"Normed Space"

Definition:

Let X is vector space over field, then the function $\|\cdot\|: X \longrightarrow R$ is called norm function if it is satisfies the following axioms:-

① $\|x\| \geq 0$ for all $x \in X$.

② $\|x\| = 0$ iff $x = 0$.

③ $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and $\lambda \in F$.

④ $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The space $(X, \|\cdot\|)$ is called normed space and denoted by X.

Remark:- If $F = \mathbb{R}$ is called real normed space and If $F = \mathbb{C}$ is called complex normed space.

Theorem; If X is normed space then:

① $\|0\| = 0$

② $\|-x\| = \|x\|$ for all $x \in X$

③ $\|x-y\| = \|y-x\|$ for all $x, y \in X$

④ $\big|\|x\| - \|y\|\big| \leq \|x-y\|$ for all $x, y \in X$.

proof:-

①, (2) and (3) it easy to show it from definition

(4) proof: $\underset{\sim}{x} = (\underset{\sim}{x} - \underset{\sim}{y}) + \underset{\sim}{y}$

$$\|\underset{\sim}{x}\| = \|(\underset{\sim}{x} - \underset{\sim}{y}) + \underset{\sim}{y}\| \le \|\underset{\sim}{x} - \underset{\sim}{y}\| + \|\underset{\sim}{y}\|$$

$$\|\underset{\sim}{x}\| - \|\underset{\sim}{y}\| \le \|\underset{\sim}{x} - \underset{\sim}{y}\| \text{ --- } (1)$$

by the same way

$$\underset{\sim}{y} = (\underset{\sim}{y} - \underset{\sim}{x}) + \underset{\sim}{x}$$

$$\|\underset{\sim}{y}\| = \|(\underset{\sim}{y} - \underset{\sim}{x}) + \underset{\sim}{x}\| \le \|\underset{\sim}{y} - \underset{\sim}{x}\| + \|\underset{\sim}{x}\|$$

$$\|\underset{\sim}{y}\| - \|\underset{\sim}{x}\| \le \|\underset{\sim}{y} - \underset{\sim}{x}\| = \|\underset{\sim}{x} - \underset{\sim}{y}\|$$

$$-\|\underset{\sim}{x} - \underset{\sim}{y}\| \le \|\underset{\sim}{x}\| - \|\underset{\sim}{y}\| \text{ ---- } (2)$$

from (1) and (2) we get

$$-\|\underset{\sim}{x} - \underset{\sim}{y}\| \le \|\underset{\sim}{x}\| - \|\underset{\sim}{y}\| \le \|\underset{\sim}{x} - \underset{\sim}{y}\|$$

$$\therefore \big|\, \|\underset{\sim}{x}\| - \|\underset{\sim}{y}\| \,\big| \le \|\underset{\sim}{x} - \underset{\sim}{y}\|$$

"Some important inequalites"

(1) Holder's inequality

If $p, q \in \mathbb{R}$ s.t $\frac{1}{p} + \frac{1}{q} = 1$ then $\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$

In spcial case, if $p = 2$ then $q = 2$ and $\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |y_i|^2\right)^{\frac{1}{2}}$ is called "Caushy – Schwar's inequality"

(2) Minkowsk's inequality:

If $p \ge 1$ then $\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$

Ex: Let $X = \mathbb{R}$ and $\|\cdot\| : X \longrightarrow \mathbb{R}$ which defined by: $\|x\| = |x|$ for all $x \in X$ be norm over X.

Sol:- ① $\|x\| = |x|$

since $|x| \geq 0 \implies \|x\| \geq 0$

② $\|x\| = 0 \iff |x| = 0 \iff x = 0$

③ Let $x \in X$ & $\lambda \in F$

$\|\lambda x\| = |\lambda . x| = |\lambda| \, |x| = |\lambda| . \|x\|$

④ Let $x, y \in X$

$\|x + y\| = |x + y| \leq |x| + |y| = \|x\| + \|y\|$

$\therefore$ $\|\cdot\|$ is normed function over $\mathbb{R}$.

$\therefore$ X is normed space.

Ex: Let $\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$ which defined by $\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $\|\cdot\|$ is norm over $\mathbb{R}^n$.

Sol:- ① Since $x_i^2 \geq 0$ for all $i = 1, \dots, n$ then $\|x\| \geq 0$

② $\|x\| = 0 \iff (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} = 0$

$\iff \sum_{i=1}^{n} x_i^2 = 0$

$\iff x_i^2 = 0$ for all $i = 1, \dots, n$

$\iff x_i = 0$ for all $i = 1, \dots, n$

$\iff x = 0$

③ Let $x \in R^n$ and $\lambda \in R$

$\lambda x = \lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$

$\|\lambda x\| = \left(\sum_{i=1}^{n} (\lambda x_i)^2\right)^{\frac{1}{2}} = |\lambda| \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$

$= |\lambda| \, \|x\|$

④ Let $x, y \in R^n$

$x + y = (x_1, \dots, x_n) + (y_1, \dots, y_n)$

$= (x_1 + y_1, \dots, x_n + y_n)$

$\|x + y\| = \left(\sum_{i=1}^{n} (x_i + y_i)^2\right)^{\frac{1}{2}}$

by using "Minkowsk's inequality"

$\|x + y\| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}} = \|x\| + \|y\|$

Ex: Let $\|.\| : L^p \longrightarrow R$, $(1 \leq p \leq \infty)$ which defined by:

$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$ for all $x = (x_1, \dots, x_n, \dots) \in L^p$, then $\|.\|$ is norm over L^p.

