Vector Algebra

Introduction

- Vectors are introduced in physics and mathematics courses in Cartesian, Spherical and Cylindrical coordinate. All three coordinate systems must be used in Electromagnetics.
- In order to distinguish vectors (quantities having magnitude and direction) from scalars (quantities having magnitude).

By use of the unit vectors a_x,a_y,a_z along the x,y,z axes of Cartesian coordinate system, vector A can be written as:

$$\vec{A} = A_x a_x + A_y a_y + A_z a_z$$

The absolute value of a vectors can be defined as:

$$|\vec{A}| = \vec{A} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

So the unit vectors in the direction of a vector A is determined by:

$$a = \frac{\vec{A}}{|\vec{A}|}$$

- **Vectors algebra**

1. Vectors may be add and subtracted.

$$\vec{A} \pm \vec{B} = (A_x a_x + A_y a_y + A_z a_z) \pm (B_x a_x + B_y a_y + B_z a_z)$$

$$(A_x \pm B_x)_{ax} + (A_y \pm B_y)_{ay} + (A_z \pm B_z)_{az}$$

2. The associative, distributive, and commutative laws apply:

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

$$k(\vec{A} + \vec{B}) = k\vec{A} + k\vec{B}$$

$$(k_1 + k_2)\vec{A} = k_1 \vec{A} + k_2 \vec{B}$$

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

3. The dot product of two vectors is, by definition

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

Where θ is smaller angle between $\vec{A}$ and $\vec{B}$.

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

which gives, in particular

$$|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$$

The dot product obeys the distributive and scalar multiplication laws

$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$ $\qquad$ $\vec{A} \cdot k\vec{B} = k(\vec{A} \cdot \vec{B})$

Example 1 .1: Using the properties of dot product prove property 3 from above.

$$\vec{A} \cdot \vec{B} = (A_x a_x + A_y a_y + A_z a_z) \cdot (B_x a_x + B_y a_y + B_z a_z)$$
$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

❖ Note

In Dot product $ax \cdot ax = ay \cdot ay = az \cdot az = 1$ when θ=zero

And $ax \cdot ax = ay \cdot ay = az \cdot az = 0$ when

4. The cross product between two vectors define as:

$$\vec{A} \times \vec{B} = (A\ B\ sin\theta)_{an}$$

Where θ is the smaller angle between A and B, and a_n is a unit vector normal to the plane determined by A and B when they are drawn from the common point. The unit vector which selected when A move towards B as shown in the figure this satisfy to right hand rule.

The commutative laws not apply to the cross product, so we get:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

❖ Note

When the two vectors are parallel the cross product =0 because the angle θ =0

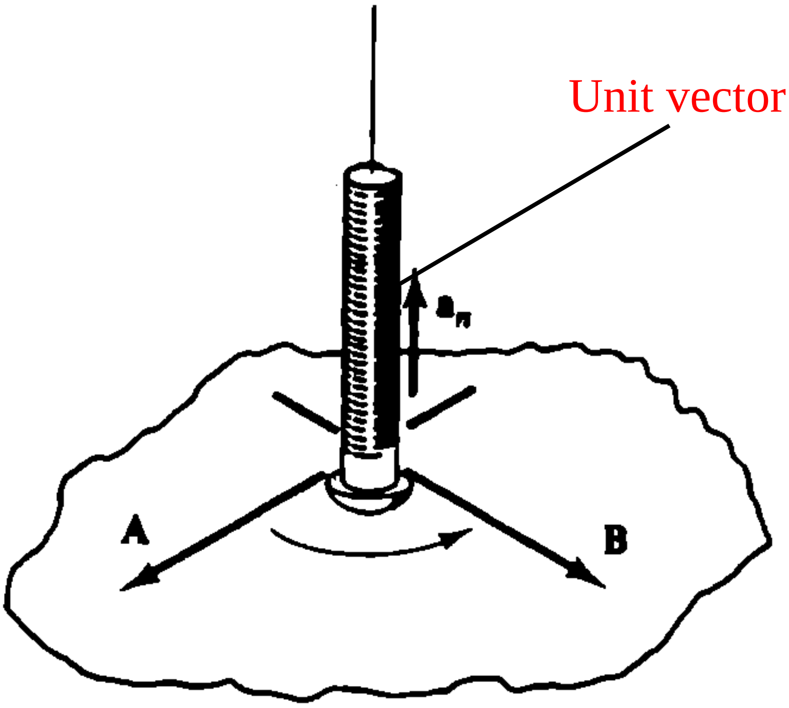

Fig 1. Right Hand Rule

Expending the cross product in component form.

