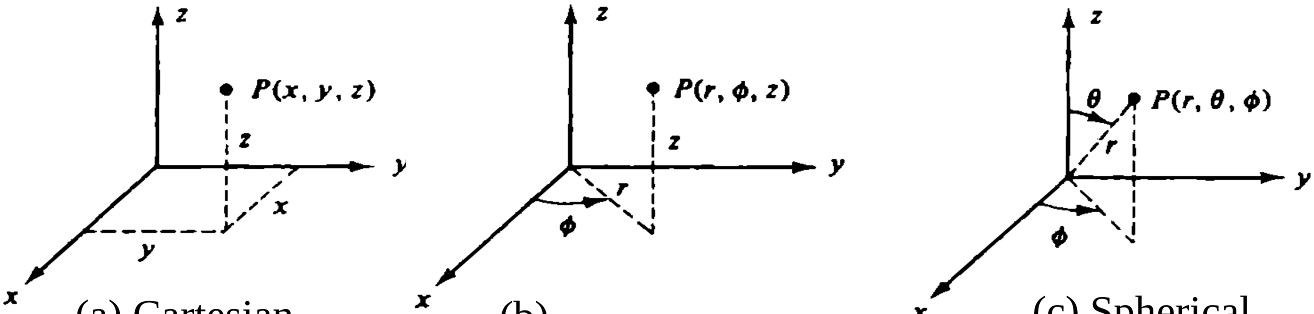
- **Coordinate Systems**

A point P is described by three coordinates, in Cartesian (x,y,z), circular cylindrical (r,ϕ,z) and spherical (r,θ,ϕ) as shown in figure.2.



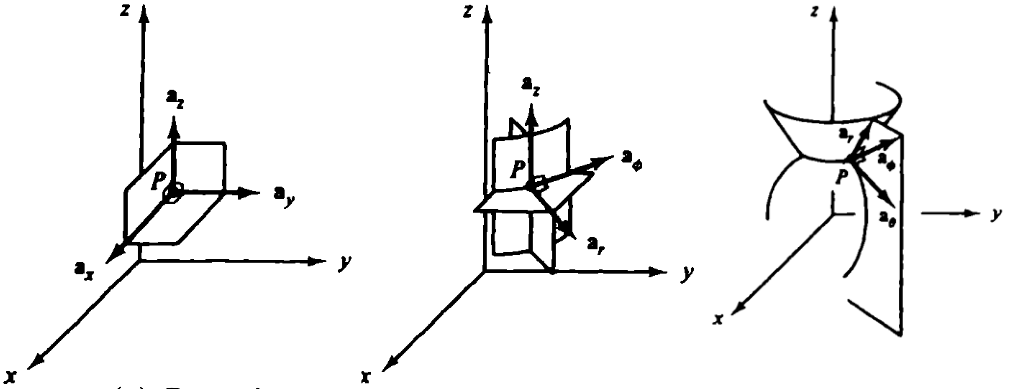
(a) Cartesian (b) (c) Spherical

Figure 2. Coordinate Systems Definition

The order of specifying coordinate is important and should be carefully followed. The angle ($\emptyset$) is the same in both cylindrical and spherical systems , but in the order coordinate ($\emptyset$) appears in second position in cylindrical (r,ϕ,z) and third position in spherical $(r,\theta, \emptyset)$. Where the angle $\emptyset$ is limited to range $0 \leq \emptyset \leq 2\pi$ and θ is limited to range $0 \leq \theta \leq \pi$. The symbol (r)is used in both cylindrical and spherical for two quite different things. In cylindrical (r) measure the distance from the z-axis plan normal to the z-axis while in spherical measure the distance from the origin to the point.

In the Cartesian system the unit vectors have fixed directions, independent of the location of P. This is not true for the other two systems (except in the case of a_z). We notice that all these three systems are satisfy to right hand rule.

$$a_x \times a_y = a_z \qquad a_r \times a_\emptyset = a_z \qquad a_r \times a_\theta = a_\emptyset$$



(a) Cartesian (b) Cylindrical (c) Spherical

The component forms of a vector in the three system are:

$$\vec{A} = A_x a_x + A_y a_y + A_z a_z \qquad cartesian$$

$$\vec{A} = A_r a_r + A_\theta a_\theta + A_z a_z \qquad cylindrical$$

$$\vec{A} = A_r a_r + A_\theta a_\theta + A_\emptyset a_\emptyset \qquad spherical$$

It should be noted that be the components A_x , $A_r, A_\emptyset, etc$ are generally not constant but are often functions of the coordinates in that particular system.

• Differential volume, surface, and line elements.

