2- Topological Spaces:

Definition 2.1: Let X be any nonempty set. The family T of subsets of X is called Topology on X if satisfay the following

① $\phi, X \in T$

② $\bigcup_i U_i \in T$ for any $\forall U_i \in T$

③ $\bigcap_{i=1}^{n} U_i \in T$ $\forall U_i \in T$ $1 \leq i \leq n$

The pair (X, T) is called topological space.

Example 2.1: Let $X=\{a,b,c,d,e,f\}$ and let $T_1=\{X, \phi, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e,f\}\}$

Then T_1 is topology on X since

① $\phi, X \in T_1$

② $\bigcup U_i \in T$ for each $U_i \in T$

③ $\bigcap_{i=1}^{n} U_i \in T$ $\forall U_i \in T$ $1 \leq i \leq n$

But if we take $T_2=\{X, \phi, \{a\}, \{c,d\}, \{a,c,e\}, \{b,c,d\}\}$ then T_2 is not topology since $\{c,d\} \cup \{a,c,e\} = \{a,c,d,e\}$. and this mean the second condition is not satisfay.

Example 2.2: Let N is ~~natural~~ Natural numbers set and let T_3 consist of N, ϕ and all finite subsets of N. Then T_3 is not topology on N since the infinite union $\{2\} \cup \{3\} \cup \cdots \cup \{n\} \cup \cdots = \{2, 3, \ldots, n, \ldots\}$ is not belong to T_3, therefore the second condition is not satisfay.

Example 2.3: Let $X=$ ~~$\{a,b\}$~~ any nonempty set and $T_4 = \{\phi, X\}$ then T_4

Let $X=\{a,b,c\}$, T is topology on X and $\{a\},\{b\},\{c\}\in T$
that T is discrete topology on X.

e 2.5: Let X be any nonempty set, let $T=\{U\subset X: U^c$ finite$\}\cup$
then T is topology on X (prove?) which called cofinite
ogy on X and denoted by T_c

e 2.6: Let $T=\{(a,b): a,b\in\mathbb{R}\}$ be a collection of all
set in $\mathbb{R}$ then T is topology on $\mathbb{R}$ which called
topology and denoted by T_u.

n, closed and clopen sets:

tion 2.2: If (X,T) is topological space then every element)
is called open set.

we will have new definition

tion 2.3: Let X be any nonempty set, T is collection
then T is topology on X if satisfay following
nd $\emptyset$ are open sets
union any number (finite or infinite) of open sets
set
intersection any number (finite) of open sets is
et.

ion 2.4: Let (X,T) is topological space, the subset $S\subseteq X$
closed set in (X,T) if S^c is open in (X,T).

2.7: In example 2.1, the closed sets is

$b,c,d,e,f\}$, $\{a,b,e,f\}$, $\{a\}$, $\{b,e,f\}$

ϕ and X is closed
) The intersection any number (finite or infinite) of the closed sets is closed set.
) The union any number (finite) of closed sets is closed se

proof:

① since the complement X and ϕ are ϕ and X, respectively then X and ϕ is closed sets.

② Let $S_1, S_2, \ldots$ are closed sets in X, then $S_1^c, S_2^c, \ldots$ are open sets in X, ~~therefore~~ $\cup S_i^c$ is open then $(\cup S_i^c)^c$ is closed

i.e $\cap S_i$ is open

③ H.W

Definition 2.5: The subset S of topological space (X, T) is called clopen set if it is open and closed in (X, T).

Example 2.8:

① In every topological space (X, T), The sets X and ϕ are clopen

② In example 2.7, the sets $\{a\}, \{b,c,d,e,f\}, \phi, X$ are clopen sets.

* Limit points and closure:

Definition 2.6: Let A be subset of topological space (X, T). The point $x \in X$ is limit point or accumulation point or cluster point for set A if ever open set U contain x, then U contain anthor point $y \in A$, such that $y \neq x$. i.e $U_x - \{x\} \cap A \neq \phi$. for each open set U contain x.

Example 2.9: Let $X = \{a, b, c, d, e\}$ and $T = \{X, \phi, \{a\}, \{c, \ldots$

roof: H.W

ample 2.10: Let (X, T) is discrete topological space and $A \subseteq X$, then A hasn't limit points since $\forall x \in X$, $\{x\}$ is open set in X.

xample 2.11: Let $A = [a, b) \subseteq \mathbb{R}$, then every point in $[a, b]$ is limit point of A.

H.W: Let (X, T) is indiscrete topological space and A is subset of X contain at least two elements, what the limit point of A? If A contain only one element then what the element points of A?

roposition 2.2: Let A be a subset of topological space (X, T). Then A is closed in (X, T) iff A contains every its limit points

proof: Let A is closed in (X, T) and p is limit point of A and $p \notin A$ then $p \in A^c$, A^c is open and contain limit point p, therefore, A^c is contains anthor point different of p but this is Cont! therefore $p \in A$.

$\Leftarrow$ Let A contains every its limit points

T.P A is closed

let $z \in A^c$ then z is not limit point of A since A has limit points, therefore, $\exists$ open set U_z contain z s.t $U_z \cap A = \phi$ and this mean $U_z \subseteq A^c$

$\Rightarrow A^c = \bigcup_{z \in A^c} U_z \Rightarrow A^c$ is open therefore A is closed.

roposition 2.3: Let A be a subset of topological space (X, T) and A' (or dA) the set of all limit poins of A then $A \cup A'$ is closed set

proof: T.P $A \cup A'$ contains every its limit points i.e every $p \in (A \cup A')^c$ is not limit point of $A \cup A'$

et $p \in (A \cup A')^c$, since $p \notin A'$

hen there exist open set U and contains p

$U \cap A = \phi$. Now let $U \cap A \neq \phi$ then $\exists x \in$ ...

d but U is open therefore U contain anthor point y differ

f from x in A and this mean $U \cap A \neq \phi \Rightarrow C!$

$\Rightarrow U \cap A' = \phi$

$U \cap (A \cup A') = \phi$

p is not limit point of $A \cup A' \Rightarrow$

$A \cup A$ is closed set by proposition 2.2.

Definition 2.7: Let A be a subset of topological space (X, T)

The set $A \cup A'$ is called ~~closs~~ closure of A and denoted by

$\overline{A}$.

