

Lecture 1

References:-

1- Operation Research, third edition by Hamdy .T.

2- Operation Research, second edition, P.Rama Murthy

3- بحوث العمليات وتطبيقاتها / د. هلال هادي

4- بحوث العمليات / د. عبد ذياب جزاع

Operation Research (OR)

Operation research (OR) seeks the determination of the best (optimum) course of action of a decision problem under the restriction of limited resources. The term (OR) quite often is associated almost exclusively with the use of mathematical techniques to model and analyze decision problem. Although mathematics and mathematical models represent a corner stone of OR.

Art of modeling

The heart of OR is the model that provides a concise framework for analyzing a decision problem in a systematic manner. In this respect, two basic components are essential for constructing a model:

1- The objective of the system.

2-The constraints imposed on the system.

Types of OR models

The general mathematical models can be written in the form:

Optimize $Z = f(x_1, x_2, \dots, x_n)$ (objective function)

Subject to $g_i(x_1, x_2, \dots, x_n) \leq b_i \quad i=1, 2, \dots, m$ (constraints)

$x_1, x_2, \dots, x_n \geq 0$ (nonnegative restriction)

Phases of (OR) study

The major phases through which the OR team would proceed in order to effect on OR study include:-

- 1- Definition of the problem.
- 2- Construction of the model.
- 3- Solution of the model.
- 4- Validation of the model.
- 5- Implementation of the final results.

Linear programming (LP)

A linear programming (LP) is a resources allocation model that seeks the best allocation of limited resources to a number of activities .LP has been applied with considerable success to a multitude of practical problems.

A linear programming problem (L.P.P):

Is an optimization problem for which we do the following?

- 1- We attempt to maximize (or min) a linear function of the decision variables. The function that is to be maximum or minimum is called the objective function.
- 2- The values of the decision variables must satisfy a set of constraints. Each constraint must be a linear equation or linear inequality.
- 3- A sign restriction is associated with each variable for any variable x_i , the restriction specifies either that x_i must be none negative ($x_i \geq 0$) or that x_i may be unrestricted in sign.

General form of (L.P.P):-

The general form of L.P.P is:

$$\text{Max (or min) } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad (\text{objective function})$$

Sub.to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n (\geq, =, \leq) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n (\geq, =, \leq) b_2$$

$$\cdot \quad \cdot \quad \quad \cdot \quad \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \quad \cdot \quad \quad \cdot \quad \cdot$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n (\geq, =, \leq) b_m$$

$$x_1, x_2, \dots, x_n \geq 0 \quad (\text{nonnegative const.})$$

$$a_{ij}, b_i, c_j \quad i = 1, 2, \dots, m \quad (\text{constant number})$$

$$j = 1, 2, \dots, m$$

Formulation problems:

Example:- A factor produced two kind of toys on three machines the first kind required (3) minutes on a machine (1) , (4) minutes on machine (2) , (2) minute on machine (3) .While the other kind required (2) minutes on machine (1), and (1) minute on machine (2), (6) minutes on machine (3), the available time for machine (1) is (6) time unit, and the other hand production planning department determine the working time for machine (2) by (8) time unit to complete the manufacturing successfully and the time for the machine (3) must be at least (12) time unit the manufacturing cost per unit are (3) diners for kind (1) and (2) diners for kind (2) .formulate the mathematical models for the problem above?

| machines | Kind (1) | Kind (2) | Available time |
|----------|----------|----------|----------------|
| I | 3 | 2 | 6 |
| II | 4 | 1 | 8 |
| III | 2 | 6 | 12 |
| Cost | 3D | 2D | |

Solution:

Let x_1 =first kind

x_2 = second kind

$$\text{Min } Z = 3 x_1 + 2 x_2$$

Sub.to

$$3 x_1 + 2 x_2 \leq 6$$

$$4 x_1 + x_2 = 8$$

$$2 x_1 + 6 x_2 \geq 12$$

$$x_1 , x_2 \geq 0$$

H.W/ Three kind of texture are process successively on two machines the manufacturing time (in hours) each kind are tabulated below for the two machines formulated the mathematical model (for the above primal to minimize the total cost).

| machines | Kind (1) | Kind (2) | Kind (3) | Available time |
|----------|----------|----------|----------|----------------|
| I | 2 | 3 | 2 | 40 |
| II | 2 | 1 | 2 | 60 |
| cost | 5D | 7D | 8D | |

Lecture 2

- Standard form of (L.P.P)

A linear programming is in standard form if all the functional constraints are written as equation and all the variables are required to be non-negative. The objective function may be max. or min. .

Example:

Mathematical form:

$$\text{Max } Z=1.5x_1+1.8x_2$$

Sub. To

$$3x_1+2x_2 \leq 15000$$

$$x_1+2x_2 \leq 10000$$

$$x_1 \leq 4000$$

$$x_2 \leq 4500$$

$$x_1, x_2 \geq 0$$

Standard form:

$$\text{Max } Z = 1.5 x_1 + 1.8x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

Sub. To

$$3x_1+2x_2 + s_1 = 15000$$

$$x_1 + 2x_2 + s_2 = 10000$$

$$x_1 + s_3 = 4000$$

$$x_2 + s_4 = 4500$$

$$x_1, x_2, s_1, s_2, s_4 \geq 0$$

Methods of solving Linear Programming Problems

1-Graphical method

Since there are two variables we can solve this problem by graphical, first we find the set of points in the plane that satisfies all the constraints and then finding which point of this set that Maximized or Minimized the value of objective function.

-steps of Graphical method

- 1- Change all the constraint to the equation.
- 2- Drawing the constraint.
- 3-Identify the feasible solution (feasible region).
- 4- Identify the point (or points) that satisfied the optimal solution.

