of vectors.

Why would one wish to have this notion? One of the reasons for hoping to have such a notion is that we can then have a notion of orthogonal projections and talk about best approximations in normed spaces. We will elaborate on this in §4.3. We will begin with a discussion of inner product spaces.

4.1 Inner product spaces

Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ is called an *inner product* on the vector space X over $\mathbb{K}$ if:

IP1 *(Positive definiteness)* For all $x \in X$, $\langle x, x \rangle \geq 0$. If $x \in X$ and $\langle x, x \rangle = 0$, then $x = 0$.

IP2 *(Linearity in the first variable)* For all $x_1, x_2, y \in X$, $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$.
For all $x, y \in X$ and all $\alpha \in \mathbb{K}$, $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

IP3 *(Conjugate symmetry)* For all $x, y \in X$, $\langle x, y \rangle = \langle y, x \rangle^*$, where $\cdot^*$ denotes complex conjugation[1].

An *inner product space* is a vector space equipped with an inner product.

It then follows that the inner product is also antilinear with respect to the second variable, that is additive, and such that

$$\langle x, \alpha y \rangle = \alpha^* \langle x, y \rangle.$$

It also follows that in the case of complex scalars, $\langle x, x \rangle \in \mathbb{R}$ for all $x \in X$, so that IP1 has meaning.

[1] If $z = x + yi \in \mathbb{C}$ with $x, y \in \mathbb{R}$, then $z^* = x - yi$.

Examples.

1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathbb{K}^n$ is an inner product space with the inner product

$$\langle x, y \rangle = x_1 y_1^* + \cdots + x_n y_n^*, \quad \text{where } x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

2. The vector space ℓ^2 of square summable sequences is an inner product space with

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n^* \quad \text{for all } x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in \ell^2.$$

It is easy to see that if

$$\sum_{n=1}^{\infty} |x_n|^2 < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} |y_n|^2 < +\infty,$$

then the series $\sum_{n=1}^{\infty} x_n y_n^*$ converges absolutely, and so it converges. Indeed, this is a consequence of the following elementary inequality:

$$|x_n y_n^*| = |x_n||y_n| \leq \frac{|x_n|^2 + |y_n|^2}{2}.$$

3. The space of continuous $\mathbb{K}$-valued functions on $[a, b]$ can be equipped with the inner product

$$\langle f, g \rangle = \int_a^b f(x) g(x)^* dx, \tag{4.1}$$

We use this inner product whenever we refer to $C[a, b]$ as an inner product space in these notes. $\Diamond$

We now prove a few 'geometric' properties of inner product spaces.

Theorem 4.1.1 (Cauchy-Schwarz inequality) *If X is an inner product space, then*

$$\text{for all } x, y \in X, \quad |\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle. \tag{4.2}$$

There is equality in (4.2) iff x and y are linearly dependent.

Proof From IP3, we have $\langle x, y \rangle + \langle y, x \rangle = 2\,\mathrm{Re}(\langle x, y \rangle)$, and so

$$\langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + 2\mathrm{Re}(\langle x, y \rangle). \tag{4.3}$$

Using IP1 and (4.3), we see that $0 \leq \langle x + \alpha y, x + \alpha y \rangle = \langle x, x \rangle + 2\mathrm{Re}(\alpha^* \langle x, y \rangle) + |\alpha|^2 \langle y, y \rangle$. Let $\alpha = re^{i\Theta}$, where Θ is such that $\langle x, y \rangle = |\langle x, y \rangle| e^{i\Theta}$, and $r \in \mathbb{R}$. Then we obtain:

$$\langle x, x \rangle + 2r|\langle x, y \rangle| + \langle y, y \rangle r^2 \geq 0 \quad \text{for all } r \in \mathbb{R}$$

and so it follows that the discriminant of this quadratic expression is ≤ 0, which gives (4.2).

Cauchy-Schwarz

Finally we show that equality in (4.2) holds iff x and y are linearly independent, by using the fact that the inner product is positive definite. Indeed, if the discriminant is zero, the equation $\langle x, x\rangle + 2r|\langle x, y\rangle| + \langle y, y\rangle r^2 = 0$ has a root $r \in \mathbb{R}$, and so there exists a number $\alpha = re^{i\Theta}$ such that $\langle x + \alpha y, x + \alpha y\rangle = 0$, from which, using IP1, it follows that $x + \alpha y = 0$. ■

We now give an application of the Cauchy-Schwarz inequality.

Example. Let $C[0, T]$ be equipped with the usual inner product. Let F be the filter mapping $C[0, T]$ into itself, given by

$$(Fu)(t) = \int_0^t e^{-(t-\tau)} u(\tau) d\tau, \quad t \in [0, T], \quad u \in C[0, T].$$

Such a mapping arises quite naturally, for example, in electrical engineering, this is the map from the input voltage u to the output voltage y for the simple RC-circuit shown in Figure 4.1. Suppose we want to choose an input $u \in X$ such that $\|u\|_2 = 1$ and $(Fu)(T)$ is maximum.

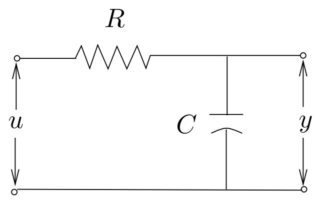

Figure 4.1: A low-pass filter.

Define $h \in C[0, T]$ by $h(t) = e^t$, $t \in [0, T]$. Then if $u \in X$, $(Fu)(T) = e^{-T}\langle h, u\rangle$. So from the Cauchy-Schwarz inequality

$$|(Fu)(T)| \le e^{-T}\|h\|\|u\|$$

with the equality being taken when $u = \alpha h$, where α is a constant. In particular, the solution of our problem is

$$u(t) = \sqrt{\frac{2}{e^{2T} - 1}} e^t, \quad t \in [0, T].$$

Furthermore,

$$(Fu)(T) = e^{-T}\sqrt{\frac{e^{2T} - 1}{2}} = \sqrt{\frac{1 - e^{-2T}}{2}}.$$

◊

If $\langle \cdot, \cdot \rangle$ is an inner product, then we define

$$\|x\| = \sqrt{\langle x, x \rangle}. \tag{4.4}$$

The function $x \mapsto \|x\|$ is then a norm on the inner product space X. Thanks to IP2, indeed we have: $\|\alpha x\| = |\alpha| \|x\|$, and using Cauchy-Schwarz inequality together with (4.3), we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(\langle x, y \rangle) \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2$$

so that the triangle inequality is also valid. Since the inner product is positive definite, the norm is also positive definite. For example, the inner product space $C[a, b]$ in Example 3 on page 48 gives rise to the norm

$$\|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{\frac{1}{2}}, \quad f \in C[a, b],$$

called the L^2-*norm*.

Note that the inner product is determined by the corresponding norm: we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(\langle x, y \rangle) \tag{4.5}$$
$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\mathrm{Re}(\langle x, y \rangle) \tag{4.6}$$

so that $\mathrm{Re}(\langle x, y \rangle) = \frac{1}{4} \left(\|x+y\|^2 - \|x-y\|^2 \right)$. In the case of real scalars, we have

$$\langle x, y \rangle = \frac{1}{4} \left(\|x+y\|^2 - \|x-y\|^2 \right). \tag{4.7}$$

In the complex case, $\mathrm{Im}(\langle x, y \rangle) = \mathrm{Re}(-i\langle x, y \rangle) = \mathrm{Re}(\langle x, iy \rangle)$, so that

$$\langle x, y \rangle = \frac{1}{4} \left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \right) = \frac{1}{4} \sum_{k=1}^{4} i^k \|x + i^k y\|^2. \tag{4.8}$$

(4.7) (respectively (4.8)) is called the *polarization formula*.

If we add the expressions in (4.5) and (4.6), we get the *parallelogram law*:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in X. \tag{4.9}$$

(The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the other two sides; see Figure 4.2.)

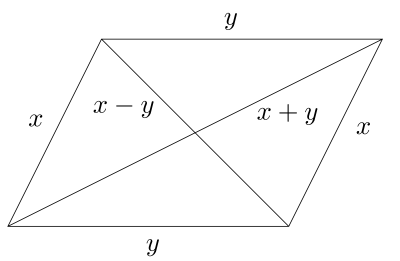

Figure 4.2: The parallelogram law.

If x and y are orthogonal, that is, $\langle x, y \rangle = 0$, then from (4.5) we obtain Pythagoras' theorem:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \quad \text{for all } x, y \text{ with } x \perp y. \tag{4.10}$$

If the norm is defined by an inner product via (4.4), then we say that the norm is *induced by an inner product*. This inner product is then uniquely determined by the norm via the formula (4.7) (respectively (4.8)).

Definition. A *Hilbert space* is a Banach space in which the norm is induced by an inner product.

The hierarchy of spaces considered in these notes is depicted in Figure 4.3.

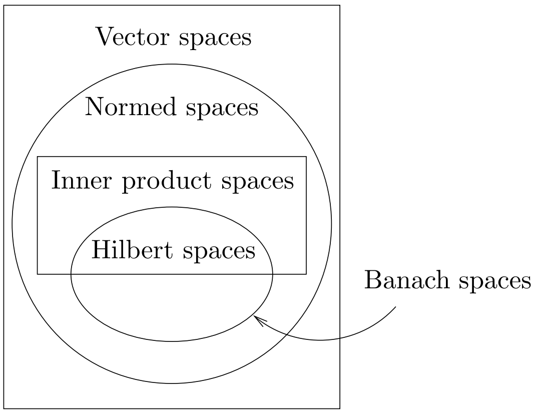

Figure 4.3: Hierarchy of spaces.

Exercises.

1. If $A, B \in \mathbb{R}^{m\times n}$, then define $\langle A, B\rangle = \operatorname{tr}(A^\top B)$, where $A^\top$ denotes the transpose of the matrix A. Prove that $\langle \cdot, \cdot\rangle$ defines an inner product on the space of $m \times n$ real matrices. Show that the norm induced by this inner product on $\mathbb{R}^{m\times n}$ is the Hilbert-Schmidt norm (see the remark in Example 1 on page 23).

2. Prove that given an ellipse and a circle having equal areas, the perimeter of the ellipse is larger.

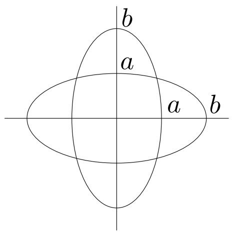

Figure 4.4: Congruent ellipses.

Hint: If the ellipse has major and minor axis lengths as $2a$ and $2b$, respectively, then observe that the perimeter is given by

$$P = \int_0^{2\pi} \sqrt{(a\cos\Theta)^2 + (b\sin\Theta)^2}d\Theta = \int_0^{2\pi} \sqrt{(a\sin\Theta)^2 + (b\cos\Theta)^2}d\Theta,$$

where the last expression is obtained by rotating the ellipse through 90°, obtaining a new ellipse with the same perimeter; see Figure 4.4. Now use Cauchy-Schwarz inequality to prove that P^2 is at least as large as the square of the circumference of the corresponding circle with the same area as that of the ellipse.

3. Let X be an inner product space, and let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be convergent sequences in X with limits x, y, respectively. Show that $(\langle x_n, y_n\rangle)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{K}$ and that
$$\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle. \tag{4.11}$$

4. $(*)$ *(Completion of inner product spaces)* If an inner product space $(X, \langle\cdot,\cdot\rangle_X)$ is not complete, then this means that there are some 'holes' in it, as there are Cauchy sequences that are not convergent–roughly speaking, the 'limits that they are supposed to converge to', do not belong to the space X. One can remedy this situation by filling in these holes, thereby enlarging the space to a larger inner product space $(\overline{X}, \langle\cdot,\cdot\rangle_{\overline{X}})$ in such a manner that:

C1 X can be identified with a subspace of $\overline{X}$ and for all x, y in X, $\langle x, y\rangle_X = \langle x, y\rangle_{\overline{X}}$.

C2 $\overline{X}$ is complete.

Given an inner product space $(X, \langle\cdot,\cdot\rangle_X)$, we now give a construction of an inner product space $(\overline{X}, \langle\cdot,\cdot\rangle_{\overline{X}})$, called the *completion* of X, that has the properties C1 and C2.

Let $\mathscr{C}$ be the set of all Cauchy sequences in X. If $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are in $\mathscr{C}$, then define the relation[2] R on $\mathscr{C}$ as follows:
$$((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) \in R \quad \text{if} \quad \lim_{n\to\infty} \|x_n - y_n\|_X = 0.$$

Prove that R is an equivalence relation on $\mathscr{C}$.