There are relatively few problems in electromagnetics that can be solved without some sort integration - along a curve ,over a surface, throughout a volume. When the coordinate of the point P are expended to:

$$x + dx, y + dy, z + d_z$$

$$r + dr, \emptyset + d\emptyset, z + dz$$

$$r + dr, \theta + d\theta, \emptyset + d\emptyset$$

A differential volume *dv* is formed. To the first order infinitesimal quantities the differential is, in all three coordinate systems, a rectangular box. The value of *dv* in each system is given by

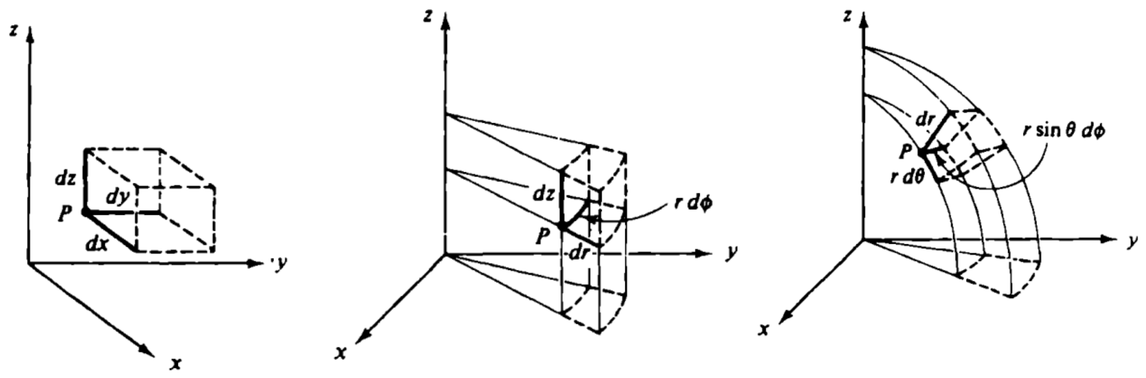


fig (a) Cartesian (b) Cylindrical (c) Spherical

Figure 4. Differential volume of the coordinate systems

So we find dv of the three coordinate system are:

$dv = dx\, dy\, dz$ — *Cartesian*

$dv = rdr\, d\emptyset\, dz$ — *Cylindrical*

$dv = r^2 \sin\theta\, dr\, d\theta\, d\emptyset$ — *spherical*

May also can be represented the area of the surface element for the coordinates is:

$ds = dx\, dy$ — Cartesian

$ds = rd\emptyset\, dz$ — *Cylindrical*

$ds = r^2 \sin\theta\; d\theta\, d\emptyset$ — *Spherical*

The differential line element $d\ell$ is the diagonal through **P**, thus:

$d\ell^2 = dx^2 + dy^2 + dz^2$ — Cartesian

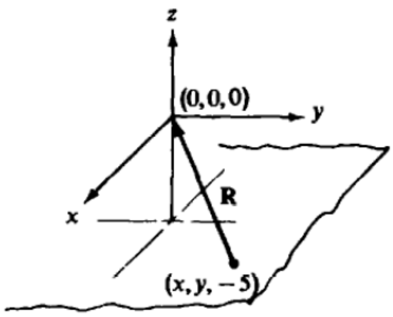
$d\ell^2 = dr^2 + rd\emptyset^2 + dz^2$ — *Cylindrical*

$d\ell^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\emptyset^2$ — *Spherical*

Problems:

1. Express the unit vector which is directed toward to the origin point from an arbitrary point on the plane $z = -5$

Since the problem in Cartesian coordinate so we can represent the vector R as:

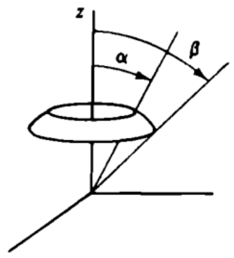


$$\vec{R} = -xa_x - ya_y + 5a_z$$

$$a_R = \frac{-xa_x - ya_y + 5a_z}{\sqrt{x^2 + y^2 + 25}}$$

2. Express the unit vector which points from z=h on the z - axis toward (r,ϕ,0) in cylindrical coordinate. H.W

3. Use the spherical coordinate system to find the area of the strip $\propto \le \theta \le \beta$ on the spherical shell of radius $r = r_0$. What results when $\propto = 0 \quad \beta = \pi$?



$$ds = r^2 \sin\theta \, d\theta \, d\emptyset$$

$$A = \int_0^{2\pi} \int_\alpha^\beta r_0^2 \sin\theta \, d\theta \, d\emptyset$$

$$A = 2\pi r_0^2 (\cos\alpha - \cos\beta)$$

When $\propto = 0 \quad \beta = \pi$

$$A = 4\pi r_0^2$$

This give us the area of the entire sphere

4. Use the cylindrical coordinate system to find the area of the curved surface of a right circular cylinder where r=2m h=5m and $30° \le \emptyset \le 120°$.