Definition (1): (Feasible solution of L.P.P)

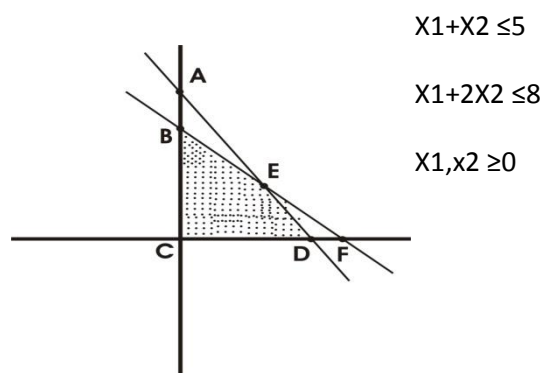
It is a vector $x = \{x_1, x_2, \dots, x_n\}$ that satisfied all the constraint of L.P problem and the non negativity condition.

Definition (2): (feasible region)

The feasible region for any LP is the set of all points satisfying all the Lp's constraints and all the Lp's sign restrictions.

Ex/

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Definition (3) : (Basic feasible solution(B.F. S)

Any basic solution to the form: Max or Min $Z = C x$

$$\text{Sub. To} \quad Ax \geq B$$

$$x \geq 0$$

In which all variables is nonnegative is a basic feasible solution (B.F.S)

Definition (4): (optimal solution)

For max problem, an optimal solution to an LP is a point in the feasible region which the largest objective function value .similarly, for a min. problem an optimal solution is a point in the smallest objective function value.

Example: Find the feasible region and the optimal solution of the following

L.P.P:

$$\text{Max } Z = 5x_1 + 4x_2$$

$$\text{Sub. To} \quad 2x_1 + 4x_2 \leq 8$$

$$-2x_1 + x_2 \leq 2$$

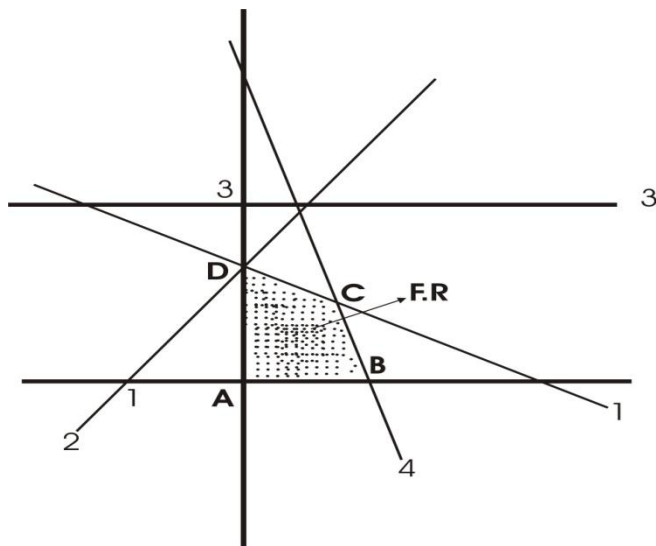
$$3x_2 \leq 9$$

$$4x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution:

- Let $2x_1 + 4x_2 = 8$... line (1) $\Rightarrow (0,2), (4,0)$
 $-2x_1 + x_2 = 2$... line (2) $\Rightarrow (0,2), (-1,0)$
 $3x_2 = 9$... line (3) $\Rightarrow (0,3)$
 $4x_1 + x_2 = 4$...line (4) $\Rightarrow (0,4), (1,0)$



The optimal solution is:

$Z = 9.714, x_1 = 4/7, x_2 = 12/7$

| Extreme Point | x_1 | x_2 | $Z = 5x_1 + 4x_2$ |
|---------------|-------|-------|-------------------|
| A | 0 | 0 | 0 |
| B | 1 | 0 | 5 |
| C | 4/7 | 12/7 | 9.714 |
| D | 0 | 2 | 8 |

H.w/ find the feasible region and the optimal solution for the following L.P.P model using Graphical method.

$$\text{Min } Z = 4x_1 + 8x_2$$

Sub. To

$$4x_1 + 2x_2 \geq 8$$

$$-7x_1 + 4x_2 \leq 28$$

$$8x_1 + 6x_2 \leq 48$$

$$4x_1 - x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Lecture 3

2-Simplex method

Simplex method was the first method developed to solve Lp in iteration where the same computational steps are repeated a number of time before the optimal solution are reached. At each iteration the simplex method selects the variable that will produce the largest change towards the min or max solution.

Steps of simplex method:

Step(0):- convert the Lp to standard form by insert the slack variables(s_i).

Step(1):- select the entering variables from among the current non basic variables.

Step(2):-select the leaving variable from among the current basic variables that must be set to zero (be come non basic) when the entering variable become basic.

Step(3):- determine the pivot element .It is results from the intersection of the entering variable .Column and the leaving variable row.

Step(4):- determine the new basic solution by making the entering variable basic and the leaving variable non basic.

Step(5):- if there is a negative number in the first row ($z-c_j$),then go to step (1) , otherwise go to step (6).

Step(6):- if there isn't a negative number in the first row ($z-c_j$),then the optimal solution has been reached.

-Standard from conditions

- 1 –all constraints are equation
- 2 - RHS for each equation must be (+ve).
- 3 - Decision variables are nonnegative.
- 4 - Objective function is max. or min. type.

Example(1):Use simplex method to solve the following linear programming problem (L.P.P).

$$\text{Max } Z = 12x_1 + 8x_2$$

Sub. To

$$8x_1 + 6x_2 \leq 2200$$

$$4x_1 + 9x_2 \leq 1800$$

$$x_1 + 2x_2 \leq 400$$

$$x_1, x_2 \geq 0$$

Solution:

The details of the simplex algorithm will be explained by using the Reddy mikks model. This will require expressing the objective function and all the constraints of the standard form as:

$$\text{Max } Z = 12x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

Sub. To

$$8x_1 + 6x_2 + s_1 = 2200$$

$$4x_1 + 9x_2 + s_2 = 1800$$

$$x_1 + 2x_2 + s_3 = 400$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Now we use simplex tableau as follows:

Table (1)

| Basic var. | X1 | X2 | S1 | S2 | S3 | R.H.S |
|------------|-----|----|----|----|----|-------|
| Z | -12 | -8 | 0 | 0 | 0 | 0 |
| S1 | (8) | 6 | 1 | 0 | 0 | 2200 |
| S2 | 4 | 9 | 0 | 1 | 0 | 1800 |
| S3 | 1 | 2 | 0 | 0 | 1 | 400 |

-Optimality condition: - (entering variable)

The entering variable in max (min) in the non basic variable with the most negative (positive) coefficient in the z-equation. A tie is broken arbitrarily when all non basic coefficients in the z-equation are nonnegative (non-positive), the optimum is reached.

