— [0,1] be a closed interval A and B

$A \subset G_{i_1} \cup \cdots \cup G_{i_m} = (A \cap H_{i_1}) \cup \cdots \cup (A \cap H_{i_m}) = A \cap (H_{i_1} \cup \cdots \cup H_{i_m}) \subset H_{i_1} \cup \cdots \cup H_{i_m}$, Thus $\{H_i\}$ has finite subcover,

$\Rightarrow$ A is compact with respect to T.

Remark: A subset of a compact space need not be compact.

Heine Borel Theorem: Every open cover of a closed and bounded interval $A = [a, b]$ has finite subcover.

Therefore, the closed interval $[0, 1]$ is compact, but $(0, 1) \subseteq [0, 1]$ is not compact by example (8.2)

Theorem 8.3: Let F be a closed subset of a compact space X. Then F is also compact.

proof:

Let $G = \{G_i\}$ be an open cover of F, i.e. $F \subset \bigcup_i G_i$. Then $X = \{\bigcup_i G_i\} \cup F^c$ that is $G^* = \{G_i\} \cup \{F^c\}$ is a cover of X.

But F^c is open since F is closed, so G^* is an open cover of X. By hypothesis, X is compact; hence G^* has finite subcover, say $X = G_{i_1} \cup \cdots \cup G_{i_m} \cup F^c$ $G_{i_k} \in G$

But F and F^c are disjoint; hence $F \subset G_{i_1} \cup \cdots \cup G_{i_m}$ $G_{i_k} \in G$

$\Rightarrow$ F is compact.

Finite intersection property:

A class $\{A_i\}$ of sets is said to have the finite intersection property if every finite sub-class $\{A_{i_1}, \ldots, A_{i_m}\}$ has a non-empty intersection, i.e. $A_{i_1} \cap \cdots \cap A_{i_m} \neq \emptyset$

Example 8.5: Consider the following of open intervals

$\mathcal{A} = \{(0,1), (0, \frac{1}{2}), (0, \frac{1}{3}), (0, \frac{1}{4}), \ldots\}$

Now, $\mathcal{A}$ has the finite intersection property

$(0, a_1) \cap \cdots (0, a_m) = (0, b)$

where $b = \min(a_1, \ldots, a_m) > 0$

Theorem 8.4 : A topological space X is compact iff every class $\{F_i\}$ of closed subsets of X which satisfies the finite intersection property has, itself, a non-empty intersection.

proof: ⟹

Let X is compact, Let $\mathcal{G}=\{F_i\}$ is class of closed sets which satisfy the finite intersection property

T.p $\bigcap F_i \neq \phi$

Let $\bigcap F_i = \phi \Rightarrow \bigcup F_i^c = X$ therefore $\{F_i^c\}$ is open cover for X, but X is compact ⟹

$\exists F_{i_1}^c, \ldots, F_{i_m}^c \in \{F_i^c\}$ s.t $X = F_{i_1}^c \cup \cdots \cup F_{i_m}^c$

$\Rightarrow \phi = X^c = (F_{i_1}^c \cup \cdots \cup F_{i_m}^c)^c = F_{i_1} \cap \cdots \cap F_{i_m} \Rightarrow C!$

⟸

T.p X is compact, then let $X \subseteq \bigcup F_i$ i.e

$X = \bigcup F_i \Rightarrow$ ~~$\phi = X^c = (\bigcup F_i)^c = \bigcap F_i^c$~~

~~since each G_i is open $\{G_i^c\}$ is a class of closed sets and, by above, has an empty~~

Let $\nexists m$ s.t $X = \bigcup_{j=1} F_{i_j}$

i.e $X \neq F_{i_1} \cup \cdots \cup F_{i_m} \quad \forall m \Rightarrow$

$\phi \neq F_{i_1}^c \cap \cdots \cap F_{i_m} \quad \forall m$

$\Rightarrow \phi \neq \bigcap F_i \Rightarrow X \neq \bigcup F_i$ but this contradiction

$\Rightarrow \exists m$ s.t $X = F_{i_1} \cup \cdots \cup F_{i_m}$.