Remark 2.1: $\overline{A}$ is closed set and by proposition 2.2 $\overline{A}$ is the least closed set contain A such that $\overline{A}$ has the following properteies

1. $\overline{A}$ is closed set
2. $\overline{A} = \bigcap_{i \in I} \{C : C \text{ is closed set contain } A\}$
3. $\overline{A}$ is the least closed set
4. if A is closed then $A = \overline{A}$

Example 2.12: Let $X = \{a,b,c,d,e\}$ and $T = \{X, \phi, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$, prove $\overline{\{b,d\}} = \{b,c,d,e\}$, $\overline{\{a,c\}} = X$ and $\overline{\{b\}} = \{b,e\}$

proof: The closed sets are $\phi, X, \{b,c,d,e\}, \{a,b,e\}, \{b,e\}, \{a\}$ therefore

1. the least closed set contain $\{b\}$ is $\{b,e\}$
2. the least closed set contain $\{a,c\}$ is X
3. the leas closed set

2.13: let Q the rational numbers then prove $\overline{Q}=R$

let $\overline{Q}\neq R$ then $\exists\, x\in R/\overline{Q}$, but $R/\overline{Q}$ is openset

ore $\exists$ open interval (a,b) s.t $x\in(a,b)\subseteq R/\overline{Q}$

for any open interval there exist rational number q

g to it (~~s~~ because the density of rational numbers)

! $\Rightarrow \overline{Q}=R$.

ition 2.8: let A be a subset of topological space (X,T)

dense in X or ~~dense~~ everywhere dense in X if $\overline{A}=X$

ples 2.14:

is dense in $\mathbb{R}$

n example 2.12 $\{a,c\}$ is dense in X

n discrete topology, every subset of X is closed

fore, ~~the~~ X is the unige dense set.

osition 2.4: Let A be a subset of topological space

) then A is dense in X iff for each $U\in T$ and

ϕ then $A\cap U\neq\phi$

of: Let A is dense in X and let $U\in T$ and $U\neq\phi$

$U\cap A\neq\phi$

$U\cap A=\phi$ then if $x\in U$ then $x\notin A$ and x is not

point for A, i.e $x\notin A'$ therefore $x\notin A\cup A'=\overline{A}$

C: since A is dense

let for each $U\in T$ and $U\neq\phi$ then $U\cap A\neq\phi$

p A is dense in X

$A=X$ then A is dense, If $A\neq X$

... for each $U\in T$ contain x, then $U\cap A$

Neighborhoods

finition 2.9: Let (X,T) is topological space, $N \subseteq X$ and $p \in N$ n N is neighborhood of p if there exist open set U (i.e $U \in T$) h that $p \in U \subseteq N$

mples 2.15:

The closed interval $[0,1]$ in R is neighborhood for $\frac{1}{2}$ since $\in (\frac{1}{4}, \frac{3}{4}) \subseteq [0,1]$

The ~~...~~ half-closed interval $(0,1]$ in R is neighborhood for . since $\frac{1}{4} \in (0, \frac{1}{2}) \subseteq (0,1]$. but $(0,1]$ is not neighborhood for 1. (prove it)

Let (X,T) any topological space and $U \in T$ then U is neighborhood for every $p \in U$.

Let (X,T) is top. space and N is neigh. of p and $N \subseteq S$ then is neigh. of p.

position 2.5:

Let $U \subseteq (X,T)$ then U is open ($U \in T$) iff $x \in U$ there exist $V \in T$ s.t $x \in V \subseteq U$ (i.e U is neighborhood out every its points)

nterior, exterior and boundary points:

finition 2.10:

Let (X,T) is top. space. ~~and~~ Let $A \subseteq X$ and $p \in A$ the interior point of A if there exist open set G ($G \in T$) $p \in G \subseteq A$. The set of all interior points of A denoted int A or A° and it is largest open in A.

mples 2.16:

The int points of $[0,1]$ in $(\mathbb{R}, T_u)$ is $(0,1)$.

en p is exterior point of A if there exist open set G s.t $G \subseteq A^c$. The set of all exterior points of A denoted by
t A or A^e.

mple 2.17:

The exterior points of $[0,1], (0,1], [0,1)$ and
$1)$ is $(-\infty, 0) \cup (1, \infty)$.

finition 2.12:

Let (X, T) is topological space. Let $A \subseteq X$ and $p \in X$
$p \in A$ or $p \notin A$). then p is boundary point of A if every open
t G contain p then $G \cap A \neq \phi \wedge G \cap A^c \neq \phi$. The set
of all boundary of A by $b(A)$.

ample 2.18:

The boundary points of $[0,1], [0,1), (0,1]$
d $(0,1)$ is $\{0,1\}$.

xioms of interior, exterior and boundary points:

terior

$\phi^\circ = \phi$ ② $X^\circ = X$ ③ $A^{\circ\circ} = A^\circ$

A° is the largest open subset in A

$(A \cap B)^\circ = A^\circ \cap B^\circ$

oof:

$$(A \cap B)^\circ = \overline{(A \cap B)^c}^c = \overline{(A^c \cup B^c)}^c = (\overline{A^c} \cup \overline{B^c})^c = (\overline{A^c} \cup \overline{B^c})^c$$
$$= \overline{A^c}^c \cap \overline{B^c}^c = A^\circ \cap B^\circ$$

$A^\circ = A \Leftrightarrow A$ is open

terior:

$\phi^e = X$ ② $A^e \subseteq A^c$ ③ $A^e = A^{e^{c^e}}$

$(A \cup B)^e = A^e \cap B^e$ (H.w)

let $p \in A$, since $bA \cap A = \emptyset$

$\Rightarrow p \notin b(A) \Rightarrow p \in (b(A))^c$

{ $b(A)$ is closed since by prop. 2.2 }

$\Rightarrow (b(A))^c$ is open and $(b(A))^c = \text{int} E \cup \text{ext} E$

$\Rightarrow p \in \text{int} E$ or $p \in \text{ext} E$ but $p \in E \Rightarrow p \notin \text{ext} E$

$\Rightarrow p \in \text{int} E \subseteq E \Rightarrow E$ is open.

$\Leftarrow$ Conv ?

Example 2.19:

Let $A = \{\frac{1}{n}, n \in N\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ find $A^\circ, A', \bar{A}$ in $(R, T_c), T_d, T_i, T_u$?

