* The functions $\sin^{-1}z, \cos^{-1}z, \sinh^{-1}z, \cosh^{-1}z$

1) $w = \sin^{-1}z \longleftrightarrow z = \sin w$

2) $w = \cos^{-1}z \longleftrightarrow z = \cos w$

3) $w = \sinh^{-1}z \longleftrightarrow z = \sinh w$

4) $w = \cosh^{-1}z \longleftrightarrow z = \cosh w$.

theorem:

1) $\sin^{-1}z = -i \log(iz + \sqrt{1-z^2})$

2) $\cos^{-1}z = -i \log(z + i\sqrt{1-z^2})$

3) $\tan^{-1}z = \frac{i}{2} \log\left[\frac{(i+z)}{(i-z)}\right]$

4) $\sinh^{-1} z = \log(z + \sqrt{z^2+1})$

5) $\cosh^{-1} z = \log(z + \sqrt{z^2-1})$

6) $\tanh^{-1} z = \frac{1}{2} \log\left[\frac{1+z}{1-z}\right]$

proof :-

2) let $w = \cos^{-1} z \Rightarrow z = \cos w$

$$z = \frac{e^{iw} + e^{-iw}}{2} \Rightarrow 2z = e^{iw} + e^{-iw}$$

$$\Rightarrow 2z e^{iw} = e^{2iw} + 1 \Rightarrow (e^{iw})^2 - 2z e^{iw} + 1 = 0$$

$\Rightarrow$ So, $e^{iw} = \frac{2z \mp \sqrt{4z^2-4}}{2} \Rightarrow e^{iw} = z \mp \sqrt{z^2-1}$ — (1)

From (1), we have if choosing $\cos^{-1} 0 = \frac{\pi}{2}$, then

$$iw = \log(z \mp \sqrt{z^2-1})$$

$$\Rightarrow w = \frac{1}{i} \log(z + \sqrt{z^2-1})$$

$$\Rightarrow \cos^{-1} z = -i \log(z + \sqrt{-(1-z^2)})$$

$$\Rightarrow \cos^{-1} z = -i \log(z + i\sqrt{1-z^2}).$$

ex// find $\sinh^{-1} i$

Sol// we have

$$\sinh^{-1} z = \log\left(z + \sqrt{z^2+1}\right)$$

$$= \log\left(i + \sqrt{i^2+1}\right)$$

$$= \log(i)$$

$$= \ln|i| + i\arg(i)$$

$$= 0 + i\left(\frac{\pi}{2} + 2k\pi\right), \quad k = 0, \mp 1, \ldots$$

$$= i\left(\frac{\pi}{2} + 2k\pi\right), \quad k = 0, \mp 1, \ldots$$

theorem ;-

1) $\frac{d}{dz}(\sin^{-1} z) = \frac{1}{\sqrt{1-z^2}}$

2) $\frac{d}{dz}(\cos^{-1} z) = -\frac{1}{\sqrt{1-z^2}}$

3) $\frac{d}{dz}(\tan^{-1} z) = \frac{1}{1+z^2}$

4) $\frac{d}{dz}(\sinh^{-1} z) = \frac{1}{\sqrt{z^2+1}}$

5) $\frac{d}{dz}(\cosh^{-1} z) = \frac{1}{\sqrt{z^2-1}}$

6) $\frac{d}{dz}(\tanh^{-1} z) = \frac{1}{1-z^2}$.

proof :- (تمرين : برهان بطريقة اخرى)

1) let $w = \sin^{-1} z \Rightarrow \sin w = z$

so $\cos w \cdot \left(\frac{dw}{dz}\right) = 1$

$$\Rightarrow \frac{dw}{dz} = \frac{1}{\cos w} = \frac{1}{\sqrt{1-\sin^2 w}} \qquad \boxed{\sin^2 z + \cos^2 z = 1}$$

$$\Rightarrow \frac{d}{dz}(\sin^{-1} z) = \frac{1}{\sqrt{1-z^2}}.$$

* Complex expon. ، أس

for complex numb. z $(z \neq 0)$ and for every α

$\alpha \in \mathbb{C}$, we define

$$z^\alpha = e^{\alpha \log z}$$

and

$$\alpha^z = e^{z \log \alpha}$$

ملاحظة // الدالة z^α تعتبر كثيرة حدود [illegible] إذا ألفا عدد صحيح

(log) والدالة α^z تعتبر كثيرة حدود [illegible] إذا z عدد صحيح

القيمة الأساسية

ex, find $(-i)^{2i}$ and find the prin. value of it

sol, $(z)^{\alpha} = e^{\alpha \log z}$

$z = -i$, $\alpha = 2i$

$(-i)^{2i} = e^{2i \log(-i)}$

$\log(-i) = \ln|-i| + i \arg(-i)$

$= i(-\frac{\pi}{2} + 2k\pi)$, $k = 0, \mp 1, \dots$

then

$(-i)^{2i} = e^{2i[i(-\frac{\pi}{2} + 2k\pi)]} = e^{(\pi - 4k\pi)}$.

and the pri. V. is

$e^{\alpha \log z} = e^{2i\,\mathrm{Ln}(-i)}$

$\mathrm{Ln}(-i) = \ln|-i| + i\,\mathrm{Arg}(-i)$

$= \frac{-\pi i}{2}$

then

$(-i)^{2i} = e^{2i \ln(-i)} = e^{2i(\frac{-\pi i}{2})} = e^{\pi}$.

note//

1) $f(z) = z^{\alpha} \Rightarrow f'(z) = \alpha z^{\alpha - 1}$

2) $f(z) = \alpha^{z} \Longrightarrow f'(z) = \alpha^{z} \log(\alpha)$. //