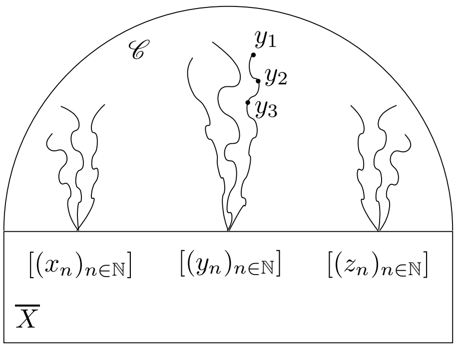

Figure 4.5: The space $\overline{X}$.

Let $\overline{X}$ be the set of equivalence classes of $\mathscr{C}$ under the equivalence relation R. Suppose that the equivalence class of $(x_n)_{n\in\mathbb{N}}$ is denoted by $[(x_n)_{n\in\mathbb{N}}]$. See Figure 4.5. Define vector addition $+ : \overline{X}\times\overline{X} \to \overline{X}$ and scalar multiplication $\cdot : \mathbb{K} \to \overline{X}$ by
$$[(x_n)_{n\in\mathbb{N}}] + [(y_n)_{n\in\mathbb{N}}] = [(x_n + y_n)_{n\in\mathbb{N}}] \quad \text{and} \quad \alpha\cdot[(x_n)_{n\in\mathbb{N}}] = [(\alpha x_n)_{n\in\mathbb{N}}].$$

[2]Recall that a *relation* on a set S is a simply a subset of the cartesian product $S\times S$. A relation R on a set S is called an *equivalence relation* if

ER1 *(Reflexivity)* For all $x \in S$, $(x, x) \in R$.

ER2 *(Symmetry)* If $(x, y) \in R$, then $(y, x) \in R$.

ER3 *(Transitivity)* If $(x, y), (y, z) \in R$, then $(x, z) \in R$.

If $x \in S$, then the *equivalence class of* x, denoted by $[x]$, is defined to be the set $\{y \in S \mid (x, y) \in R\}$. It is easy to see that $[x] = [y]$ iff $(x, y) \in R$. Thus equivalence classes are either equal or disjoint. They partition the set S, that is the set can be written as a disjoint union of these equivalence classes.

Show that these operations are well-defined. It can be verified that $\overline{X}$ is a vector space with these operations.

Define $\langle \cdot, \cdot \rangle_{\overline{X}} : \overline{X} \times \overline{X} \to \mathbb{K}$ by

$$\langle [(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}] \rangle_{\overline{X}} = \lim_{n\to\infty} \langle x_n, y_n \rangle_X .$$

Prove that this operation is well-defined, and that it defines an inner-product on $\overline{X}$.

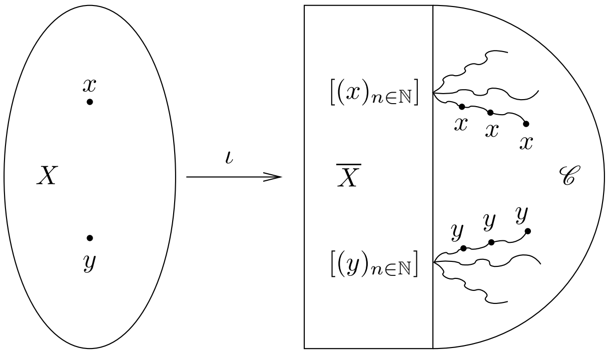

Figure 4.6: The map ι.

Define the map $\iota : X \to \overline{X}$ as follows:

$$\text{if } x \in X, \text{ then } \iota(x) = [(x)_{n\in\mathbb{N}}],$$

that is, ι takes x to the equivalence class of the (constant) Cauchy sequence $(x, x, x, \dots)$. See Figure 4.6. Show that ι is an injective bounded linear transformation (so that X can be identified with a subspace of $\overline{X}$), and that for all x, y in X, $\langle x, y \rangle_X = \langle \iota(x), \iota(y) \rangle_{\overline{X}}$.

We now show that $\overline{X}$ is a Hilbert space. Let $([(x_1^k)_{n\in\mathbb{N}}])_{k\in\mathbb{N}}$ be a Cauchy sequence in $\overline{X}$. For each $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ such that for all $n, m \geq n_k$,

$$\|x_n^k - x_m^k\|_X < \frac{1}{k}.$$

Define the sequence $(y_k)_{k\in\mathbb{N}}$ by $y_k = x_{n_k}^k$, $k \in \mathbb{N}$. See Figure 4.7. We claim that $(y_k)_{k\in\mathbb{N}}$ belongs to $\mathscr{C}$.

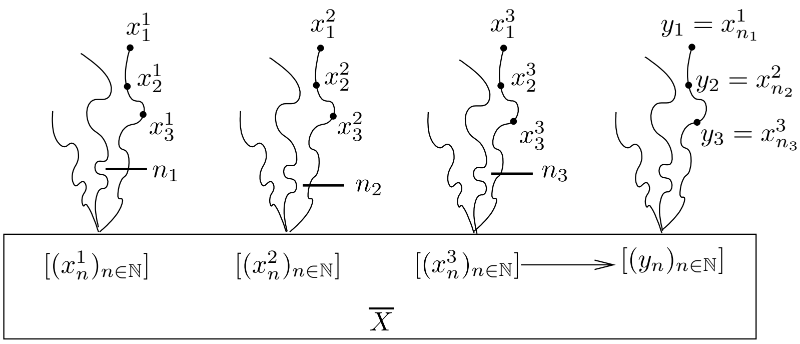

Figure 4.7: Completeness of $\overline{X}$.

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Let $K_1 \in \mathbb{N}$ be such that for all $k, l > K_1$,

$$\|[(x_n^k)_{n\in\mathbb{N}}] - [(x_n^l)_{n\in\mathbb{N}}]\|_{\overline{X}} < \epsilon,$$

that is,

$$\lim_{n\to\infty} \|x_n^k - x_n^l\|_X < \epsilon.$$

Define $K = \max\{N, K_1\}$. Let $k, l > K$. Then for all $n > \max\{n_k, n_l\}$,

$$\begin{aligned}
\|y_k - y_l\|_X &= \|y_k - x_n^k + x_n^k - x_n^l + x_n^l - y_l\|_X \\
&\leq \|y_k - x_n^k\|_X + \| + x_n^k - x_n^l\|_X + \|x_n^l - y_l\|_X \\
&\leq \frac{1}{k} + \| + x_n^k - x_n^l\|_X + \frac{1}{l} \\
&\leq \frac{1}{K} + \| + x_n^k - x_n^l\|_X + \frac{1}{K} \\
&< \epsilon + \|x_n^k - x_n^l\|_X + \epsilon.
\end{aligned}$$

So

$$\|y_k - y_l\|_X \leq \epsilon + \lim_{n\to\infty} \|x_n^k - x_n^l\|_X + \epsilon < \epsilon + \epsilon + \epsilon = 3\epsilon.$$

This shows that $(y_n)_{n\in\mathbb{N}} \in \mathscr{C}$, and so $[(y_n)_{n\in\mathbb{N}}] \in \overline{X}$. We will prove that $([(x_n^k)_{n\in\mathbb{N}}])_{k\in\mathbb{N}}$ converges to $[(y_n)_{n\in\mathbb{N}}] \in \overline{X}$.

Given $\epsilon > 0$, choose K_1 such that $\frac{1}{K_1} < \epsilon$. As $(y_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, there exists a $K_2 \in \mathbb{N}$ such that for all $k, l > K_2$, $\|y_k - y_l\|_X < \epsilon$. define $K = \max\{K_1, K_2\}$. Then for all $k > K$ and all $m > \max\{n_k, K\}$, we have

$$\begin{aligned}
\|x_m^k - y_m\|_X &\leq \|x_m^k - x_{n_k}^k\|_X + \|x_{n_k}^k - y_m\|_X \\
&< \frac{1}{k} + \|y_k - y_m\|_X \\
&< \frac{1}{K} + \epsilon \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

Hence

$$\|[(x_m^k)_{m\in\mathbb{N}}] - [(y_m)_{m\in\mathbb{N}}]\|_{\overline{X}} = \lim_{m\to\infty} \|x_m^k - y_m\|_X \leq 2\epsilon.$$

This completes the proof.

5. $(*)$ *(Incompleteness of $C[a,b]$ and $L^2[a,b]$)* Prove that $C[0,1]$ is not a Hilbert space with the inner product defined in Example 3 on page 48.

   HINT: The functions f_n in Figure 4.8 form a Cauchy sequence since for all $x \in [0,1]$ we have $|f_n(x) - f_m(x)| \leq 2$, and so

$$\|f_n - f_m\|^2 = \int_{\frac{1}{2}}^{\frac{1}{2}+\max\left\{\frac{1}{n},\frac{1}{m}\right\}} |f_n(x) - f_m(x)|^2 dx \leq 4\max\left\{\frac{1}{n}, \frac{1}{m}\right\}.$$

   But the sequence does not converge in $C[0,1]$. For otherwise, if the limit is $f \in C[0,1]$, then for any $n \in \mathbb{N}$, we have

$$\|f_n - f\|^2 = \int_0^{\frac{1}{2}} |f(x)|^2 dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |f_n(x) - f(x)|^2 dx + \int_{\frac{1}{2}+\frac{1}{n}}^1 |1 - f(x)|^2 dx.$$

   Show that this implies that $f(x) = 0$ for all $x \in [0, \frac{1}{2}]$, and $f(x) = 1$ for all $x \in (\frac{1}{2}, 1]$. Consequently,

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

   which is clearly discontinuous at $\frac{1}{2}$.

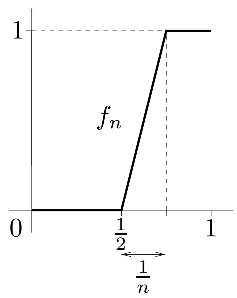

Figure 4.8: Graph of f_n.

The inner product space in Example 3 on page 48 is *not* complete, as demonstrated in above. However, it can be completed by the process discussed in Exercise 4 on 52. The completion is denoted by $L^2[a, b]$, which is a Hilbert space. One would like to express this new inner product also as an integral, and it this can be done by extending the ordinary Riemann integral for elements of $C[a, b]$ to the more general *Lebesgue* integral. For continuous functions, the Lebesgue integral is the same as the Riemann integral, that is, it gives the same value. However, the class of Lebesgue integrable functions is much larger than the class of continuous functions. For instance it can be shown that the function

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \\ 1 & \text{if } x \in [0, 1] \cap \mathbb{Q} \end{cases}$$

is Lebesgue integrable, but not Riemann integrable on $[0, 1]$. For computation aspects, one can get away without having to go into technical details about Lebesgue measure and integration.

However, before we proceed, we also make a remark about related natural Hilbert spaces arising from Probability Theory. The space of random variables X on a probability space $(\Omega, \mathscr{F}, P)$ for which $\mathbb{E}(X^2) < +\infty$ (here $\mathbb{E}(\cdot)$ denotes expectation), is a Hilbert space with the inner product $\langle X, Y \rangle = \mathbb{E}(XY)$.

6. Let X be an inner product space. Prove that

$$\text{for all } x, y, z \in X, \quad \|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\left\|z - \frac{1}{2}(x + y)\right\|^2.$$

(This is called the *Appollonius identity.*) Give a geometric interpretation when $X = \mathbb{R}^2$.

7. Let X be an inner product space over $\mathbb{C}$, and let $T \in \mathscr{L}(X)$ be such that for all $x \in X$, $\langle Tx, x \rangle = 0$. Prove that $T = 0$.

   HINT: Consider $\langle T(x + y), x + y \rangle$, and also $\langle T(x + iy), x + iy \rangle$. Finally take $y = Tx$.

4.2 Orthogonal sets

Two vectors in $\mathbb{R}^2$ are perpendicular if their dot product is 0. Since an inner product on a vector space is a generalization of the notion of dot product, we can talk about perpendicularity (henceforth called *orthogonality*[3]) in the general setting of inner product spaces.

Definitions. Let X be an inner product space. Vectors $x, y \in X$ are said to be *orthogonal* if $\langle x, y \rangle = 0$. A subset S of X is said to be *orthonormal* if for all $x, y \in S$ with $x \neq y$, $\langle x, y \rangle = 0$ and for all $x \in S$, $\langle x, x \rangle = 1$.

[3]The prefix 'ortho' means straight or erect.

Examples.