Sol:

) in $(\mathbb{R}, T_i)$

$A^\circ = \emptyset$ since for each $a \in A$ there is not open set $G \in T_i$ s.t $a \in G \subseteq A$

) (R, T_d)

$A^\circ = A$, since for each $a \in A$ there exist open $\{a\}$ s.t $a \in \{a\} \subseteq A$

) $(\mathbb{R}, T_c)$

$A^\circ = \emptyset$ since every point $a \in A$ there is no open set contai it G s.t $a \in G \subseteq A$ i.e there is no open which it complete finite containing in A.

) $(\mathbb{R}, T_u)$

$A^\circ = \emptyset$ since for example for any $a \in A$ then the open is $(a+\epsilon, a-\epsilon) \not\subseteq N$.

$(\mathbb{R}, T_i)$

en $A \subseteq \mathbb{R}$ since

② in $(\mathbb{R}, T_d)$
then $\underline{A'} = \emptyset$ since for each $a \in X$, $\{a\} / a \cap N = \emptyset$
$\Rightarrow \overline{A} = \emptyset \cup A = A$

③ in $(\mathbb{R}, T_c)$
$A' = \mathbb{R}$ since $R - \{2\} / \{1\} \cap A \neq \emptyset$
$R - \{2\} / \{1/2\} \cap A \neq \emptyset$
$\vdots$

$\Rightarrow \overline{A} = A \cup \mathbb{R} = \mathbb{R}$

④ in $(\mathbb{R}, T_u)$
$A' = \{0\}$ since $(\frac{1}{2}, 2) . / \{1\} \cap A = \emptyset$
$(\frac{1}{3}, \frac{2}{3}) \mid \{\frac{1}{2}\} \cap A = \emptyset$
$(-1, \frac{1}{n}) \mid \{0\} \cap A \neq \emptyset$

$\Rightarrow \overline{A} = A \cup \{0\} = A$

Lecture three

Basis and Subbasis

efinition 3.1: Let (X, T) is topological space. The famil
$\beta \subset T$ in X is called basis for the top. X if $\forall G \in T$
is the union of member of β.

mark 3.1: if β is basis for T then the subset $U \subseteq X$ is
pen iff U is the union of member of β.

xamples 3.1:

In $(\mathbb{R}, T_d)$ then $\beta_1 = T_d$ trivial Base

$\beta_2 =$ set of all Singletone $= \{\{1\}, \{2\}, \cdots$

.e $\beta_2 = \{\{x\} : x \in \mathbb{R}\}$

Let $X = \{a, b, c, d, e, f\}$ and $T_1 = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\},$
$b, c, d, e, f\}\}$
then $\beta = \{\{a\}, \{c, d\}, \{b, c, d, e, f\}\}$ is base for T_1 since
$\beta \subseteq T_1$ and every element in T_1 may be ~~rewritten as~~
write as union elements of β (ϕ is empty union of β
and So T_1 is ~~basis~~ base for T_1.

Let $\beta = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ then β is ~~basis~~ base f
he usual top. on $\mathbb{R}$.

let $X = \{a, b, c\}$, $\beta = \{\{a\}, \{c\}, \{a, b\}, \{b, c\}\}$ then β is
se for any top. on X.

1:

t β is ~~top~~ base for top. T on X, T is consist of all un
in β. i.e $T = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}\}$

sition 3.1: Let X be a nonempty set and $\mathcal{B}$ is family subsets of X. Then $\mathcal{B}$ is base for topology some T on X

$= \bigcup_{B \in \mathcal{B}} B$

$B_1, B_2 \in \mathcal{B}$ then $B_1 \cap B_2$ is union elements of $\mathcal{B}$

f: $\Longrightarrow$

is basis for T then X is openset $\Rightarrow X = \bigcup_{B \in \mathcal{B}} B$

so for any two $B_1, B_2 \in \mathcal{B}$ then $B_1, B_2 \in T$

$B_1 \cap B_2 \in T$

$B_1 \cap B_2$ is openset which is union elements of $\mathcal{B}$

~~suppose $\mathcal{B}$ is a class of subsets of X~~ Let T be the class of all subsets of X which are unions of members of $\mathcal{B}$

$\Leftarrow$ Let $\mathcal{B}$ has the properties (a) and (b)

p $\mathcal{B}$ is base for topology T on X.

T is collection of all subsets of X which unions all elements of $\mathcal{B}$ then < T.p T is topology on X >

nce $X = \bigcup_{B \in \mathcal{B}} B \Rightarrow X \in T$ and since ϕ is the union all empty sets in $\mathcal{B} \Rightarrow \phi \in T$

is satisfied.

The property (b) of topology is < T.p $\bigcup B_i \in T$ >

$B_1, B_2, \ldots \in T$ then $B_i = \bigcup_j B_j^*$ s.t $B_j^* \in \mathcal{B}$

y $B_i \in T$ then

$\bigcup_i B_i = \bigcup_i \{ \bigcup_j B_j^* : j \in J \}$ and this mean the union

ents of T is union elements of $\mathcal{B} \Rightarrow$

(c) of topology is satisfied.

$\cdot D = (\bigcup_{k\in K} B_k) \cap (\bigcup_{j\in J} B_j) = \bigcup_{k\in K, j\in J} (B_j \cap B_k)$

$B_k \cap B_j$ $\forall k\in K$ and $j \in J$, is union elements of $\mathcal{B}$

$\cdot C \cap D \in \mathcal{T}$.

...ample 3.2: Let $\mathcal{B}$ is family of open rectangle

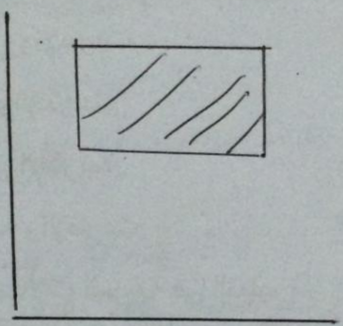
$\{(x,y) : (x,y) \in \mathbb{R}^2, a < x < b, c < y < d\}$ in space s.t

...ery side is parallel for axis X or Y axis

...te:

...all a, b, c, d are

...varying.



~~Note that (x,y) is change but a, b, c, d are constant.~~

Then $\mathcal{B}$ is basis for Euclidean topology on plane $\mathbb{R}^2$

...nce

The plane $\mathbb{R}^2$ is union of all open rectangles

the intersection of any two rectangles is rectangle

therefore the conditions of proposition 3.1 are satisfi...

