Matrix

An $m \times n$ Matrix $A$ is a rectanglar arry of $mn$ real or (complex) numbers arranged in $m$ horizontal rows and $n$ vertical columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the $i$-th row of $A$ is $[a_{i1} \; a_{i2} \cdots a_{in}]$ $\quad 1 \le i \le m$

The $j$-th column of $A$ is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, $1 \le j \le n$

Note: If $m = n$ the we say that $A$ is a squre matrix of order $n$ and the main daigonal of $A$ is $[a_{11} \; a_{22} \cdots a_{nn}]$

Def:-

A squar matrix $A = [a_{ij}]_{n \times n}$ for which every term of main diagonal is zero that is $a_{ij} = 0$ for $i \ne j$ is called diagonal matrix

for example $A = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

Def:

A diagonal matrix $A = [a_{ij}]$ for which are the element in the main daigonal are equal, that is $a_{ij} = c$ where $i = j$ and $a_{ij} = 0$ where $i \neq j$ is called a Scalar matrix

for example: $I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Def:

The two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if $a_{ij} = b_{ij}$, $1 \le i \le m$, $1 \le j \le n$,

that is, If corresponding elements are agree

forexample $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{bmatrix}$

are equal if $w = -1$, $x = -3$, $y = 0$, and $z = 5$

Matrix addition

Def: If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then the same of A and B is

the $m\times n$ matrix $C = [c_{ij}]_{m\times n}$

s.t $c_{ij} = a_{ij} + b_{ij}$ $(1 \le i \le m,\ 1 \le j \le n)$

for example: let $A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 1 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 2 & -4 \\ 1 & 3 & 1 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 4 \end{bmatrix}$$

Matrix multipliaction

Def:

If $A = [a_{ij}]_{m\times p}$ and $B = [b_{ij}]_{p\times n}$

the AB is $m\times n$ matrix $C = [c_{ij}]$ defined by

$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj}$ $(1 \le i \le m,\ 1 \le j \le n)$

for example: let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}_{3\times 2}$

$$AB = \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}$$

Note!

$AB \ne BA$ in the matrices

## Scalar multipliaction

suppose that $r$ be a real number then $rA = [ra_{ij}]_{m\times n}$

for example: let $A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$ then $2A = \begin{bmatrix} 4 & 0 \\ 2 & 8 \end{bmatrix}$

## The transpose of the matrix

Def:

If $A = [a_{ij}]_{m\times n}$ then its transpose is $A^T = [a_{ji}]_{n\times m}$

That is mean $a^T_{ij} = a_{ji}$ $(1 \le i \le m\ ;\ 1 \le j \le n)$

for example:

let $A = \begin{bmatrix} 4 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}$ then $A^T = \begin{bmatrix} 4 & 3 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}$

Theorem: let $A, B$ are matrices of $m\times n$. then

① $A + B = B + A$

② $A + (B + C) = (A + B) + C$

③ $A + 0 = 0 + A = A$ s.t $0$ is zero matrix

Theorem:

If $r$ is a scalar and $A, B$ are matrices then:

① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

③ $(AB)^T = B^T A^T$

④ $(rA)^T = r(A^T)$

Def:

A matrix $A = [a_{ij}]$ is called symmetric matrix if $A^T = A$

For example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$\therefore A^T = A$

So that A is a symmetric matrix

"Type of matrices"

① Row matrix

A matrix which has exactly one row is called row matrix

$$A = [1 \quad 2 \quad 3]_{1\times 3}$$

② Column matrix

A matrix which has exactly one columns is called column matrix

$$A = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}_{3\times 1}$$

③ Null matrix:

A matrix which has all elements are zero is called a null matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

④ Identity matrix

A diagonal matrix whose mine diagonal elements are all equal to 1 and denoted by $I_n$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

⑤ Traingular matrix

A square matrix whose element $a_{ij} = 0$ when $j > i$ then the matrix is called lower traingular matrix.

similarly a square matrix whose $a_{ij} = 0$ when $i < j$ is called upper traingular matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix}$$

lower traingular matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

upper traingular matrix

Determinants

Introduction

To every square matrix that is assigned a specific number called the determinant to the matrix

