heorem 6.1: $[C_1]$ is Topological property.

proof:

Let $f:(X,T) \longrightarrow (Y,S)$ is homeomorphism between top. spaces (X,T) and (Y,S)

Let (X,T) is $[C_1]$ then we want to prove (Y,S) is $[C_1]$

let $y \in Y \Rightarrow \exists x \in X$ s.t $f(x)=y$ (f is onto)

since (X,T) is $[C_1] \Rightarrow \exists$ countable local base $\mathcal{B}_n(x)$ then $f(\mathcal{B}_n(x))$ is also local Base in Y since $\{f$ is open and f is countable $(1-1)\}$

$\Rightarrow (Y,S)$ is $[C_1]$

H.W: prove $(\mathbb{R}, T_u)$ is $[C_1]$.

Second Countable spaces:

A topological space (X,T) is called a second countable space if it satisfi the following axiom,

$[C_2]$ There exists a countable base $\mathcal{B}$ for the topology T

This axiom is Called the second axiom of countability.

Example 6.4: The class $\mathcal{B}$ of open intervals (a,b) with rational endp: i.e $a,b \in \mathbb{Q}$ is countable and is a base for usual top. on $\mathbb{R} \Rightarrow (\mathbb{R},$ satisfies $[C_2]$

Example 6.5: $(\mathbb{R}, T_d)$ discret top. is not satisfaies $[C_2]$ since the Base $\mathcal{B}=\{\{a\}: a \in \mathbb{R}\}$ and $\mathbb{R}$ is not countable.

Example 6.6: $(\mathbb{R}^2, T_u)$ is satisfy $[C_2]$ since the $\mathcal{B}=\{\mathcal{B}_\epsilon(x) : x=(x_1,x_2)\ x_1,x$ and $\epsilon \in \mathbb{Q}\}$ is Countable and is base for usual top on $\mathbb{R}$.

Example 6.7: show that every subspace of a second countable spac is second countable.

Sol: Let $\mathcal{B}=\{B_n : n \in N\}$ be a countable base for the Second

... Let Y is subspace of X then B ...

-- X is a T_0-space

osition 6.3: A second countable space is also first countable

roof: ?

Is the converse is right?

Lindelofs theorems:

Let $A \subset X$ and let $\mathcal{A}$ be a class of subsets of X such that

$$A \subset \bigcup\{E : E \in \mathcal{A}\}$$

Then $\mathcal{A}$ is called a cover of A. If each member of $\mathcal{A}$ is an open of X the $\mathcal{A}$ is called an open cover of A. Furthermore, if $\mathcal{A}$ contains a countable subclass which also is a cover of A, then $\mathcal{A}$ is said to be reducible to a countable cover or $\mathcal{A}$ is said to contain a countable subcover.

Theorem 6.2 (Lindelof 1): Let A be any subset of a second countable space X. If $\mathcal{G}$ is an open cover of A, then $\mathcal{G}$ is reducible to a countabl cover.

proof: Let $\mathcal{B}$ be a countable base for X. Since $A \subset \bigcup\{G : G \in \mathcal{G}\}$, for every $p \in A$, $\exists G_p \in \mathcal{G}$ s.t $p \in G_p$. Since $\mathcal{B}$ is a base for X, for every $p \in A$

$\exists B_p \in B$ s.t $p \in B_p \subset G_p$

Hence $A \subset \bigcup\{B_p : p \in A\}$. But $\{B_p : p \in A\} \subset \mathcal{B}$, so it countabl

hence $\{B_p : p \in A\} = \{B_n : n \in N\}$

where N is a countable index set. For each $n \in N$ choose one set $G_n \in \mathcal{G}$ s.t $B_n \subset G_n$ then $A \subset \bigcup\{B_n : n \in N\} \subset \bigcup\{G_n : n \in N\}$ and so $\{G_n : n \in N\}$ is a countable subcover of

Theorem 6.3 (Lindelof 2): Let $\mathcal{G}$ be a base for a second countal space X. Then $\mathcal{G}$ is reducible to a countable base for X.

proof:

Since X is second countable, X has a countable base $\mathcal{B} = \{B_n : n \in N\}$. Since $\mathcal{G}$ is also a base for X, for each $n \in N$ $B_n = \{\bigcup G : G \in \mathcal{G}_n^*\}$ with $\mathcal{G}_n^* \subset \mathcal{G}$

$\mathcal{G}_n^*$ is an open cover of B_n and by (Theorem 6.2) is redu