...: Circulated the above idea on plane $\mathbb{R}^n$ and descrip...

Euclidean topology.

tion 3.2: Let (X, T) be a top. space. The family $\mathcal{B}$ of ubsets of X is base for topology T iff $\forall x \in X$ and $U \in T$

then $\exists B \in \mathcal{B}$ s.t $x \in B \subseteq U$

: Let $\mathcal{B}$ is base for T and let $x \in U \in T$.

$\exists B \in \mathcal{B}$ s.t $x \in B \subseteq U$ >

: $\mathcal{B}$ is base for T and $U \in T$ then $U = \bigcup_{j \in J} B_j$, $B_j \in \mathcal{B}$

$\exists j$ s.t $x \in B_j \subseteq U$.

$\forall x \in X$, $U \in T$ ~~then~~ and $x \in U \Rightarrow \exists B \in \mathcal{B}$ s.t

$\subseteq U$

T.P $\mathcal{B}$ is base for T >

T.P for each V open set is union elements o

V is open set $\Rightarrow \forall x \in V, \exists B_x \in B$ s.t

$B_x \subseteq V \Rightarrow V = \bigcup_{x \in V} B_x \Rightarrow \mathcal{B}$ is base for T.

---

sition 3.3: Let $\mathcal{B}_1$ and $\mathcal{B}_2$ two bases for topology T_1 T_2, respectively, on nonempty set X. Then $T_1 = T_2$

$\forall B \in \mathcal{B}_1$ and $x \in B$, $\exists B' \in \mathcal{B}_2$ s.t $x \in B' \subseteq B$

$B \in \mathcal{B}_2$ and $x \in B$ $\exists B' \in \mathcal{B}_1$ s.t $x \in B' \subseteq B$

: Let $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases for T_1 and T_2, res

, s.t $T_1 = T_2$

① and ②

clear

$\mathcal{B}$ and $\mathcal{B}_2$ satisfy ① and ②

$\mathcal{B} \subseteq T \cup$

$\Rightarrow$

~~$\mathcal{B}_2$ give T_1 and T_2 $\Rightarrow T_1 \subseteq T_2$.~~ since $\mathcal{B}$ is union

~~from ② then $T_2 \subseteq T_1$ $\Rightarrow T_1 = T_2$~~ elements of $\mathcal{B}$

proof: suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ satisfy ① and ②

$\forall B \in \mathcal{B}_1$ is open in $(X, T_2) \Rightarrow \mathcal{B}_1 \subseteq T_2$

every element $V \in T_1$ is union elements of T

, $T_1 \subseteq T_2$ and by ② then $T_2 \subseteq T_1 \Rightarrow T_1 =$

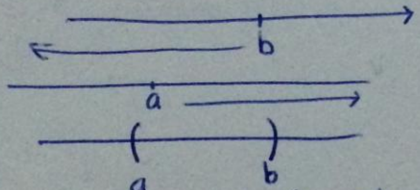
finition 3.2: Let (X,T) be a T.S. and let $S \subseteq T$, S

ubbase for $T \overset{iff}{\Longleftrightarrow}$ the finite intersection for S is a Ba

T.

mple 3.3: (R, T_u) then $S = \{(-\infty, b), (a, \infty), a, b \in R$

$b\}$ is subBase for (R, T_u)



finition 3.3: Let (X,T) be a T.S. and $p \in X$ th

mily $\beta_p \subseteq T$ is local Base for p iff $\forall G \in$

$p \in G$ then $\exists G_p \in \beta_p$ s.t $p \in G_p \subseteq G$.

ple 3.4: In $(\mathbb{R}, T_d)$ for any $a \in \mathbb{R}$ then there e

l base for a is $\{a\}$.

ition 3.4: Let (X,T) be a T.S. $X' \subset X$ then T'

ology on X' called the relative topology or induced

ology on X' called the relative topology or induced

$.5$: Let $X = \{a,b,c,d,e\}$
$= \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{b,c,d\}, \{a,c,d\}, \{a,$
$\{b,c,d,e\}\}$

$,c,e\}$
$\cap X' = \{\phi, X', \{a\}, \{c\}, \{a,c\}, \{c,e\}\}$
pology on X', $T' \not\subset T$.
ion 3.4: Let (X^*, T^*) be a relative topology fro
), $A \subset X^* \subset X$ prove that if A is open in T then
in T^*?
$T^* = \{G^* : G^* = G \cap X^*, G \in T\}$
T and $A \subset X^*$
$A \in T^*$ ~~since~~.

Continuity

Definition 4.1: The function $f: X \to Y$, from topological space X to top. space Y is continuous if for each open set U in Y, $f^{-1}(U)$ is open set in X.

Examples 4.1 ① Let $f:(\mathbb{R}, T_u) \to (\mathbb{R}, T_u)$ s.t $f(x) = x$ $\forall x \in \mathbb{R}$ then for any open set U in $\mathbb{R}$ then $f^{-1}(U) = U$ is open in $\mathbb{R}$
$\therefore f$ is continuous.

② Let $f:(\mathbb{R}, T_u) \to (\mathbb{R}, T_u)$ define as following $f(x) = c$ s.t c is constant $\forall x \in \mathbb{R}$. then for any open set U in $\mathbb{R}$ then

$$f^{-1}(U) = \begin{cases} \mathbb{R} & \text{if } c \in U \\ \phi & \text{if} \notin U \end{cases}$$ and $\mathbb{R}, \phi$ are open set in $\mathbb{R}$

$\Rightarrow f$ is continuous mapping.

③ Let $f:(X, T_d) \to (Y, T)$, then f is continuous since for any open $U \in T$ then $f^{-1}(U) \subseteq X$ is open since the top. is discrete.

④ Let $f:(X, T) \to (Y, T_i)$ then f is continuous since for any open U in Y is Y or ϕ therefore $f^{-1}(U) = \begin{cases} X & \text{if } U = Y \\ \phi & \text{if } U = \phi \end{cases}$
ϕ and X are open set in $X \Rightarrow f$ is continuous.

Lemma 4.1: Let (X, T) and (Y, T') are topological spaces th and let $f: (X, T) \to (Y, T')$ is mapping then f is continuous $\forall x \in X$ and $\forall U \in T'$ and $f(x) \in U$ there exist $V \in T$ such th $x \in V$ and $f(V) \subseteq U$.