1. Let e_n denote the sequence with nth term equal to 1 and all others equal to 0. The set $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal set in ℓ^2.
2. Let $n \in \mathbb{Z}$, and let $T_n \in C[0,1]$ denote the *trigonometric polynomial* defined as follows:
$$T_n(x) = e^{2\pi i n x}, \quad x \in [0,1].$$
The set $\{T_n \mid n \in \mathbb{Z}\}$ is an orthonormal sequence in $C[0,1]$ with respect to the inner product defined by (4.1). Indeed, if $m, n \in \mathbb{Z}$ and $n \neq m$, then we have
$$\int_0^1 T_n(x)T_m(x)^* dx = \int_0^1 e^{2\pi i n x} e^{-2\pi i m x} dx = \int_0^1 e^{2\pi i (n-m)x} dx = 0.$$
On the other hand, for all $n \in \mathbb{Z}$,
$$\|T_n\|_2^2 = \langle T_n, T_n\rangle = \int_0^1 e^{2\pi i n x} e^{-2\pi i n x} dx = \int_0^1 1 dx = 1.$$
Hence the set of trigonometric polynomials is an orthonormal set in $C[0,1]$ (with the usual inner product (4.1)). ♢

Orthonormal sets are important, since they have useful properties. With vector spaces, linearly independent spanning sets (called *Hamel bases*) were important because every vector could be expressed as a linear combination of these 'basic' vectors. It turns out that when one has infinite dimensional vector spaces with not just a purely algebraic structure, but also has an inner product (so that one can talk about distance and angle between vectors), the notion of Hamel basis is not adequate, as Hamel bases only capture the algebraic structure. In the case of inner product spaces, the orthonormal sets play an analogous role to Hamel bases: if a vector which can be expressed as a linear combination of vectors from an orthonormal set, then there is a special relation between the coefficients, norms and inner products. Indeed if
$$x = \sum_{k=1}^n \alpha_k u_k$$
and the u_k's are orthonormal, then we have
$$\langle x, u_j\rangle = \left\langle \sum_{k=1}^n \alpha_k u_k, u_j \right\rangle = \sum_{k=1}^n \alpha_k \langle u_k, u_j\rangle = \alpha_j,$$
and
$$\|x\|^2 = \langle x, x\rangle = \left\langle \sum_{k=1}^n \langle x, u_k\rangle u_k, x \right\rangle = \sum_{k=1}^n \langle x, u_k\rangle\langle u_k, x\rangle = \sum_{k=1}^n |\langle x, u_k\rangle|^2.$$

Theorem 4.2.1 *Let X be an inner product space. If S is an orthonormal set, then S is linearly independent.*

Proof Let $x_1, \dots, x_n \in S$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ be such that $\alpha_1 x_1 + \dots + \alpha_n x_n = 0$. For $j \in \{1, \dots, n\}$, we have $0 = \langle 0, x_j\rangle = \langle \alpha_1 x_1 + \dots + \alpha_n x_n, x_j\rangle = \alpha_1\langle x_1, x_j\rangle + \dots + \alpha_n \langle x_n, x_j\rangle = \alpha_j$. Consequently, S is linearly independent. ■

Thus every orthonormal set in X is linearly independent. Conversely, given a linearly independent set in X, we can construct an orthonormal set such that span of this new constructed orthonormal set is the same as the span of the given independent set. We explain this below, and this algorithm is called the *Gram-Schmidt orthonormalization process*.

Theorem 4.2.2 (Gram-Schmidt orthonormalization) *Let $\{x_1, x_2, x_3, \dots\}$ be a linearly independent subset of an inner product space X. Define*

$$u_1 = \frac{x_1}{\|x_1\|} \quad \textit{and} \quad u_n = \frac{x_n - \langle x_n, u_1\rangle u_1 - \cdots - \langle x_n, u_{n-1}\rangle u_{n-1}}{\|x_n - \langle x_n, u_1\rangle u_1 - \cdots - \langle x_n, u_{n-1}\rangle u_{n-1}\|} \textit{ for } n \geq 2.$$

Then $\{u_1, u_2, u_3, \dots\}$ is an orthonormal set in X and for $n \in \mathbb{N}$,

$$\operatorname{span}\{x_1, \dots, x_n\} = \operatorname{span}\{u_1, \dots, u_n\}.$$

Proof As $\{x_1\}$ is a linearly independent set, we see that $x_1 \neq 0$. We have $\|u_1\| = \frac{\|x_1\|}{\|x_1\|} = 1$ and $\operatorname{span}\{u_1\} = \operatorname{span}\{x_1\}$.

For some $n \geq 1$, assume that we have defined u_n as stated above, and proved that $\{u_1, \dots, u_n\}$ is an orthonormal set satisfying $\operatorname{span}\{x_1, \dots, x_n\} = \operatorname{span}\{u_1, \dots, u_n\}$. As $\{x_1, \dots, x_{n+1}\}$ is a linearly independent set, x_{n+1} does not belong to $\operatorname{span}\{x_1, \dots, x_n\} = \operatorname{span}\{u_1, \dots, u_n\}$. Hence it follows that

$$x_{n+1} - \langle x_{n+1}, u_1\rangle u_1 - \cdots - \langle x_{n+1}, u_n\rangle u_n \neq 0.$$

Clearly $\|u_{n+1}\| = 1$ and for all $j \leq n$ we have

$$\begin{aligned} \langle u_{n+1}, u_j\rangle &= \frac{1}{\|x_{n+1} - \langle x_{n+1}, u_1\rangle u_1 - \cdots - \langle x_{n+1}, u_n\rangle u_n\|}\left(\langle x_{n+1}, u_j\rangle - \sum_{k=1}^{n}\langle x_{n+1}, u_k\rangle\langle u_k, u_j\rangle\right) \\ &= \frac{1}{\|x_{n+1} - \langle x_{n+1}, u_1\rangle u_1 - \cdots - \langle x_{n+1}, u_n\rangle u_n\|}\left(\langle x_{n+1}, u_j\rangle - \langle x_{n+1}, u_j\rangle\right) \\ &= 0, \end{aligned}$$

since $\langle u_k, u_j\rangle = 0$ for all $k \in \{1, \dots, n\} \setminus \{j\}$. Hence $\{u_1, \dots, u_{n+1}\}$ is an orthonormal set. Also,

$$\operatorname{span}\{u_1, \dots, u_n, u_{n+1}\} = \operatorname{span}\{x_1, \dots, x_n, u_{n+1}\} = \operatorname{span}\{x_1, \dots, x_n, x_{n+1}\}.$$

By mathematical induction, the proof is complete. ■

Example. For $n \in \mathbb{N}$, let x_n be the sequence $(1, \dots, 1, 0, \dots)$, where 1 occurs only in the first n terms. Then the set $\{x_n \mid n \in \mathbb{N}\}$ is linearly independent in ℓ^2, and the Gram-Schmidt orthonormalization process gives $\{e_n \mid n \in \mathbb{N}\}$ as the corresponding orthonormal set. ◊

Exercises.

1. (a) Show that the set $\{1, x, x^2, x^3, \dots\}$ is linearly independent in $C[-1, 1]$.

   (b) Let the orthonormal sequence obtained via the Gram-Schmidt orthonormalization process of the sequence $\{1, x, x^2, x^3, \dots\}$ in $C[-1, 1]$ be denoted by $\{u_0, u_1, u_2, u_3, \dots\}$. Show that for $n = 0, 1$ and 2,

   $$u_n(x) = \sqrt{\frac{2n+1}{2}}P_n(x), \tag{4.12}$$

   where

   $$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2 - 1)^n. \tag{4.13}$$

   In fact it can be shown that the identity (4.12) holds for all $n \in \mathbb{N}$. The polynomials P_n given by (4.13) are called *Legendre polynomials*.

2. Find an orthonormal basis for $\mathbb{R}^{m \times n}$, when it is equipped with the inner product $\langle A, B\rangle = \operatorname{tr}(A^\top B)$, where $A^\top$ denotes the transpose of the matrix A.

4.3 Best approximation in a subspace

Why are orthonormal sets useful? In this section we give one important reason: it enables one to compute the best approximation to a given vector in X from a given subspace. Thus, the following optimization problem can be solved:

$$\boxed{\begin{array}{l}\text{Let } Y = \text{span}\{y_1, \dots, y_n\}, \text{ where } y_1, \dots, y_n \in X.\\ \text{Given } x \in X, \text{ minimize } \|x-y\|, \text{ subject to } y \in Y.\end{array}} \tag{4.14}$$

See Figure 4.9 for a geometric depiction of the problem in $\mathbb{R}^3$.

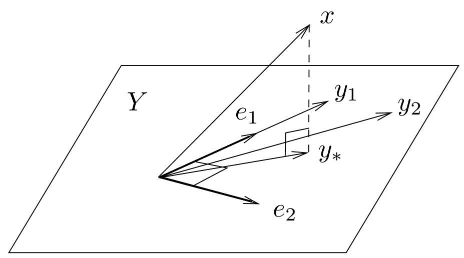

Figure 4.9: Best approximation in a plane of a vector in $\mathbb{R}^3$.

Theorem 4.3.1 *Let X be an inner product space. Suppose that $Y = \text{span}\{y_1, \dots, y_n\}$, where $y_1, \dots, y_n$ are linearly independent vectors in X. Let $\{u_1, \dots, u_n\}$ be an orthonormal set such that $\text{span}\{u_1, \dots, u_n\} = Y$. Suppose that $x \in X$, and define*

$$y_* = \sum_{k=1}^{n} \langle x, u_k \rangle u_k.$$

Then y_* is the unique vector in Y such that for all $y \in Y$, $\|x - y\| \geq \|x - y_*\|$.

Proof If $y \in Y = \text{span}\{y_1, \dots, y_n\} = \text{span}\{u_1, \dots, u_n\} = Y$, then $y = \displaystyle\sum_{k=1}^{n} \langle y, u_k \rangle u_k$, and so

$$\begin{aligned}
\langle x - y_*, y \rangle &= \left\langle x - \sum_{k=1}^{n} \langle x, u_k \rangle u_k, y \right\rangle \\
&= \langle x, y \rangle - \sum_{k=1}^{n} \langle x, u_k \rangle \langle u_k, y \rangle \\
&= \langle x, y \rangle - \left\langle x, \sum_{k=1}^{n} \langle y, u_k \rangle u_k \right\rangle \\
&= \langle x, y \rangle - \langle x, y \rangle \\
&= 0.
\end{aligned}$$

Thus by Pythagoras' theorem (see (4.10)), we obtain

$$\|x - y\|^2 = \|x - y_* + y_* - y\|^2 = \|x - y_*\|^2 + \|y_* - y\|^2 \geq \|x - y_*\|^2,$$

with equality iff $y_* - y = 0$, that is, $y = y_*$. This completes the proof. ∎

Example. Least square approximation problems in applications can be cast as best approximation problems in appropriate inner product spaces. Suppose that f is a continuous function on $[a, b]$ and we want to find a polynomial p of degree at most m such that the 'error'

$$\mathscr{E}(p) = \int_a^b |f(x) - p(x)|^2 dx \tag{4.15}$$

is minimized. Let $X = C[a, b]$ with the inner product defined by (4.1), and let $\mathscr{P}_m$ be the subspace of X comprising all polynomials of degree at most m. Then Theorem 4.3.1 gives a method of finding such a polynomial p_*.

Let us take a concrete case. Let $a = -1$, $b = 1$, $f(x) = e^x$ for $x \in [-1, 1]$, and $m = 1$. As

$$\mathscr{P}_m = \text{span}\{1, x, x^2\},$$

by a Gram-Schmidt orthonormalization process (see Exercise 1b on page 57), it follows that

$$\mathscr{P}_m = \text{span}\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \frac{3\sqrt{10}}{4}\left(x^2 - \frac{1}{3}\right)\right\}.$$

We have

$$\langle f, u_0\rangle = \frac{1}{\sqrt{2}}\left(e - \frac{1}{e}\right), \quad \langle f, u_1\rangle = \frac{\sqrt{6}}{e}, \quad \langle f, u_2\rangle = \frac{\sqrt{5}}{\sqrt{2}}\left(e - \frac{7}{e}\right),$$

and so from Theorem 4.3.1, we obtain that

$$p_*(x) = \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{e}x + \frac{15}{4}\left(e - \frac{7}{e}\right)\left(x^2 - \frac{1}{3}\right).$$

This polynomial p_* is the unique polynomial of degree at most equal to 2 that minimizes the L_2-norm error (4.15) on the interval $[-1, 1]$ when f is the exponential function.

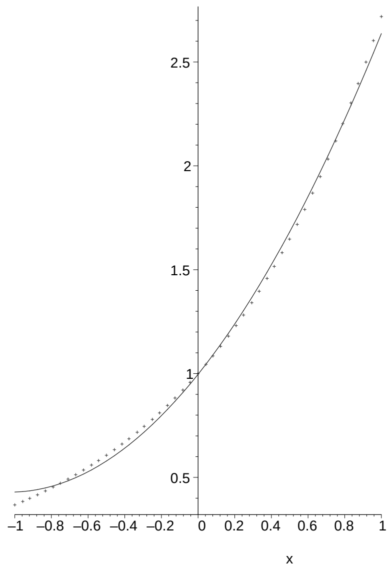

Figure 4.10: Best quadratic approximation the exponential function in the interval $[-1, 1]$: here the dots indicate points (x, e^x), while the curve is the graph of the polynomial p_*.

