الثالث

[redacted]

<u>Proof:</u>

1) $\arg z_1 = \theta_1$, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\arg z_2 = \theta_2$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 \cdot z_2 = [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]$$

$$= r_1 r_2 [((\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1)]$$

$$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$\arg z_1 z_2 = \theta_1 + \theta_2$$
$$= \arg z_1 + \arg z_2$$

2) H.W.

3) $$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1}{r}\left[\frac{1}{\cos\theta + i\sin\theta}\right] \cdot \left[\frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}\right]$$

$$= \frac{\cos\theta - i\sin\theta}{r[[\cos^2\theta + \sin^2\theta] + i[\cos\theta\sin\theta - \cos\theta\sin\theta]}$$

$$= \frac{1}{r}[\cos\theta - i\sin\theta]$$

$$= \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$$

$$\arg\frac{1}{z} = -\theta = -\arg z .$$

4) H.W.

Note; the above theorem is Not true if but Arg z, for example

$$\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$$

if $z_1 = 3i$, $z_2 = -2$

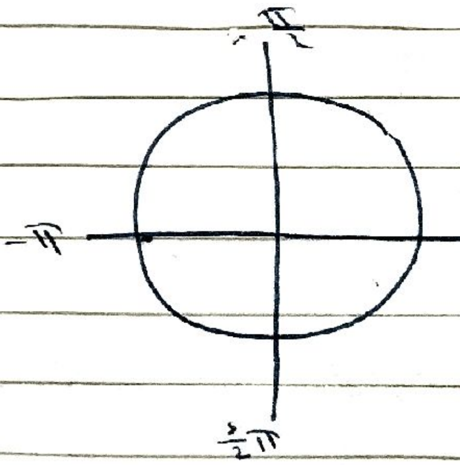
$z_1 = 3i = 0 + 3i$

$\text{Arg}\, z_1 = \frac{\pi}{2}$

$z_2 = -2 + i0$

$\text{Arg}\, z_2 = \pi$

$z_1 z_2 = -6i \Rightarrow \text{Arg}(z_1 \cdot z_2) = \frac{3\pi}{2}$



$$-\pi < \text{Arg}(z_1 \cdot z_2) \leq \pi$$

$\text{Arg}(z_1 z_2) = -\frac{\pi}{2}$

$\text{Arg}(z_1) + \text{Arg}(z_2) = \frac{3}{2}\pi$

$\therefore \text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$.

By generalitation the property (1) of above theorem

For $z_1, z_2, \dots, z_n$

$$z_1 \cdot z_2 \cdots z_n = r_1 r_2 \cdots r_n [\cos(\theta_1+\theta_2+\dots+\theta_n) + i\sin(\theta_1+\theta_2+\dots+\theta_n)]$$

$$\arg(z_1 \cdot z_2 \cdots z_n) = \theta_1+\theta_2+\dots+\theta_n$$

$$= \arg z_1 + \arg z_2 + \dots + \arg z_n$$

If $z_1 = z_2 = \dots = z_n = z$, then

$$\underbrace{z \cdot z \cdots z}_{n\ \text{term}} = r \cdot r \cdots r \cdot ((\cos(\underbrace{\theta+\theta+\dots+\theta}_{n\ \text{term}}) + i\sin(\underbrace{\theta+\theta+\dots+\theta}_{n\ \text{term}}))$$

$$z^n = r^n ((\cos(n\theta) + i\sin(n\theta))$$

$$\Rightarrow (r(\cos\theta + i\sin\theta))^n = r^n ((\cos n\theta + i\sin n\theta))$$

$$\Rightarrow ((\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

this formula defind by (De-Moiver's) Theorem.

also $\frac{1}{z^n} = z^{-n} = \frac{1}{r^n}[\cos(-n\theta) + i\sin(-n\theta)] = (\frac{1}{z})^n$

example:

Show that

1) $\cos^2\theta - \sin^2\theta = \cos 2\theta$.
$2\sin\theta\cos\theta = \sin 2\theta$.

2) $\sin 3\theta = 3\sin\theta - 4\sin\theta$.
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

proof:

Hint:
$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

such that $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$\therefore (a+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \cdots + b^n$$

* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) = \cos 2\theta + i\sin 2\theta$

$[\cos^2\theta - \sin^2\theta] + i(\sin\theta\cos\theta + \sin\theta\cos\theta) = \cos 2\theta + i\sin 2\theta$

$[\cos^2\theta - \sin^2\theta] + 2i\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$.

$2\sin\theta\cos\theta = \sin 2\theta$.