Sol:- ① $\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} \geq 0 \implies \|x\| \geq 0$

② ~~$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$~~

$\|x\| = 0 \iff \sum_{i=1}^{\infty} (|x_i|^p)^{\frac{1}{p}} = 0$

$\iff \sum_{i=1}^{n} |x_i|^p = 0 \quad \forall i = 1, \dots, n, \quad 1 < p \leq \infty$

$\iff \sum_{i=1}^{n} |x_i| = 0 \quad \forall i = 1, \dots, n$

$\iff |x_i| = 0 \quad \forall i = 1, \dots, n$

$\iff x_i = 0 \quad \forall i = 1, \dots, n$

$\iff x = 0$

(3) Let $y, x \in L^p$ and $\lambda \in R$

$$\underset{\sim}{x}+\underset{\sim}{y} = (x_1 + \underset{\sim}{y}_1, \dots, x_n + \underset{\sim}{y}_n, \dots)$$

$$\|x + \underset{\sim}{y}\| = \left(\sum_{i=1}^{\infty} |x_i + \underset{\sim}{y}_i|^p \right)^{\frac{1}{p}}$$

by using "Minkowsk's inequality"

$$\|x+y\| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |y_i|^p \right)^{\frac{1}{p}} = \|x\| + \|y\|$$

$\|.\|$ is norm over L^p.

Ex: Let $\|.\|: \ell^\infty \longrightarrow R$ which defined $\|x\| = \sup_i |x_i|$. then $\|.\|$ is norm over ℓ^∞.

Ex: Let $X = C[0,1]$ and Let $\|.\|: X \longrightarrow R$ which defined by: $\|f\| = \max\{|f(x)| : 0 \leq x \leq 1\}$ for all $x \in f$ then $\|.\|$ is norm over X.

"Equivalent Norms"

Definition:

Let $\|.\|_1$ & $\|.\|_2$ are two norm over vector space X. we Say that $\|.\|_1$ is equivalent to the norm $\|.\|_2$ or $\|.\|_1$ & $\|.\|_2$ are equivalent and write by $\|.\|_1 \sim \|.\|_2$ if there exist a postive integer number a and b s.t $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$ for all $x \in X$.

Ex: Suppose $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ $\forall x = (x_1, \dots, x_n) \in R^n$

$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$ $\forall x = (x_1, \dots, x_n) \in R^n$

Then $\|.\|_1 \sim \|.\|_2$.

proof:-

from Cauchy-Schwar's inequality

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} y_i\right)^{\frac{1}{2}} \quad , \forall x_i, y_i \in R$$

put $y_i = 1$ for all $i = 1, \dots, n$

$$\sum_{i=1}^{n} |x_i| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} 1\right)^{\frac{1}{2}}$$

$$\sum_{i=1}^{n} |x_i| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} 1^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{i=1}^{n} (x_i)^2\right)^{\frac{1}{2}} \cdot (n)^{\frac{1}{2}}$$

$\|x\|_1 \leq \sqrt{n} \, \|x\|_2$

$\therefore \frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2$

put $a = \frac{1}{\sqrt{n}} \implies a\|x\|_1 \leq \|x\|_2$

$\therefore \|x\|_2 \leq \|x\|_1$

Choose $b = 1$

$\frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2 \leq \|x\|_1$

$\therefore a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$

$\therefore \|x\|_1 \sim \|x\|_2$

"Linear combination inequality"

Let $\{x_1, \dots, x_n\}$ is Linearly independent set of normed space X. Then there exist a postive integer number c s.t $\|\sum_{i=1}^{n} \lambda_i x_i\| \geq c \sum_{i=1}^{n} |\lambda_i|$ for all $\lambda_i \in F$, $i = 1, \dots, n$.

Theorem: Every norms over finitely-dimensional vector space are equivalent.

proof:-

Suppose X is finitely-dimensional vector space and $\dim X = n > 0$ and Let $\|\cdot\|_1$, $\|\cdot\|_2$ are two norm over X.

we must proof that $\|\cdot\|_1 \sim \|\cdot\|_2$

Let $\{x_1, \dots, x_n\}$ is a basis for X (because of $\dim X = n$)

Let $x \in X$. There exist a unique form

$$x = \sum_{i=1}^{n} \lambda_i x_i \quad , \lambda_i \in F. \quad \text{---} (1)$$

$$\|x\|_1 = \|\sum_{i=1}^{n} \lambda_i x_i\|_1 \leq \sum_{i=1}^{n} |\lambda_i| \|x_i\|_1 \quad \text{---} (2)$$

put $K = \max\{\|x_1\|_1, \dots, \|x_n\|_1\}$ for all $i = 1, \dots, n$

$$\sum_{i=1}^{n} |\lambda_i| \|x_i\|_1 \leq K \sum_{i=1}^{n} |\lambda_i| \quad \text{----} (3)$$

from (2) and (3) we get

$$\|x\|_1 \leq K \sum_{i=1}^{n} |\lambda_i| \quad \text{----} (4)$$

since $\{x_1, \dots, x_n\}$ is Linearly independent set by using Linear Combination inequality. There exist $c>0$ such that:

$\|x\|_2 = \|\sum_{i=1}^{n} \lambda_i x_i\|_2 \geq c \sum_{i=1}^{n} |\lambda_i|$ --- (5)

from (1) and (5) we get.