-Feasibility condition :- (leaving variable)

For both the max and min problems the leaving variable is basic variable having the smallest ratio (with positive denominator). A tie is broken arbitrarily.

By applying the optimality and feasibility conditions. We find the entering variable (x1) with most negative coefficient in z-equation (since the problem is max) then the leaving variable is (s1) since the min. ratio of (2200/8, 1800/4 , 400/1)=(2200/8)=275

And the pivot element is (8).

The next iteration (new basic solution) is determined by applying Gauss-Jordan method. The method effects a change in basic by using two types of computations.

Type (1):- (pivot equation)

$$\text{New pivot equation} = (\text{old equation} / \text{pivot element})$$

Type (2):- (all other equations)

$$\text{New equation} = \text{all equation} - [(\text{its entering column coefficient}) * (\text{new pivot equation})].$$

Table (2)

| B.v | X1 | X2 | S1 | S2 | S3 | R.H.S |
|-----|----|-----|------|----|----|-------|
| Z | 0 | 1 | 3/2 | 0 | 0 | 3300 |
| X1 | 1 | 6/8 | 1/8 | 0 | 0 | 275 |
| S2 | 0 | 6 | -1/2 | 1 | 0 | 700 |
| S3 | 0 | 5/4 | -1/8 | 0 | 1 | 125 |

Now the solution in table (2) is optimal since $z-c_j \geq 0$ (z-equation) and R.H.S ≥ 0 . Hence the optimal solution in this problem is:

$$z=3300, x_1=275, x_2=0$$

Example (2):-

$$\text{Min } Z = x_1 - 3x_2 - 2x_3$$

Sub. To

$$3x_1 - x_2 + 2x_3 \leq 7$$

$$-2x_1 + 4x_2 \leq 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$

$$x_1, x_2, x_3 \geq 0$$

Solution:-

The standard form of this problem is:

$$\text{Min } Z = x_1 - 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\text{Sub. To } 3x_1 - x_2 + 2x_3 + s_1 = 7$$

$$-2x_1 + 4x_2 + s_2 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Table (1)

| B.V | X1 | X2 | X3 | S1 | S2 | S3 | R.H.S |
|-----|----|-----|----|----|----|----|-------|
| Z | -1 | 3 | 2 | 0 | 0 | 0 | 0 |
| S1 | 3 | -1 | 2 | 1 | 0 | 0 | 7 |
| S2 | -2 | (4) | 0 | 0 | 1 | 0 | 12 |
| S3 | -4 | 3 | 8 | 0 | 0 | 1 | 10 |

Table (2)

| B.V | X1 | X2 | X3 | S1 | S2 | S3 | R.H.S |
|-----|------|----|-----|----|------|----|-------|
| Z | 1/2 | 0 | 2 | 0 | -3/4 | 0 | -9 |
| S1 | 5/2 | 0 | 2 | 1 | 1/4 | 0 | 10 |
| X2 | -1/2 | 1 | 0 | 0 | 1/4 | 0 | 3 |
| S3 | 5/2 | 0 | (8) | 0 | -3/4 | 1 | 1 |

The solution in table (2) is not optimal since z-equation is not negative (min problem), now we choose x_3 the entering variable (most positive coefficient in z-equation), and the leaving variable is (s_3).

H.W/ Find the new table and the optimal solution of above problem.

Lecture 4

Special Cases in Simplex Techniques:

- 1- Multiple Optimal Solutions
- 2- Infeasible Problem
- 3- Unbounded solution.
- 4- Degeneracy.

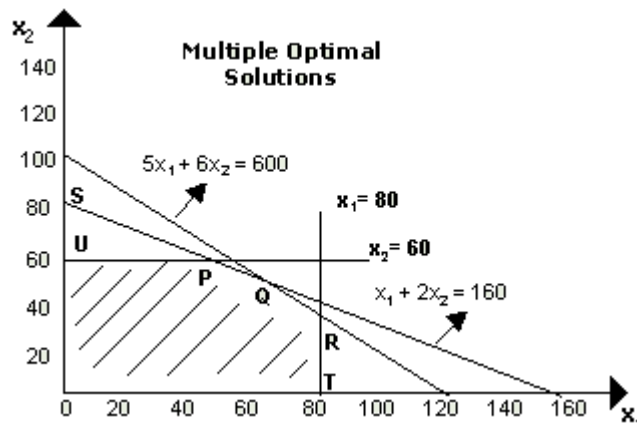
1. Multiple Optimal Solutions

The linear programming problems discussed in the previous section possessed unique solutions. This was because the optimal value occurred at one of the extreme points (corner points). But situations may arise, when the optimal solution obtained is not unique. This case may arise when the line representing the objective function is parallel to one of the lines bounding the feasible region. The presence of multiple solutions is illustrated through the following example.