* Euler's Formula صيغة أويلر *

The formula $e^{i\theta} = \cos\theta + i\sin\theta$ is called Euler's formula, same can rewrite the number z by using Euler's formula by:

$z = r(\cos\theta + i\sin\theta)$

$\therefore z = re^{i\theta}$

~~then~~ So, we can write if $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

then $z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

and

$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

and so, if $z = re^{i\theta}$, then $\frac{1}{z} = \frac{1}{r} e^{-i\theta}$

* Note, we can prove that the properties of argument by using Euler's formula like

1) If $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$ where $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

then $z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\arg(z_1 \cdot z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

* Powers and Roots

القوى والجذور

The powers of the complex number $z = re^{i\theta}$, $z \neq 0$ can be find by using (De Moiver's theorem) as following:

$$z = re^{i\theta}$$

$$z^n = (re^{i\theta})^n \quad , n \in \mathbb{Z}^+$$

$$= r^n (e^{i\theta})^n$$

$$= r^n (\cos\theta + i\sin\theta)^n$$

$$= r^n (\cos n\theta + i\sin n\theta)$$

$$= r^n e^{in\theta}$$

Note// the formula is true of all $n \in \mathbb{Z}$.

ex: ① Find $(1+i)^8$, $(1+i)^{-8}$

② find $(1+i\sqrt{3})^{-4}$, $(1+i\sqrt{3})^4$

Solu/ $(1+i)^8$, $n=8$, $z=1+i$

$r=|z|=\sqrt{x^2+y^2}=\sqrt{1+1}=\sqrt{2}$

$1+i=r(\cos\theta+i\sin\theta)$

$1+i=\sqrt{2}(\cos\theta+i\sin\theta)$

$\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}=\cos\theta+i\sin\theta$

So, $\cos\theta=\frac{1}{\sqrt{2}}$, $\sin\theta=\frac{1}{\sqrt{2}}$

$\theta=\frac{\pi}{4}$ — θ تقع في الربع الأول

$z=re^{i\theta}=\sqrt{2}\,e^{i\frac{\pi}{4}}$

$z^8=(1+i)^8=\left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^8=\left(2^{\frac{1}{2}}\right)^8\left(e^{i\frac{\pi}{4}}\right)^8$

$=2^4\,e^{i2\pi}$

So, $(1+i)^8=16(\cos 2\pi+i\sin 2\pi)$

$=16(1+i0)=16$

and

$(1+i)^{-8}=\frac{1}{(1+i)^8}=\frac{1}{16}$.

(2) H.W.

*** The roots of complex numbers**

* إيجاد الجذور العقدية

To find the roots of complex numbers, we using

$z^n = r^n e^{in\theta}$

Now, to extracting the nth roots of complex number, we have two cases:

1) find the nth roots of the eq. $z^n = 1$, $n \in \mathbb{Z}^+$

$z^n = 1$, $r = |z|$, $\theta = \arg(z)$, $z = re^{i\theta}$

so

$$z = re^{i\theta} = 1$$

$$(re^{i\theta})^n = 1e^{i0} \Rightarrow r^n e^{in\theta} = 1e^{i0}$$

Hence

$$n\theta = 0 + 2k\pi \quad , k = 0, \mp 1, \mp 2, \ldots \quad , r^n = 1$$

((If $z_1 = z_2 \Rightarrow |z_1| = |z_2|$ and $\theta_1 = \theta_2 + 2k\pi$)

((إذا تساوى عددين مركبين فإن مقياسيهما متساوية وسعتيهما تختلف بمقدار $2\pi k$))

then

$$\theta = \frac{2k\pi}{n} \quad , r = 1 \quad , k = 0, \mp 1, \mp 2, \ldots$$

so the solution of equ. is

$$\boxed{Z_k = e^{i\left(\frac{2k\pi}{n}\right)}} \quad , k = 0, \mp 1, \mp 2, \ldots$$

* Now, let $w_n = e^{\frac{2\pi i}{n}}$, then by using (De Moivre's) theorem, the $\underline{n}$th roots of $z^n = 1$ are

$1, w_n, w_n^2, w_n^3, \ldots, w_n^{n-1}$

such that

$Z_0 = 1, Z_1 = w_n, Z_2 = w_n^2, \ldots, Z_{n-1} = w_n^{n-1}$

* If we have Z_0, then we can find $Z_1 = Z_0 w_n, Z_2 = Z_0 w_n^2, \ldots$ s.t. $w_n = e^{\frac{2\pi i}{n}}$

(2) we can generalazed the ($\frac{2\pi i}{n}$) to find the $\underline{n}$th roots for any complex number w i.e $z^n = w$ as following

$z = re^{i\theta}$, s.t $|z| = r$, $\theta = \arg(z)$

$w = \rho e^{i\phi}$ s.t, $\rho = |w|$, $\phi = \arg w$

$z^n = w \Rightarrow (re^{i\theta})^n = \rho e^{i\phi}$

$\Rightarrow r^n e^{in\theta} = \rho e^{i\phi} \Rightarrow r^n = \rho$ and $n\theta = \phi + 2k\pi$, $k = 0, \mp 1, \ldots$

then $r = \sqrt{\rho}$ and $\theta = \dfrac{\phi + 2k\pi}{n}$

then

the solution of equ $z^n = w$ is

$z_k = \sqrt[n]{\rho}\, e^{i\left(\frac{\phi + 2k\pi}{n}\right)}$, $k = 0, \mp 1, \mp 2,$

ex:

① find the roots of the equ $z^6 = 1$.