In Figure 4.10, we have graphed the resulting polynomial p_* and e^x together for comparison. $\Diamond$

Exercises. *(The least squares regression line)* The idea behind the technique of least squares is as follows. Suppose one is interested in two variables x and y, which are supposed to be related via a linear function $y = mx + b$. Suppose that m and b are unknown, but one has measurements $(x_1, y_1), \dots, (x_n, y_n)$ from an experiment. However there are some errors, so that the measured values y_i are related to the measured values x_i by $y_i = mx_i + b + e_i$, where e_i is the (unknown) error in measurement i. How can we find the 'best' approximate values of m and b? This is a very common situation occurring in the applied sciences.

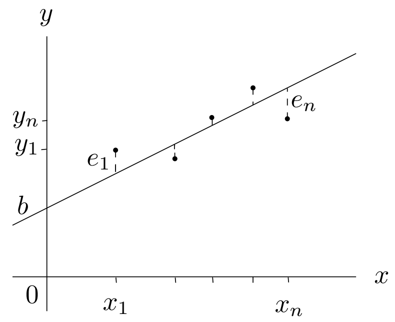

Figure 4.11: The least squares regression line.

The most common technique used to find approximations for a and b is as follows. It is reasonable that if the m and b we guessed were correct, then most of the errors $e_i := y_i - mx_i + b$ should be reasonably small. So to find a good approximation to m and b, we should find the m and b that make the e_i's collectively the smallest in some sense. So we introduce the error

$$\mathscr{E} = \sqrt{\sum_{i=1}^{n} e_i^2} = \sqrt{\sum_{i=1}^{n} (y_i - mx_i - b)^2}$$

and seek m, b such that $\mathscr{E}$ is minimized. See Figure 4.11.

Convert this problem into the setting of (4.14).

Month	Mean temperature (°C)	Inland energy consumption (million tonnes coal equivalent)
January	2.3	9.8
February	3.8	9.3
March	7.0	7.4
April	12.7	6.6
May	15.0	5.7
June	20.9	3.9
July	23.4	3.0
August	20.0	4.1
September	17.9	5.0
October	11.6	6.3
November	5.8	8.3
December	4.7	10.0

Table 4.1: Data on energy consumption and temperature.

The Table 4.1 shows the data on energy consumption and mean temperature in the various months of the year.

1. Draw a scatter chart and fit a regression line of energy consumption on temperature.
2. What is the intercept, and what does it mean in this case?
3. What is the slope, and how could one interpret it?
4. Use the regression line to forecast the energy consumption for a month with mean temperature 9°C.

4.4 Fourier series

Under some circumstances, an orthonormal set can act as a 'basis' in the sense that every vector can be decomposed into vectors from the orthonormal set. Before we explain this further in Theorem 4.4.1, we introduce the notion of *dense*-ness.

Definition. Let X be a normed space. A set D is *dense* in X if for all $x \in X$ and all $\epsilon > 0$, there exists a $y \in D$ such that $\|x - y\| < \epsilon$.

Examples.

1. Let c_{00} denote the set of all sequences with at most finitely many nonzero terms. Clearly, c_{00} is a subspace of ℓ^2. We show that c_{00} is dense in ℓ^2. Let $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$, and suppose that $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |x_n|^2 < \epsilon^2.$$

Defining $y = (x_1, \dots, x_N, 0, 0, 0, \dots) \in c_{00}$, we see that $\|x - y\|_2 < \epsilon$.

2. c_{00} is not dense in ℓ^∞. Consider the sequence $x = (1, 1, 1, \dots) \in \ell^\infty$. Suppose that $\epsilon = \frac{1}{2} > 0$. Clearly for any $y \in c_{00}$, we have

$$\|x - y\|_\infty = 1 > \frac{1}{2} = \epsilon.$$

Thus c_{00} is not dense in ℓ^∞. ◊

Theorem 4.4.1 *Let* $\{u_1, u_2, u_3, \dots\}$ *be an orthonormal set in an inner product space* X *and suppose that* $\mathrm{span}\{u_1, u_2, u_3, \dots\}$ *is dense in* X*. Then for all* $x \in X$*,*

$$x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n \tag{4.16}$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2. \tag{4.17}$$

Furthermore, if $x, y \in X$*, then*

$$\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x, u_n\rangle \langle y, u_n\rangle^*. \tag{4.18}$$

Proof Let $x \in X$, and $\epsilon > 0$. As $\text{span}\{u_1, u_2, u_3, \dots\}$ is dense in X, there exists a $N \in \mathbb{N}$ and scalars $\gamma_1, \dots, \gamma_N \in \mathbb{K}$ such that

$$\left\| x - \sum_{k=1}^{N} \gamma_k u_k \right\| < \epsilon. \tag{4.19}$$

Let $n > N$. Then

$$\sum_{k=1}^{N} \gamma_k u_k \in \text{span}\{u_1, \dots, u_n\},$$

and so from Theorem 4.3.1, we obtain

$$\left\| x - \sum_{k=1}^{n} \langle x, u_k \rangle u_k \right\| \leq \left\| x - \sum_{k=1}^{N} \gamma_n u_n \right\|. \tag{4.20}$$

Combining (4.19) and (4.20), we have shown that given any $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $n > N$,

$$\left\| x - \sum_{k=1}^{n} \langle x, u_k \rangle u_k \right\| < \epsilon,$$

that is, (4.16) holds. Thus if $x, y \in X$, we have

$$\begin{aligned}
\langle x, y \rangle &= \left\langle \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, u_k \rangle u_k, \lim_{n\to\infty} \sum_{k=1}^{n} \langle y, u_k \rangle u_k \right\rangle \\
&= \lim_{n\to\infty} \left\langle \sum_{k=1}^{n} \langle x, u_k \rangle u_k, \sum_{k=1}^{n} \langle y, u_k \rangle u_k \right\rangle \qquad \text{(see (4.11) on page 52)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, u_k \rangle \left\langle u_k, \sum_{j=1}^{n} \langle y, u_j \rangle u_j \right\rangle \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, u_k \rangle \langle y, u_k \rangle^* \\
&= \sum_{n=1}^{\infty} \langle x, u_n \rangle \langle y, u_n \rangle^*,
\end{aligned}$$

and so we obtain (4.18). Finally with $y = x$ in (4.18), we obtain (4.17). ■

Remarks.

1. Note that Theorem 4.4.1 really says that if an inner product space X has an orthonormal set $\{u_1, u_2, u_3, \dots\}$ that is dense in X, then computations in X can be done as if one is in the space ℓ^2: indeed the map $x \mapsto (\langle x, u_1 \rangle,)$ is in injective bounded linear operator from X to ℓ^2 that preserves inner products.

2. (4.17) and (4.18) are called *Parseval's identities*.

3. We observe in Theorem 4.4.1 that every vector in X can be expressed as an 'infinite linear combination' of the vectors $u_1, u_2, u_3, \dots$. Such a 'spanning' orthonormal set $\{u_1, u_2, u_3, \dots\}$ is therefore called an *orthonormal basis*.

Example. Consider the Hilbert space $C[0, 1]$ with the usual inner product. It was shown in Example 2 on page 56 that the trigonometric polynomials T_n, $n \in \mathbb{Z}$, form an orthonormal set.

Furthermore, it can be shown that the set $\{T_1, T_2, T_3, \dots\}$ is dense in $C[0,1]$. Hence by Theorem 4.4.1, it follows that every $f \in C[0,1]$ has the following Fourier expansion:

$$f = \sum_{k=-\infty}^{\infty} \langle f, T_k \rangle T_k.$$

It must be emphasized that the above means that the sequence of functions (f_n) given by

$$f_n = \sum_{k=-n}^{n} \langle f, T_k \rangle T_k$$

converges to f in the $\|\cdot\|_2$-norm: $\displaystyle\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|^2 dx = 0.$ $\Diamond$

Exercises.

1. $(*)$ Show that an inner product space X is dense in its completion $\overline{X}$.
2. $(*)$ A normed space X is said to be *separable* if it has a countably dense subset $\{u_1, u_2, u_3, \dots\}$.
   (a) Prove that ℓ^p is separable if $1 \le p < +\infty$.
   (b) What happens if $p = +\infty$?
3. *(Isoperimetric theorem)* Among all simple, closed piecewise smooth curves of length L in the plane, the circle encloses the maximum area.

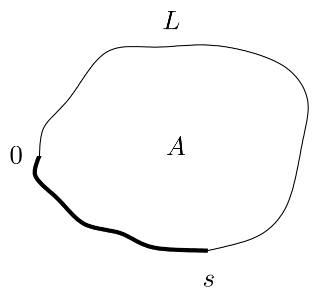

Figure 4.12: Parameterization of the curve using the arc length as a parameter.

This can be proved by proceeding as follows. Suppose that (x, y) is a parametric representation of the curve using the arc length s, $0 \le s \le L$, as a parameter. Let $t = \frac{s}{L}$ and let

$$\begin{aligned} x(t) &= a_0 + \sqrt{2}\sum_{n=1}^{\infty}(a_n \cos(2\pi n t) + b_n \sin(2\pi n t)) \\ y(t) &= c_0 + \sqrt{2}\sum_{n=1}^{\infty}(c_n \cos(2\pi n t) + d_n \sin(2\pi n t)) \end{aligned}$$

be the Fourier series expansions for x and y on the interval $0 \le t \le 1$.

It can be shown that

$$L^2 = \int_0^1 \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt = \sum_{n=1}^{\infty} 4\pi^2 n^2 (a_n^2 + b_n^2 + c_n^2 + d_n^2)$$

and the area A is given by

$$A = \int_0^1 x(t)\frac{dy}{dt}(t)dt = \sum_{n=1}^{\infty} 2\pi n(a_n d_n - b_n c_n).$$

Prove that $L^2 - 4\pi A \geq 0$ and that equality holds iff

$$a_1 = d_1, \quad b_1 = -c_1, \text{ and } a_n = b_n = c_n = d_n = 0 \text{ for all } n \geq 2,$$

which describes the equation of a circle.

4. *(Riesz-Fischer theorem)* Let X be a Hilbert space and $\{u_1, u_2, u_3, \dots\}$ be an orthonormal sequence in X. If $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of scalars such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < +\infty$, then $\sum_{n=1}^{\infty} \alpha_n u_n$ converges in X.

   HINT: For $m \in \mathbb{N}$, define $x_m = \sum_{n=1}^{m} \alpha_n u_n$. If $m > l$, then

$$\|x_m - x_l\|^2 = \sum_{n=l+1}^{m} |\alpha_n|^2.$$

   Conclude that $(x_m)_{m\in\mathbb{N}}$ is Cauchy, and hence convergent.

4.5 Riesz representation theorem

If X is a Hilbert space, and $x_0 \in X$, then the map

$$x \mapsto \langle x, x_0\rangle : X \to \mathbb{K}$$

is a bounded linear functional on X. Indeed, the linearity follows from the linearity property of the inner product in the first variable (IP2 on page 47), while the boundedness is a consequence of the Cauchy-Schwarz inequality:

$$|\langle x, x_0\rangle| \leq \|x_0\|\|x\|.$$

Conversely, it turns out that every bounded linear functional on a Hilbert space arises in this manner, and this is the content of the following theorem.

Theorem 4.5.1 (Riesz representation theorem) *If $T \in \mathscr{L}(X, \mathbb{K})$, then there exists a unique $x_0 \in X$ such that*

$$\forall x \in X, \quad Tx = \langle x, x_0\rangle. \tag{4.21}$$

Proof We prove this for a Hilbert space X with an orthonormal basis $\{u_1, u_2, u_3, \dots\}$.

