Chapter eight

Compactness and Connectedness:

1- Compactness:

Definition 8.1: Let $\mathcal{A}=\{G_i\}$ be a class of subsets of X such that $A\subset \bigcup_i G_i$ for some $A\subseteq X$. Recall that $\mathcal{A}$ is then called a cover of A, and an open cover if each G_i is open. Furthermore if a finite subclass of $\mathcal{A}$ is also a cover of A i.e.

$\exists\, G_{i_1},\dots,G_{i_m}\in\mathcal{A}$ s.t $A\subset G_{i_1}\cup\dots\cup G_{i_m}$

then $\mathcal{A}$ is said to be contains a finite subcover

Definition 8.2: A subset A in top. space X is compact if for each open cover for A ~~there exist~~ has a finite subcover

In otherwords, if A is compact and $A\subset \bigcup_i G_i$ where G_i are open sets then there exist $\{G_{i_1},\dots,G_{i_n}\}$ s.t $A\subset \bigcup_{j=1}^{n} G_{i_j}$

Example 8.1: Let A be any finite subset of a top. space X A is compact,

Say $A=\{a_1,\dots,a_m\}$ then

Let $\mathcal{G}=\{G_\ell\}$ is open cover for A then each point $a_i\in A$ there exist $G_i\in\mathcal{G}$ s.t $a_i\in G_{\ell_i}$ say $a_1\in G_{\ell_1}$, $a_2\in G_{\ell_2}$, $\dots$, $a_m\in G_{\ell_m}$ $\Rightarrow A\subset G_{i_1}\cup G_{i_2}\cup\dots\cup G_{i_m}$.

$\therefore A$ is compact.

Example 8.2: The open interval $A=(0,1)$ is not compact when The topology on $\mathbb{R}$ is usual.

Let $\mathcal{G}=\{(\frac{1}{3},1),(\frac{1}{4},\frac{1}{2}),(\frac{1}{5},\frac{1}{3}),\dots\}$ s.t the intervals in $\mathcal{G}$ is of form $(\frac{1}{n+2},\frac{1}{n})$ therefore $A=\bigcup_{n=1}^{\infty} G_n$

$\mathcal{G}$ is open cover for A. But $\mathcal{G}$ contains no finite subcover. For let $\mathcal{G}^*=\{(a_1,b_1),(a_2,b_2),\dots,(a_m,b_m)\}$ be any finite subclass of $\mathcal{G}$. If $\epsilon=\min(a_1,\dots,a_m)$ then $\epsilon>0$ and

$(a_1,b_1)\cup\dots\cup(a_m,b_m)\subset(\epsilon,1)$

But $(0,\epsilon]\cap(\epsilon,1)=\emptyset$ here $\mathcal{G}^*$ is not cover of A

not compact

Example 8.3:
Let T be the cofinite top on any set X. Show that (X, T) is a compact space.

Sol:
Let $G = \{G_i\}$ be an open cover of X. Choose $G_0 \in G$. Since T is the cofinite top then G_0^c is a finite set say $G_0^c = \{a_1, \dots, a_m\}$ since G is cover for X then $\forall a_k \in G_0^c$ $\Rightarrow \exists G_{i_k} \in G$ s.t $a_k \in G_{i_k} \Rightarrow G_0^c \subset G_{i_1} \cup \dots \cup G_{i_m}$
Sine $X = G_0 \cup G_0^c = G_0 \cup G_{i_1} \cup \dots \cup G_{i_m} \Rightarrow X$ is compact.

Example 8.4: ~~Indesercte~~ In indescrete top. space (X, T) then X is not compact if X is infinite