$\|x\|_2 \geq c \sum_{i=1}^{n} |\lambda_i|$ --- (6) $\Rightarrow \sum_{i=1}^{n} |\lambda_i| \leq \frac{1}{c} \|x\|_2$

$\|x\|_1 \leq K \cdot \sum |\lambda_i| \leq \frac{K}{c} \|x\|_2$

from (4) and (6) we get

$\|x\|_1 \leq \frac{K}{c} \|x\|_2$

put $a = \frac{c}{K}$ and we get $a\|x\|_1 \leq \|x\|_2$ --- (7)

$\|x\|_2 \leq K \sum_{i=1}^{n} |\lambda_i|$ --- (8)

$\|x\|_1 \geq c \sum_{i=1}^{n} |\lambda_i|$ --- (9) $\Rightarrow c\sum |\lambda_i| \leq \|x\|_1$

from (8) and (9) we get

$\|x\|_2 \leq \frac{K}{c} \|x\|_1$

put $b = \frac{K}{c}$ and we get

$\|x\|_2 \leq b\|x\|_1$ --- (10)

from (7) and (10) we get

$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$

So that $\|\cdot\|_1 \sim \|\cdot\|_2$

"Concepts of metric in Normed space"

Definition:

Let X is anonempty set and the function $d: X \times X \to R$ is called metric function over X if it is satisfies the following axioms:

1- $d(x,y) \geq 0$ for all $x,y \in X$

2- $d(x,y) = 0 \iff x = y$

3- $d(x,y) = d(y,x)$ for all $x,y \in X$

4- $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$

The metric space is (X,d) s.t ~~X~~ is anonempty set and d is metric function over X

Theorem: Every normed space is metric space.

proof:- Let X is normed space and $d: X \times X \to R$ metric function which defined by $d(x,y) = \|x-y\|$ $\forall x,y \in X$

① let $x,y \in X \Rightarrow x-y \in X$ (because of X is vectorspace)

$\Rightarrow \|x-y\| \geq 0 \Rightarrow d(x,y) \geq 0$

② $d(x,y) = 0 \iff \|x-y\| = 0 \iff x-y = 0 \iff x = y$

③ let $x,y \in X \Rightarrow d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$

④ Let $x,y,z \in X \Rightarrow \|x-y\| = \|(x-z)+(z-y)\| \leq \|x-z\| + \|z-y\|$

$$d(x,y) \leq d(x,z) + d(z,y)$$

So that d is metric function

Then (X,d) is metric space.

~~Remark: not necessary to be every metric space is normed space~~

Definition: Induced metric function by norm is the function that satisfies the following conditions:

① $d(x+z, z+y) = d(x,y)$, $\forall x,y,z \in X$

② $d(\lambda x, \lambda y) = |\lambda| d(x,y)$, $\forall x,y \in X$ & $\lambda \in F$.

③ $d(x,0) = \|x-0\| = \|x\|$, $\forall x \in X$

i.e.; norm x is the dimension between x and zero."

Ex: Let $X = C[0,1]$ and $f,g \in C[0,1]$ s.t

$d(f,g) = \sup\{|f(x)-g(x)|, x\in[0,1]\}$

Does d induced function by norm or not?

proof:- ① Let $f,g,h \in C[0,1]$

$$\begin{aligned} d(f+h, g+h) &= \sup\{|(f+h)(x)-(g+h)(x)| : x\in[0,1]\} \\ &= \sup\{|f(x)+h(x)-g(x)-h(x)| : x\in[0,1]\}, 0\le x\le 1\} \\ &= \sup\{|f(x)-g(x)| : 0\le x\le 1\} \\ &= d(f,g) \end{aligned}$$

② Let $f,g \in C[0,1]$ and $\lambda \in F$

$$\begin{aligned} d(\lambda f, \lambda g) &= \sup\{|(\lambda f)(x)-(\lambda g)(x)| : 0\le x\le 1\} \\ &= \sup\{|\lambda||f(x)-g(x)| : 0\le x\le 1\} \\ &= |\lambda| \sup\{|f(x)-g(x)| : 0\le x\le 1\} \\ &= |\lambda| d(f,g) \end{aligned}$$

(3) Let $f \in C[0,1]$

$d(f,0) = \sup\{|f(x)-0(x)| : 0 \le x \le 1\}$

$\quad = \sup\{|f(x)| : 0 \le x \le 1\} = \|f\|$

Q: If d is metric function over X. Does d be induced function by norm?

ans: not necessary

Ex: Let X is none-zero vector space ($X \neq \{0\}$) and Let

$d : X \times X \to \mathbb{R}$ which defined by $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

It is easy to show that d is metric function over X and there is no norm over X s.t that d be induced from it.

because if we supposed that there exist a norm over X s.t d is induced from it

Let $\lambda \in F$ and Let $\lambda \neq 0$

$\because \|\lambda x\| = |\lambda| \|x\|$

$\because d$ is induced from norm

$\therefore \|\lambda x\| = d(\lambda x, \lambda . 0) = |\lambda| d(x,0)$

$\therefore \|\lambda x\| = \|x\|$ C!

or $\|\lambda x\| = d(\lambda x, y) = \begin{cases} 0 & \lambda x = y \\ 1 & \lambda x \neq y \end{cases}$

$|\lambda| . \|x\| = |\lambda| d(x,y) = |\lambda| \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

if $|\lambda| \neq 1$

$\therefore \|\lambda x\| \neq |\lambda| \|x\|$

Definition:-

Let X be normed space and let $x_0 \in X$. if r a postive integer number then the set $\{x \in X : \|x - x_0\| < r\}$ is called open ball and x_0 is called center of the ball, r is radius of the ball. we denote to the open ball which it is center x_0 and radius is r by $B_r(x_0)$, So $B_r(x_0) = \{x \in X : \|x - x_0\| < r\}$

The closed ball which it is center is x_0 and it is radius r and denoted by $\overline{B_r}(x_0)$ which defined by: $\overline{B_r}(x_0) = \{x \in X : \|x - x_0\| \leq r\}$

In spcial Case: $B_1(0) = \{x \in X : \|x\| < 1\}$ is called open uint ball and $\overline{B_1}(0) = \{x \in X : \|x\| \leq 1\}$ is call closed uint ball.