$$\vec{A} \times \vec{B} = \left(A_x a_x + A_y a_y + A_z a_z\right) \times \left(B_x a_x + B_y a_y + B_z a_z\right)$$

$$\vec{A} \times \vec{B} = \left(A_y B_z + A_z B_y\right) a_x + (A_z B_x + A_x B_z) a_y + \left(A_x B_y + A_z B_x\right) a_z$$

Which is conveniently expressed as a determinant:

$$\vec{A} \times \vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example 2: Given $\vec{A} = 2a_x + 4a_y - 3a_z \quad \vec{B} = a_x - a_y$ find $\vec{A} \cdot \vec{B}$ *and* $\vec{A} \times \vec{B}$.

$$\vec{A} \cdot \vec{B} = (2)(1) + (4)(-1) - (3)(0) = -2$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ 2 & 4 & -3 \\ 1 & 1 & 0 \end{vmatrix} = -3a_x - 3a_x - 6a_z$$

Problems:

1. What is the angle between the two vectors $\vec{A} = 3a_x + a_y - 4a_z \quad \vec{B} = 2a_x + 14a_y + 5a_z$.

$$\vec{A} = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

$$\vec{A} = \sqrt{3^2 + 1^2 + (-4^2)} = \sqrt{26}$$

$$\vec{B} = \sqrt{2^2 + 14^2 + 5^2} = \sqrt{225}$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{\vec{A}\vec{B}} = \frac{(3)(2) + (14)(1) + (-4)(5)}{\sqrt{26}\ \sqrt{225}} = 0$$

This mean θ=90° and $A \perp B \quad (\vec{A} \cdot \vec{B}) = 0$

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2. What is the angle between the two vectors $\vec{A} = a_x - a_z \quad \vec{B} = -a_y + a_z$. H.W

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3. If we have $\vec{A} = -3a_x + a_y + 4a_z \quad \vec{B} = a_x + a_y + a_z$. find the component of the vectors A towards the vector B.

$$\vec{A} \cdot \vec{B} = B(A\ \cos\theta)$$

$$A\cos\theta = \frac{\vec{A}\cdot\vec{B}}{B} = \frac{-3+1+4}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

4. Find the value of X in which the vectors $\vec{A} = a_x - 6a_y$is perpendicular of the vectors $\vec{B} = 3a_x + 2a_y$. H.W

5. Effective force of a particle is $\vec{F} = 7a_x + 5a_y$ it caused displacement $\vec{r} = 3a_x - 4a_y$ Calculate the work done by this force.

$$W = \vec{F}.\vec{r} = (7a_x + 5a_y).(3a_x - 4a_y$$
$$W = 21 + 20 = 41\ N.m$$

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6. If $A = 2a_x + 6a_y - 27a_z$ and $B = a_x + 3a_y - \frac{27}{2}a_z$
Find $A \times B$.

$$\vec{A}\times\vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ 2 & 6 & -27 \\ 1 & 3 & \frac{-27}{2} \end{vmatrix} = 0$$

This mean AB sin θ=0 and θ=180° or θ=0°.

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7. Find a vector that is vertical of the two vectors $\vec{A} = 7a_x - 4a_y - 3a_z$ $\vec{B} = -a_x + a_y + a_z$ and how can you prove this vector is vertical?? H.W

8. Calculate the area of parallelogram if there side are $\vec{A} = 3a_x - 4a_y + 5a_z$ and $\vec{B} = a_x - 6a_z$.

Parallelogram area=$|\vec{A}\times\vec{B}|$

$$\vec{A}\times\vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ 3 & -4 & 5 \\ 1 & 0 & -6 \end{vmatrix} = 24a_x + 23a_y + 4a_z$$
$$|\vec{A}\times\vec{B}| = \sqrt{(24)^2 + (23)^2 + (4)^2}$$

$$\sqrt{1121} = 33.48$$

9. Mass of particle is 2 kg and its velocity $\vec{v} = 2a_x + 3a_y - 2a_z$ when the particle position is $\vec{r} = a_x - 2a_y - a_z$ calculate the Angular movement of the particle in the origin point. H.W