...ore, A and B are not separated since each $p \in B$ ...mit point of A and so $A \cup B$ is connected. But ...B is not arcwise connected in fact there exists no path from any point in A to any point in B.

Now, Note:

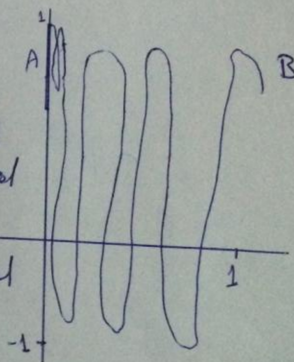
<u>Theorem:</u> A set is connected iff it is not the union of two non-empty separated sets

<u>Theorem:</u> If A and B are connected set which are not separated then $A \cup B$ is connected

<u>Example 10.5:</u> Consider the following subsets of the plane $\mathbb{R}^2$

$A = \{(0,y) : \frac{1}{2} < y \leq 1\}$

$B = \{(x,y) : y = \sin(1/x) \quad 0 < x \leq 1\}$



Now A and B are continuous images ...f intervals and are therefore connected Moreover, A and B are not separated ...ets and so $A \cup B$ is connected ...ut $A \cup B$ is not arcwise connected ...fact there exists no path ...m a point in A to a point in B.

<u>...heorem 10.3:</u> A continuous image of arcwise connected set ...arcwise connected.

...oof: Let $E \subset X$ be arcwise connected and let $f: X \to Y$ ... continuous we claim that $f(E)$ is arcwise connected ... let $p, q \in f[E]$. Then $\exists\, p^*, q^* \in E$ s.t. $f(p^*) = p$ ...d $f(q^*) = q$. But E is arcwise connected and so

e composition of Continuous functions is
[illegible]nuous and so $f \circ g : I \to Y$ is continuous, Furthermore
$(f \circ g)(0) = f(p^*) = p$, $f \circ g(1) = f(q^*) = q$ and $(f \circ g)(I) = f(g(I)) \subset$
$f(E)$. Thus $f[E]$ is arcwise connected.

Example 10.6: show that the open Disk in $\mathbb{R}^2$ is arcwise conne[cted]

sol: Let $p = (a_1, b_1)$ and $q = (a_2, b_2)$ then the function $f : I \to$
defined by $f(t) = (a_1 + t(a_2 - a_1), b_1 + t(b_2 - b_1))$ is path fro[m]
p to q which is contained in D.

Theorem 10.4: Let E be a non-empty open connected subset of the plane $\mathbb{R}^2$ then E is arcwise connected

proof:

Let $p \in E$ and let G consist of those points in E which can be joined to p by a path in E. We claim that G is open. For let $q \in G \subset E$. Now E is open and so $\exists$ an open disc with center q such that $q \in D \subset E$. But D is arcwise connected; hence each point $x \in D$ can be joined to q which can be joined to p. Hence each point $x \in D$ can be joined to p, and so $q \in D \subset G$. Accor[dingly]
G is open.

Now set $H = E \setminus G$, i.e. H consists of those points in E [w]hich cannot be joined to E by a path in E. We claim [t]hat H is open. For let $q^* \in H \subset E$. Since E is open, $\exists$ [o]pen disc D^* with center q^* such that $q^* \in D^* \subset E$. Sin[ce]
D^* is arcwise connected, each $x \in D^*$ cannot be join[ed]
[t]o p with a path in E, and so $q^* \in D^* \subset H$. H[...]