Lemma 4.2: Let $(X, T), (Y, T')$ and (Z, T'') are topologic spaces then if $f:(X, T) \to (Y, T')$ and $g:(Y, T') \to (Z$ are continuous mappings then $g \circ f: (X, T) \to (Z, T'')$ continuous function

.p $g \circ f$ is continuous then

t prove $(g \circ f)^{-1}(U)$ is open in T for any open set in

ince U is open and g is cont then $g^{-1}(U)$ is open set

' and since $g^{-1}(U)$ is open in T' and f is cont

$f^{-1}(g^{-1}(U))$ is open in T $\Rightarrow$ $g \circ f$ is cont.

m 4.1: Let (X, T) and (Y, T') are top. spaces and $f:(X,T)$

$T')$ is cont iff each closed set F in Y then $f^{-1}(F)$ is

l in X

Let f is cont and F is closed in Y < T.p $f^{-1}(F)$ is

d in X >

F is closed set in Y $\Rightarrow$ F^c is open set in Y

$f^{-1}(F^c)$ is open in X but $f^{-1}(F^c) = (f^{-1}(F))^c$

$(F))^c$ is open $\Rightarrow$ $f^{-1}(F)$ is closed in in X

et G is open in Y $\Rightarrow$ G^c is closed in Y

$f^{-1}(G^c)$ is closed in Y but $f^{-1}(G^c) = (f^{-1}(G))^c$ $\Rightarrow$

$))^c$ is closed $\Rightarrow$ $f^{-1}(G)$ is open.

ion 4.2: Let (X, T) and (Y, T') are two top. spaces then

$T) \rightarrow (Y, T')$ is open (close) if for each open

(closed set G) in X then $f(U)$ is open in

$G)$ is closed in Y).

4.2: $f:(\mathbb{R}, T_u) \rightarrow (\mathbb{R}, T_u)$ s.t $f(x) = x$

is open function and so closed.

a) $\Rightarrow$ (b)

$^{-1}$ is cont then T.p f is open then

is open in X < T.p $f(G)$ is open >

G is open in X and f^{-1} is cont then $(f^{-1})^{-1}(G)$ is

$\Rightarrow$ $f(G)$ is open in Y.

$\Rightarrow$(C) H.W

$\Rightarrow$ (a) H.W

nition 4.3: Let $f:(X,T) \to (Y,T')$ be a function bet topological spaces (X,T) and (Y,T') then f is home n if satisfay the following

cont. 2- f^{-1} is cont. 3 f is 1-1 4 f is ont

, the two topological spaces is called homeomorphic.

les 4.3:

t $f:(X,T) \longrightarrow (X,T)$ define as following $f(x)=x$ th omeomorphism.

$(\mathbb{R},T_u) \longrightarrow (\mathbb{R},T_u)$ s.t $f(x)=x^2$ then f is not homeom

f is not open, i.e ~~$(f^{-1})^{-1}$ is~~ f^{-1} is not cont

f has $A=(-1,1)$ then $f(A)=[0,1)$

n 4.3: Let $f:(X,T) \to (X',T')$ and $g:(X',T') \to (X$

eomorphism then $g \circ f$ is homeomorphism.

f: H.W.

ition 4.4: Let p be any property, we said p is t

T is finer of T' ($T' \subseteq T$)

f: $\Rightarrow$:

Let i is cont and let $V \in T'$

and since i is cont $\Rightarrow i^{-1}(V) \in T \Rightarrow V \in T \Rightarrow$

$T' \subseteq T$

$\Leftarrow$ Let $T' \subseteq T$ and let $V \in T' \Rightarrow i^{-1}(V) \in T$

$\Rightarrow V \in T \Rightarrow i$ is cont.

Example 4.4: Let T is topology generated by the base on $\mathbb{R}$

$= \{ [a,b) : a<b , a,b \in \mathbb{R} \}$ and let T_u is usual topology on

$\mathbb{R}$ and let $i : (\mathbb{R}, T) \longrightarrow (\mathbb{R}, T_u)$ then, is i is cont?

Proof:

Let $(a,b) \in \beta_u$, $(a,b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b)$

$a + \frac{1}{n} \longrightarrow a$ when $n \to \infty$

$\Rightarrow (a,b) \in T \Rightarrow \beta_u \subseteq T \Rightarrow T_u \subseteq T$

هل العكس صحيح ؟

Theorem 4.5: Let $f : (X, T) \longrightarrow (Y, T')$ is continuous func

let (A, T_A) is subspace of (X, T) then $f \restriction_A : (A, T_A) \to$

(Y, T') is cont. fun.

Proof: let $V \in T'$ and since f is cont then $f^{-1}(V) \in$

$f^{-1}\restriction_A(V) \in T_A$ then $f^{-1}\restriction_A(V) = f^{-1}(V) \cap A$

$f^{-1}\restriction_A(V)$ is open $\Rightarrow f^{-1}\restriction_A(V)$ is open $\Rightarrow$

let $f^{-1}(V) \cap A$ is open $\Rightarrow f^{-1}\restriction_A(V)$ is open $\Rightarrow$

$f\restriction_A$ is cont fun.

Example 4.5: Let $f : (\mathbb{R}, T_u) \longrightarrow (\mathbb{R}, T_u)$ s.t $f(x) = 2x$

f is homeomorphism?

i.p f is ... $\Rightarrow$ ...
let $2x+1=y \Rightarrow x=\frac{y-1}{2}\in\mathbb{R}$ $\Rightarrow$ ...
$\exists$ $\frac{y-1}{2}$ s.t $f(\frac{y-1}{2})=y$. $\Rightarrow$ f is onto.

f is cont since $\mathbb{R}$ is metric space
since $2x+1=y \Rightarrow x=\frac{y-1}{2} \Rightarrow f^{-1}(y)=\frac{y-1}{2}$
~~$f^{-1}(y)=y$~~ nf then f^{-1} is cont
$\Rightarrow$ f is homeomorphism.