Example: Maximize $z = x_1 + 2x_2$

subject to

$$\begin{aligned}x_1 &\leq 80 \\x_2 &\leq 60 \\5x_1 + 6x_2 &\leq 600 \\x_1 + 2x_2 &\leq 160 \\x_1, x_2 &\geq 0.\end{aligned}$$



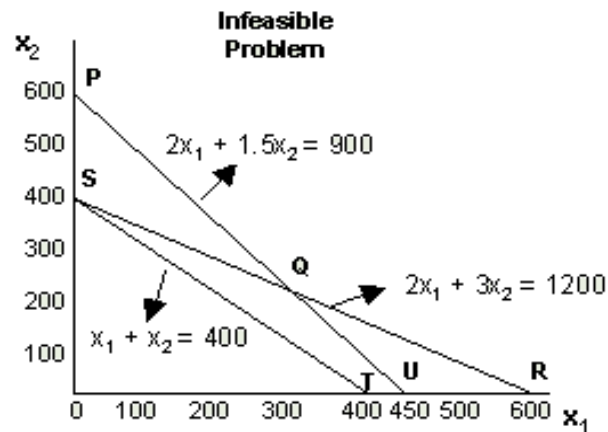
In the above figure, there is no unique outer most corners cut by the objective function line. All points from P to Q lying on line PQ represent optimal solutions and all these will give the same optimal value (maximum profit) of result 160. This is indicated by the fact that both the points P with coordinates (40, 60) and Q with coordinates (60,50) are on the line $x_1 + 2x_2 = 160$. Thus, every point on the line PQ maximizes the value of the objective function and the problem has multiple solutions.

2. Infeasible Problem

In some cases, there is no feasible solution area, i.e., there are no points that satisfy all constraints of the problem. An infeasible LP problem with two decision variables can be identified through its graph. For example, let us consider the following linear programming problem.

Example: Minimize $z = 200x_1 + 300x_2$

$$\begin{aligned} \text{subject to} \quad & 2x_1 + 3x_2 \geq 1200 \\ & x_1 + x_2 \leq 400 \\ & 2x_1 + 1.5x_2 \geq 900 \\ & x_1, x_2 \geq 0 \end{aligned}$$



The region located on the right of PQR includes all solutions, which satisfy the first and the third constraints. The region located on the left of ST includes all solutions, which satisfy the second constraint. Thus, the problem is infeasible because there is no set of points that satisfy all the three constraints.

3. Unbounded Solutions

It is a solution whose objective function is infinite. If the feasible region is unbounded then one or more decision variables will increase indefinitely without violating feasibility, and the value of the objective function can be made arbitrarily large. Consider the following model:

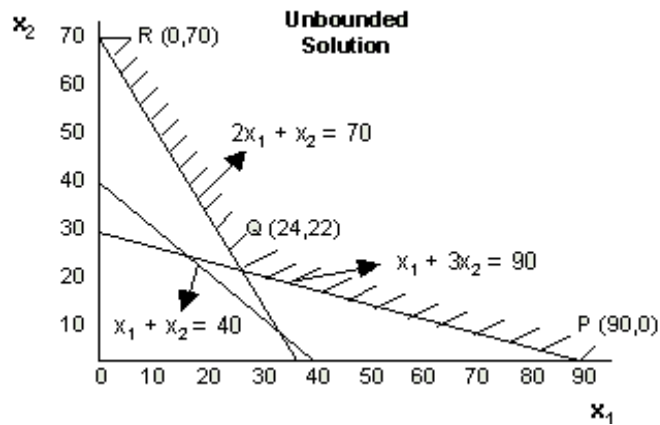
$$\text{Minimize } z = 40x_1 + 60x_2$$

$$\text{subject to } 2x_1 + x_2 \geq 70$$

$$x_1 + x_2 \geq 40$$

$$x_1 + 3x_2 \geq 90$$

$$x_1, x_2 \geq 0$$



The point (x_1, x_2) must be somewhere in the solution space as shown in the figure by shaded portion. The three extreme points (corner points) in the finite plane are:

$$P = (90, 0); Q = (24, 22) \text{ and } R = (0, 70)$$

The values of the objective function at these extreme points are:

$$Z(P) = 3600, Z(Q) = 2280 \text{ and } Z(R) = 4200$$

In this case, no maximum of the objective function exists because the region has no boundary for increasing values of x_1 and x_2 . Thus, it is not possible to maximize the objective function in this case and the solution is unbounded.

4. Degeneracy

It is one of the special cases of the simplex method to solve linear programming occur when a tie for the min. ratio may be broken arbitrarily for the purpose of determining the leaving variable. When this happens, however, one or more of the basic variables will necessarily equal zero in the next iteration. In this case, we say that the new solution is degenerate.

Example: Maximize $Z = 3x_1 + 9x_2$

$$\begin{aligned} \text{Subject to } & x_1 + 4x_2 \leq 8 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

| B.V | X1 | X2 | S1 | S2 | R.H.S |
|-----|----|----|----|----|-------|
| Z | -3 | -9 | 0 | 0 | 0 |
| S1 | 1 | 4 | 1 | 0 | 8 |
| S2 | 1 | 2 | 0 | 1 | 4 |

| B.V | X1 | X2 | S1 | S2 | R.H.S |
|-----|------|----|------|----|-------|
| Z | -3/4 | 0 | 9/4 | 0 | 18 |
| X2 | 1/4 | 1 | 1/4 | 0 | 2 |
| S2 | 1/2 | 0 | -1/2 | 1 | 0 |

| B.V | X1 | X2 | S1 | S2 | R.H.S |
|-----|----|----|-----|------|-------|
| Z | 0 | 0 | 3/2 | 3/2 | 18 |
| X2 | 0 | 1 | 1/2 | -1/2 | 2 |
| X1 | 1 | 0 | -1 | 2 | 0 |

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Lecture 5

- Artificial Variable Techniques:

These techniques will be used when slack variable (S_i) doesn't provide basic feasible solution in case when at least of the constraint is the type ($=$) or (\geq).

There are three methods based on the artificial variable:

1-Big-M Techniques

2-Two Phases Method

3-Dual Simplex Method

- The Big-M Techniques:

These techniques have the following steps:

1-Written L.P problem in a standard form.

2-Adding (R_i) variables to each constraint of type ($=$) or (\geq) where slack variable will be negative ($-S_i$).

3-Adding (R_i) multiply by ($-M$) for Max problem and ($+M$) for Min problem to the objective function.