STEP 1. First we show that $\sum_{n=1}^{\infty} |Tu_n|^2 < +\infty$. For $m \in \mathbb{N}$, define

$$y_m = \sum_{n=1}^{m} (T(u_n))^* u_n.$$

Since $\{u_1, u_2, u_3, \dots\}$ is an orthonormal set,

$$\|y_m\|^2 = \langle y_m, y_m\rangle = \sum_{n=1}^{m} |T(u_n)|^2 = k_m \text{ (say)}.$$

Since

$$T(y_m) = \sum_{n=1}^{m} (T(u_n))^* T(u_n) = k_m,$$

and $|T(y_m)| \leq \|T\|\|y_m\|$, we see that $k_m \leq \|T\|\sqrt{k_m}$, that is, $k_m \leq \|T\|^2$. Letting $m \to \infty$, we obtain

$$\sum_{n=1}^{\infty} |T(u_n)|^2 \leq \|T\|^2 < +\infty.$$

STEP 2. From the Riesz-Fischer theorem (see Exercise 4 on page 64), we see that the series

$$\sum_{n=1}^{\infty} (T(u_n))^* u_n$$

converges in X to x_0 (say). We claim that (4.21) holds. Let $x \in X$. This has a Fourier expansion

$$x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n.$$

Hence

$$T(x) = \sum_{n=1}^{\infty} \langle x, u_n\rangle T(u_n) = \sum_{n=1}^{\infty} \langle x, (T(u_n))^* u_n\rangle = \left\langle x, \sum_{n=1}^{\infty} (T(u_n))^* u_n \right\rangle = \langle x, x_0\rangle.$$

STEP 3. Finally, we prove the uniqueness of $x_0 \in X$. If $x_1 \in X$ is another vector such that

$$\forall x \in X, \quad Tx = \langle x, x_1\rangle,$$

then letting $x = x_0 - x_1$, we obtain $\langle x_0 - x_1, x_0\rangle = T(x_0 - x_1) = \langle x_0 - x_1, x_1\rangle$, that is,

$$\|x_0 - x_1\|^2 = \langle x_0 - x_1, x_0 - x_1\rangle = 0.$$

Thus $x_0 = x_1$. ■

Thus the above theorem characterizes linear functionals on Hilbert spaces: they are precisely inner products with a fixed vector!

Exercise. In Theorem 4.5.1, show that also $\|T\| = \|x_0\|$, that is, the norm of the functional is the norm of its representer.

HINT: Use Theorem 4.1.1.

4.6 Adjoints of bounded operators

With every bounded linear operator on a Hilbert space, one can associate another operator, called its adjoint, which is geometrically related. In order to define the adjoint, we prove the following result. (Throughout this section, X denotes a Hilbert space with an orthonormal basis $\{u_1, u_2, u_3, \dots\}$.)

Theorem 4.6.1 *Let X be a Hilbert space. If $T \in \mathscr{L}(X)$, then there exists a unique operator $T^* \in \mathscr{L}(X)$ such that*

$$\forall x, y \in X, \quad \langle Tx, y \rangle = \langle x, T^* y \rangle. \tag{4.22}$$

Proof Let $y \in X$. The map L_y given by $x \mapsto \langle Tx, y \rangle$ from X to $\mathbb{K}$ is a linear functional. We verify this below.

Linearity: If $x_1, x_2 \in X$, then

$$L_y(x_1 + x_2) = \langle T(x_1 + x_2), y \rangle = \langle Tx_1 + Tx_2, y \rangle = \langle Tx_1, y \rangle + \langle Tx_2, y \rangle = L_y(x_1) + L_y(x_2).$$

Furthermore, if $\alpha \in \mathbb{K}$ and $x \in X$, then $L_y(\alpha x) = \langle T(\alpha x), y \rangle = \langle \alpha Tx, y \rangle = \alpha \langle Tx, y \rangle = \alpha L_y(x)$.

Boundedness: For all $x \in X$,

$$|L_y(x)| = |\langle Tx, y \rangle| \leq \|Tx\| \|y\| \leq \|T\| \|y\| \|x\|,$$

and so $\|L_y\| \leq \|T\| \|y\| < +\infty$.

Hence $L_y \in \mathscr{L}(X, \mathbb{K})$, and by the Riesz representation theorem, it follows that there exists a unique vector, which we denote by $T^* y$, such that for all $x \in X$, $L_y(x) = \langle x, T^* y \rangle$, that is,

$$\forall x \in X, \quad \langle Tx, y \rangle = \langle x, T^* y \rangle,$$

In this manner, we get a map $y \mapsto T^* y$ from X to X. We claim that this is a bounded linear operator.

Linearity: If $y_1, y_2 \in X$, then for all $x \in X$ we have

$$\langle x, T^*(y_1 + y_2) \rangle = \langle Tx, y_1 + y_2 \rangle = \langle Tx, y_1 \rangle + \langle Tx, y_2 \rangle = \langle x, T^* y_1 \rangle + \langle x, T^* y_2 \rangle = \langle x, T^* y_1 + T^* y_2 \rangle.$$

In particular, taking $x = T^*(y_1 + y_2) - (T^* y_1 + T^* y_2)$, we conclude that $T^*(y_1 + y_2) = T^* y_1 + T^* y_2$. Furthermore, if $\alpha \in \mathbb{K}$ and $y \in X$, then for all $x \in X$,

$$\langle x, T^*(\alpha y) \rangle = \langle T(x), \alpha y \rangle = \alpha^* \langle Tx, y \rangle = \alpha^* \langle x, T^* y \rangle = \langle x, \alpha T^* y \rangle.$$

In particular, taking $x = T^*(\alpha y) - \alpha(T^* y)$, we conclude that $T^*(\alpha y) = \alpha(T^* y)$. This completes the proof of the linearity of T^*.

Boundedness: From the Exercise in §4.5, it follows that for all $y \in X$, $\|T^* y\| = \|L_y\| \leq \|T\| \|y\|$. Consequently, $\|T^*\| \leq \|T\| < +\infty$.

Hence $T^* \in \mathscr{L}(X)$. Finally, if S is another bounded linear operator on X satisfying (4.22), then for all $x, y \in X$, we have

$$\langle x, T^* y \rangle = \langle Tx, y \rangle = \langle x, Sy \rangle,$$

and in particular, taking $x = T^* y - Sy$, we can conclude that $T^* y = Sy$. As this holds for all y, we obtain $S = T^*$. Consequently T^* is unique. ■

Definition. Let X be a Hilbert space. If $T \in \mathscr{L}(X)$, then the unique operator $T^* \in \mathscr{L}(X)$ satisfying

$$\forall x, y \in X, \quad \langle Tx, y \rangle = \langle x, T^* y \rangle,$$

is called the *adjoint of* T.

Before giving a few examples of adjoints, we will prove a few useful properties of adjoint operators.

Theorem 4.6.2 *Let X be a Hilbert space.*

1. *If $T \in \mathscr{L}(X)$, then $(T^*)^* = T$.*
2. *If $T \in \mathscr{L}(X)$, then $\|T^*\| = \|T\|$.*
3. *If $\alpha \in \mathbb{K}$ and $T \in \mathscr{L}(X)$, then $(\alpha T)^* = \alpha^* T$.*
4. *If $S, T \in \mathscr{L}(X)$, then $(S+T)^* = S^* + T^*$.*
5. *If $S, T \in \mathscr{L}(X)$, then $(ST)^* = T^* S^*$.*

Proof 1. For all $x, y \in X$, we have

$$\langle x, (T^*)^* y\rangle = \langle T^* x, y\rangle = \langle y, T^* x\rangle^* = \langle Ty, x\rangle^* = \langle x, Ty\rangle,$$

and so $(T^*)^* y = Ty$ for all y, that is, $(T^*)^* = T$.

2. From the proof of Theorem 4.6.1, we see that $\|T^*\| \leq \|T\|$. Also, $\|T\| = \|(T^*)^*\| \leq \|T^*\|$. Consequently $\|T\| = \|T^*\|$.

3. For all $x, y \in X$, we have

$$\langle x, (\alpha T)^* y\rangle = \langle (\alpha T)x, y\rangle = \langle \alpha(Tx), y\rangle = \alpha\langle Tx, y\rangle = \alpha\langle x, T^* y\rangle = \langle x, \alpha^*(T^* y)\rangle = \langle x, (\alpha^* T^*)y\rangle,$$

and so it follows that $(\alpha T)^* = \alpha^* T^*$.

4. For all $x, y \in X$, we have

$$\begin{aligned}
\langle x, (S+T)^* y\rangle &= \langle (S+T)x, y\rangle \\
&= \langle Sx + Tx, y\rangle \\
&= \langle Sx, y\rangle + \langle Tx, y\rangle \\
&= \langle x, S^* y\rangle + \langle x, T^* y\rangle \\
&= \langle x, S^* y + T^* y\rangle \\
&= \langle x, (S^* + T^*)y\rangle,
\end{aligned}$$

and so $(S+T)^* = S^* + T^*$.

5. For all $x, y \in X$, we have

$$\langle x, (ST)^* y\rangle = \langle (ST)x, y\rangle = \langle S(Tx), y\rangle = \langle Tx, S^* y\rangle = \langle x, T^*(S^* y)\rangle = \langle x, (T^* S^*)y\rangle,$$

and so it follows that $(ST)^* = T^* S^*$. ∎

The adjoint operator T is geometrically related to T. Before we give this relation in Theorem 4.6.3 below, we first recall the definitions of the kernel and range of a linear transformation, and also fix some notation.

Definitions. Let U, V be vector spaces and $T : U \to V$ a linear transformation.

1. The *kernel of* T is defined to be the set $\ker(T) = \{u \in U \mid T(u) = 0\}$.
2. The *range of* T is defined to be the set $\operatorname{ran}(T) = \{v \in V \mid \exists u \in U \text{ such that } T(u) = v\}$.

Notation. If X is a Hilbert space, and S is a subset of X, then $S^\perp$ is defined to be the set of all vectors in X that are orthogonal to each vector in S:

$$S^\perp = \{x \in X \mid \forall y \in S, \langle x, y\rangle = 0\}.$$

We are now ready to give the geometric relation between the operators T and T^*.

Theorem 4.6.3 *Let X be a Hilbert space, and suppose that $T \in \mathscr{L}(X)$. Then* $\ker(T) = (\operatorname{ran}(T^*))^\perp$.

Proof If $x \in \ker(T)$, then for all $y \in X$, we have $\langle x, T^*y\rangle = \langle Tx, y\rangle = 0$, and so $x \in (\operatorname{ran}(T^*))^\perp$. Consequently $\ker(T) \subset (\operatorname{ran}(T^*))^\perp$.

Conversely, if $x \in (\operatorname{ran}(T^*))^\perp$, then for all $y \in X$, we have $0 = \langle x, T^*y\rangle = \langle Tx, y\rangle$, and in particular, with $y = Tx$, we obtain that $\langle Tx, Tx\rangle = 0$, that is $Tx = 0$. Hence $x \in \ker(T)$. Thus $(\operatorname{ran}(T^*))^\perp \subset \ker(T)$ as well. ∎

We now give a few examples of the computation of adjoints of some operators.

Examples.

1. Let $X = \mathbb{C}^n$ with the inner product given by

$$\langle x, y\rangle = \sum_{i=1}^n x_i y_i^*.$$

Let $T_A : \mathbb{C}^n \to \mathbb{C}^n$ be the bounded linear operator corresponding to the $n \times n$ matrix A of complex numbers:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}.$$

What is the adjoint T_A^*? Let us denote by A^* the matrix obtained by transposing the matrix A and by taking the complex conjugates of each of the entries. Thus

$$A^* = \begin{bmatrix} a_{11}^* & \dots & a_{n1}^* \\ \vdots & & \vdots \\ a_{1n}^* & \dots & a_{nn}^* \end{bmatrix}.$$

We claim that $T_A^* = T_{A^*}$. Indeed, for all $x, y \in \mathbb{C}^n$,

$$\begin{aligned} \langle T_A x, y\rangle &= \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}x_j\right) y_i^* \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_j y_i^* \\ &= \sum_{j=1}^n x_j \sum_{i=1}^n a_{ij} y_i^* \\ &= \sum_{j=1}^n x_j \left(\sum_{i=1}^n a_{ij}^* y_i\right)^* \\ &= \langle x, T_{A^*} y\rangle. \end{aligned}$$

2. Recall the left and right shift operators on ℓ^2:

$$R(x_1, x_2, \dots) = (0, x_1, x_2, \dots), \quad L(x_1, x_2, \dots) = (x_2, x_3, x_4, \dots).$$

We will show that $R^* = L$. Indeed, for all $x, y \in \ell^2$, we have

$$\begin{aligned} \langle Rx, y\rangle &= \langle (0, x_1, x_2, \dots), (y_1, y_2, y_3, \dots)\rangle \\ &= \sum_{n=1}^{\infty} x_n y_{n+1}^* \\ &= \langle (x_1, x_2, x_3, \dots), (y_2, y_3, y_4, \dots)\rangle \\ &= \langle x, Ly\rangle. \end{aligned}$$

$\Diamond$

Exercises.

1. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a bounded sequence of scalars, and consider the diagonal operator $D : \ell^2 \to \ell^2$ defined by (2.18) on page 24. Determine D^*.
2. Consider the anticlockwise *rotation* through angle Θ in $\mathbb{R}^2$, given by the operator $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to the matrix
$$A = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix}$$
What is T_A^*? Give a geometric interpretation.

Let X be a Hilbert space with an orthonormal basis $\{u_1, u_2, u_3, \dots\}$.