Note:

we denoted to the determinant by $|A|$

The methods to find the determinant

① Matrix of dimension $2\times2$

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = ad - cb$$

② Matrix of order $3\times3$

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

we can find $|A|$ by

$$|A| = a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32}$$
$$-\left(a_{13}\,a_{22}\,a_{31} + a_{11}\,a_{23}\,a_{32} + a_{12}\,a_{21}\,a_{33}\right)$$

③ by this method we can find the determinant of matrix by any dimension. we will apply to matrix of $3\times3$.

let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Ex: Suppose that

$A = \begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}$ find $|A|$

$\therefore |A| = 3 \times 4 - ((-2) \times 1) = 12 + 2 = 14$

Ex:

Suppose that $A = \begin{pmatrix} 2 & 9 & 3 \\ 1 & -4 & 5 \\ 7 & -3 & 5 \end{pmatrix}$ find $|A|$

$|A| = 335$

The properties of determinants

① $\det(A) = \det(A^T)$

where $A^T$ is the transpose of A.

② If any two rows (or two columns) of a matrix are interch-anging the value of determinant is multiply by $-1$

③ If all elements of rows or columns of a matrix are zero then $\det(A) = 0$

④ If two parallel columns or rows of a matrix are equal then $\det(A) = 0$

⑤ If all the elements of one row or columns of determinant are multiplied by the same factor k, the value of new determinant is k times the given determinant.

Examples:-

① $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , $\therefore |A| = 4 - 6 = -2$

since $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ , $\therefore |A| = 4 - 6 = -2$

$\therefore |A| = |A^T|$

② let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\therefore |A| = -2$

If we interchange row1 by row2

$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $\therefore |B| = 6 - 4 = 2$

$\therefore |A| = -|B|$ by 2

③ let $A = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$

$|A| = 0(3) - 0(1) = 0$ by $\underline{\underline{3}}$

④ let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore |A| = 1 - 1 = 0$ by $\underline{4}$

⑤ let $A = \begin{bmatrix} 4 & 6 & 1 \\ 3 & -9 & 2 \\ -1 & 12 & 3 \end{bmatrix}$

$|A| = -243$

$$A = \begin{bmatrix} 4 & 6 & 1 \\ 3 & -9 & 2 \\ -1 & 12 & 3 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 4 & 2 & 1 \\ 3 & -3 & 2 \\ -1 & 4 & 3 \end{bmatrix}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

Ex:

Solve the system

$3x_1 - x_2 = 9$

$x_1 + 2x_2 = -4$

sol:-

The system can put in the form

$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \end{pmatrix}$$

$$\therefore D = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 7$$

$$x_1 = \frac{\begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix}}{D} = \frac{14}{7} = 2$$

$$x_2 = \frac{\begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix}}{D} = \frac{-21}{7} = -3$$

Ex:-

Solve the system by Gramer's rule

$x_1 + 3x_2 - 2x_3 = 11$

$4x_1 - 2x_2 + x_3 = -15$

$3x_1 + 4x_2 - x_1 = 3$

Sol:-

we can write the system in the form

$$\begin{pmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ -15 \\ 3 \end{pmatrix}$$

Since

$$D = \begin{vmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -25$$

$$x_1 = \frac{\begin{vmatrix} 11 & 3 & -2 \\ -15 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix}}{-25} = \frac{50}{-25} = -2$$

$$x_2 = \frac{\begin{vmatrix} 1 & 11 & -2 \\ 4 & -15 & 1 \\ 5 & 3 & -1 \end{vmatrix}}{-25} = \frac{-25}{25} = -5$$

$$x_3 = \frac{\begin{vmatrix} 1 & 3 & 11 \\ 4 & -2 & -15 \\ 3 & 4 & 3 \end{vmatrix}}{-25} = \frac{125}{-25} = -5$$

The Gauss - Jordan reduction

Solve the system by using Gauss-Jordan.

$$x + 2y + 3z = 9$$
$$2x - y + z = 8$$
$$3x \quad - z = 3$$

So we can write the system in the form

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \underset{-3R_1 + R_3}{\overset{-2R_1 + R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right] \overset{-\frac{1}{5}R_2}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right] \overset{6R_2 + R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \overset{-\frac{1}{4}R_3}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{-2R_2 + R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \underset{-R_3 + R_2}{\overset{-R_3 + R_1}{\sim}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$x = 2$$
$$y = -1$$
$$z = 3$$