...ple 4.6: Let $X=\{a,b,c,d\}$
$T=\{\phi, X,\{a\},\{b\},\{a,b\}\}$ is $T\cong T'$?
$T'=\{\phi,X,\{c\}\}$
if $(X,T)\cong(X,T')$ then there exist homeomorphism
$(X,T')\to(X,T)$ s.t since $\{a,b\}\in T$ then
$f^{-1}\{a,b\}=\phi \Rightarrow C!$ since f is
$\{a,b\}\in T'$ then if
if $f^{-1}\{a,b\}=X$ or $c \Rightarrow C!$ since f is 1-1
$(X,T)\ncong(X,T')$

$X=\{a,b,c,d\}$
$=\{\phi,X,\{a\},\{b\},\{a,b\}\}$, $T'=\{\phi,X,\{c\},\{b\},\{b,c\}\}$
$(X,T')\cong(X,T)$?

...ple 4.7: for any interval in $\mathbb{R}$ is homeomorphic ...
...val in $\mathbb{R}$; i.e, $(a,b)\cong(c,d)$ for all a,b,c,d
let $h:(a,b)\longrightarrow(c,d)$
$h(a)=c$ and $h(b)=d$
... $mx+k$

$-ma = d-c \Rightarrow m(b-a) = d-c \Rightarrow$

$\frac{d-c}{b-a}$ in ① then $\frac{d-c}{b-a}a + k = c \Rightarrow$

$- \frac{ad-ac}{b-a} = \frac{cb \mp ca - ad + ac}{b-a} = \frac{cb-ad}{b-a}$

$h(x) = \left(\frac{d-c}{b-a}\right)x + \left(\frac{cb-ad}{b-a}\right)$

h is 1-1, ont, cont and h^{-1} is cont since h is linear
tion.

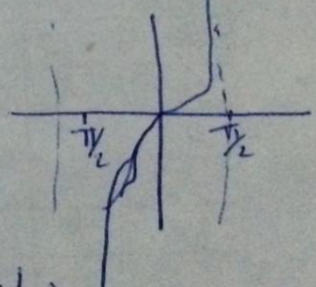
: $(2,3) \cong (0,1)$

ple 4.8: Is $(0,1) \cong \mathbb{R}$?

let $g: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ define as following

$= \tan x \quad \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

g is homeomorphism ~~since~~ and this its clear
graph



So for any two intervals in $\mathbb{R}$
homeomorphism $\Rightarrow (-1,1) \cong \mathbb{R}$

es 4.1:

t $f: (X,T) \to (Y,T')$ is homeomorphism
$^{-1}$ is homeomorphism

image of open set under the cont fun. is not
ry is open but if the f is homeomorphism then the
is open.

Metric space: Lecture Five

Definition 5.1: Let X be a nonempty set and $d: X \times X \to R^+$ such that $\forall a,b \in X$ then

1- $d(a,b) \geq 0$, $d(a,b) = 0$ iff $a=b$

2- $d(a,b) = d(b,a)$

3- $d(a,b) \leq d(a,c) + d(c,b) \quad \forall a,b,c \in X$

Then d is called metric on X and (X,d) is called Metric space and so $d(a,b)$ is called the distance between a and b.

Example 5.1: Let $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is define as following $d(a,b) = |a-b|$ s.t $a,b \in \mathbb{R}$ and d is called Euclidean metric on R

Example 5.2: Let $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is define as following $d((a_1,a_2),(b_1,b_2)) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$ and d is called Euclidean metric on $\mathbb{R}$.

Example 5.3: Let X be anonempty set and $d: X \times X \to \mathbb{R}^+$ is define $d(a,b) = \begin{cases} 0 & a=b \\ 1 & a \neq b \end{cases}$ then d is the distance (metric) on X and is called the discrete metric.

Example 5.4: Let $C[0,1]$ is the set of all continuous functions on $[0,1]$ the we define d on $C[0,1]$ as following

$d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}$ then d is metric on $C[0,1]$

Definition 5.2: Let (X,d) is Metric space and $r \in R$ then open Ball $B_r(a) = \{x \in X : d(a,x) < r\}$

Example 5.5: In $\mathbb{R}$ with Euclidean space, then $B_r(a)$ is open interval $(a-r, a+r)$

Example 5.6: In $\mathbb{R}^2$ with Euclidean, $B_r(a)$ is open disk which radius r and it Center is a

Proposition 5.1: Let (X,d) be a metric space and B_1 and B_2 are open Balls then $B_1 \cap B_2$ is union open Balls in (X,d).

Proposition 5.2: Let (X,d) be a metric space. Then family all open Balls in (X,d) is base for topology on X.

Proof:

- The union of open Balls $\bigcup B_r(a) = X \quad \forall a \in X$ and $r \in R$

- by pro 5.1 then $B_1 \cap B_2$ is union open Balls in (X,d)

topology on X by the distance d. The subset $U \subseteq X$ is open in (X,T) iff $\forall a \in U$ there exist $\epsilon > 0$ s.t the open Ball $B_\epsilon(a)$ is subset of U.

proof: Let $U \in T$ and let $a \in U$ then by proposition (3.2) there exist $b \in X$ and $\delta > 0$ s.t $a \in B_\delta(b) \subseteq U$ let $\epsilon = \delta - d(a$

then $a \in B_\epsilon(a) \subseteq U$.

$\Leftarrow$ let $U \subseteq X$ s.t for each $a \in U$ there exist $\epsilon > 0$ then $\exists B_\epsilon(a)$ s.t

$B_\epsilon(a) \subseteq U$

by pro(Let $\mathcal{B}$ is base for X then the subset $U \subseteq X$ is open iff $\forall x \in U, \exists B \in$ s.t $x \in B \subseteq U$) $\Rightarrow$ U is open.