Ex: prove the following:

① $B_r(x_0) = x_0 + r B_1(0)$

② $\overline{B_r}(x_0) = x_0 + r \overline{B_1}(0)$

proof:- ① $x_0 + r B_1(0) = x_0 + r\{x \in X : \|x\| < 1\}$

$$= x_0 + \{rx \in X : \|x\| < 1\}$$

$$= x_0 + \{y = rx : x \in X, \|x\| < 1\}$$

$$= x_0 + \{y \in X : \|\tfrac{y}{r}\| < 1\}$$

$$= x_0 + \{y \in X : \|y\| < r\}$$

$$= \{x_0 + y : y \in X, \|y\| < r\}$$

$$= \{Z = x_0 + y : y \in X, \|y\| < r\}$$

Ex: If X is normed space then the open ball and the closed ball are convex sets.

proof:- Now, we must proof that $B_r(x_0)$ which center is x_0 and radius r is convex set.

Let $x, y \in B_r(x_0)$ and $0 \leq \lambda \leq 1$

$\|x - x_0\| < r$, $\|y - x_0\| < r$

Now, we must proof $\lambda x + (1-\lambda) y \in B_r(x_0)$

$$\begin{aligned}
\|\lambda x + (1-\lambda) y - x_0\| &= \|\lambda x + (1-\lambda) y - x_0 + \lambda x_0 - \lambda x_0\| \\
&= \|(\lambda x - \lambda x_0) + (1-\lambda) y - (x_0 - \lambda x_0)\| \\
&= \|\lambda(x - x_0) + (1-\lambda) y - (1-\lambda) x_0\| \\
&= \|\lambda(x - x_0) + (1-\lambda)(y - x_0)\| \\
&\leq \|\lambda(x - x_0)\| + \|(1-\lambda)(y - x_0)\| \\
&\leq |\lambda| \|x - x_0\| + |1-\lambda| \|y - x_0\| \\
&< \lambda r + (1-\lambda) r \\
&< \cancel{\lambda r} + r - \cancel{\lambda r} = r
\end{aligned}$$

$\|\lambda x + (1-\lambda) y - x_0\| < r$

$\therefore \lambda x + (1-\lambda) y \in B_r(x_0)$

So that $B_r(x_0)$ is convex set

Definition:

Let A is subset of normed space X. we say that A is open set in X if for all $x \in A$ there exist $r > 0$ s.t $B_r(x) \subset A$

Ex: Every open ball is open set.

Theorem:- Let X is normed space then:

① X and ϕ are open set

② The intersection of finitely open sets be open set

③ The union of finitely and infinitely open sets be open set

④ If $A \subseteq X$, then A be open set iff A is equal to union of open sets.

proofs:-

② Let X is normed space

~~Let A_i are open sets in X~~

Let A_i, $i = 1, \dots, n$ are open sets in X

To proof $\bigcap_{i=1}^{n} A_i$ is open set.

$\because$ A_i is open set for all $i = 1, \dots, n$

$\therefore$ There exist r_i, $i = 1, \dots, n$ s.t $B_{r_i}(x) \subseteq A_i$

Let $r = \min\{r_1, \dots, r_n\}$

$\therefore B_r(x) \subseteq A_i \quad \forall i = 1, \dots, n$

$\therefore B_r(x) \subset \bigcap_{i=1}^{n} A_i$

$\therefore \bigcap_{i=1}^{n} A_i$ is open set

Theorem: Let X is normed space then:-

① X and ϕ are closed sets

② the union of finitely closed sets be closed set.

③ The intersection of finitely and infinitely closed sets be closed sets.

proof:- ① To proof that ϕ and X are closed sets

we must proof that X^c and ϕ^c are open sets

since $X^c = \phi$ and ϕ is open set

Then X^c is open set

since $X = \phi^c$ and X is open set

then ϕ^c is open set

So that X^c and ϕ^c are open set

and so X and ϕ are closed set.

② suppose A_i, $i=1, \ldots, n$ are closed set

To proof that $\bigcup_{i=1}^{n} A_i$ is closed set

we must proof that $\left(\bigcup_{i=1}^{n} A_i\right)^c$ is open set

[illegible] $= \bigcap A_i^c$

since A_i, are closed set for all $i=1, \ldots, n$

then A_i^c are open set for all $i=1, \ldots, n$

Since The intersection of finitely open set be open set

$\therefore \bigcap_{i=1}^{n} A_i^c$ is open set , since $(\bigcup_{i=1}^{n} A_i)^c = \bigcap_{i=1}^{n} A_i^c$

$\therefore (\bigcup_{i=1}^{n} A_i)^c$ is open set

So that $\bigcup_{i=1}^{n} A_i$ is closed set.

③ Suppose A_i are closed sets s.t $i \in I$

To proof that $\bigcap_{i \in I} A_i$ is closed set

we must proof that $(\bigcap_{i \in I} A_i)^c$ is open set.

Since $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c$

Since A_i are closed sets s.t $i \in I$

then A_i^c are open sets s.t $i \in I$

Since the union of finitely and infinitely open sets be open set.

then $\bigcup_{i \in I} A_i^c$ is open set

$\therefore (\bigcap_{i \in I} A_i)^c$ is open set

So that $\bigcap_{i \in I} A_i$ is closed set.