4-Use (R_i) and (S_i) variables as a basic feasible solution.

5-Procceed the steps of the simplex methods.

Example:

Maximize $z = 5x_1 + x_2$

Subject to $x_1 \leq 10$
 $x_1 - 2x_2 \geq 3$
 $x_1 + x_2 = 12$
 $x_1, x_2 \geq 0$

Solution:-

Put the problem in the standard form:

Max $Z = 5x_1 + x_2 + 0s_1 + 0s_2$
 Sub. To $x_1 + s_1 = 10$
 $x_1 - 2x_2 - s_2 = 3$
 $x_1 + x_2 = 12$
 $x_1, x_2, s_1, s_2 \geq 0$

Now we add (Ri) variables to each constraints of type (\geq) and (=) in the primal problems:

Max $Z - 5x_1 - x_2 - 0s_1 - 0s_2 + MR_2 + MR_3 = 0$
 Sub. To $x_1 + s_1 = 10$
 $x_1 - 2x_2 - s_2 + R_2 = 3$
 $x_1 + x_2 + R_3 = 12$
 $x_1, x_2, s_1, s_2, R_2, R_3 \geq 0$

| B.V | X1 | X2 | S1 | S2 | R2 | R3 | R.H.S |
|-----|----|----|----|----|----|----|-------|
| Z | -5 | -1 | 0 | 0 | +M | +M | 0 |
| S1 | 1 | 0 | 1 | 0 | 0 | 0 | 10 |
| R2 | 1 | -2 | 0 | -1 | 1 | 0 | 3 |
| R3 | 1 | 1 | 0 | 0 | 0 | 1 | 12 |

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The equation of R_2 and R_3 :

$$R_2 = 3 - x_1 + 2x_2 + s_2$$

$$R_3 = 12 - x_1 - x_2$$

And the objective function thus becomes:

$$\begin{aligned} Z &= 5x_1 + x_2 - M(3 - x_1 + 2x_2 + s_2) - M(12 - x_1 - x_2) \\ &= 5x_1 + x_2 - 3M + Mx_1 - 2Mx_2 - Ms_2 - 12M + Mx_1 + Mx_2 \\ &= (5 + 2M)x_1 + (1 - M)x_2 - Ms_2 - 15M \end{aligned}$$

$$Z - (5 + 2M)x_1 - (1 - M)x_2 + Ms_2 = -15M$$

Table (1)

| B.V | X1 | X2 | S1 | S2 | R2 | R3 | R.H.S |
|-----|-------|------|----|----|----|----|-------|
| Z | -5-2M | -1+M | 0 | M | 0 | 0 | -15M |
| S1 | 1 | 0 | 1 | 0 | 0 | 0 | 10 |
| R2 | (1) | -2 | 0 | -1 | 1 | 0 | 3 |
| R3 | 1 | 1 | 0 | 0 | 0 | 1 | 12 |

Now by using simplex Algorithm we obtain the new table as follows:

Table (2)

| B.V | X1 | X2 | S1 | S2 | R2 | R3 | R.H.S |
|-----|----|--------|----|------|------|----|-------|
| Z | 0 | -11-3M | 0 | -5-M | 5+2M | 0 | 15-9M |
| S1 | 0 | 2 | 1 | 1 | -1 | 0 | 7 |
| X1 | 1 | -2 | 0 | -1 | 1 | 0 | 3 |
| R3 | 0 | (3) | 0 | 1 | -1 | 1 | 9 |

Table (3)

| B.V | X1 | X2 | S1 | S2 | R2 | R3 | R.H.S |
|-----|----|----|----|-------|-------|--------|-------|
| Z | 0 | 0 | 0 | -4/3 | 4/3+M | 11/3+M | 48 |
| S1 | 0 | 0 | 1 | (1/3) | -1/3 | -2/3 | 1 |
| X1 | 1 | 0 | 0 | -1/3 | 1/3 | 2/3 | 9 |
| X2 | 0 | 1 | 0 | 1/3 | -1/3 | 1/3 | 3 |

Table (4)

| B.V | X1 | X2 | S1 | S2 | R2 | R3 | R.H.S |
|-----|----|----|----|----|----|-----|-------|
| Z | 0 | 0 | 4 | 0 | M | 1+M | 52 |
| S2 | 0 | 0 | 3 | 1 | -1 | -2 | 3 |
| X1 | 1 | 0 | 1 | 0 | 0 | 0 | 10 |
| X2 | 0 | 1 | -1 | 0 | 0 | 1 | 2 |

Hence the solution in table (4) is optimal since Z-equation ≥ 0 and R.H.S is (+v), therefore the optimal solution in the problem is:

Max $Z=52$, $X_1=10$, $X_2=2$

H.W / Solve the following problem using Big-M method:

$$\text{Min } Z = 2x_1 + x_2$$

Sub. to

$$3x_1 + x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Note :(Min $Z=2x_1+x_2+0s_2+0s_3+MR_1+MR_2$)

The optimal solution in 3 iteration is ($z=12/5$, $x_1=3/5$, $x_2=6/5$)

Lecture 6

- Dual Simplex Method:

The steps of dual simple method are:

1- Construct the standard form.

2-Choose a leaving var. (most negative number in R.H.S Column).

3-Choose an entering var. where:

Entering var. = Min | Coefficients of Z-equation/Coefficients of row Leaving var.|

4- Use simplex method to obtain a new table.

5-Stop if $Z-c_j \geq 0$, (R.H.S +ve) in Max problem or $Z-c_j \leq 0$, (R.H.S +ve) in Min problem.