Remark 5.1: Every set X and every metric d on X is give induced topology on X

Definition 5.3: A topological space (X,T) is Hausdroff space or (T_2-space) if $\forall a,b \in X$ $\exists$ $U,V \in T$ s.t $a \in U$ and $b \in V$ s.t $U \cap V = \phi$

proposition 5.4: Let (X,d) be a metric space and T is induced topology on X by d then (X,T) is Hausdroff Space (T_2-space)

proof: Let $a,b \in X$ s.t $a \neq b$ then $d(a,b) > 0$ then let $d(a,b) = \epsilon$ then consider $B_{\frac{\epsilon}{2}}(a)$ and $B_{\frac{\epsilon}{2}}(b)$ (open Balls). then $B_{\frac{\epsilon}{2}}(a), B_{\frac{\epsilon}{2}}(b) \in T$ and $B_{\frac{\epsilon}{2}}(a) \cap B_{\frac{\epsilon}{2}}(b) = \phi$.

suppose $x \in B_{\frac{\epsilon}{2}}(a) \cap B_{\frac{\epsilon}{2}}(b) \Rightarrow d(x,a) < \frac{\epsilon}{2}$ and $d(x,b) < \frac{\epsilon}{2}$

$\Rightarrow$ But d is metric, therefore

$d(a,b) \leq d(a,x) + d(x,b) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \Rightarrow d(a,b) < \epsilon$

$\Rightarrow$ C! since $d(a,b) = \epsilon$

1- Finite product:

Definition 1: Let $(X_1, T_1), \ldots, (X_n, T_n)$ be a collection of $n \geq 2$ topological spaces. Then their Cartesian product $X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) : x_i \in X_i, i=1, \ldots, n\}$ becomes a topological space with the product topology $T_{X_1 \times \cdots \times X_n}$ defined as the topology generated by the basis

$$\mathcal{B}_{X_1 \times \cdots \times X_n} = \{U_1 \times \cdots \times U_n \subset X_1 \times \cdots \times X_n : U_i \in T_i, i=1, \ldots, n\}$$

The space $(X_1 \times \cdots \times X_n, T_{X_1 \times \cdots \times X_n})$ is called the product of $(X_1, T_1), \ldots, (X_n, T_n)$ or simply a product space. we shall refer to the spaces $(X_i, T_i), i=1, \ldots, n$ as the factors of $(X_1 \times \cdots \times X_n, T_{X_1 \times \cdots \times X_n})$. The functions $\Pi_i : X_1 \times \cdots \times X_n \to X_i$ s.t. $\Pi_i(x_1, \ldots, x_n) = x_i$ shall be referred to as the projection maps or simply as the projections.

Since $X_i \in T_i$ for each $i=1, \ldots, n$, we see that from definition the basis above then $X_1 \times X_2 \times \cdots \times X_n \in \mathcal{B}_{X_1 \times \cdots \times X_n}$ therefore the first condition of ~~definition topology~~ proposition 3.1 is satisfied.

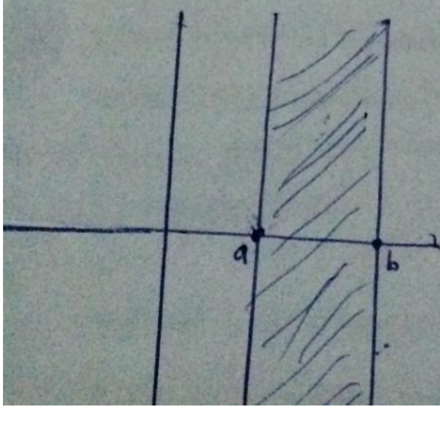
As for the second, Let $U_i, V_i \in T_i$ for each $i=1, \ldots, n$ [th]en let $(x_1, \ldots, x_n) \in (U_1 \times \cdots \times U_n) \cap (V_1 \times \cdots \times V_n)$ be any [p]oint then $(U_1 \times \cdots \times U_n) \cap (V_1 \times \cdots \times V_n) = (U_1 \cap V_1) \times \cdots \times (U_n \cap V_n)$ [a]nd because $U_i \cap V_i \in T_i$ we see that $(U_1 \times \cdots \times U_n) \cap (V_1 \times \cdots \times V_n)$ [∈] $\mathcal{B}_{X_1 \times \cdots \times X_n}$ and therefore indeed, $\mathcal{B}_{X_1 \times \cdots \times X_n}$ is a basis [o]f a topology.

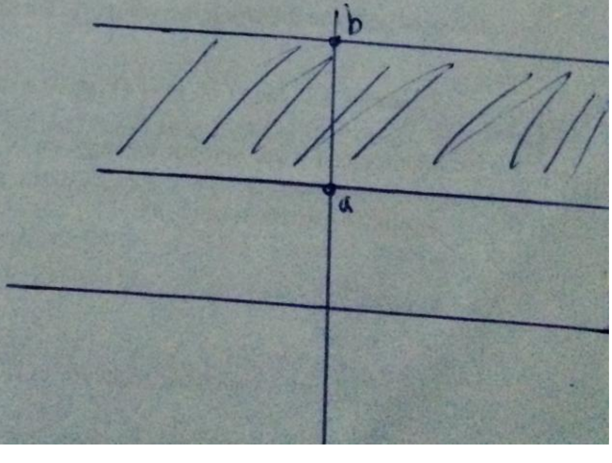
[R]emark 1: It may be tempting, but incorrect, to assume [tha]t every open set in $(X \times Y, T_{X \times Y})$ is a product of [ope]n sets from X and Y.

~~rather~~, according to definition, every element $W \in$
$T_{X \times Y}$ is a union of such products:
$W = \bigcup_{i \in I} U_i \times V_i$ with $U_i \in T_X$ and $V_i \in T_Y$, $i \in I$
for some indexing set I.

Example 9.1: Consider the topological spaces $(\mathbb{R}, T_p)$ and $(\mathbb{R}, T_q)$ where T_p and T_q are the included ~~point~~ all point topologies associated to the points $p, q \in \mathbb{R}$. An open set $W \subset X \times Y$ is a union of open set $U \times V$ with $U \in T_p$ and $V \in T_q$. In particular, if U and V are non-empty, then $p \in U$ and $q \in V$ so that $(p,q) \in W$. The converse of this is false, i.e., a subset W of $X \times Y$ that contains (p,q) may not be open. An example is any set $W = \{(p,q), (x,y)\}$ with $x \neq p$ and $y \neq q$, is not union of elements $U_i \times V_i$ with $p \in U_i$ and $q \in V_i$. projection map Def.

Example 9.2: Consider Cartesian plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Recall that the inverse $\pi_1^{-1}(a,b)$ and $\pi_2^{-1}(a,b)$ are infinite open strips which form subbase for the usual top on $\mathbb{R}^2$.





we take the inverse ...

al the union the strips gives the product topology.

Lemma 9.1: Let X and Y be two topological spaces and let $A \subset X$ and $B \subset Y$ be two non-empty subsets.

ⓐ $A \times B$ is an open subset of $X \times Y$ with respect to the product topology $T_{X \times Y}$ if and only if A and B are both open.