3. If $T \in \mathscr{L}(X)$ is such that it is invertible, then prove that T^* is also invertible and that $(T^*)^{-1} = (T^{-1})^*$.
4. An operator $T \in \mathscr{L}(X)$ is called *self-adjoint* (respectively *skew-adjoint*) if $T = T^*$ (respectively $T = -T^*$). Show that every operator can be written as a sum of a self-adjoint operator and a skew-adjoint operator.
5. *(Projections)* A bounded linear operator $P \in \mathscr{L}(X)$ is called a *projection* if it is idempotent (that is, $P^2 = P$) and self-adjoint ($P^* = P$). Prove that the norm of P is at most equal to 1.
Hint: $\|Px\|^2 = \langle Px, Px\rangle = \langle Px, x\rangle \le \|Px\|\|x\|$.
Consider the subspace $Y = \text{span}\{u_1, \dots, u_n\}$. Define $P_n : X \to X$ as follows:
$$P_n x = \sum_{k=1}^{n} \langle x, u_k\rangle u_k, \quad x \in X.$$
Show that P_n is a projection. Describe the kernel and range of P_n.
Suppose that $\{u_1, u_2, u_3, \dots\}$ is an orthonormal basis for X. Prove that
$$\forall x \in X, \quad P_n x \stackrel{n\to\infty}{\longrightarrow} x \text{ in } X.$$
6. Let X be a Hilbert space. Let $A \in \mathscr{L}(X)$ be fixed. We define $\Lambda : \mathscr{L}(X) \to \mathscr{L}(X)$ by
$$\Lambda(T) = A^*T + TA, \quad T \in \mathscr{L}(X).$$
Show that $\Lambda \in \mathscr{L}(\mathscr{L}(X))$. Prove that if T is self-adjoint, then so is $\Lambda(T)$.

Chapter 5

Compact operators

In this chapter, we study a special class of linear operators, called compact operators. Compact operators are useful since they play an important role in the numerical approximation of solutions to operator equations. Indeed, compact operators can be approximated by finite-rank operators, bringing one in the domain of finite-dimensional linear algebra.

In Chapter 2, we considered the following problem: given y, find x such that

$$(I - A)x = y. \tag{5.1}$$

It was shown that if $\|A\| < 1$, then the unique x is given by a Neumann series. It is often the case that the Neumann series cannot be computed. If A is a compact operator, then there is an effective way to construct an approximate solution to the equation (5.1): we replace the operator A by a sufficiently accurate finite rank approximation and solve the resulting finite system of equations! We will elaborate on this at the end of this chapter in §5.2.

We begin by giving the definition of a compact operator.

5.1 Compact operators

Definition. Let X be an inner product space. A linear transformation $T : X \to X$ is said to be *compact* if

$$\boxed{\forall \text{ bounded sequence } (x_n)_{n\in\mathbb{N}} \text{ contained in } X,\ (Tx_n)_{n\in\mathbb{N}} \text{ has a convergent subsequence.}}$$

Before giving examples of compact operators, we prove the following result, which says that the set of compact operators is contained in the set of all bounded operators.

Theorem 5.1.1 *Let X be an inner product space. If $T : X \to X$ is a compact operator, then $T \in \mathscr{L}(X)$.*

Proof Suppose that there does not exist a $M > 0$ such that

$$\forall x \in X \text{ such that } \|x\| \leq 1, \quad \|Tx\| \leq M.$$

Let $x_1 \in X$ be such that $\|x_1\| \leq 1$ and $\|Tx_1\| > 1$. If $x_n \in X$ has been constructed, then let $x_{n+1} \in X$ be such that $\|x_{n+1}\| \leq 1$ and

$$\|Tx_{n+1}\| > 1 + \max\{\|Tx_1\|, \dots, \|Tx_n\|\}.$$

Clearly $(x_n)_{n\in\mathbb{N}}$ is bounded (indeed for all $n \in \mathbb{N}$, $\|x_n\| \leq 1$). However, $(Tx_n)_{n\in\mathbb{N}}$ does not have any convergent subsequence, since if $n_1, n_2 \in \mathbb{N}$ and $n_1 < n_2$, then

$$\|Tx_{n_1} - Tx_{n_2}\| \geq \|Tx_{n_2}\| - \|Tx_{n_1}\| > 1 + \max\{\|Tx_1\|, \dots, \|Tx_{n_2-1}\|\} - \|Tx_{n_1}\| \geq 1.$$

So T is compact. ■

However, the converse of Theorem 5.1.1 above is not true, as demonstrated by the following Example which says that the identity operator in any *infinite*-dimensional inner product space is not compact. Later on we will see that all finite-rank operators are compact, and in particular, the identity operator in a *finite*-dimensional inner product space is compact.

Example. Let X be an infinite-dimensional inner product space. Then we can construct an orthonormal sequence $u_1, u_2, u_3, \dots$ in X (take any countable infinite independent set and use the Gram-Schmidt orthonormalization procedure). Consider the identity operator $I : X \to X$. The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded (for all $n \in \mathbb{N}$, $\|u_n\| = 1$). However, the sequence $(Iu_n)_{n\in\mathbb{N}}$ has no convergent subsequence, since for all $n, m \in \mathbb{N}$ with $n \neq m$, we have

$$\|Iu_n - Iu_m\| = \|u_n - u_m\| = \sqrt{2}.$$

Hence I is not compact. However, I is clearly bounded ($\|I\| = 1$). ♢

It turns out that all finite rank operators are compact. Recall that an operator T is called a *finite rank operator* if its range, $\operatorname{ran}(T)$, is a finite-dimensional vector space.

Theorem 5.1.2 *Let X be an inner product space and suppose that $T \in \mathscr{L}(X)$. If $\operatorname{ran}(T)$ is finite-dimensional, then T is compact.*

Proof Let $\{u_1, \dots, u_m\}$ be an orthonormal basis for $\operatorname{ran}(T)$. If $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in X, then for all $n \in \mathbb{N}$ and each $k \in \{1, \dots, m\}$, we have

$$|\langle Tx_n, u_k\rangle| \leq \|Tx_n\|\|u_k\|^2 \leq \|T\|\|x_n\| \leq \|T\| \sup_{n\in\mathbb{N}} \|x_n\|. \tag{5.2}$$

Hence $(\langle Tx_n, u_1\rangle)_{n\in\mathbb{N}}$ is a bounded sequence. By the Bolzano-Weierstrass theorem, it follows that it has a convergent subsequence, say $(\langle Tx_n^{(1)}, u_1\rangle)_{n\in\mathbb{N}}$. From (5.2), it follows that $(\langle Tx_n^{(1)}, u_2\rangle)_{n\in\mathbb{N}}$ is a bounded sequence, and again by the Bolzano-Weiertrass theorem, it has a convergent subsequence, say $(\langle Tx_n^{(2)}, u_2\rangle)_{n\in\mathbb{N}}$. Proceeding in this manner, it follows that $(x_n)_{n\in\mathbb{N}}$ gas a subsequence $(x_n^{(m)})_{n\in\mathbb{N}}$ such that the sequences

$$(\langle Tx_n^{(m)}, u_1\rangle)_{n\in\mathbb{N}}, \dots, (\langle Tx_n^{(m)}, u_m\rangle)_{n\in\mathbb{N}}$$

are all convergent, with limits say $\alpha_1, \dots, \alpha_m$, respectively. Then

$$\left\|Tx_n^{(m)} - \sum_{k=1}^{m} \alpha_k u_k\right\|^2 = \left\|\sum_{k=1}^{m} \langle Tx_n^{(m)}, u_k\rangle u_k - \sum_{k=1}^{m} \alpha_k u_k\right\|^2 = \sum_{k=1}^{m} |\langle Tx_n^{(m)}, u_k\rangle - \alpha_k|^2 \xrightarrow{n\to\infty} 0,$$

and so it follows that $(Tx_n^{(m)})_{n\in\mathbb{N}}$ is a convergent subsequence (with limit $\sum_{k=1}^{m} \alpha_k u_k$) of the sequence $(Tx_n)_{n\in\mathbb{N}}$. Consequently T is compact, and this completes the proof. ■

Next we prove that if X is a Hilbert space, then the limits of compact operators are compact. Thus the subspace $\mathscr{C}(X)$ of $\mathscr{L}(X)$ comprising compact operators is *closed*, that is $\mathscr{C}(X) = \overline{\mathscr{C}(X)}$. Hence $\mathscr{C}(X)$, with the induced norm from $\mathscr{L}(X)$ is also a Banach space.

Theorem 5.1.3 *Let X be a Hilbert space. Suppose that $(T_n)_{n\in\mathbb{N}}$ is a sequence of compact operators such that it is convergent in $\mathscr{L}(X)$, with limit $T \in \mathscr{L}(X)$. Then T is compact.*

Proof We have

$$\lim_{n\to\infty} \|T - T_n\| = 0.$$

Suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that for all $n \in \mathbb{N}$, $\|x_n\| \leq M$. Since T_1 is compact, $(T_1x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(T_1x_n^{(1)})_{n\in\mathbb{N}}$, say. Again, since $(x_n^{(1)})_{n\in\mathbb{N}}$ is a bounded sequence, and T_2 is compact, $(T_2x_n^{(1)})_{n\in\mathbb{N}}$ contains a convergent subsequence $(T_2x_n^{(2)})_{n\in\mathbb{N}}$. We continue in this manner:

$$\begin{array}{|cccc}
\hline
x_1 & x_2 & x_3 & \dots \\
x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \dots \\
x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & \dots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Consider the diagonal sequence $(x_{n+1}^{(n)})_{n\in\mathbb{N}}$, which is a subsequence of $(x_n)_{n\in\mathbb{N}}$. For each $k \in \mathbb{N}$, $(T_kx_n^{(n)})_{n\in\mathbb{N}}$ is convergent in X. For $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
\|Tx_n^{(n)} - Tx_m^{(m)}\| &\leq \|Tx_n^{(n)} - T_kx_n^{(n)}\| + \|T_kx_n^{(n)} - T_kx_m^{(m)}\| + \|T_kx_m^{(m)} - Tx_m^{(m)}\| \\
&\leq \|T - T_k\|\|x_n^{(n)}\| + \|T_kx_n^{(n)} - T_kx_m^{(m)})\| + \|T_k - T\|\|x_m^m\| \\
&\leq 2M\|T - T_k\| + \|T_kx_n^{(n)} - T_kx_m^{(m)})\|.
\end{aligned}$$

Hence $(Tx_n^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in X and since X is complete, it converges in X. Hence T is compact. ∎

We now give an important example of a compact operator.

Example. Let

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots \\ k_{21} & k_{22} & k_{23} & \dots \\ k_{31} & k_{32} & k_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

be an infinite matrix such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |k_{ij}|^2 < +\infty.$$

Then K defines a compact linear operator on ℓ^2.

If $x = (x_1, x_2, x_3, \dots) \in \ell^2$, then

$$Kx = \left(\sum_{j=1}^{\infty} k_{1j}x_j, \sum_{j=1}^{\infty} k_{2j}x_j, \sum_{j=1}^{\infty} k_{3j}x_j \right).$$

As

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |k_{ij}|^2 < +\infty,$$

it follows that for each $i \in \mathbb{N}$, $\displaystyle\sum_{j=1}^{\infty} |k_{ij}|^2 < +\infty$, and so $k_i := (k_{i1}, k_{i2}, k_{i3}, \dots) \in \ell^2$. Thus

$$\langle k_i x \rangle = \sum_{j=1}^{\infty} k_{ij} x_j$$

converges. Hence Kx is a well-defined sequence. Moreover, we have

$$\begin{aligned}
\|Kx\|_2^2 &= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} k_{ij} x_j \right|^2 \leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |k_{ij}||x_j| \right)^2 \\
&\leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |k_{ij}|^2 \right) \left(\sum_{j=1}^{\infty} |x_j|^2 \right) \qquad \text{(Cauchy-Schwarz)} \\
&= \left(\sum_{j=1}^{\infty} |x_j|^2 \right) \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |k_{ij}|^2 \right) = \|x\|^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |k_{ij}|^2.
\end{aligned}$$

This shows that $Kx \in \ell^2$ and that $K \in \mathscr{L}(\ell^2)$, with

$$\|K\| \leq \sqrt{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |k_{ij}|^2}.$$

Define the operator $K_n \in \mathscr{L}(\ell^2)$ as follows:

$$K_n x = \left(\sum_{j=1}^{\infty} k_{1j} x_j, \dots, \sum_{j=1}^{\infty} k_{nj} x_j, 0, 0, 0, \dots \right), \quad x \in \ell^2.$$

This is a finite rank operator corresponding to the matrix

$$K_n = \left[\begin{array}{cccc}
k_{11} & \dots & k_{1n} & \dots \\
\vdots & & \vdots & \\
k_{n1} & \dots & k_{nn} & \dots \\
\hline
0 & \dots & 0 & \dots \\
\vdots & & \vdots &
\end{array} \right]$$

and is simply the operator $P_n K$, where P_n is the projection onto the subspace $Y = \text{span}\{e_1, \dots, e_n\}$ (see Exercise 4.6 on page 69). As K_n is finite rank, it is compact. We have

$$K - K_n = \left[\begin{array}{cccc}
0 & \dots & 0 & \dots \\
\vdots & & \vdots & \\
0 & \dots & 0 & \dots \\
\hline
k_{(n+1)1} & \dots & k_{(n+1)n} & \dots \\
\vdots & & \vdots &
\end{array} \right]$$

and so

$$\|K - K_n\|^2 \leq \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty} |k_{ij}|^2 \overset{n \to \infty}{\longrightarrow} 0.$$

Thus K is compact. This operator is the discrete analogue of the integral operator (2.17) on page 23, which can be also shown to be a compact operator on the Lebesgue space $L^2(a, b)$. $\Diamond$

It is easy to see that if S, T are compact operators then $S+T$ is also a compact operator. Furthermore, if $\alpha \in \mathbb{K}$ and T is a compact operator, then αT is also a compact operator. Clearly the zero operator 0, mapping every vector $x \in X$ to the zero vector in X is compact. Hence the set of all compact operators is a subspace of $\mathscr{C}(X)$.