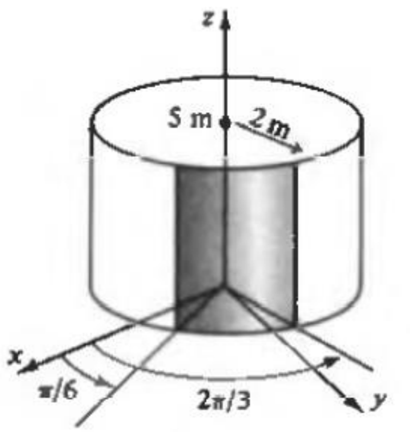
The differential surface element is

$$ds = r \, d\emptyset \, dz$$

then

$$A = \int_0^5 \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} 2 d\emptyset dz$$

$$A = 5\pi \, m^2$$



5. Transform the vector $\vec{A} = y a_x + x a_y + \frac{x^2}{\sqrt{x^2+y^2}} a_z$ from Cartesian to Cylindrical coordinate.

$$x = r\cos\emptyset \quad y = r\sin\emptyset \quad r = \sqrt{x^2 + y^2}$$

$$\vec{A} = r\sin\emptyset a_x + r\cos\emptyset \, a_y + r \, cos^2\emptyset \, a_z$$

Now the projection of the Cartesian unit vector on $a_r, a_\emptyset \; and \; a_z$ are obtained:

$a_x \cdot a_r = \cos \emptyset \qquad a_x \cdot a_\emptyset = -\sin \emptyset \qquad a_x \cdot a_z = 0$

$a_y \cdot a_r = \sin \emptyset \qquad a_y \cdot a_\emptyset = \cos \emptyset \qquad a_y \cdot a_z = 0$

$a_z \cdot a_r = 0 \qquad a_z \cdot a_\emptyset = 0 \qquad a_z \cdot a_z = 1$

Therefor:

$$a_x = \cos \emptyset \, a_r - \sin \emptyset \; a_\emptyset$$

$$a_y = \sin \emptyset \, a_r + \cos \emptyset \; a_\emptyset$$

$$a_z = a_z$$

And $\vec{A} = 2r \sin \emptyset \cos \emptyset \, a_r + (r \cos^2 \emptyset - r\, sin^2 \emptyset \,) a_\emptyset + r \cos^2 \emptyset \; a_z$

6. Transform the same vector in point 5 to the spherical coordinate. H.W

7. Given point P(-2, 6, 3) and vector A = ya_x + (x + z)a_y, express P and A in cylindrical and spherical coordinates.

At point P

$x = -2 \quad y = 6 \quad z = 3$

$r = \sqrt{x^2 + y^2} \qquad \sqrt{4 + 36} = 6.32$

$\emptyset = \tan^{-1} \dfrac{y}{x} = \dfrac{6}{-2} = 108.43°$

$z = 3$

$r = \sqrt{x^2 + y^2 + z^2} \qquad \sqrt{4 + 36 + 9} = 7$

$\theta = \tan^{-1} \dfrac{\sqrt{x^2 + y^2}}{z} = \dfrac{6.32}{3} = 64.62°$

Thus,

P(x,y,z)= (-2, 6, 3)

P(r, θ ,z)= (6.32, 108.43° , 3)

P(r, θ , ∅)= (7, 64.62°,108.43°)

In the Cartesian system, A at P is:

$\vec{A} = 6a_x + a_y$

Ax = y, Ay = x + z, Az = 0.

Hence, in the cylindrical system

$$Ar = y\cos\emptyset + (x+z)\sin\emptyset$$

$$A\emptyset = -y\sin\emptyset + (x+z)\cos\emptyset$$

$$Az = 0$$

But x=r cos ∅, y=r sin ∅ we substituted these in equations we get:

$$\begin{bmatrix} Ar \\ A\emptyset \\ Az \end{bmatrix} = \begin{bmatrix} \cos\emptyset & \sin\emptyset & 0 \\ -\sin\emptyset & \cos\emptyset & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

or

$$\vec{A} = (Ar, A\emptyset, Az)$$
$$= (r\sin\emptyset\cos\emptyset + (r\cos\emptyset + z)\sin\emptyset)a_r$$
$$+ (-r\,sin^2\,\emptyset + (r\cos\emptyset + z)\cos\emptyset)a_\emptyset$$

At P

$$r = \sqrt{40} \qquad \tan\emptyset = \frac{6}{-2} \qquad \cos\emptyset = \frac{-2}{\sqrt{40}} \qquad \sin\emptyset = \frac{6}{\sqrt{40}}$$

$$\vec{A} = \left[\sqrt{40}\ \cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + (\sqrt{40}\ \cdot\frac{-2}{\sqrt{40}} + 3)\cdot\frac{6}{\sqrt{40}}\right]a_r$$
$$+ \left[-\sqrt{40}\ \cdot\frac{36}{40} + (\sqrt{40}\ \cdot\frac{-2}{\sqrt{40}} + 3)\cdot\frac{-2}{\sqrt{40}}\right]a_\emptyset$$

$$\vec{A} = \frac{-6}{\sqrt{40}}a_r - \frac{38}{\sqrt{40}}a_\emptyset = -0.9487a_r - 6.008\,a_\emptyset$$