Definition:-

Let X is normed space, $A \subseteq X$. we say that x is interior point in A if there exist $r>0$ s.t $B_r(x) \subset A$ and denoted to set of interior points by $int(A)$ or (A°)

~~Definition:-~~

~~Let A is a closed of normed space X. we say~~

Remark:-

① $int(A) \subset A$

② $int(A)$ is open set

③ A is open set iff $int(A)=A$

④ $int(int(A)) = int(A)$

⑤ $int(A)=$ The union of all open sets which contained in A and so that $int(A)$ is Largest open set Contained in A.

⑥ $int(A)=\{x \in A : \exists r>0, x+rB_1(0) \subset A\}$.

Definition:

Let X is normed space, $A \subseteq X$, we say that A is is closed set in X if A^c is open set in X.

Ex: Every closed set in normed space be closed set.

X

Theorem: Let X is normed space then:

① X, ϕ are closed sets

~~② The finitely~~

② The union of finitely closed set be closed set

③ The intersection of finitely and infinitely closed set be closed set.

*It is easy to show that

① $A \subseteq \bar{A}$, $A' \subseteq \bar{A}$

② $x \in \bar{A}$ iff for all $r>0$ there exist $y \in A$ s.t $\|x-y\|<r$

③ $\bar{A}$ is closed set.

④ A is closed set iff $\bar{A} = A$

⑤ $\bar{\bar{A}} = A$

⑥ $\bar{A}$ = intersection of all closed sets which contains A and so that $\bar{A}$ is smallest closed set contains A.

⑦ $\bar{A} = \bigcap_{r>0} (A + r\overline{B_1(0)})$

Theorem: If M is subspace of normed space X. Then $\bar{M}$ is subspace in X.

proof:-

Since $0 \in M$ & $M \subset \bar{M}$ $\Rightarrow 0 \in \bar{M}$

$\therefore \bar{M} \neq \phi$

let $x, y \in \bar{M}$ & $\alpha, \beta \in F$

we must proof that $\alpha x + \beta y \in \bar{M}$

Let $r > 0$,

There is two cases

Case 1: If $\alpha \neq 0$ & $\beta \neq 0$

$\therefore \frac{r}{2|\alpha|} > 0$ & $\frac{r}{2|\beta|} > 0$

Let $a, b \in M$ ~~&~~

$\because$ M is subspace $\Longrightarrow$ $\alpha a + \beta b \in M$ s.t $\|x - a\| < \frac{r}{2|\alpha|}$

and $\|y - b\| < \frac{r}{2|\beta|}$

$((\alpha x + \beta y) - (\alpha a + \beta b)) = \alpha(x - a) + \beta(y - b)$

$\|(\alpha x + \beta y) - (\alpha a + \beta b)\| = \|\alpha(x - a) + \beta(y - b)\|$

$\leq |\alpha| \|x - a\| + |\beta| \|y - b\|$

$< |\alpha| \cdot \frac{r}{2|\alpha|} + |\beta| \frac{r}{2|\beta|} = r$

$\therefore \alpha x + \beta y \in \overline{M}$

So $\overline{M}$ is subspace

Case 2: If $\alpha = 0$ & $\beta = 0$ then $\alpha x + \beta y = 0 \in \overline{M}$

$\therefore \alpha x + \beta y \in \overline{M}$

So $\overline{M}$ is subspace

Case 3: If $\alpha = 0$ or $\beta = 0$

$\alpha x + \beta y = \beta y \in \overline{M}$ or $\alpha x + \beta y = \alpha x \in \overline{M}$

$\therefore \alpha x + \beta y \in \overline{M}$

So $\overline{M}$ is subspace

Note: The union of infinitely closed sets doesn't be closed set.

Ex: In real normed space $A_n = [1 - \frac{1}{n}, -1 + \frac{1}{n}], n \in Z^+$

we note that $\bigcup_{n \in Z^+} A_n = (1, -1)$ is open interval

So that doesn't be closed set.

Theorem: If X is normed space. Then every set which contains only one element be closed set in X.

proof:-

Let $A = \{x\}$

To proof that A is closed set

we must proof that A^c is open set.

Let $y \in A^c \Rightarrow y \notin A \Rightarrow y \neq x \Rightarrow \|x - y\| > 0$

put $r = \|x - y\| \Rightarrow r > 0$

Since $\|x - y\| < r \Rightarrow x \notin B_r(y)$

$\Rightarrow A \cap B_r(y) = \phi$

$\Rightarrow B_r(y) \subset A^c$

$\Rightarrow A^c$ is open set

$\Rightarrow A$ is closed set.

Corollarly :-

Every finitely set in normed space be closed set.

proof:-

Let A is finitely set in normed space X, i.e.; $A = \{x_1, \dots x_n\}$

$\Rightarrow A = \bigcup_{i=1}^{n} \{x_i\}$

Since $\{x_i\}$ is closed set for all $i = 1, \dots, n$

"by Last theorem"

Since the finitely union of closed sets be closed set.

$\therefore A = \bigcup_{i=1}^{n} \{x_i\}$ is closedset.

Definition:

Let A is subset of normed space X. we say that $x \in X$

نقطة تجمع ، نقطة نهاية

is accumulation point or Limit point to the set A if

for all open set G in X Contains x, Contains another point

$(x \neq y)$

i.e.; If G is open set in X and $x \in G$, then $A \cap (G \setminus \{x\}) \neq \phi$

we denoted to the set which all elements are accumulation points for A. by A' and called derivative of A.

