The metric spaces:

Definition(4.1):

An order pair (X, d) is called a metric space if X is a non-empty set and d is a function

$$d: X \times X \to R$$

Satisfies:

1) $d(x, y) \geq 0 \quad \forall\, x, y \in X$
2) $d(x, y) = 0 \quad iff \quad x = y \quad \forall x, y \in X$
3) $d(x, y) = d(y, x) \qquad \forall x, y \in X$
4) $d(x, y) \leq d(x, z) + d(z, y) \qquad \forall x, y, z \in X$

d is called the distance function, and the elements of X are called the element of the space.

Examples(4.2):

1) (R, d); R the set of real numbers and $d: R \times R \to R$

is defined by $d(x, y) = |x - y| \qquad \forall x, y \in R$

1. $d(x, y) = |x - y| \geq 0 \quad \forall x, y \in R$
2. $d(x, y) = |x - y| = 0 \; iff \; x - y = 0 \; iff \; x = y \quad \forall x, y \in R$
3. $d(x, y) = |x - y| = |-(y - x)| = |(-1)(y - x)| = |y - x| = d(y, x) \qquad \forall x, y \in R$
4. $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y|$

$$\leq d(x, z) + d(z, y) \qquad \forall x, y, z \in R$$

$\therefore$ (R, d) is a metric space.

2) If $X = R^n$ such that

$R^n = \{x = (x_1, x_2, \cdots, x_n) : x_i \in R\}$.

If $x = (x_1, x_2, \cdots, x_n) \in R^n$, $y = (y_1, y_2, \cdots, y_n) \in R^n$.

Defined: $d: R^n \times R^n \to R$ by:

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \|x - y\| \qquad \forall x, y \in R^n$$

1. $\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \geq 0$

2. $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = 0 \quad iff \quad \sum_{i=1}^{n}(x_i - y_i)^2 = 0$

$iff \quad (x_i - y_i)^2 = 0 \quad iff \quad x_i = y_i \quad \forall\, i = 1,2,\cdots,n \quad iff \quad x = y$

3. $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = d(y,x)$

To prove (4) we need the following:

Lemma (4.3): The Cauchy - Schwarz inequality

For each real numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ we have:

$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

Lemma (4.4):

For each real numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ we have:

$$|(a_1 + b_1)^2 + (a_2 + b_2)^2 + \cdots + (a_n + b_n)^2| \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} + \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

Proof:

$$(a_1+b_1)^2+(a_2+b_2)^2+\cdots+(a_n+b_n)^2=(a_1^2+a_2^2+\cdots+a_n^2)+ \\ 2(a_1b_1+a_2b_2+\cdots+a_nb_n)+(b_1^2+b_2^2+\cdots+b_n^2)$$

By lemma (4.3)

$$\leq (a_1^2+a_2^2+\cdots+a_n^2)+2\sqrt{a_1^2+a_2^2+\cdots+a_n^2}\cdot\sqrt{b_1^2+b_2^2+\cdots+b_n^2}$$

$$+(b_1^2+b_2^2+\cdots+b_n^2)$$

$$\therefore \sqrt{(a_1+b_1)^2+(a_2+b_2)^2+\cdots+(a_n+b_n)^2} \\ \leq \sqrt{a_1^2+a_2^2+\cdots+a_n^2}+\sqrt{b_1^2+b_2^2+\cdots+b_n^2}$$

4. $d(x,y)=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$

Let $z=(z_1,z_2,\cdots,z_n)$

$$=\sqrt{\sum_{i=1}^{n}(x_i-z_i+z_i-y_i)^2}\leq\sqrt{\sum_{i=1}^{n}(x_i-z_i)^2}+\sqrt{\sum_{i=1}^{n}(z_i-y_i)^2}$$

[By lemma (4.4)]

$\therefore d(x,y)\leq d(x,z)+d(z,y)$.

3) Let X is a non-empty set define:

$$d: X\times X\to R$$

By:

$$d(x,y)=\begin{cases}0 & if \quad x=y\\ \dfrac{1}{3} & if \quad x\neq y\end{cases}$$

1) $d(x,y)\geq 0 \quad \forall x,y\in X$

2) $d(x,y)=0 \quad iff \quad x=y \quad \forall x,y\in X$

3) $d(x,y) = d(y,x)$ $\quad \forall\ x,y \in X$

$\frac{1}{3} = \frac{1}{3}$ $\quad or \quad 0 = 0$

4) $d(x,y) \le d(x,z) + d(z,y)$ $\quad \forall x,y,z \in X$

4) If $X = R^2$ such that

$R^2 = \{x = (x_1, x_2): x_1, x_2 \in R\}$

. Defined $d: R^2 \times R^2 \to R$ by:

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

$$x = (x_1, x_2), \qquad y = (y_1, y_2)$$

1) $d(x,y) = |x_1 - y_1| + |x_2 - y_2| \ge 0$

2) $d(x,y) = |x_1 - y_1| + |x_2 - y_2| = 0$ $\quad iff$
$|x_1 - y_1| = 0 \quad and \quad |x_2 - y_2| = 0$ $\quad iff$
$x_1 = y_1 \quad and \quad x_2 = y_2$

3) $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$
$= |y_1 - x_1| + |y_2 - x_2| = d(y,x)$

4) $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

Let $z = (z_1, z_2)$

$= |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2|$

$\le |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2|$

$\le d(x,z) + d(z,y)$

<u>**H.W:**</u> If $X = R^2$. Defined $d: R^2 \times R^2 \to R$ by:

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

$$x = (x_1, x_2), \qquad y = (y_1, y_2)$$

Is (X, d) a metric space?