In fact, $\mathscr{C}(X)$ is an ideal of the algebra $\mathscr{L}(X)$, and we prove this below in Theorem 5.1.4.

Definition. An *ideal* I of an algebra R is a subset I of R that has the following three properties:

I1 $0 \in I$.

I2 If $a, b \in I$, then $a+b \in I$.

I3 If $a \in I$ and $r \in R$, then $ar \in I$ and $ra \in I$.

Theorem 5.1.4 *Let X be a Hilbert space.*

1. *If $T \in \mathscr{L}(X)$ is compact and $S \in \mathscr{L}(X)$, then TS is compact.*
2. *If $T \in \mathscr{L}(X)$ is compact, then T^* is compact.*
3. *If $T \in \mathscr{L}(X)$ is compact and $S \in \mathscr{L}(X)$, then ST is compact.*

Proof 1. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. As S is a bounded linear operator, it follows that $(Sx_n)_{n\in\mathbb{N}}$ is also a bounded sequence. Since T is compact, there exists a subsequence $(T(Sx_{n_k}))_{k\in\mathbb{N}}$ that is convergent. Thus TS is compact.

2. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. From the part above, it follows that TT^* is compact, and so there exists a subsequence $(TT^*x_{n_k})_{k\in\mathbb{N}}$ that is convergent. Hence

$$\|T^*x_{n_k} - T^*x_{n_l}\|^2 = \langle TT^*(x_{n_k} - x_{n_l}), (x_{n_k} - x_{n_l})\rangle \leq \|TT^*x_{n_k} - TT^*x_{n_l}\|\|x_{n_k} - x_{n_l}\|.$$

hence $(T^*x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence and as X is a Hilbert space, it is convergent. This shows that T^* is compact.

3. As T is compact, it follows that T^* is also compact. Since $S^* \in \mathscr{L}(X)$, it follows that T^*S^* is compact. From the previous part, we obtain that $(T^*S^*)^* = ST$ is compact. ■

Summarizing, the set $\mathscr{C}(X)$ is a closed ideal of $\mathscr{L}(X)$.

Exercises.

1. Let X be an infinite-dimensional Hilbert space. If $T \in \mathscr{L}(X)$ is invertible, then show that T cannot be compact.
2. Let X be an infinite-dimensional Hilbert space. Show that if $T \in \mathscr{L}(X)$ is such that T is self-adjoint and T^n is compact for some $n \in \mathbb{N}$, then T is compact.

   HINT: First consider the case when $n = 2$.
3. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a bounded sequence of scalars, and consider the diagonal operator $D : \ell^2 \to \ell^2$ defined by (2.18) on page 24. Show that D is compact iff $\lim\limits_{n\to\infty} \lambda_n = 0$.

4. Let X be a Hilbert space. Let $A \in \mathscr{L}(X)$ be fixed. We define $\Lambda : \mathscr{L}(X) \to \mathscr{L}(X)$ by
$$\Lambda(T) = A^*T + TA, \quad T \in \mathscr{L}(X).$$
Show that the subspace of compact operators is Λ-invariant, that is,
$$\{\Lambda(T) \mid T \in \mathscr{C}(X)\} \subset \mathscr{C}(X).$$

5. Let X be a Hilbert space with an orthonormal basis $\{u_1, u_2, u_3, \dots\}$. An operator $T \in \mathscr{L}(X)$ is called *Hilbert-Schmidt* if
$$\sum_{n=1}^{\infty} \|Tu_n\|^2 < +\infty.$$
   (a) Let $T \in \mathscr{L}(X)$ be a Hilbert-Schmidt operator. If $m \in \mathbb{N}$, then define $T_m : X \to X$ by
$$T_m x = \sum_{n=1}^{m} \langle x, u_n \rangle T u_n, \quad x \in X.$$
   Prove that $T_m \in \mathscr{L}(X)$ and that
$$\|(T - T_m)x\| \le \|x\| \sqrt{\sum_{n=m+1}^{\infty} \|Tu_n\|^2}. \tag{5.3}$$
   Hint: In order to prove (5.3), observe that
$$\|(T - T_m)x\| = \left\| \sum_{n=m+1}^{\infty} \langle x, u_n \rangle T u_n \right\| \le \sum_{n=m+1}^{\infty} |\langle x, u_n \rangle| \|Tu_n\|,$$
   and use the Cauchy-Schwarz inequality in ℓ^2.

   (b) Show that every Hilbert-Schmidt operator T is compact.
   Hint: Using (5.3), conclude that T is the limit in $\mathscr{L}(X)$ of the sequence of finite rank operators T_m, $m \in \mathbb{N}$.

5.2 Approximation of compact operators

Compact operators play an important role since they can be approximated by finite rank operators. This means that when we want to solve an operator equation involving a compact operator, then we can replace the compact operator by a sufficiently good finite-rank approximation, reducing the operator equation to an equation involving finite matrices. Their solution can then be found easily using tools from linear algebra. In this section we will prove Theorem 5.2.3, which is the basis of the Projection, Sloan and Galerkin methods in numerical analysis.

In order to prove Theorem 5.2.3, we will need Lemma 5.2.2, which relies on the following deep result.

Theorem 5.2.1 (Uniform boundedness principle) *Let X be a Banach space and Y be a normed space. If $\mathscr{F} \subset \mathscr{L}(X, Y)$ is a family of bounded linear operators that is pointwise bounded, that is,*
$$\forall x \in X, \quad \sup_{T \in \mathscr{F}} \|Tx\| < +\infty,$$
then the family is uniformly bounded, that is, $\sup_{T \in \mathscr{F}} \|T\| < +\infty$.

Proof See Appendix B on page 81. ∎

Lemma 5.2.2 *Let X be a Hilbert space. Suppose that $(T_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{L}(X)$, $T \in \mathscr{L}(X)$ and $S \in \mathscr{C}(X)$. If*

$$\forall x \in X, \quad T_n x \stackrel{n\to\infty}{\longrightarrow} Tx \text{ in } X,$$

then $T_n S \stackrel{n\to\infty}{\longrightarrow} TS$ in $\mathscr{L}(X)$.

Proof Step 1. Suppose that $T_nS - TS$ does not converge to 0 in $\mathscr{L}(X)$ as $n \to \infty$. This means that

$$\neg\left[\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \text{ such that } \forall n > N, \forall x \in X \text{ with } \|x\| = 1, \;\; \|(T_nS - TS)x\| \leq \epsilon\right].$$

Thus there exists an $\epsilon > 0$ such that

$$\forall N \in \mathbb{N}, \;\; \exists n > N, \;\; \exists x_n \in X \text{ with } \|x_n\| = 1 \text{ and } \|(T_nS - TS)x_n\| > \epsilon.$$

So we can construct a sequence $(x_{n_k})_{k\in\mathbb{N}}$ in X such that $\|x_{n_k}\| = 1$ and $\|(T_{n_k}S - TS)x_{n_k}\| > \epsilon$.

Step 2. As $(x_{n_k})_{k\in\mathbb{N}}$ is bounded and S is compact, there exists a subsequence, say $(Sx_{n_{k_l}})_{l\in\mathbb{N}}$, of $(S_{n_k})_{k\in\mathbb{N}}$, that is convergent to y, say. Then we have

$$\begin{aligned} \epsilon < \|(T_{n_{k_l}}S - TS)x_{n_{k_l}}\| &= \|(T_{n_{k_l}} - T)y + (T_{n_{k_l}} - T)(Sx_{n_{k_l}} - y)\| \\ &\leq \|T_{n_{k_l}}y - Ty\| + \|T_{n_{k_l}} - T\|\|Sx_{n_{k_l}} - y\|. \end{aligned} \tag{5.4}$$

Choose $L \in \mathbb{N}$ large enough so that if $l > L$, then

$$\|T_{n_{k_l}}y - Ty\| < \frac{\epsilon}{2} \quad \text{and} \quad \|Sx_{n_{k_l}} - y\| < \frac{\epsilon}{2(M + \|T\|)},$$

where $M := \sup_{n\in\mathbb{N}} \|T_n\|$ ($< \infty$, by Theorem 5.2.1). Then (5.4) yields the contradiction that $\epsilon < \epsilon$. This completes the proof. ∎

Theorem 5.2.3 *Let X be a Hilbert space and let K be a compact operator on X. Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of projections of finite rank and let $K_n^P = P_nK$, $K_n^S = KP_n$, $K_n^G = P_nKP_n$, $n \in \mathbb{N}$. If*

$$\forall x \in X, \quad P_n x \stackrel{n\to\infty}{\longrightarrow} x \text{ in } X,$$

then the operators K_n^P, K_n^S, K_n^G all converge to K in $\mathscr{L}(X)$ as $n \to \infty$.

Proof From Lemma 5.2.2, it follows that $K_n^P \stackrel{n\to\infty}{\longrightarrow} K$ in $\mathscr{L}(X)$. As $P_n = P_n^*$, and K^* is compact, we also have similarly that $P_n^*K^* \stackrel{n\to\infty}{\longrightarrow} K^*$ in $\mathscr{L}(X)$, that is, $\lim\limits_{n\to\infty} \|P_n^*K^* - K^*\| = 0$. Since

$$\|P_n^*K^* - K^*\| = \|(P_n^*K^* - K^*)^*\| = \|KP_n - K\| = \|K_n^S - K\|,$$

we obtain that $\lim\limits_{n\to\infty} \|K_n^S - K\| = 0$, that is, $K_n^S \stackrel{n\to\infty}{\longrightarrow} K$ in $\mathscr{L}(X)$. Finally,

$$\begin{aligned} \|K_n^G - K\| &= \|P_nKP_n - P_nK + P_nK - K\| \\ &\leq \|P_n(KP_n - K)\| + \|P_nK - K\| \\ &\leq \|P_n\|\|K_n^S - K\| + \|K_n^P - K\| \end{aligned}$$

which tends to zero as $n \to \infty$, since $\|P_n\| \leq 1$. ∎

Theorem 5.2.4 *Let X be a Hilbert space and K be a compact operator on X such that $I-K$ is invertible. Let $K_0 \in \mathscr{L}(X)$ satisfy*

$$\epsilon := \|(K-K_0)(I-K)^{-1}\| < 1.$$

Then for given $y, y_0 \in X$, there are unique $x, x_0 \in X$ such that

$$x - Kx = y \text{ and } x_0 - K_0 x_0 = y_0,$$

and

$$\|x - x_0\| \leq \frac{(I-K)^{-1}}{1-\epsilon}(\epsilon\|y\| + \|y-y_0\|). \tag{5.5}$$

Proof As X is a Hilbert space and $\|(K-K_0)(I-K)^{-1}\| = \epsilon < 1$, it follows from Theorem 2.4.3 that

$$I - K_0 = (I-K) + K - K_0 = (I + (K-K_0)(I-K)^{-1})(I-K)$$

is invertible with inverse $(I-K)^{-1}(I+(K-K_0)(I-K)^{-1})^{-1}$, and that

$$\begin{aligned} \|(I-K_0)^{-1}\| &= \|(I-K)^{-1}(I+(K-K_0)(I-K)^{-1})^{-1}\| \\ &\leq \|(I-K)^{-1}\|\|(I+(K-K_0)(I-K)^{-1})^{-1}\| \\ &\leq \frac{\|(I-K)^{-1}\|}{1-\|(K-K_0)(I-K)^{-1}\|} = \frac{\|(I-K)^{-1}\|}{1-\epsilon}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} (I-K)^{-1} - (I-K_0)^{-1} &= (I-K_0)^{-1}((I-K_0)(I-K)^{-1} - I) \\ &= (I-K_0)^{-1}((I-K_0) - (I-K))(I-K)^{-1} \\ &= (I-K_0)^{-1}(K-K_0)(I-K)^{-1}, \end{aligned}$$

and so

$$\|(I-K)^{-1} - (I-K_0)^{-1}\| \leq \|(I-K_0)^{-1}\|\|(K-K_0)(I-K)^{-1}\| \leq \frac{\|(I-K)^{-1}\|}{1-\epsilon}\epsilon.$$

Let $y, y_0 \in X$. Since $I-K$ and $I-K_0$ are invertible, there are unique $x, x_0 \in X$ such that

$$x - Kx = y \text{ and } x_0 - K_0 x_0 = y_0.$$

Also, $x - x_0 = (I-K)^{-1}y - (I-K_0)^{-1}y_0 = [(I-K)^{-1} - (I-K_0)^{-1}]y + (I-K_0)^{-1}(y-y_0)$. Hence

$$\|x - x_0\| \leq \frac{\epsilon\|(I-K)^{-1}\|}{1-\epsilon}\|y\| + \frac{\|(I-K)^{-1}\|}{1-\epsilon}\|y-y_0\|,$$

as desired. ∎

Remark. Observe that as K_0 and y_0 become closer and closer to K and y, respectively, the error bound on $\|x-x_0\|$ (left hand side of (5.5)) converges to 0. In particular, if we take $K_0 = P_n K P_n$ and $y_0 = P_n y$, where P_n is a projection as in Theorem 5.2.3, then we see that the approximate solutions x_0 converge to the actual solution x. We illustrate this procedure in a specific case below.

