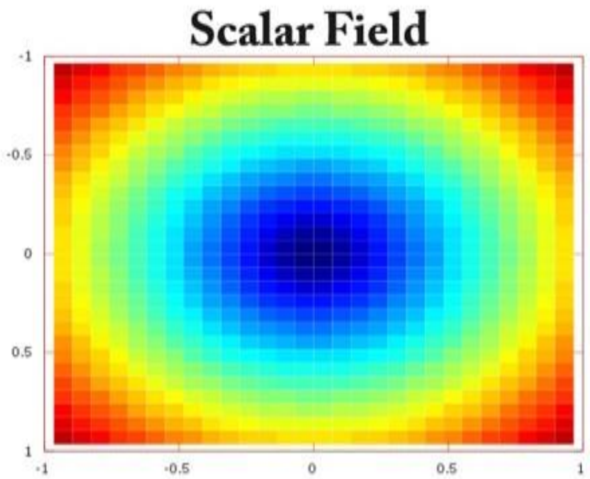
Gradient of Scalar Field

Gradient is vector that represents both the magnitude and direction of the maximum space rate of increase of scalar field. Suppose, we have field F(x,y,z) expressed in rectangular coordinates. A differential charge in this function with differential charge in each coordinate is given by:



$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

$$\overrightarrow{d\ell} = space\ vector$$

$$\overrightarrow{d\ell} = dx\ a_x + dy\ a_y + dz\ a_z$$

$$dF = (\frac{\partial F}{\partial x}a_x + \frac{\partial F}{\partial y}a_x + \frac{\partial F}{\partial z}a_x) \cdot \overrightarrow{d\ell}$$

The quantity between the brackets represent the function gradient.

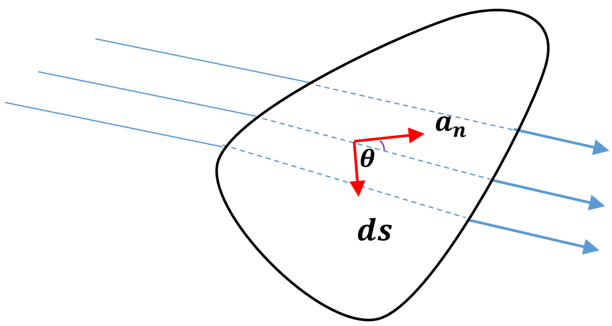
grad F or $\nabla F = \frac{\partial F}{\partial x}a_x + \frac{\partial F}{\partial y}a_x + \frac{\partial F}{\partial z}a_x$

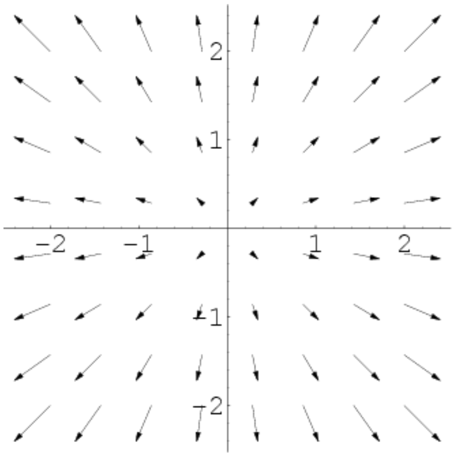
$$dF = \nabla F \cdot \overrightarrow{d\ell}$$

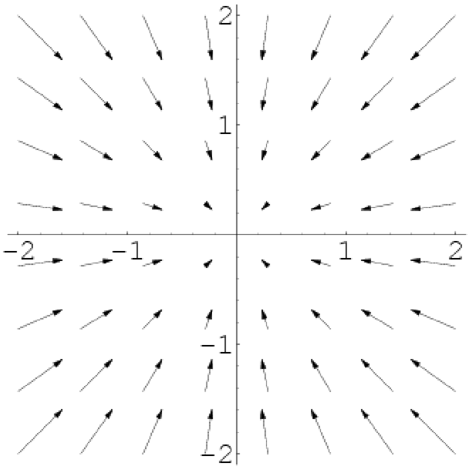
Divergence of vector field

Divergence is a vector operator that produces a scalar field, giving the quantity of a vector field's source at each point. More technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point. The flow of the flux of vector field through some surface is defined as :

$$\varphi = \oint \vec{F} \cdot \overrightarrow{ds}$$







This expansion of fluid flowing with velocity field F is captured by the divergence of F, which we denote divF

The divergence of the above vector field is positive since the flow is expanding.

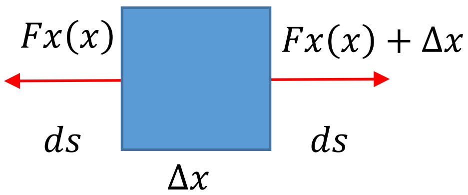
In contrast, the vector field represents fluid flowing so that it compresses as it moves toward the origin. Since this compression of fluid is the opposite of expansion, the divergence of this vector field is negative

- In open surface has two sides; this will give Ambiguous of the normal of the surface. So we deal with closed surface.
- Normal component in computing net flux; the net outward flux of vector field form closed surface provide a measure of magnitude of source or sinks.
    1. If flux is positive this mean sources exceeds sinks strength.
    2. If flux is negative this means sinks exceeds sources strength.

$$div\ F\ or\ \vec{\nabla} \cdot \vec{F} = \lim_{\Delta v \to 0} \oint \frac{F \cdot ds}{\Delta v}$$

Note: This integral to lines of field in unit volume. When the volume become zero.

