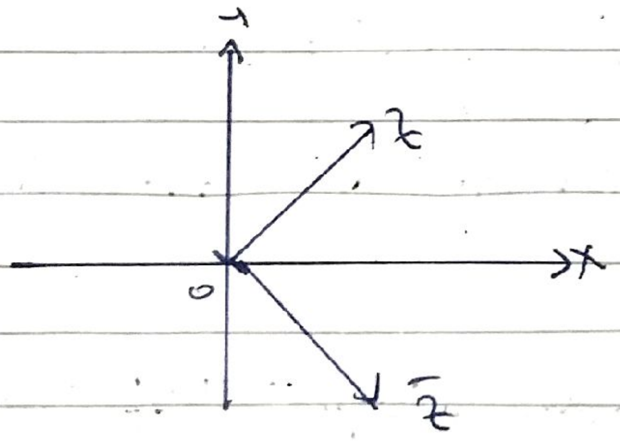
Complex Conjugates

المرافقات المركبة

The complex conjugate of the complex number $z = x+iy$ is the number $\bar{z} = x - iy$

انعكاس

Geometrically the conjugates of z is the reflection of z in the axis of real



Properties of the complex Conjugates

1) $\overline{(z_1 \mp z_2)} = \bar{z}_1 \mp \bar{z}_2 \quad \forall z_1, z_2 \in \mathbb{C}$

2) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

3) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

4) $\bar{z} = 0 \Longleftrightarrow z = 0$

5) $\bar{\bar{z}} = z$

6) If $z = x + iy$, then $z\bar{z} = x^2 + y^2$

7) $z + \bar{z} = 2\,Re(z) \Rightarrow Re(z) = \dfrac{z + \bar{z}}{2}$

8) $z - \bar{z} = 2i\,Im(z) \Rightarrow Im(z) = \dfrac{z - \bar{z}}{2i}$

9) If z is imaginary number then $\bar{z} = -z$.

10) If z is real number then $\bar{z} = z$.

The proof of the all a properties are ((H.W)).

Modulus or Absolute Values ((المقياس)) القيمة المطلقة

We can define the absolute values of the complex number $z = x + iy$ as a non-negative real number $\sqrt{x^2 + y^2}$ and denoted by $|z|$, such that

$|z| = \sqrt{x^2 + y^2}$

Properties of the absolut value

خصائص القيمة المطلقة

1) $|z| = \sqrt{z\bar{z}} \Rightarrow |z|^2 = z\bar{z}$.

2) $|z| = |\bar{z}|$

3) $|z_1 - z_2| = |z_2 - z_1|$.

4) $Re(z) \leq |Re(z)| \leq |z|$.

5) $Im(z) \leq |Im(z)| \leq |z|$.

6) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

7) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$.

8) $|z_1 + z_2| \leq |z_1| + |z_2|$.

9) $|z_1 - z_2| \geq |z_1| - |z_2|$.

10) $||z_1| - |z_2|| \leq |z_1 - z_2|$.

proof:

1) $z = x+iy$, $\bar{z} = x-iy$

$z\bar{z} = (x+iy)(x-iy) = x^2+y^2+i(-yx+xy) = x^2+y^2$

$= |z|^2$

4) $Re(z) \leq |Re(z)| \leq |z|$

$\sqrt{x^2+y^2} = \sqrt{(Re(z))^2+(Im(z))^2}$

$\Rightarrow |z|^2 = (Re(z))^2+(Im(z))^2$

Since $(Im(z))^2 \geq 0$, then

$(Re(z))^2 \leq |z|^2$

$|Re(z)| \leq |z|$ —— ①

$(Re(z)) \leq |Re(z)|$ —— ②

$(Re(z)) \leq |Re(z)| \leq |z|$.

8) from (7), we get

$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + z_1\bar{z}_2 + \bar{\bar{z}}_2\bar{z}_1 + |z_2|^2$

$= |z_1|^2 + z_1\bar{z}_2 + \overline{\bar{z}_2 z_1} + |z_2|^2$

$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2$

from (4), we get

$\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$

$= |z_1|^2 + 2|z_1|\cdot|\bar{z}_2| + |z_2|^2$

$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$

$= (|z_1|+|z_2|)^2$

$\therefore |z_1+z_2| \leq |z_1|+|z_2|.$

The Geometric Representation of the Complex number

التمثيل الهندسي للعدد العقدي

Geometrically, the absolute value of z is the length of the vector z; it is the distance of the point z from the origin:

Consequently, $|z_1 - z_2|$ is the distance between the point z_1 and z_2, this is also show

$$|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)|$$
$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

example:- find the geometric representation of the equations.

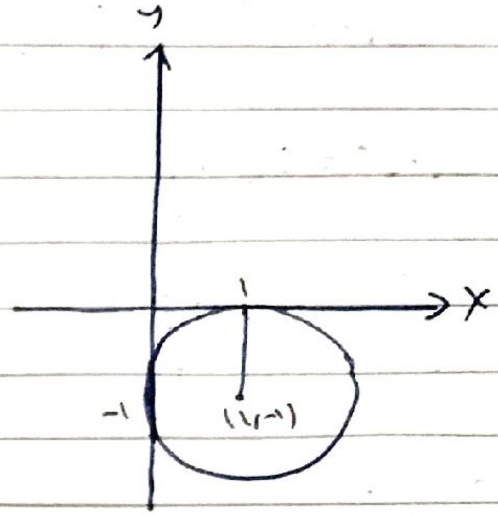
1) $|z - 1 + i| = 1$

2) $Re(\bar{z} - i) = 2$

3) $|z - i| = |z + i|$

Proof:

