Theorem 8.1: A Continuous image of Compact

proof,

Let (X, T) is top. space and So (Y, S)

Let $f: X \to Y$ is continuos mapping and $A \subseteq X$ is compact

T.P $<f(A)$ is Compact$>$

Let $f(A) \subset \bigcup G_i$ s.t $G_i \in S$

then $A \subseteq f^{-1}[f(A)] \subset f^{-1}(\bigcup_i G_i) = \bigcup f^{-1}(G_i)$

Hence $\mathcal{H} = \{f^{-1}(G_i)\}$ is cover for A, but A is compact

therefore $\mathcal{H}$ has finite subcover for A s.t

$A \subset f^{-1}(G_{i_1}) \cup \cdots \cup f^{-1}(G_{i_m}) \Rightarrow f(A) \subset f(f^{-1}(G_{i_1}) \cup \cdots \cup f^{-1}(G_{i_m}))$

$\subset G_{i_1} \cup \cdots \cup G_{i_m}$. $\Rightarrow f(A)$ is compact.

Theorem 8.2: Let A be a subset of top. space (X, T). Then the following are equivalent.

i- A is compact with respect to T

ii- A is compact with respect to the relative top. T_A on A

proof: i $\Rightarrow$ ii

Let $\{G_i\}$ be a T_A-open cover of $A \Rightarrow \exists H_i \in T$ s.t

$G_i = A \cap H_i \subset H_i \Rightarrow$ Hence $A \subset \bigcup_i G_i \subset \bigcup H_i$

$\Rightarrow \{H_i\}$ is open cover for A since A is compact with respe

to $T \Rightarrow \{H_i\}$ have a finite subcover for A s.t

$A \subset H_{i_1} \cup \cdots \cup H_{i_m}$ but $A \subset A \cap (H_{i_1} \cup \cdots \cup H_{i_m}) =$

~~A~~ $(A \cap H_{i_1}) \cup \cdots \cup (A \cap H_{i_m}) = G_{i_1} \cup \cdots \cup G_{i_m}$

Thus $\{G_i\}$ contains a finite subcover $\{G_{i_1}, \ldots, G_{i_m}\}$ and

(A, T_A) is compact.

(ii) $\Rightarrow$ (i)

Let $\{H_i\}$ is open cover for A s.t $A \subset \bigcup H_i$

$\Rightarrow A \subset A \cap (\bigcup H_i) = \bigcup_i (A \cap H_i) = \bigcup_i G_i$

$\{G_i\}$ is open cover for A, but A is compact with respect

$-[0,1]$ be a closed interval ...

$\Rightarrow A \subset G_{i_1} \cup -- \cup G_{i_m} = (A \cap H_{i_1}) \cup -- \cup (A \cap H_{i_m}) = A \cap (H_{i_1} \cup -- \cup H_{i_m}) \subset H_{i_1} \cup -- \cup H_{i_m}$, Thus $\{H_i\}$ has finite subcover

$\Rightarrow A$ is compact with respect to T.

Remark: A subset of a compact space need not be compact.

Heine Borel Theorem: Every open cover of a closed and bounded interval $A = [a,b]$ has finite subcover.

Therefore, the closed interval $[0,1]$ is compact, but $(0,1) \subseteq [0,1]$ is not compact by example (8.2)

Theorem 8.3 : Let F be a closed subset of a compact space X. Then F is also compact.

Proof:

Let $\mathcal{G} = \{G_i\}$ be an open cover of F, i.e. $F \subset \cup_i G_i$. Then $X = (\cup_i G_i) \cup F^c$ that is $\mathcal{G}^* = \{G_i\} \cup \{F^c\}$ is a cover of X.

But F^c is open since F is closed, so $\mathcal{G}^*$ is an open cover of X. By hypothesis, X is compact; hence $\mathcal{G}^*$ has finite subcover, say $X = G_{i_1} \cup --- \cup G_{i_m} \cup F^c \quad G_{i_k} \in \mathcal{G}$

But F and F^c are disjoint; hence $F \subset G_{i_1} \cup -- \cup G_{i_m} \quad G_{i_k} \in \mathcal{G}$

$\Rightarrow F$ is compact.

Finite intersection property:

A class $\{A_i\}$ of sets is said to have the finite intersection property if every finite sub-class $\{A_{i_1}, ---, A_{i_m}\}$ has a non-empty intersection, i.e. $A_{i_1} \cap -- \cap A_{i_m} \neq \emptyset$

Example 8.5 : Consider the following of open intervals

$\mathcal{A} = \{(0,1), (0,\frac{1}{2}), (0,\frac{1}{3}), (0,\frac{1}{4}), ---\}$

Now, $\mathcal{A}$ has the finite intersection property

$(0,a_1) \cap --- (0,a_m) = (0,b)$

where $b = \min(a_1, ---, a_m) > 0$