Every metric space X is Hausdorff space

Sol:

Let $a, b \in X$ ad $a \neq b \Rightarrow d(a,b) = \epsilon > 0$

we take open balls s.t $a \in B_{\epsilon/3}(a)$ ad $b \in B_{\epsilon/3}(b)$

$\langle$T.p $B_{\epsilon/3}(a) \cap B_{\epsilon/3}(b) = \phi \rangle$

let $p \in B_{\epsilon/3}(a) \cap B_{\epsilon/3}(b) \Rightarrow d(a,p) < \epsilon/3$ ad $d(b,p) < \epsilon/3$

$\therefore d(a,b) \leq d(a,p) + d(p,b) < \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \frac{2}{3}\epsilon$.

$\Rightarrow$ C! since $d(a,b) = \epsilon \Rightarrow B_{\epsilon/3}(a) \cap B_{\epsilon/3}(b) = \phi$.

~~Theorem 7.2: Every T_2-space is T_1-space but the Conversers~~.

~~Proof: Let (X, T) is T_2-space $\langle$T.p (X, T) is T_1-space$\rangle$~~

~~late~~

Theorem 7.2: Let (X, T) be a Hausdorff space. Then every convergent sequence in X has a unique limit.

proof:

let $\langle a_1, a_2, \ldots \rangle$ convergent to a and b ad suppose $a \neq b$.

since X is Hausdorff $\Rightarrow \exists$ open sets G and H s.t

$a \in G$, $b \in H$ ad $G \cap H = \phi$

since $\langle a_n \rangle \to a \Rightarrow \exists n_0$ s.t $n > n_0$, $a_n \in G$

i.e G contains all ~~expect~~ except a finite number of the terms of the sequence. But G and H are disjoint $\Rightarrow H$ can only contain those terms of the sequence which do not belong to G, i.e H contain Finite number of sequence

$\Rightarrow \langle a_n \rangle \not\to b \Rightarrow$ C! since $\langle a_n \rangle \to b \Rightarrow$

$a = b$.

Theorem 7.3: Let (X, T) is first countable space then the following are equivalent:

(i) X is Hausdorff (ii) Every convergent sequence has a unique limit.

Sol: (i) $\Rightarrow$ (ii) (Theorem 7.2)

T.p (ii) $\Rightarrow$ (i)

Let (X, T) is not Hausd. Then $\exists a, b \in X$, $a \neq b$

s.t $\forall G, H$ ~~of~~ open sets $a \in G$ ad $b \in H$ then

$G \cap H \neq \phi$

$\cdots H_n \neq \phi$ ... nested bases at a al b then $\forall n \in N$ then ~~the~~ ∃ sequence $\langle a_1, a_2, \ldots \rangle$

s.t $a_1 \in G_1 \cap H_1$, $a_2 \in G_2 \cap H_2$, ... $\Rightarrow (X, T)$ is Hausdorff

$\Rightarrow \langle a_n \rangle \rightarrow$ a al b $\Rightarrow$ C! space.

Example 7.6: Let T is topology on R generated by open-closed $(a, b]$ then (R, T) is Hausdorff space

Proof:

Let $a, b \in R$ al $a \neq b$ al $a < b$ ~~so~~ choose $G =$ $(a-1, a]$ al $H = (a, b] \Rightarrow a \in G$ al $b \in H$

$G \cap H = \phi$

[R] or Regular space: $\forall x \in X$, $F \subseteq X$ is closed set al $x \notin F$ $\exists$ G al H open sets s.t $x \in G$ al $F \subseteq H$

$G \cap H = \phi$

Example 7.7: Let $X = \{a, b, c\}$ al $T = \{X, \phi, \{a\}, \{b, c\}\}$,

then the ~~closed~~ closed sets are ϕ, X, $\{b, c\}$, $\{a\}$

then (X, T) is regular space.

$[T_3]$ A regular space which satisfy $[T_1]$ is called T_3-space. i.e. $[T_3]$ is $[R] + [T_1]$

Remark: A regular space not necessary satisfy $[T_1]$ as ~~examp~~ in example (7.7).

[N] If F_1 and F_2 are disjoint closed subsets of X, i.e. $\forall F_1, F_2 \subseteq X$ s.t F_1, F_2 are closed sets al $F_1 \cap F_2 = \phi$ then there exist two open sets G, H s.t $F_1 \subseteq G$ al $F_2 \subseteq H$ al $G \cap H = \phi$

Theorem (7.4): Let X be a top. space. Then the following conditions are equivalent (i) X is normal (ii) If H is an open ~~set F⊆H~~ al F is closed s.t $F \subseteq H$ then there exist open set G s.t $F \subseteq G \subseteq \overline{G} \subseteq H$

proof: (i) $\Rightarrow$ (ii)

Let H is open al F is closed s.t $F \subseteq H$

... ad $F \cap H^c = \emptyset$ But X is normal

$\Rightarrow \exists$ open sets G, G^* s.t $F \subset G$ ad $H^c \subset G^*$

$G \cap G^* = \emptyset$

since $G \cap G^* = \emptyset \Rightarrow G \subset G^{*c}$ and $H^c \subset G^*$

$\Rightarrow G^{*c} \subset H$

$\therefore \quad F \subset G \subset \overline{G} \subset G^{*c} \subset H$

(ii) $\Rightarrow$ (i)

Let F_1 ad F_2 be disjoint closed sets, Then $F_1 \subset F_2^c$ ad F_2^c is open $\Rightarrow \exists$ open set G s.t $F_1 \subset G \subset \overline{G} \subset F_2^c$

$\overline{G} \subset F_2^c \Rightarrow F_2 \subset \overline{G}^c$ ad $G \subset \overline{G} \Rightarrow G \cap \overline{G}^c = \emptyset$

Furthermore, $\overline{G}^c$ is open. Thus $F_1 \subset G$ ad $F_2 \subset \overline{G}^c$ with $G \cap \overline{G}^c = \emptyset$

$\therefore X$ is normal

[T_4] A normal space X which also satisfies [T_1] is called [T_4]. i.e. [T_4] is [N] + [T_1]

Example 7.8: Let $X = \{a, b, c\}$ ad $T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ the closed sets are $X, \emptyset, \{b, c\}, \{a, c\}, \{c\}$. If F_1 ad F_2 are disjoint closed subsets then one of them, say F_1, $F_1 = \emptyset$ Therefore (X, T) is [N]. But (X, T) is not T_1-space since the singletone $\{a\}$ is not closed

Theorem (7.5): Let $\mathcal{B}$ be a base for T_4-space. Show that for each $G_i \in \mathcal{B}$ and any $p \in G_i$, $\exists\, G_j \in \mathcal{B}$ s.t $p \in \overline{G_j} \subseteq G_i$

proof:

Since X is T_1-space $\Rightarrow \{p\}$ is closed.; hence G_i is open ~~of closed set~~ $\{p\}$ ad $\{p\} \subseteq G_i$.

Then by (Th 7.4), $\exists\, G$ is open set s.t $\{p\} \subset G \subset \overline{G} \subset G_i$

Since $p \in G$, $\exists\, G_j \in \mathcal{B}$ s.t $p \in G_j \subset G$ ad so $p \in \overline{G_j} \subset \overline{G}$, but $\overline{G} \subset G_i \Rightarrow p \in \overline{G_j} \subset G_i$