Example. Consider the following operator on ℓ^2:

$$K = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{3} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We observe that

$$\|Kx\|^2 = \sum_{n=1}^{\infty} \left| \frac{x_{n+1}}{n+1} \right|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} |x_{n+1}|^2 \leq \frac{1}{4}\|x\|^2,$$

and so $\|K\| \leq \frac{1}{2}$. Consequently $I - K$ is invertible. As

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |k_{ij}|^2 = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} < +\infty,$$

it follows that K is compact.

Let $y = (\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots) \in \ell^2$. To find approximate solutions of the equation $x - Kx = y$, we fix an $n \in \mathbb{N}$, and solve $x - P_nKP_nx = P_ny$, that is, the system

$$\left[\begin{array}{ccccc|cc} 1 & -\frac{1}{2} & & & & & \\ & 1 & -\frac{1}{3} & & & & \\ & & 1 & \ddots & & & \\ & & & \ddots & -\frac{1}{n} & & \\ & & & & 1 & & \\ \hline & & & & & 1 & \\ & & & & & & \ddots \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \\ \hline x_{n+1} \\ \vdots \end{array}\right] = \left[\begin{array}{c} \frac{1}{3} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{n+2} \\ \hline 0 \\ \vdots \end{array}\right].$$

The approximate solutions for $n = 1, 2, 3, 4, 5$ are given (correct up to four decimal places) by

$$\begin{aligned} x^{(1)} &= (0.3333, 0, 0, 0, \dots) \\ x^{(2)} &= (0.4583, 0.2500, 0, 0, 0, \dots) \\ x^{(3)} &= (0.4917, 0.3167, 0.2000, 0, 0, 0, \dots) \\ x^{(4)} &= (0.4986, 0.3306, 0.2417, 0.1667, 0, 0, 0, \dots) \\ x^{(5)} &= (0.4998, 0.3329, 0.2488, 0.1952, 0.1428, 0, 0, 0, \dots), \end{aligned}$$

while the exact unique solution to the equation $(I - K)x = y$ is given by $x := \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right) \in \ell^2$. $\Diamond$

5.3 Appendix A: Bolzano-Weierstrass theorem

We used the Bolzano-Weierstrass theorem in the proof of Theorem 5.1.2. In this appendix, we give a proof of this theorem, which says that every bounded sequence in $\mathbb{R}$ has a convergent subsequence. In order to prove this result, we need two auxiliary results, which we prove first.

Lemma 5.3.1 *If a sequence in $\mathbb{R}$ is monotone and bounded, then it is convergent.*

Proof

$\underline{1}^\circ$ Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence. Since $(a_n)_{n\in\mathbb{N}}$ is bounded, it follows that the set

$$S = \{a_n \mid n \in \mathbb{N}\}$$

has an upper bound and so $\sup S$ exists. We show that in fact $(a_n)_{n\in\mathbb{N}}$ converges to $\sup S$. Indeed given $\epsilon > 0$, then since $\sup S - \epsilon < \sup S$, it follows that $\sup S - \epsilon$ is not an upper bound for S and so $\exists a_N \in S$ such that $\sup S - \epsilon < a_N$, that is

$$\sup S - a_N < \epsilon.$$

Since $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence, for $n > N$, we have $a_N \leq a_n$. Since $\sup S$ is an upper bound for S, $a_n \leq \sup S$ and so $|a_n - \sup S| = \sup S - a_n$, Thus for $n > N$ we obtain

$$|a_n - \sup S| = \sup S - a_n \leq \sup S - a_N < \epsilon.$$

$\underline{2}^\circ$ If $(a_n)_{n\in\mathbb{N}}$ is a decreasing sequence, then clearly $(-a_n)_{n\in\mathbb{N}}$ is an increasing sequence. Furthermore if $(a_n)_{n\in\mathbb{N}}$ is bounded, then $(-a_n)_{n\in\mathbb{N}}$ is bounded as well ($|-a_n| = |a_n| \leq M$). Hence by the case considered above, it follows that $(-a_n)_{n\in\mathbb{N}}$ is a convergent sequence with limit

$$\sup\{-a_n \mid n \in \mathbb{N}\} = -\inf\{a_n \mid n \in \mathbb{N}\} = -\inf S,$$

where $S = \{a_n \mid n \in \mathbb{N}\}$. So given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n > N$, $|-a_n - (-\inf S)| < \epsilon$, that is,

$$|a_n - \inf S| < \epsilon.$$

Thus $(a_n)_{n\in\mathbb{N}}$ is convergent with limit $\inf S$. ■

Lemma 5.3.2 *Every sequence in $\mathbb{R}$ has a monotone subsequence.*

We first give an illustration of the idea behind this proof. Assume that $(a_n)_{n\in\mathbb{N}}$ is the given sequence. Imagine that a_n is the height of the hotel with number n, which is followed by hotel $n+1$, and so on, along an infinite line, where at infinity there is the sea. A hotel is said to have the *seaview property* if it is higher than all hotels following it. See Figure 5.1. Now there are only

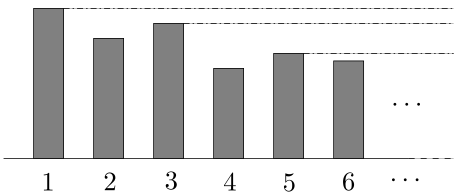

Figure 5.1: The seaview property.

two possibilities:

$\underline{1}^\circ$ There are infinitely many hotels with the seaview property. Then their heights form a decreasing subsequence.

$\underline{2}^\circ$ There is only a finite number of hotels with the seaview property. Then after the last hotel with the seaview property, one can start with any hotel and then always find one that is at least as high, which is taken as the next hotel, and then finding yet another that is at least as high as that one, and so on. The heights of these hotels form an increasing subsequence.

Proof Let

$$S = \{m \in \mathbb{N} \mid \text{for all } n > m,\ a_n < a_m\}.$$

Then we have the following two cases.

$\underline{1}^\circ$ S is infinite. Arrange the elements of S in increasing order: $n_1 < n_2 < n_3 < \dots$. Then $(a_{n_k})_{k\in\mathbb{N}}$ is a decreasing subsequence of $(a_n)_{n\in\mathbb{N}}$.

$\underline{2}^\circ$ S is finite. If S empty, then define $n_1 = 1$, and otherwise let $n_1 = \max S + 1$. Define inductively

$$n_{k+1} = \min\{m \in \mathbb{N} \mid m > n_k \text{ and } a_m \geq a_{n_k}\}.$$

(The minimum exists since the set $\{m \in \mathbb{N} \mid m > n_k \text{ and } a_m \geq a_{n_k}\}$ is a nonempty subset of $\mathbb{N}$: indeed otherwise if it were empty, then $n_k \in S$, and this is not possible if S was empty, and also impossible if S was not empty, since $n_k > \max S$.) ∎

The Bolzano-Weiertrass theorem is now a simple consequence of the above two lemmas.

Theorem 5.3.3 (Bolzano-Weierstrass theorem.) *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence. Then there exists a $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \leq M$. From Lemma 5.3.2 above, it follows that the sequence $(a_n)_{n\in\mathbb{N}}$ has a monotone subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Then clearly for all $k \in \mathbb{N}$, $|a_{n_k}| \leq M$ and so the sequence $(a_{n_k})_{k\in\mathbb{N}}$ is also bounded. Since $(a_{n_k})_{k\in\mathbb{N}}$ is monotone and bounded, it follows from Lemma 5.3.1 that it is convergent. ∎

5.4 Appendix B: uniform boundedness principle

In this appendix, we give a proof of the uniform boundedness principle.

Theorem 5.4.1 (Uniform boundedness principle) *Let X be a Banach space and Y be a normed space. If $\mathscr{F} \subset \mathscr{L}(X, Y)$ is a family of bounded linear operators that is pointwise bounded, that is,*

$$\forall x \in X, \quad \sup_{T\in\mathscr{F}} \|Tx\| < +\infty,$$

then the family is uniformly bounded, that is, $\sup\limits_{T\in\mathscr{F}} \|T\| < +\infty$.

Proof We will assume that $\mathscr{F}$ is pointwise bounded, but not uniformly bounded, and obtain a contradiction. For each $x \in X$, define

$$M(x) = \sup_{T\in\mathscr{F}} \|Tx\|.$$

Our assumption is that for every $x \in X$, $M(x) < +\infty$. Observe that if $\mathscr{F}$ is not uniformly bounded, then for any pair of positive numbers ϵ and C, there must exist some $T \in \mathscr{F}$ with $\|T\| > \frac{C}{\epsilon}$, and hence some $x \in X$ with $\|x\| = \epsilon$, but $\|Tx\| > C$. We can therefore choose sequences $(T_n)_{n\in\mathbb{N}}$ in $\mathscr{F}$ and $(x_n)_{n\in\mathbb{N}}$ in X as follows. First choose T_1 and x_1 so that

$$\|x_1\| = \frac{1}{2} \quad \text{and} \quad \|T_1 x_1\| \geq 2$$

($\epsilon = \frac{1}{2}$, $C = 2$ case). Having chosen $x_1, \dots, x_{n-1}$ and $T_1, \dots, T_{n-1}$, choose x_n and T_n to satisfy

$$\|x_n\| = \frac{1}{2^n \sup_{k<n} \|T_k\|} \quad \text{and} \quad \|T_n x_n\| \geq \sum_{k=1}^{n-1} M(x_k) + 1 + n. \tag{5.6}$$

Now let

$$x = \sum_{n=1}^{\infty} x_n.$$

The sum converges, since

$$\sum_{k=n+1}^{m} \|x_k\| \leq \frac{1}{\|T_1\|} \sum_{k=n+1}^{m} \frac{1}{2^k} \leq \frac{1}{\|T_1\|} \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{\|T_1\|} \frac{1}{2^n} \xrightarrow{n\to\infty} 0,$$

and since X is complete. For any $n \geq 2$,

$$\|T_n x\| = \left\| T_n x_n + \sum_{k\neq n} T_n x_k \right\| \geq \|T_n x_n\| - \left\| \sum_{k\neq n} T_n x_k \right\|.$$

Using (5.6), we bound the subtracted norm above:

$$\begin{aligned} \left\| \sum_{k\neq n} T_n x_k \right\| &\leq \sum_{k=1}^{n-1} \|T_n x_k\| + \sum_{k=n+1}^{\infty} \|T_n\| \|x_k\| \\ &\leq \sum_{k=1}^{n-1} M(x_k) + \sum_{k=n+1}^{\infty} \|T_n\| \frac{1}{2^k \sup_{j<k} \|T_j\|} \\ &\leq \sum_{k=1}^{n-1} M(x_k) + \sum_{k=n+1}^{\infty} \frac{1}{2^k} \leq \sum_{k=1}^{n-1} M(x_k) + 1. \end{aligned}$$

Consequently, $\|T_n x\| \geq n$, contradicting the assumption that $\mathscr{F}$ is pointwise bounded. This completes the proof. ■

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