From define:



$$\oint_{x=0} F \cdot ds = -Fx\Delta y\Delta z$$

$$\oint_{x=\Delta x} F \cdot ds = Fx\Delta y\Delta z + \frac{\partial Fx}{\partial x}\Delta x\Delta y\Delta z$$

The net flux through the volume enclosed by the surface.

$$\oint_{x=\Delta x} F \cdot ds + \oint_{x=0} F \cdot ds = Fx\Delta y\Delta z + \frac{\partial Fx}{\partial x}\Delta x\Delta y\Delta z - Fx\Delta y\Delta z$$
$$= \frac{\partial Fx}{\partial x}\Delta x\Delta y\Delta z$$

In similar way:

$$\frac{\partial Fy}{\partial y}\Delta y\Delta x\Delta z$$

$$\frac{\partial Fz}{\partial z}\Delta z\Delta x\Delta y$$

$$\vec{\nabla} \cdot \vec{F} = \lim_{\Delta v \to 0} \frac{\left(\frac{\partial Fx}{\partial x} + \frac{\partial Fy}{\partial y} + \frac{\partial Fz}{\partial z}\right)\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z}$$

$\vec{\nabla} \cdot \vec{F} = \left(\frac{\partial Fx}{\partial x} + \frac{\partial Fy}{\partial y} + \frac{\partial Fz}{\partial z}\right)$ Cartisian

$\vec{\nabla} \cdot \vec{F} = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta} \cdot \frac{\partial F_\theta \sin\theta}{\partial \theta} + \frac{1}{r\sin\theta} \cdot \frac{\partial F_\emptyset}{\partial \emptyset}$ Spherical

$\vec{\nabla} \cdot \vec{F} = \frac{1}{r} \cdot \frac{\partial}{\partial r}(rF_r) + \frac{1}{r} \cdot \frac{\partial F_\emptyset}{\partial \emptyset} + \frac{\partial F_z}{\partial z}$ Cylindrical

Problems:

1. Given $\vec{A} = e^{-y}(\cos x \, a_x - \sin x \, a_y)$ Find $\vec{\nabla} \cdot \vec{A}$?

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}(e^{-y}\cos x) + \frac{\partial}{\partial y}(-e^{-y}\sin x)$$

$$\vec{\nabla}\cdot\vec{A} = -e^{-y}\sin x + e^{-y}\sin x = 0$$

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2. Given $\vec{A} = r\sin\emptyset\ a_r + 2r\cos\emptyset\ a_\emptyset + 2Z^2 a_z$ Find $\vec{\nabla}\cdot\vec{A}$?

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\cdot\frac{\partial}{\partial r}(r^2\sin\emptyset) + \frac{1}{r}\cdot\frac{\partial}{\partial\emptyset}(2r\cos\emptyset + \frac{\partial}{\partial z}(2Z^2)$$

$$\vec{\nabla}\cdot\vec{A} = 2\sin\emptyset - 2\sin\emptyset + 4\,Z = 4Z$$

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3. Given $\vec{A} = \frac{5}{r^2}\sin\theta\ a_r + r\cot\theta\ a_\theta + r\sin\theta\cos\emptyset\, a_\emptyset$ Find $\vec{\nabla}\cdot\vec{A}$?

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2\frac{5}{r^2}\sin\theta) + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\theta}(r\sin\theta\cot\ \theta) + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\emptyset}(r\sin\theta\cos\emptyset)$$

$$\vec{\nabla}\cdot\vec{A} = -1 - \sin\emptyset$$

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4.Given $\vec{A} = 10\sin^2\emptyset\, a_r + r\ a_\emptyset + \left[\left(\frac{z^2}{r}\cos^2\emptyset\right)a_z\right]$ find $\vec{\nabla}\cdot\vec{A}$ at (2, ∅,5).

$$\vec{\nabla}\cdot\vec{A} = \frac{10\sin^2\emptyset + 2z\cos^2\emptyset}{r}$$

$$\vec{\nabla}\cdot\vec{A}\ \ at\ (2,\emptyset,5) = 5$$

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H.W

1. If $\vec{F} = e^{5x}a_x + 2\cos y\ \ a_y + 2\sin z\ \ a_z$ Find $\vec{\nabla}\cdot\vec{F}$.

2. $\vec{F} = ra_r - r^2\cot\theta\ \ a_\theta$ Find $\vec{\nabla}\cdot\vec{F}$.

3.Find the gradient of the following $a)\ F(r,\emptyset,z) = 2\sin\emptyset\ - rz + 4$ $b)\ F(r,\theta,\emptyset) = 2r\cos\ \theta - 5\emptyset + 2$
