Theorem (Urysohn's Lemma) [illegible]
subsets of a normal space X. Then there exists a continuous function $f: X \to [0,1]$ such that

$f[F_1] = \{0\}$ and $f[F_2] = \{1\}$.

proof:

By hypothesis, $F_1 \cap F_2 = \phi$; hence $F_1 \subset F_2^c$. In particular, since F_2 is a closed set, F_2^c is an open superset of the closed set F_1. By theorem 7.4, there exists an open set $G_{1/2}$ such that

$$F_1 \subset G_{1/2} \subset \overline{G_{1/2}} \subset F_2^c$$

observe that $G_{1/2}$ is an open superset of the closed set F_1, and F_2^c is an open superset of the closed set $\overline{G_{1/2}}$. Hence by theorem 7.4, there exist open sets $G_{1/4}$ and $G_{3/4}$ s.t.

$$F_1 \subset G_{1/4} \subset \overline{G_{1/4}} \subset G_{1/2} \subset \overline{G_{1/2}} \subset G_{3/4} \subset \overline{G_{3/4}} \subset F_2^c$$

We continue in this manner and obtain for each $t \in D$, where D is the set of dyadic fractions in $[0,1]$, an open set G_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$ then $\overline{G_{t_1}} \subset G_{t_2}$.

Define the function f on X as follows

$$f(x) = \begin{cases} \inf\{t : x \in G_t\} & \text{if } x \notin F_2 \\ 1 & \text{if } x \in F_2 \end{cases}$$

observe that, for every $x \in X$, $0 \leq f(x) \leq 1$, i.e. f maps X into $[0,1]$. observe also that $F_1 \subset G_t$ for all $t \in D$; hence $f[F_1] = \{0\}$. Moreover, by definition, $f[F_2] = \{1\}$. consequently, the only thing left for us to prove is that f is continuous.

Now f is continuous if the inverse of sets $[0,b)$ and $(a,1]$ a[illegible] open subsets of X

(Let f be a function from a topological space X into the unit interval $[0,1]$. Show that if $f^{-1}[(a,1]]$ and $f^{-1}[[0,b)]$ are open subsets of X for all $0<a$, $b<1$, then f is continuous).

$$f^{-1}[0,a) = \cup\{G_t : t<a\}$$

$$f^{-1}(b,1] = \cup\{\overline{G_t}^c : t>b\}$$

~~We claim that~~

$$\cancel{f^{-1}[[0,b)] = \cup\{G_t : t<b\}}$$

Completely Regular spaces:

A topological space X is Completely regular iff it satisfies the following axiom:

$[CR]$ If F is a closed subset of X and $p \in X$ does not belong to F, then there exists a continuous function $f: X \to [0,1]$ such that $f(p)=0$ and $f[F]=\{1\}$.

proposition : A completely regular space is also regular.

proof: Let F be a closed subset of X and $p \in X$ such that $p \notin F$. By hypothesis, X is completely regular; hence there exists a continuous function $f: X \to [0,1]$ s.t $f(p)=0$ and $f[F]=\{1\}$. But $\mathbb{R}$ and its subspace $[0,1]$ are Hausdorff spaces; hence there are disjoint open sets G and H containing 0 and 1 respectively. Accordingly, their inverse $f^{-1}[G]$ and $f^{-1}[H]$ are disjoint, open and contain p and F respectively $\Rightarrow X$ is regular.

A completely regular X which also satisfies $[T_1]$, i.e. a completely regular T_1-space is called a Tychonoff space also called $T_{3\frac{1}{2}}$-space