Definition: Let A is subset of normed space X. Then the set $A \cup A'$ is called closure of A and denoted by $\overline{A}$

i.e.; $\overline{A} = A \cup A'$

The product space

.suppose that X, Y be two sets. then the product product of X and Y which defined by

$X \times Y = \{(x,y) : x \in X, y \in Y\}$

It is clear that $X \times Y \neq Y \times X$, and we note that if X and Y are nonempty sets, then $X \times Y$ be a nonempty set

finally, we can define the addition operation and Scalar multipliaction over $X \times Y$. by:

① $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

for all $(x_1, y_1), (x_2, y_2) \in X \times Y$

② $\lambda(x, y) = (\lambda x, \lambda y)$, $\forall (x, y) \in X \times Y$, and $\lambda \in F$.

Ex:-

Ex:

Suppose that $(X, \|.\|_1)$, and $(Y, \|.\|_2)$ be two norms, then $(X \times Y, \|.\|)$ be normed space s.t $\|(x,y)\| = \max\{\|x\|_1, \|y\|_2\}$ for all $(x,y) \in X \times Y$.

Sol:-

① Since $\|x\|_1 \geq 0$ for all $x \in X$, and $\|y\|_2 \geq 0$ for all $y \in Y$, then $\|(x,y)\| \geq 0$ for all $(x,y) \in X \times Y$.

② $\|(x,y)\| = 0 \iff \max\{\|x\|_1, \|y\|_2\} = 0 \iff \|x\|_1 = 0, \|y\|_2 = 0$

$\iff x = 0, \underset{\sim}{y} = 0 \iff (x,y) = \bar{0}$

③ let $(x,y) \in X \times Y$, and $\lambda \in F$, then $\lambda(x,y) = (\lambda x, \lambda y)$

$\|\lambda(x,y)\| = \max\{\|\lambda x\|_1, \|\lambda y\|_2\} = \max\{|\lambda|\|x\|_1, |\lambda|\|y\|_2\}$

$= |\lambda| \max\{\|x\|_1, \|y\|_2\} = |\lambda| \|(x,y)\|$

④ let $(x_1,y_1), (x_2,y_2) \in X \times Y$, then $(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2)$

$$\begin{aligned}\|(x_1,y_1) + (x_2,y_2)\| &= \max\{\|x_1 + x_2\|_1, \|\underset{\sim}{y_1} + \underset{\sim}{y_2}\|_2\} \\ &\leq \max\{\|x_1\|_1 + \|x_2\|_1, \|y_1\|_2 + \|y_2\|_2\} \\ &\leq \max\{\|x_1\|_1, \|y_1\|_2\} + \max\{\|x_2\|_1, \|y_2\|_2\} \\ &= \|(x_1,y_1)\| + \|(x_2,y_2)\|\end{aligned}$$

then $\|.\|$ be norm over $X \times Y$.

~~Definition: Let X is normed space and let A⊆X~~

"Convergence in Normed Space"

Let X is anonempty set. the function that it's domain is anatural number set N and it's range is the set X is called sequence in X.

i.e.; $f : N \longrightarrow X$

$\forall n \in N$, $\exists\, x_n \in X \ni f(n) = x_n$

Ex: If $\langle x_n \rangle = (-1)^n$, $n \in N$ is a sequence defined in R

Then $\langle x_n \rangle = \{1, -1, 1, -1, \dots\}$

$R = \{-1, 1\}$

Definition: Let $\{x_n\}$ is sequence in normed space X. we Said that $\{x_n\}$ is Convergent Sequence in X if there exist $x \in X$ s.t for all $\epsilon > 0$ there is $k \in Z^+$ s.t $\|x_n - x\| < \epsilon \quad \forall n > k$.

Theorem: If $\{x_n\}$ is asequence in normed space and $\lim_{n\to\infty} x_n = x_1$, $\lim_{n\to\infty} x_n = x_2$ then $x_1 = x_2$

Theorem: the Limit point in normed space is a unique.

proof:- Let $\{x_n\}$ is a sequence in normed space X.

Let $x_1, x_2 \in X$ s.t $\lim_{n\to\infty} x_n = x_1$ and $\lim_{n\to\infty} x_n = x_2$ and $x_1 \neq x_2$

then $x_1 - x_2 \neq 0 \Rightarrow \|x_1 - x_2\| > 0$

put $\|x_1 - x_2\| = \epsilon \Rightarrow \epsilon > 0$

Since, $\lim_{n\to\infty} x_n = x_1$

$\forall \epsilon > 0 \ \exists k_1 \in Z^+ \ni \|x_n - x_1\| \leq \frac{\epsilon}{2} \quad \forall n > k_1$

Since, $\lim_{n\to\infty} x_n = x_2$

$\forall \epsilon > 0 \ \exists k_2 \in Z^+ \ni \|x_n - x_2\| \leq \frac{\epsilon}{2} \quad \forall n > k_2$

put $k = \max\{k_1, k_2\}$

$\|x_n - x_1\| < \frac{\epsilon}{2}$ and $\|x_n - x_2\| < \frac{\epsilon}{2}$

$$\epsilon = \|x_1 - x_2\| = \|x_1 - x_2 + x_n - x_n\|$$

$$= \|(x_n - x_1) + (x_n - x_2)\|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad C!$$

So that $x_1 = x_2$

Example:-

Let $X = R$, $\|x\| = |x|$ for $x \in R$ is $\{x_n\} = \{\frac{1}{n}, n \in Z^+\}$ is Convergent or not.