Example:

$$\text{Min } Z = 2x_1 + x_2$$

$$\begin{aligned} \text{Sub. to} \quad & 3x_1 + x_2 \geq 3 \\ & 4x_1 + 3x_2 \geq 6 \\ & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution:-

$$\text{Min } Z - 2x_1 - x_2 = 0$$

$$\begin{aligned} \text{Sub. to} \quad & -3x_1 - x_2 \leq -3 \\ & -4x_1 - 3x_2 \leq -6 \\ & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Now we change the problem to the standard form:

$$\text{Min } Z - 2x_1 - x_2 - 0s_1 - 0s_2 - 0s_3 = 0$$

$$\text{Sub. to } -3x_1 - x_2 + s_1 = -3$$

$$-4x_1 - 3x_2 + s_2 = -6$$

$$x_1 + 2x_2 + s_3 = 3$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Table (1)

| B.V | X1 | X2 | S1 | S2 | S3 | R.H.S |
|-----|----|------|----|----|----|-------|
| Z | -2 | -1 | 0 | 0 | 0 | 0 |
| S1 | -3 | -1 | 1 | 0 | 0 | -3 |
| S2 | -4 | (-3) | 0 | 1 | 0 | -6 |
| S3 | 1 | 2 | 0 | 0 | 1 | 3 |

Table (2)

| B.V | X1 | X2 | S1 | S2 | S3 | R.H.S |
|-----|--------|----|----|------|----|-------|
| Z | -2/3 | 0 | 0 | -1/3 | 0 | 2 |
| S1 | (-5/3) | 0 | 1 | -1/3 | 0 | -1 |
| X2 | 4/3 | 1 | 0 | -1/3 | 0 | 2 |
| S3 | -5/3 | 0 | 0 | 2/3 | 1 | -1 |

Table (3)

| B.V | X1 | X2 | S1 | S2 | S3 | R.H.S |
|-----|----|----|------|------|----|-------|
| Z | 0 | 0 | -2/5 | -1/5 | 0 | 12/5 |
| X1 | 1 | 0 | -3/5 | 1/5 | 0 | 3/5 |
| X2 | 0 | 1 | 4/5 | -3/5 | 0 | 6/5 |
| S3 | 0 | 0 | -1 | 1 | 1 | 0 |

The solution in table(3) is optimal since $z-c_j \leq 0$ and $R.H.S \geq 0$

H.W/ using dual simplex method to solve the following L.P problems:

1) $\text{Min } Z = 3x_1 + 2x_2 + 3x_3$

Sub. to $x_1 + 4x_2 + x_3 \geq 7$
 $2x_1 + x_2 + x_4 \geq 10$
 $x_1, x_2, x_3, x_4 \geq 0$

2) $\text{Min } Z = 5x_1 + 6x_2$

Sub. to $x_1 + 9x_2 \leq 60$
 $2x_1 + 3x_2 \leq 45$
 $5x_1 + 2x_2 \geq 20$
 $x_1, x_2 \geq 0$

Lecture 7

-Dual Problem

The dual is an auxiliary Lp problem defined directly and systematically from the original or primal Lp model. In most Lp treatments, the dual is defined for various forms of the primal depending on the types of the constraints, the signs of the variables and the sense of optimization.

The general standard form of the primal is defined as:

$$\left(\begin{array}{l} \text{Max.} \\ \text{or} \\ \text{Min.} \end{array} \right) Z = \sum_{j=1}^n c_j x_j$$

$$\text{Sub.to} \quad \sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, 2, \dots, m$$

$$x_j \geq 0 \quad j = 1, 2, \dots, n$$

The standard form of the dual is defined as:

$$\left(\begin{array}{l} \text{Min.} \\ \text{or} \\ \text{Max.} \end{array} \right) w = \sum_{i=1}^m b_i y_i$$

$$\text{Sub.to} \quad \sum_{i=1}^m a_{ij} y_i \geq c_j \quad j = 1, 2, \dots, n$$

$$y_i \geq 0 \quad i = 1, 2, \dots, m$$

Note that the n variables, y_i , includes the surplus and slacks.

The diagram shows that the dual is obtained symmetrically from the primal according to the following rules:

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$$\text{Max} Z = C^T x$$

$$\text{Sub.to} \quad Ax \leq b \\ x \geq 0$$

Where $x^T = [x_1, x_2, \dots, x_n]$ and $c^T = [c_1, c_2, \dots, c_n]$ are vectors in \mathbb{R}^n , and $b^T = [b_1, b_2, \dots, b_m] \in \mathbb{R}^m$, $A = [a_{ij}]$ is a matrix in $\mathbb{R}^{n \times m}$.

The dual of above problem is:

$$\text{Min} Z = b^T y$$

$$\text{Sub.to} \quad A^T y \geq c \\ y \geq 0$$

Example (1)

$$\text{Max } Z = 3x_1 + 5x_2 \quad (\text{Primal Model})$$

$$\text{Sub. to} \quad -x_1 - x_2 \leq 8 \\ 2x_1 + 4x_2 \leq 7 \\ x_1, x_2 \geq 0$$

$$\text{Min } W = 8y_1 + 7y_2 \quad (\text{dual model})$$

$$\text{Sub. to} \quad -y_1 + 2y_2 \geq 3 \\ -y_1 + 4y_2 \geq 5 \\ y_1, y_2 \geq 0$$

Example (2)

$$\text{Max } Z = 5x_1 + 12x_2 + 4x_3 \quad (\text{Primal Model})$$

$$\text{Sub. to} \quad x_1 + 2x_2 + x_3 \leq 5 \\ 2x_1 - 2x_2 + 3x_3 = 2 \\ x_1, x_2, x_3 \geq 0$$

$$\text{Max } Z = 5x_1 + 12x_2 + 4x_3$$

$$\text{Sub. to} \quad x_1 + 2x_2 + x_3 \leq 5 \quad \leftarrow y_1 \\ 2x_1 - 2x_2 + 3x_3 \leq 2 \quad \leftarrow y_2^+$$

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$$-2x_1 + 2x_2 - 3x_3 \leq -2 \leftarrow y_2^-$$

$$x_1, x_2, x_3 \geq 0$$

The dual model is:

$$\text{Min } W = 5y_1 + 2(y_2^+ - y_2^-)$$

$$\text{Sub. to } \begin{aligned} y_1 + 2(y_2^+ - y_2^-) &\geq 5 \\ 2y_1 - 2(y_2^+ - y_2^-) &\geq 12 \end{aligned}$$

$$\begin{aligned} y_1 + 3(y_2^+ - y_2^-) &\geq 4 \\ y_1 &\geq 0, y_2 \text{ unrestricted } (y_2^+, y_2^- \geq 0) \end{aligned}$$

