* Limits

Def: The function $f(z)$ has a limit w_0 as z approaches z_0 denoted by

$$\lim_{z \to z_0} f(z) = w_0,$$

i.e, $\forall \epsilon > 0, \exists \delta > 0$ suchthat

$$|f(z) - w_0| < \epsilon, \forall \; 0 < |z - z_0| < \delta.$$

Note, using mapping representation $(u, v) = f(x, y)$ and note that

$|f(z) - w_0|$ = Euclidean distance between the points $f(z)$ and w_0.

We see that the above is equivalent to

$$\lim_{(x,y) \to (x_0, y_0)} (u(x,y), v(x,y)) \longrightarrow (u_0, v_0)$$

Where $w_0 = u_0 + i v_0$, $z_0 = x_0 + i y_0$

ex:- If $f(z) = \frac{zi}{2}$ defined on the disk $|z| < 1$ prove that by using definition of Limit

1

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

proof: To show $\forall \epsilon > 0, \exists \delta > 0$ s.t

$$\left|f(z) - \frac{i}{2}\right| < \epsilon \quad \text{whene} \quad |z-1| < \delta$$

i.e.,

$$\left|f(z) - \frac{i}{2}\right| = \left|\frac{iz}{2} - \frac{i}{2}\right| = \left|\frac{i}{2}(z-1)\right|$$

$$\therefore \left|f(z) - \frac{i}{2}\right| < \epsilon \Rightarrow \frac{1}{2}|z-1| < \epsilon$$

$$\Rightarrow |z-1| < 2\epsilon$$

So, we can choose $\delta = 2\epsilon$.

ex// Show that $\lim_{z \to 0} \frac{\bar{z}}{z}$ is not existing.

proof:

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x+iy \to 0+i0} \frac{x-iy}{x+iy}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x-iy}{x+iy}.$$

whene $z \to 0$ on the real axis $(y=0)$.

Then $\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x-iy}{x+iy} = \lim_{x \to 0} \frac{x}{x} = 1 = L_1.$

where $z \to 0$ on imaginary axis $(x=0)$

then $\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy} = \lim_{y\to 0} \frac{-iy}{iy} = -1 = L_2$.

$L_1 \neq L_2$.

Note// If we put in the function $f(z)=w$ for each z by $\frac{1}{z}$, we get a new function $g(z)=\frac{1}{z}$ where $z \to \infty$ then $\frac{1}{z} \to 0$, to find the Limit of function where $z \to 0$ this enough to find the Limit of the function $g(z)=f(\frac{1}{z})$ where $z=0$

So $\boxed{\lim_{z\to\infty} f(z) = w_0 \Rightarrow \lim_{z\to 0} f(\frac{1}{z})}$.

i.e// $\forall \epsilon > 0, \exists \delta > 0$ s.t $|f(z)-w_0| < \epsilon$, where $|z| > \frac{1}{\delta}$.

Hint// $|z-z_0| < \delta$

$z \to \infty$, $\frac{1}{z} \to 0$

$\therefore |\frac{1}{z} - 0| < \delta \Rightarrow |\frac{1}{z}| < \delta \Rightarrow |z| > \frac{1}{\delta}$.

* Limits involving the point at infinity

(theorem): If z_0 and w_0 are points in the z and w-planes respectively, then

① $\lim_{z \to z_0} f(z) = \infty$ if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$.

② $\lim_{z \to \infty} f(z) = w_0$ if $\lim_{z \to 0} f(\frac{1}{z}) = w_0$.

③ $\lim_{z \to \infty} f(z) = \infty$ if $\lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$.

Proof: ①

$\lim_{z \to z_0} \frac{1}{f(z)} = 0 \iff \forall \epsilon > 0, \exists \delta > 0$, s.t.

$\left| \frac{1}{f(z)} - 0 \right| < \epsilon$, $\forall\ 0 < |z - z_0| < \delta$.

$\iff \forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(z)| > \frac{1}{\epsilon}$,

$\forall\ 0 < |z - z_0| < \delta$.

$\iff \lim_{z \to z_0} f(z) = \infty$.

②, ③ H.W.

ex: By using def. of Limit show that

$\lim_{z \to \infty} \frac{1}{z^2} = 0$?

Proof:

$f(z) = \frac{1}{z^2}$, $w_0 = 0$

$\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$|f(z) - w_0| < \epsilon$ where $|z| > \frac{1}{\delta}$

$|f(z) - w_0| = \left|\frac{1}{z^2} - 0\right| = \left|\frac{1}{z^2}\right|$

then $|f(z) - w_0| < \epsilon \Rightarrow \left|\frac{1}{z^2}\right| < \epsilon$

$\Rightarrow \left|\frac{1}{z^2}\right| < \epsilon \Rightarrow |z|^2 > \frac{1}{\epsilon} \Rightarrow |z| > \frac{1}{\sqrt{\epsilon}}$

$\therefore |f(z) - w_0| < \epsilon$ where $|z| > \frac{1}{\sqrt{\epsilon}}$

So, we can choose $\delta = \sqrt{\epsilon}$.

ex: Find $\lim_{z \to \infty} \frac{1}{z^2}$?

Sol: $\lim_{z \to \infty} \frac{1}{z^2} = \lim_{z \to 0} \frac{1}{\left(\frac{1}{z}\right)^2} = \lim_{z \to 0} z^2 = 0$.

ex: Find $\lim_{z \to -1} \frac{zi+3}{z+1}$?

Sol:

$$\lim_{z \to -1} \frac{1}{\left(\frac{zi+3}{z+1}\right)} = \lim_{z \to -1} \frac{z+1}{iz+3} = 0.$$

$$\therefore \lim_{z \to -1} \frac{zi+3}{z+1} = \infty.$$