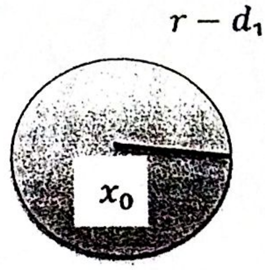
Let $(X, d)$ be a metric space and, $S \subseteq X$, $S$ is called an open set if for each $x_0 \in S$ there exists $r > 0$ , $(r \in R)$, , such that:

$$B_r(x_0) \subseteq S$$

**Examples:**

1) Every ball in any metric space is an open set.




$$B_r(x_0) = \{ x \in X : d(x, x_0) < r\}$$

**Proof:** let $y \in B_r(x_0)$

$0 < d(y, x_0) = d_1 < r$

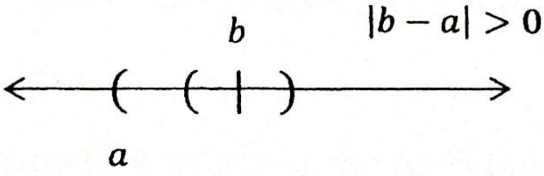
Take $\epsilon = r - d_1 > 0$ , to proof $B_\epsilon(y) \subseteq B_r(x_0)$

Let $z \in B_\epsilon(y) \overset{?}{\Rightarrow} z \in B_r(x_0)$

$d(z, y) < \epsilon$ *given* , to prove $d(z, x_0) < r$ ?

$$d(z, x_0) \leq d(z, y) + d(y, x_0)$$
$$< \epsilon + d_1$$
$$= r - d_1 + d_1$$
$$= r$$

In particular every open interval in R is an open set, $(a, \infty)$, $(-\infty, a)$ are open sets.

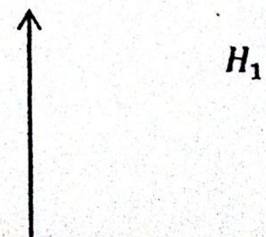



$\forall\, b \neq a$ , $\exists\, d = |b - a|$

$(b - \epsilon,\ b + \epsilon) \subseteq (a, \infty)$

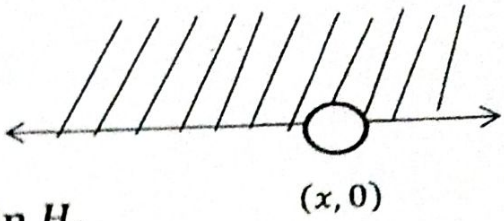
$[a, b)$ is not an open set.

$\exists\,(a - \epsilon, a + \epsilon) \not\subseteq [a, b)$



2) $H_1 = \{ (x,y) \in R^2 : x \in R \ , \ y \geq 0\}$

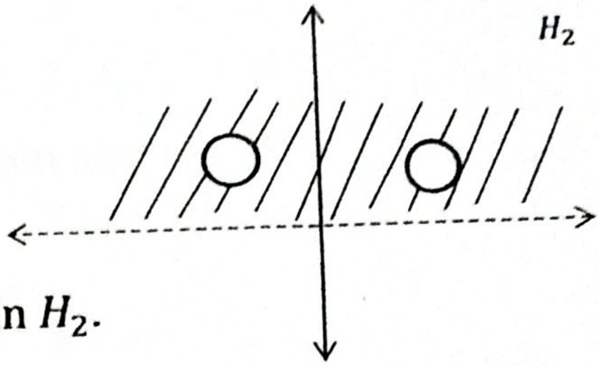
is not open subset in $R^2$.




Since the ball with center $(x, 0)$ is not contain in $H_1$.

$H_2 = \{ (x,y) \in R^2 : x \in R \ , \ y > 0\}$

is open subset in $R^2$




Since the ball with center $(x, y)$ is contain in $H_2$.

3) The set of rational (irrational) number is not open set.

Since any interval in $Q$ with center $\frac{a}{b} \in Q$, doesn't contain rational only (by density of irrational).

Also any interval in $Q'$, doesn't contain irrational only because of the density of rational number) not open.