Example (3)

Primal $\text{Max } Z = 5x_1 + 6x_2$

Sub .to $x_1 + 2x_2 = 5$

$$-x_1 + 5x_2 \geq 3$$

$$4x_1 + 7x_2 \leq 8$$

$$x_1 \text{ unrestricted, } x_2 \geq 0$$

The standard primal:

Let $x_1 = x_1' - x_1''$ where $x_1', x_1'' \geq 0$ then the standard primal becomes:

$$\text{Max } Z = 5(x_1' - x_1'') + 6x_2$$

Sub .to $(x_1' - x_1'') + 2x_2 = 5$

$$-(x_1' - x_1'') + 5x_2 - S_2 = 3$$

$$4(x_1' - x_1'') + 7x_2 + S_3 = 8$$

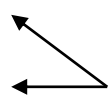
$$x_1', x_1'', x_2, S_2, S_3 \geq 0$$

The dual is:

$$\text{Min } W = 5y_1 + 3y_2 + 8y_3$$

Sub.to $y_1 - y_2 + 4y_3 \geq 5$

$$-y_1 + y_2 - 4y_3 \geq -5$$

 imply that $y_1 - y_2 + 4y_3 = 5$

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$$2y_1 + 5y_2 + 7y_3 \geq 6$$

$$-y_2 \geq 0 \quad \longleftarrow \text{implies that } y_2 \leq 0$$

$$y_3 \geq 0$$

y_1, y_2, y_3 unrestricted

H.W / Find the dual problem of the following L.p primal problem:

1) $\text{Max } Z = 5x_1 + 12x_2 + 4x_3$

Sub.to $x_1 + 2x_2 + x_3 \leq 10$

$$2x_1 - x_2 + 3x_3 = 8$$

$$x_1, x_2, x_3 \geq 0$$

2) $\text{Max } Z = x_1 - x_2$

Sub.to $2x_1 + x_2 = 5$

$$3x_1 - x_2 = 6$$

x_1, x_2 unrestricted

Example:- write the dual of the primal problem given and solve the both and interpret the results.

1-Primal problem:

$$\text{Max } Z = 5x_1 + 20x_2$$

s.t $5x_1 + 2x_2 \leq 20$

$$x_1 + 2x_2 \leq 8$$

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$$x_1 + 6x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

The standard form:

$$\text{Max } Z = 5x_1 + 20x_2 + 0s_1 + 0s_2 + 0s_3$$

$$\text{s.t } 5x_1 + 2x_2 + s_3 = 20$$

$$x_1 + 2x_2 + s_2 = 8$$

$$x_1 + 6x_2 + s_3 = 12$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Now we solve the primal problem by using simplex method:

Table (1)

| B.V | X1 | X2 | S1 | S2 | S3 | R.H.S |
|-----|----|-----|----|----|----|-------|
| Z | -5 | -20 | 0 | 0 | 0 | 0 |
| S1 | 5 | 2 | 1 | 0 | 0 | 20 |
| S2 | 1 | 2 | 0 | 1 | 0 | 8 |
| S3 | 1 | (6) | 0 | 0 | 1 | 12 |

Table (2)

| B.V | X1 | X2 | S1 | S2 | S3 | R.H.S |
|-----|--------|----|----|----|------|-------|
| Z | -10/6 | 0 | 0 | 0 | 10/3 | 40 |
| S1 | (14/3) | 0 | 1 | 0 | -1/3 | 16 |
| S2 | 2/3 | 0 | 0 | 1 | -1/3 | 4 |
| X2 | 1/6 | 1 | 0 | 0 | 1/6 | 2 |

Table (3)

| B.V | X1 | X2 | S1 | S2 | S3 | R.H.S |
|-----|----|----|------|----|-------|-------|
| Z | 0 | 0 | 5/14 | 0 | 45/14 | 300/7 |
| X1 | 1 | 0 | 3/14 | 0 | -1/14 | 24/7 |

| | | | | | | |
|-----------|----------|----------|--------------|----------|-------------|-------------|
| S3 | 0 | 0 | -1/7 | 1 | -2/7 | 12/7 |
| X2 | 0 | 1 | -1/28 | 0 | 5/28 | 10/7 |

Hence the solution in tableau is optimal: $x_1=24/7$, $x_2=10/7$, $Z=300/7$

Now let us solve the dual of the above.

2- Dual of the given problem:

$$\text{Min } W=20y_1+8y_2+12y_3$$

I.e. $\text{Max } Z=-20y_1-8y_2-12y_3$

Sub.to $5y_1+y_2+y_3 \geq 5$

$$2y_1+2y_2+6y_3 \geq 20$$

$$y_1, y_2, y_3 \geq 0$$

To solve this problem we use Big-M method:

The standard form is:

$$\text{Max } Z=-20y_1-8y_2-12y_3+0s_1+0s_2-MR_1-MR_2$$

s.t $5y_1+y_2+y_3-s_1+R_1=5$

$$2y_1+2y_2+6y_3-s_2+R_2=20$$

$$y_1, y_2, y_3, s_1, s_2, R_1, R_2 \geq 0$$

Then we solve it by using simplex tableau to find the optimal solution.

Lecture 8

Sensitivity or post optimal analysis

You will discover the following:

1- Infeasibility of the current solution can arise only if we change the availability of resources (right side of constraints) and / or add new constraints.

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2- Non optimality of the current solution can occur only if we change objective function and/or certain elements of the left sides of constraints. It can also occur if a new activity is added to the model.

Based on the discussion above, the general procedure for carrying out sensitivity analysis can be summarized as follows:

Step (1): solve the original Lp model and obtain its optimal simplex tableau, go to step2.