1) $|z-1+i|=1$, $z=x+iy$



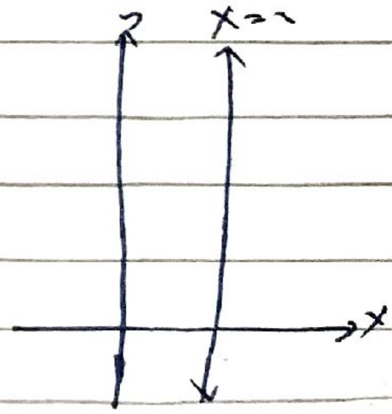
$|x+iy-1+i|=1$

$|(x-1)+i(y+1)|=1$

$\Rightarrow \sqrt{(x-1)^2+(y+1)^2}=1$

$(x+1)^2+(y+1)^2=1$

This is circle equation and radius is (1,-1) and it center



2) $Re(\bar{z}-i)=2$

$\bar{z}=x-iy$, $z=x+iy \Rightarrow \bar{z}-i=x-iy-1$

$z-i=x-i(y+1) \Rightarrow Re(z-i)=x \Rightarrow 2=x$

this is a lieun $x=2$

3) H.W.

example:

prove that if $|z|=2$, then

1) $|z^2+2z-1| \leq 9$

2) $\dfrac{1}{|z^3-1|} \leq \dfrac{1}{7}$.

proof:

1) $|z^2+2z-1| \leq |z^2| + |2z| + 1$

$\leq |z|^2 + 2|z| + 1$, $|z|=2$

$= 4+4+1 = 9$

2) $\dfrac{1}{|z^3-1|} \leq \dfrac{1}{7}$

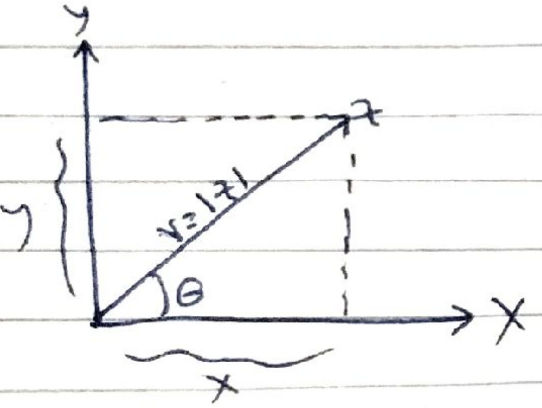
$|z^3-1| \geq \big||z^3| - |1|\big|$, $|z|=2$

$\geq \big||z|^3 - 1\big| \Rightarrow |8-1| = 7$

So $|z^3-1| \geq 7 \Rightarrow \dfrac{1}{|z^3-1|} \leq \dfrac{1}{7}$.

polar Coordinates الإحداثيات القطبية

Let r and θ the polar coordinates of the point represting z. then



$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$r = |z| = \sqrt{x^2+y^2}$

The complex number z can be written $z = r(\cos\theta + i\sin\theta)$

• This is polar form of z.

then the polar form of the complex number $z = x+iy$

$$z = r(\cos\theta + i\sin\theta)$$

The angle θ, also called the argument of z, we obtaind from the formula $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

the argument $\theta = \arg z$, is multiple-valued. this is because for any given value of θ. we could take instead $\theta + 2\pi$ and arrive at the same complex No. z.

We defined the principal valued of $\theta = \arg z$ and defined by $\operatorname{Arg} z$ such that $-\pi < \operatorname{Arg} z \leq \pi$. and defined by the formula:

$$\arg z = \operatorname{Arg} z + 2k\pi \quad , k = 0, \mp 1, \ldots$$

ملاحظات

* زوايا الربع الثاني / تحسب بالعلاقة $\pi - \theta_0$ حيث θ_0 في الربع الأول

* زوايا الربع الثالث / تحسب بالعلاقة $\pi + \theta_0$ 〃 〃 〃 〃 〃 〃

* زوايا الربع الرابع / 〃 〃 $2\pi - \theta_0$ 〃 〃 〃 〃 〃

example; write the following complex numbers by Polar Coordinates!

1) $z = 1 + i\sqrt{3}$, 2) $z = 1 - i\sqrt{3}$, 3) $z = 2$

Solutions:

1) $z = r(\cos\theta + i\sin\theta)$

$x + iy = r(\cos\theta + i\sin\theta)$

$\frac{x}{r} + i\frac{y}{r} = \cos\theta + i\sin\theta$

$z = 1 + i\sqrt{3} \Rightarrow x = 1, y = \sqrt{3}$

$r = |z| = \sqrt{x^2 + y^2} = \sqrt{1+3} = \sqrt{4} = 2$

$1 + i\sqrt{3} = 2(\cos\theta + i\sin\theta)$

$\frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\theta + i\sin\theta$

$\Rightarrow \cos\theta = \frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2}$

θ تقع في الربع الاول وتساوي $\frac{\pi}{3}$

$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

or

$1+i\sqrt{3} = 2\left(\cos\left(\frac{\pi}{3}+2k\pi\right)+i\sin\left(\frac{\pi}{3}+2k\pi\right)\right) \quad k=0,\pm1,\pm2,\dots$

②, ③ H.W.

2) $|z| = 2$, $\cos\theta = \frac{1}{2}$, $\sin\theta = -\frac{\sqrt{3}}{2}$.

زاويا تقع في الربع الرابع بشاري

$2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$1-i\sqrt{3} = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$.

Some properties of ang z

1) $\arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2$.

2) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

3) $\arg\left(\frac{1}{z}\right) = -\arg z$.

4) $\arg(\bar{z}) = -\arg z$.