**Proposition(4.8):**

Let $(X, d)$ be a metric space, and $T$ be a collection of all open subset of X, then $T$ satisfies the following:

1) $X, \emptyset \in T$.

2) The union of any number of open sets is open. (i.e The union of any element of $T$ is again in $T$.

3) The intersection of a finite number of element of $T$ is again in $T$.

**Proof:** 2) Let $\{T_n\}$ be any number of open sets in $T$.

To prove $\cup_n T_n \in T$ (i.e is open in $T$).

Let $x \in \bigcup_n T_n$ , $\therefore \exists\, k \in N$ s.t $x \in T_k$.

$\because T_k$ *is open* , $\therefore \exists\, r > 0$ , s.t $B_r(x) \subseteq T_k$

$\therefore B_r(x) \subseteq \bigcup_n T_n$

$\therefore \bigcup_n T_n$ is open

3) Let $T_1, T_2, \cdots, T_n$ be a finite number of open sets in .

To prove $\bigcap_{i=1}^{n} T_i$ is open in $T$.

Let $x \in \bigcap_{i=1}^{n} T_i$ , $\therefore x \in T_i \quad \forall\, i = 1,2,\cdots,n$.

$\because T_i$ *is open* , $\forall\, i = 1,2,\cdots,n$

$\therefore \exists\, r_1 \in R$ , s.t $B_{r_1}(x) \subseteq T_1$ , $\exists\, r_2 \in R$ , s.t $B_{r_2}(x) \subseteq T_2$ ,$\cdots$

Take $r = \min\{r_1, r_2, \cdots, r_n\}$

$\therefore B_r(x) \subseteq \bigcap_{i=1}^{n} T_i$

$\therefore \bigcap_{i=1}^{n} T_i$ is open

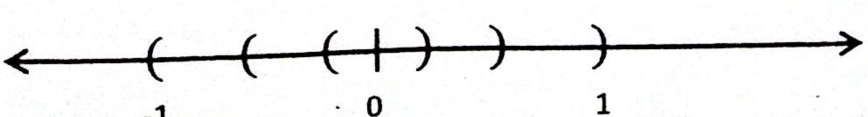
<u>Remark(4.9):</u>

The intersection of infinite number of open sets needn't be open. As the following example shows:

<u>Example:</u>

$\forall\, n \in N$ , let $A_n = \left(\frac{-1}{n}, \frac{1}{n}\right) \subseteq R$ , $\bigcap_n A_n = \{0\}$

Let $x \in \bigcap A_n$



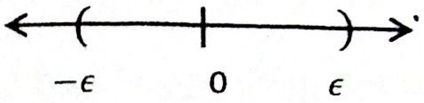

If $\exists\, x \neq 0$ , $x > 0 \Rightarrow \exists\, k \in N$ s.t $\frac{1}{k} < x$ , $\therefore x \notin \left(\frac{-1}{k}, \frac{1}{k}\right)$.

If $\exists x \neq 0$ , $x < 0$ , $0 < -x \Rightarrow \exists t \in N$ s.t $\frac{1}{t} < -x \Rightarrow$ $\frac{-1}{t} > x$ , $\therefore x \notin \left(\frac{-1}{t}, \frac{1}{t}\right) \Rightarrow x \notin \cap_n A_n$

$\therefore \cap_n A_n$ is only zero.

Note:

$\{0\}$ is not open, since. $\forall \epsilon > 0$, $B_\epsilon(0) = (-\epsilon, \epsilon) \nsubseteq \{0\}$ .




Remark:

If $(X, d)$ is a metric space, then we can define a topological space from this metric space by taking $T =$ the set of all open subsets of $X$ and by proposition (4.8) we easily seen that $(X, T)$ is a topological space

But if $(X, T)$ is a topological space, then in general we couldn't get a metric space from this topological space as the following example shows:-

Example:

Let $X = \{a, b, c, d, e, f, \dots, z\}$ *and* $T = \{X, \emptyset\}$.

$(X, T)$ is a topological space

But we cannot define a distance between the elements of $X$.

Proposition(4,12):-