② " " " " " " $z^5 = \sqrt{3} + i$

③ find $(1)^{1/4}$, $(1+i)^{1/4}$

④ If $w \neq 1$ is a nth root of the number 1

prove that $1 + w + w^2 + \dots + w^{n-1} = 0$

⑤ If $y \neq 1$, what is the roots of the equ.

$y^6 + y^5 + y^4 + y^3 + y^2 + y + 1 = 0$?

⑥ If z_k , $k = 0, 1, 2, 3 \dots$ are representation the roots

of equ. $z^4 - 1 = \sqrt{3}\, i$, prove that

$z_3 + z_1 = 0$, $z_2 + z_0 = 0$.

sol//

① From $z^n = 1$, we have $n = 6$.

So, the solutions of equ $z^6 = 1$ are

$= 0 \Rightarrow z_0 = e^0 = 1$

$k=1 \Rightarrow z_1 = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k=2 \Rightarrow z_2 = e^{\frac{2\pi i}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= \cos(\pi - \frac{\pi}{3}) + i\sin(\pi - \frac{\pi}{3})$

$= -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k=3 \Rightarrow z_3 = e^{\pi i} = \cos\pi + i\sin\pi = -1$

$k=4 \Rightarrow z_4 = e^{\frac{4\pi i}{3}} = \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi$

$= \cos(\pi + \frac{\pi}{3}) + i\sin(\pi + \frac{\pi}{3})$

$= \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$k=5 \Rightarrow z_5 = e^{\frac{5\pi i}{3}} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$.

② from $z^n = w$, $n=5$, $w = \sqrt{3} + i$

so, the solutions of the eqn. $z^5 = \sqrt{3} + i$ are

$z_k = \sqrt[5]{\rho}\, e^{i\left(\frac{\phi + 2k\pi}{5}\right)}$, $k = 0, 1, 2, 4$

$\phi = \arg w$, $\rho = |w|$

then $\rho = |w| = \sqrt{3+1} = 2$

$\sqrt{3} + i = 2(\cos\phi + i\sin\phi)$

$\cos\phi = \frac{\sqrt{3}}{2}$, $\sin\phi = \frac{1}{2}$, $\phi = \frac{\pi}{6}$

So, the Solutions are

$z_k = \sqrt[5]{2}\, e^{i\left(\frac{\frac{\pi}{6}+2k\pi}{5}\right)}$, $k = 0,1,2,3,4$

$k=0 \Rightarrow z_0 = \sqrt[5]{2}\, e^{i\left(\frac{\frac{\pi}{6}}{5}\right)} = \sqrt[5]{2}\, e^{i\frac{\pi}{30}}$

$k=1 \Rightarrow z_1 = \sqrt[5]{2}\, e^{\frac{13\pi i}{30}}$

$k=2 \Rightarrow z_2 = \sqrt[5]{2}\, e^{\frac{25\pi i}{30}}$

$k=3 \Rightarrow z_3 = \sqrt[5]{2}\, e^{\frac{49\pi i}{30}}$

or by using $w_n = e^{\frac{2\pi i}{n}} \Rightarrow w_5 = e^{\frac{2\pi i}{5}}$, we have $z_0 = \sqrt[5]{2}\, e^{i\frac{\pi}{3}}$
then

$z_1 = z_0 w_5 = \sqrt[2]{2}\, e^{\frac{i\pi}{30}} \cdot e^{\frac{2\pi i}{5}}$

$z_2 = z_0 w_5^2 = \vdots$

) let $S = 1 + w + \dots + w^{n-1}$

$wS = w + w^2 + \dots + w^n$

then $S - wS = 1 - w^n$

$\Rightarrow S(1-w) = 1 - w^n$

$\Rightarrow S = \dfrac{1-w^n}{1-w}$, since $w^n = 1$, then

$S = \dfrac{1-1}{1-w} = 0.$

(5) $S = y^6 + y^5 + \dots + y + 1 = 0$

$yS = y^7 + y^6 + \dots + y^2 + y$

So $S - yS = y^7 - 1 \Rightarrow S = \dfrac{y^7 - 1}{1-y}$

$1 - y \neq 0$, since $y \neq 1$

$S = 0 \Rightarrow \dfrac{y^7 - 1}{1-y} = 0 \Rightarrow y^7 = 1$

Now, to find the roots of $y^7 = 1$

$k = 6, \dots, 0$

$y_k = e^{\frac{2\pi i k}{7}}$

Now, we find $y_0, y_1, y_2, y_3, y_4, y_5, y_6$..