ⓑ $A \times B$ is a closed subset of $X \times Y$ with respect to the product topology $T_{X \times Y}$ if and only if A and B are both closed.

proof: ⓐ If A and B are both open then $A \times B$ is open simply by definition of $T_{X \times Y}$. suppose conversely that $A \times B$ is open. Then we can write $A \times B = \bigcup_{i \in I} U_i \times V_i$ with $U_i \subset X$ and $V_i \subset Y$ each open. But then $A = \bigcup_{i \in I} U_i$ and $B = \bigcup_{i \in I} V_i$ showing that they are both unions open sets and thus open themselves.

ⓑ $\Rightarrow$ Suppose that $A \subset X$ and $B \subset Y$ are closed subsets and note that $(X \times Y) - (A \times B) = X \times (Y - B) \cup (X - A) \times Y$

Both of the sets on the right hand side of the above are open subsets of $X \times Y$ showing that $(X \times Y) - (A \times B)$ is open and thus that $A \times B$ is closed.

$\Leftarrow$ Suppose that $A \times B \subset X \times Y$ is closed and let $x \in X - A$ be any point. pick $y \in B$ aribitrarily and observe that then (x, y) lie in $(X \times Y) - (A \times B)$ which is an open set. Thus there must be neighborhoods $U_x \subset X$ of x and $V_y \subset Y$ of y such that $U_x \times V_y$ $(X \times Y) - (A \times B)$. We claim that U_x is disjoint from A.

Let $z \in U_x \cap A \Rightarrow (z, y) \in (U_x \times V_y) \cap (A \times B) \Rightarrow C!$

Since $(U_x \times V_y) \cap (A \times B) = \phi \Rightarrow U_x \cap A = \phi$

$\Rightarrow X - A = \bigcup_{x \in X - A} U_x \Rightarrow X - A$ is open $\Rightarrow A$ is closed s...

basc

Let $\{X_i : i \in I\}$ be a collection of topological spaces and let X denoted ~~by~~ the product space, i.e., $X = \prod_i X_i$. If G_{j_0} is an open subset of the coordinate space X_{j_0}, then $\pi_{j_0}^{-1}(G_{j_0})$ consists of all points $p = \langle a_i : i \in I \rangle$ in X such that $\pi(p) \in G_{j_0}$.

In otherwords

$$\pi_{j_0}^{-1}(G_{j_0}) = \prod \{X_i : i \neq j_0\} \times G_{j_0}$$

In particular, if we have collection of topological spaces, say $\{X_1, X_2, \cdots\}$ then the product space

$X = \prod X_n = X_1 \times X \times \cdots$ consists of all sequences

$p = \langle a_1, a_2, \cdots \rangle$ where $a_n \in X_n$ and furthermore

$$\pi_{j_0}^{-1}(G_{j_0}) = X_1 \times \cdots \times X_{j_0-1} \times G_{j_0} \times X_{j_0+1} \times \cdots$$

~~Theorem 1: The class of subsets of product space $X = \prod_i X_i$ of the form~~

Theorem 9.2: The class of subsets of product space $X = \prod_i X_i$ of the form $\pi_{j_0}^{-1}(G_{j_0}) = \prod\{X_i : i \neq j_0\} \times G_{j_0}$ where G_{j_0} is an open subset of the coordinate space X_{j_0}, is a subbase and is called the defining subbase for the product topology.

Furthermore, since finite intersections of the subbase elements form a base for the topology, we also have

Theorem 9.3: The class of subsets of a product space $X = \prod_i X_i$ of the form $\pi_{j_1}^{-1}[G_{j_1}] \cap \dots \cap \pi_{j_m}^{-1}[G_{j_m}] = \prod\{X_i : i \neq j_1, \dots, j_m\} \times G_{j_1} \times \dots \times G_{j_m}$

where G_{j_k} is an open subset of the coordinate space X_{j_k}, is a base and is called the defining base for the product topology.

Theorem 9.4: A function $f : Y \longrightarrow X$ from topological space Y into a product space $X = \prod_i X_i$ is continuous iff, for every projection $\pi_i : X \longrightarrow X_i$ the composition $\pi_i \circ f : Y \longrightarrow X_i$ is continuous.

~~Solution~~ proof:

Since all projections are continuous (by def of prod)

$\Rightarrow$ and so f is cont. by hypothesis $\Longrightarrow \pi_i \circ f$ is cont.

$\Leftarrow$ let the function $\pi_i \circ f : Y \longrightarrow X_i$ is cont. $\forall i$.

~~let~~ <T.P f is cont>

$f : Y \longrightarrow X$.

Let G be an open subset of X_i, then, by continuity

is an open set in Y. ... where ... an open subset of X_i is the
$\pi_i^{-1}(G)$ where G is an open subset of X_i is the
defining subbase for the product topology on X.
Since their inverses under f are open subsets of Y, f is a continuous function by theorem (Let S be a subbase for a topological space Y. then a function $f: X \to Y$ is continuous iff the inverse of each member of S is an open subset of X.

Theorem 9.5: Every projection $\pi_i : X \to X_i$ on a product space $X = \Pi_i X_i$ is both open and continuous i.e bicontinuous.

proof:
By definition of the product space, all projections are continuous. So we need only show that they are open.
Let G be an open subset of the product space $X = \Pi X_i$. For every $p \in G$ there is B of defining base of the product top. s.t $p \in B \subset G$. Thus for any projection $\pi_i : X \to X_i \Rightarrow \pi_i(p) \in \pi_i[B] \subset \pi_i(G)$
then $\pi_i(B)$ is an open set i.e every point $\pi_i(p) \in \pi_i(G)$ belongs to an open set $\pi_i(B)$ which contained in $\pi_i(G) \Rightarrow \pi_i(G)$ is open.

Theorem: Let B be a member of the defining base for a product space $X = \Pi X_i$ then the projection of B into any coordinate space is open.

proof:
Since B belongs to the defining base for X
$$B = \Pi\{X_i : i \neq j_1, \ldots, j_m\} \times G_{j_1} \times \cdots \times G_{j_m}$$
where G_{j_k} is an open subset of X_{j_k}. So for any projection $\pi_\alpha : X \to X_\alpha$
$$\pi_\alpha(B) = \begin{cases} X_\alpha & \text{if } \alpha \neq j_1, \ldots, j_m \\ G_\alpha & \text{if } \alpha \in \{j_1, \ldots, j_m\} \end{cases} \Rightarrow \pi_\alpha(B) \text{ is open set.}$$