...ally Compact spaces:

Definition 8.3: A top. space X is locally compact iff every point in X has a compact neighborhood.

Example 8.6: Let $(\mathbb{R}, T_u)$ is usual topology, then every $p \in \mathbb{R}$ is interior to closed interval $[p-\delta, p+\delta]$ and by Heine Borel this interval is compact.

Theorem 8.5: Every compact space is locally compact

Remark: The converse is not true in general.

Example 8.7: Let $(\mathbb{R}, T_d)$ is discrete topology. then every $p \in \mathbb{R}$, $\{p\}$ is open and compact but $\mathbb{R}$ is not compact.

...nnectedness:

...eparated sets: Two subsets A and B of top. space X are said to be separated if

(i) A and B are disjoint

(ii) $A\cap\overline{B}=\emptyset$ and $\overline{A}\cap B=\emptyset$

Example 8.8: Let $A=(0,1)$ and $B=(1,2)$ and $C=[2,3)$ in $(\mathbb{R}, T_u)$.

A and B are separated since $\overline{A}=[0,1]$ and $\overline{B}=[1,2]$ and so $A\cap\overline{B}=\emptyset$, $\overline{A}\cap B=\emptyset$.

B and C are not separated since $2\in C$ is a limit point of B thus $\overline{B}\cap C=[1,2]\cap[2,3)=\{2\}\neq\emptyset$

Definition 8.4: A subset A of top space X is disconnected if there exist open subsets G and H of X s.t $A\cap G$ and $A\cap H$ are disjoint non-empty sets whose union is A. A set is connected if it is not disconnected.

i.e $A=(A\cap G)\cup(A\cup H)$

$\emptyset=(A\cap G)\cap(A\cap H)$

Example 8.9: Let $X=\{a,b,c,d,e\}$ and $T=\{X,\emptyset,\{a,b,c\},\{c,d,e\},\{c\}\}$

Now, $A=\{a,d,e\}$ is disconnected. Let $G=\{a,b,c\}$ and $H=\{c,d,e\}$ then $A\cap G=\{a\}$ and $A\cap H=\{d,e\}$ are non-empty disjoint sets whose union is A.

Theorem 8.6: ~~sh~~ if A and B are non-empty separated sets, then $A\cup B$ is disconnected

proof:

since A and B are separated, $A\cap\overline{B}=\emptyset$ and $\overline{A}\cap B=\emptyset$

Let $G=\overline{B}^c$ and $H=\overline{A}^c$ then G and H are open sets

$\Rightarrow (A\cup B)\cap G=A$ and $(A\cup B)\cap H=B$ are non-empty disjoint sets whose union is $A\cup B$. ~~Thus~~

~~G and H form a~~

Theorem 8.7: A set A is connected iff A is not the union of two non-empty separated sets

Sol: we show, equivalently, that A is disconnected iff A is the union of two non-empty separated sets. Suppose A is disconnected, ~~and let G∪H be a d.~~ $\Rightarrow$ A is the union of non-empty sets $A\cap G$ and $A\cap H$

Connected spaces: Let (X,T) be a top space; X is dis connected

Definition 8.5: Let (X,T) be a top space; X is dis connected if there exist two open subsets G and H s.t

$G \cup H = X$ and $G \cap H = \emptyset$

Theorem 8.7: Let A be a subset of a top. space (X,T) and let T_A be a relative top. on A then A is connected in X iff (A, T_A) is connected.

proof: $\Rightarrow$

Suppose A is disconnected then there exist $G, H \in T$ s.t

$(A\cap G) \cup (A \cap H) = A$ and

$(A \cap G) \cap (A \cap H) = \emptyset$

But $A\cap G, A\cap H \in T_A$ therefore, let $G_A = A\cap G, H_A = A\cap H$

$\Rightarrow G_A \cap H_A = \emptyset$ and $G_A \cup H_A = A$

$\Rightarrow (A, T_A)$ is disconnected

$\Leftarrow$ Let (A, T_A) is disconnected then there exist $G_A, H_A \in T_A$ s.t $G_A \cup H_A = A$ and $G_A \cap H_A = \emptyset$

$(G\cap A) \cup (H \cap A) = A$

$(G \cap A) \cap (H \cap A) = \emptyset$ s.t $G_A = G\cap A$ and $H_A = H \cap A$

$G, H \in T$.

Theorem 8.8: Let X be a top. space. Show that the following conditions are equivalent:

i) X is disconnected

ii) There exists a non-empty proper subset of X which is both open and closed.

proof, (i) $\Rightarrow$ (ii)

suppose $X = G \cup H$ where G and H are non-empty open sets since $G = H^c \Rightarrow G$ is both open and closed.

(ii) $\Rightarrow$ (i)

suppose A is open and closed $\Rightarrow A^c$ is open and clos[ed]

therefore $X = A \cup A^c$

Theorem 8.9: A top. space (X,T) is connected iff X and [$\emptyset$] are the only subsets of X which are both open a[nd] closed.