Chapter Six

Countability:

First Countable Spaces:

A topological space X is called a first countable space if it satisfies the following axiom,

$[C_1]$ For each point $p \in X$ there exists countable class $\mathcal{B}_p$ of open sets containing p such that every open set G containing p also contains a member of $\mathcal{B}_p$.

i.e: In otherwords, a top. space X is a first countable space iff there exist a countable local base at every point $p \in X$.

Example 6.1: Let X be a metric space and let $p \in X$. Recall that the countable class of open spheres $\{S(p,1), S(p,\frac{1}{2}), S(p,\frac{1}{3}), ---\}$ with center p is a local base at p. Hence every metric space satisfies the first axiom of countability.

Example 6.2: Let X be any discrete space. Now the singletone set $\{p\}$ is open and is contained in every open set G containing $p \in X$. Hence every discrete space satisfies $[C_1]$. The following example explain $[C_1]$ is hereditary property.

Example 6.3: Show that any subspace (Y, T_Y) of a first countable space (X, T) is also first countable.

Sol:
Let $p \in Y$, since $Y \subseteq X \Rightarrow p \in X$. By hypothesis (X, T) is a first countable space, so $\exists$ a countable T-local base $\mathcal{B}_p = \{B_n : n \in N\}$ at p then $\mathcal{B}_p^* = \{Y \cap B_n : n \in N\}$ is T_Y-local base at p. Since $\mathcal{B}_p^*$ is countable
$\Rightarrow (Y, T_Y)$ satisfies $[C_1]$.

proposition 6.1: Let $\mathcal{B}_p = \{B_1, B_2, ---\}$ be nested local base at $p \in X$ and let $\langle a_1, a_2, ---\rangle$ be a sequence such that $a_1 \in B_1$, $a_2 \in B_2$, --- show that $\langle a_n \rangle$

Let G be an open set containing p. Since $\mathcal{B}_p$ is a local base at $p \Rightarrow \exists n_0 \in N$ s.t $B_{n_0} \subset G$
But $\mathcal{B}_p$ is nested; hence $n > n_0$ implies $a_n \in B_{n_0} \subset G$ and So $a_n \to p$.

proposition 6.2: Let T be the cofinite topology on the real line R, i.e T contains ϕ and the complements of finite sets. Show that (R, T) does not satisfy the first axiom of countability.

proof:

Suppose that (R, T) does satisfy $[C_1]$, then $1 \in R$ possesses a countable open local base $\mathcal{B}_1 = \{B_n : n \in N\}$. Since each B_n is T-open, its complement B_n^c is T-closed and hence finite. Accordingly, $A = \cup \{B_n^c : n \in N\}$ is the countable union of finite sets and its therefore countable. But R is not countable; hence there exists a point $p \in R$ different from 1 which does not belong to A, i.e $p \in A^c$

Now, by DeMorgan's Law we have
$p \in A^c = (\cup \{B_n^c : n \in N\})^c = \cap \{B_n^{cc} : n \in N\} = \cap \{B_n : n \in N\}$
Hence $p \in B_n \quad \forall n \in N$. on other hand, $\{p\}^c$ is T-open set since it is the complement of a finite set, and $\{p\}^c$ contains 1 since p is different from 1. Since $\mathcal{B}_1$ is a local base, there exists a member $B_{n_0} \in \mathcal{B}_1$ s.t
$B_{n_0} \subset \{p\}^c \Longrightarrow p \notin B_{n_0}$. But this C!
$\Rightarrow (R, T)$ is not satisfy $[C_1]$.