Sol:- $\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

Let $x = 0 \in R$

Let $\epsilon > 0 \ \exists k \in Z^+$ s.t $k \cdot \epsilon > 1 \Rightarrow \frac{1}{k} < \epsilon$

$\| \frac{1}{n} - 0 \| = \| \frac{1}{n} \| = | \frac{1}{n} | = \frac{1}{n}$

So that for all $n > k \Rightarrow \frac{1}{n} < \frac{1}{k}$

$\therefore \| \frac{1}{n} - 0 \| = \frac{1}{n} < \frac{1}{k} < \epsilon$

Then $\frac{1}{n} \to 0$ and $\{x_n\}$ is convergent to Zero

theorem:

Let A is subset in normed space X. then $x \in \bar{A}$ iff there exist a sequence $\{x_n\}$ in A S.t $x_n \to x$

proof:-

Let $x \in \bar{A}$ and since $\bar{A} = A \cup A' \Rightarrow x \in A \cup A' \Rightarrow x \in A$ or $x \in A'$

If $x \in A$ then $\{x_n\}$ in A and $x_n \to x$

either $x \notin A$ then $x \in A'$

let $n \in Z^+ \Rightarrow r = \frac{1}{n} > 0 \Rightarrow B_r(x)$ is open set and Contains x

since $x \in A' \Rightarrow A \cap (B_r(x) / \{x\}) \neq \phi$

$\Rightarrow x_n \in A \cap (B_{\frac{1}{n}}(x) / \{x\}) \quad \forall n \in Z^+$

we note that $\{x_n\}$ is a sequence in A.

finally, we must proof that $x_n \to x$

Let $\epsilon > 0$ and by using Archamedis property

$\exists K \in Z^+ \ni \frac{1}{K} < \epsilon$

Since $x_n \in B_{\frac{1}{n}}(x) \implies \|x_n - x\| < \frac{1}{n} \quad \forall n \in \mathbb{Z}^+$

Let $n > K \implies \frac{1}{n} < \frac{1}{k} \implies \|x_n - x\| < \frac{1}{k} < \epsilon$

$\implies x_n \to x$

Conversly:

Let $\{x_n\}$ is a sequence in A s.t $x_n \to x$

we must proof that $x \in \overline{A}$

i.e.; $x \in A \cup A'$

If $x \in A$ then the theorem is proved

and, if $x \notin A$ we suppose that G is an open set in X

and contains x then there enist $r > 0$ s.t $B_r(x) \subset G$

Since $r > 0$, $x_n \to x \implies \exists k \in \mathbb{Z}^+$ s.t $\|x_n - x\| < r \quad \forall n > k$

$\implies x_n \in B_r(x) \quad \forall n > K$

since $x_n \in A$ for all $n \in \mathbb{Z}^+$

then, $A \cap (G \setminus \{x\}) \neq \phi$

then $x \in A'$ and then $x \in \overline{A}$

Theorem: Let $\{x_n\}, \{y_n\}$ are two sequence in normed space X s.t $x_n \to x$ and $y_n \to y$ then:

① $x_n + y_n \longrightarrow x + y$

② $x_n - y_n \longrightarrow x - y$

③ $\lambda x_n \longrightarrow \lambda x \quad \forall \lambda \in F$

④ $\|x_n\| \longrightarrow \|x\|$

⑤ $\|x_n - y_n\| \longrightarrow \|x - y\|$

⑥ $\| \frac{x_n}{y_n} \| \longrightarrow \| \frac{x}{y} \|$

proof:-

① Let $\{x_n\}$ & $\{y_n\}$ are two sequence in normed space X

s.t $x_n \to x$ & $y_n \to y$.

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\|$$

$$\leq \|x_n - x\| + \|y_n - y\|$$

$\because x_n \to x$ then $\|x_n - x\| \longrightarrow 0$

$\because y_n \longrightarrow y$ then $\|y_n - y\| \longrightarrow 0$

$\therefore \|(x_n + y_n) - (x + y)\| \longrightarrow 0$

$\therefore x_n + y_n \longrightarrow x + y$

③ proof:-

$\| \lambda x_n - \lambda x \| = |\lambda| \, \|x_n - x\|$

$\because x_n \to x \quad \forall n \in Z^+$

$\therefore \|x_n - x\| \to 0$

$\therefore \| \lambda x_n - \lambda x \| \to 0$

$\therefore \lambda x_n \to \lambda x$

④ proof:-

$\because | \|x_n\| - \|x\| | \leq \|x_n - x\|$

$\because x_n \to x \quad \forall n \in Z^+$

$\therefore \| x_n - x \| \to 0 \quad \forall n \in Z^+$

$\therefore | \|x_n\| - \|x\| | \to 0$

$\therefore \|x_n\| \to \|x\|$

⑤ proof:-

$| \|x_n - y_n\| - \|x - y\| | \leq \|(x_n - y_n) - (x - y)\| = \|(x_n - x) - (y_n - y)\|$

$\leq \|x_n - x\| + \|y_n - y\|$

Since $x_n \to x$ then $\|x_n - x\| \to 0$

Since $y_n \to y$ then $\|y_n - y\| \to 0$

So that $| \|x_n - y_n\| - \|x - y\| | \to 0$

So $\|x_n - y_n\| \to \|x - y\|$

$$\left|\,\left\|\frac{x_n}{y_n}\right\| - \left\|\frac{x}{y}\right\|\,\right| \le \left\|\frac{x_n}{y_n} - \frac{x}{y}\right\|$$

⑥ proof:- $\left\|\frac{x_n}{y_n} - \frac{x}{y}\right\| = \left\|\frac{x_n y - x y_n}{y_n y}\right\| = \left\|\frac{x_n y - x y_n - x_n y_n + x_n y_n}{y_n y}\right\|$

$$= \left\|\frac{x_n(y - y_n) + y_n(x_n - x)}{y_n y}\right\| \le \left\|\frac{x_n(y - y_n)}{y_n y}\right\| + \left\|\frac{y_n(x_n - x)}{y_n y}\right\|$$

$$\le \frac{\|x_n\|\,\|y_n - y\|}{\|y_n\|\,\|y\|} + \frac{\cancel{\|y_n\|}\,\|x_n - x\|}{\cancel{\|y_n\|}\,\|y\|}$$

Since $x_n \to x \implies \|x_n - x\| \to 0$

Since $y_n \to y \implies \|y_n - y\| \to 0$

So that $\frac{\|x_n\|\,\|y_n - y\|}{\|y_n\|\,\|y\|} + \frac{\|x_n - x\|}{\|y\|} \to 0$

$\left\|\frac{x_n}{y_n} - \frac{x}{y}\right\| \to 0$ then $\frac{x_n}{y_n} \to \frac{x}{y}$.

Definition:

Let $\{x_n\}$ is Sequence in normed space X. we say that $\{x_n\}$ is Caushy Sequece in X. If for all $\epsilon > 0$ there exist $k \in Z^+$ s.t $\|x_n - x_m\| < \epsilon$ for all $n, m > k$

theorem: Every convergenent Sequence in norm space be Caushy Sequence.

proof:- Let $\{x_n\}$ is convergent sequence in normed space X s.t $\exists x \in X$ s.t $x_n \to x$ for all $n \in Z^+$

Since $x_n \to x$ for all $n \in Z^+$

for all $\epsilon > 0$ there exist $k \in Z^+$ s.t $\|x_n - x\| < \frac{\epsilon}{2}$ $\forall n > k$

Let $n, m > K$

$$\|x_n - x_m\| = \|(x_n - x) + (x - x_m)\| \leq \|x_n - x\| + \|x_m - x\|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n, m > K$$

then $\{x_n\}$ is Caushy Sequence.

Remark: Not every Caushy Sequence is Convergent

example:- Let $X = C[0,1]$ with $d(x_n - x_m) = |x_n - x_m|$ and $\{x_n\} = \{\frac{1}{n}\} \in X$

$\{x_m\} = \{\frac{1}{m}\} \in X$

$\{x_n\} = \{\frac{1}{n}\}$ is Caushy Sequence; because

$d(x_n - x_m) = |\frac{1}{n} - \frac{1}{m}| \longrightarrow 0$ as $n, m \longrightarrow \infty$

However; $d(x_n, 0) = |\frac{1}{n} - 0| \longrightarrow 0$ as $n \longrightarrow \infty$

$= |\frac{1}{n}| \longrightarrow 0$ as $n \longrightarrow \infty$

and $0 \notin X$

Therefore, $\{x_n\}$ is Caushy Sequence in X, which is not Convergent in it.

Definition:

Let $\{x_n\}$ is a sequence in normed space X. we say that $\{x_n\}$ is bounded sequence in X if there exist a postive integer number M s.t $\|x_n\| \leq M$ for all $n \in Z^+$

Theorem:

If $\{x_n\}$ is Caushy _ Sequence in normed space X then the sequence $\{x_n\}$ is bounded.

proof:- Let $\{x_n\}$ is Caushy sequence in X

Let $\epsilon = 1$

There exist $K \in Z^+$ s.t $\|x_n - x_m\| < 1$ for all $n, m > K$

put $m = K+1 \Rightarrow \|x_n - x_{K+1}\| < 1$

Since $\|x_n\| - \|x_{K+1}\| \leq \|x_n - x_{K+1}\| < 1$

then $\|x_n\| \leq 1 + \|x_{K+1}\| \quad \forall n > K$

$M = max\{\|x_1\|, \|x_2\|, \ldots, \|x_{K+1}\| + 1\}$

$\therefore \|x_n\| \leq M \quad \forall n > K^+$

$\therefore \{x_n\}$ is bounded sequence in X.

Remark: The Converse of Last thereom is not necessary.

Theorem: If $\{x_n\}$ is Convergence sequence in normed space X, then $\{x_n\}$ be bounded sequence in X.

proof: since $\{x_n\}$ is converges sequence in X, and since every converges sequence be Cauchy Sequence and every Cauchy sequence be bounded. that is, $\{x_n\}$ be bounded Sequence in X.

Def:- Let $\{x_n\}$ is a sequence in normed space X. we say that $\{x_n\}$ is bounded sequence in X if there exist a postive integer number M s.t $\|x_n\| \leq M$ for all $n \in Z^+$.

Theorem:- If $\{x_n\}$ is Caushy-Sequence in normed space X. then $\{x_n\}$ is bounded sequence.

proof:- Let $\{x_n\}$ is Caushy-Sequence in X

Let $\epsilon = 1$, There exist $k \in Z^+$ s.t $\|x_n - x_m\| < 1 \quad \forall n, m > k$

put $m = k+1$ then $\|x_n - x_{k+1}\| < 1$

$\therefore \|x_n\| - \|x_{k+1}\| \leq \|x_n - x_{k+1}\| < 1 \quad \forall n > k$

$\therefore \|x_n\| - \|x_{k+1}\| \leq 1 \quad \forall n > k$

$\therefore \|x_n\| \leq 1 + \|x_{k+1}\| \quad \forall n > k$

$M = \max\{\|x_1\|, \|x_2\|, \ldots, \|x_{k+1}\| + 1\}$

$\therefore \|x_n\| \leq M \quad \forall n > k^+$

$\therefore \{x_n\}$ is bounded sequence in X.

Remark:- The Converse of Last theorem is not necessary