Step (2): for the proposed change(s) in the original model, recomputed the new elements of the current optimal tableau by using the primal-dual computations. Go to step 3.

Step (3): if the new tableau is nonoptimal, go to step 4. If it is infeasible, go to step5, otherwise, record the solution in the new tableau as the new optimum, and stop.

Step (4): apply the regular simplex method to the new tableau to obtain a new optimal solution stop.

Step (5): apply the dual simplex method to the new tableau to recover feasibility (or indicate that no feasible solution exists).stop.

To summarize the sensitivity analysis includes:

(A) Changes affecting feasibility:

- 1) Changes in the right hand side of constraints.
- 2) Addition of a new constraint.

(B) Changes affecting optimality:

- 1) Change in the objective function.
- 2) Change in the activities of resources.
- 3) Addition a new activity.

(A) Changes affecting feasibility:

- (1) Changes in the Right side of constraints (bi).**

Example (1):- let the following L.p.p:

$$\begin{aligned} \text{Max } Z &= 3x_1 + 2x_2 \\ \text{s.t } \quad x_1 + 2x_2 &\leq 6 \\ 2x_1 + x_2 &\leq 8 \\ -x_1 + x_2 &\leq 1 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

And the primal optimal tableau is:

| B.V | X1 | X2 | S1 | S2 | S3 | S4 | R.H.S |
|-----|----|----|------|------|----|----|-------|
| Z | 0 | 0 | 1/3 | 4/3 | 0 | 0 | 38/3 |
| X2 | 0 | 1 | 2/3 | -1/3 | 0 | 0 | 4/3 |
| X1 | 1 | 0 | -1/3 | 2/3 | 0 | 0 | 10/3 |
| S3 | 0 | 0 | -1 | 1 | 1 | 0 | 3 |
| S4 | 0 | 0 | -2/3 | 1/3 | 0 | 1 | 2/3 |

If we change the right hand side of the first constraint from (6) to (7).How the current solution affected?

Solution:

$$\begin{pmatrix} x_2 \\ x_1 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

Since the new right-side elements remain nonnegative, the current basic variables remain unchanged. Only their new values become: $x_1=3$, $x_2=2$ the new values of $Z=3(3) +2(2) =13$.

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Let us now consider an example of what happens when the current basic variables becomes infeasible, suppose that the right

side changed from $\begin{pmatrix} 6 \\ 8 \\ 1 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 7 \\ 4 \\ 1 \\ 2 \end{pmatrix}$, then the right side of the tableau is computed as follows :

$$\begin{pmatrix} x_2 \\ x_1 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 1/3 \\ -2 \\ -4/3 \end{pmatrix}$$

Now we use simplex tableau since there are negative values in the right side

| B.V | X₁ | X₂ | S₁ | S₂ | S₃ | S₄ | RHS |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|------------|
| Z | 0 | 0 | 1/3 | 4/3 | 0 | 0 | 23/3 |
| X₂ | 0 | 1 | 2/3 | -1/3 | 0 | 0 | 10/3 |
| X₁ | 1 | 0 | -1/3 | 2/3 | 0 | 0 | 1/3 |
| S₃ | 0 | 0 | (-1) | 1 | 1 | 0 | -2 |
| S₄ | 0 | 0 | -2/3 | 1/3 | 0 | 1 | -4/3 |

The solution in the above table is optimal since $(Z-C_j) \geq 0$ but infeasible (at least one basic variable is negative). Thus we must use dual simple method to recover feasibility, the application of the dual simplex method shows that the leaving and entering variables are S_3 and S_1 .this leads to the following tableau:

| B.V | X₁ | X₂ | S₁ | S₂ | S₃ | S₄ | R.H.S |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|--------------|
| Z | 0 | 0 | 0 | 5/3 | 1/3 | 0 | 7 |
| X₂ | 0 | 1 | 0 | 1/3 | 2/3 | 0 | 2 |
| X₁ | 1 | 0 | 0 | 1/3 | -1/3 | 0 | 1 |
| S₁ | 0 | 0 | 1 | -1 | -1 | 0 | 2 |
| S₄ | 0 | 0 | 0 | -1/3 | -2/3 | 1 | 0 |

The above tableau is both optimal and feasible. The new solution is:

$$x_1 = 1, x_2 = 2, Z = 7$$

(2) Addition of a new Constraint.

The addition of a new constraint can result in one of two conditions:

- 1-The constraint is satisfied by the current solution, in which case the constraint is either nonbinding or redundant and its addition will thus not change the solution.
- 2-The constraint is not satisfied by the current solution. It will thus become binding and a new solution is obtained by using the dual simplex method

Example (2):

If we add a new constraint $x_1 \leq 4$ to the model problem in example (1). Since the current solution ($x_1 = 10/3, x_2 = 4/3$) obviously satisfies the new constraint, it is nonbinding and the current solution remains unchanged.

Now suppose that the new constraint is $x_1 \leq 3$ add in the problem in example (1), which is not satisfied by the current solution $x_1 = 10/3$, and $x_2 = 4/3$.

Here is what we do to recover feasibility. First, put the new constraint in the standard form by adding a slack variable or surplus variable if necessary.

Using S_5 as a slack, we find that the standard form of $x_1 \leq 3$ is $x_1 + S_5 = 3, S_5 \geq 0$, and add to the optimal table :

| B.V | X₁ | X₂ | S₁ | S₂ | S₃ | S₄ | S₅ | R.H.S |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|--------------|
| Z | 0 | 0 | 1/3 | 4/3 | 0 | 0 | 0 | 38/3 |
| X₂ | 0 | 1 | 2/3 | -1/3 | 0 | 0 | 0 | 4/3 |
| X₁ | 1 | 0 | -1/3 | 2/3 | 0 | 0 | 0 | 10/3 |
| S₃ | 0 | 0 | -1 | 1 | 1 | 0 | 0 | 3 |
| S₄ | 0 | 0 | -2/3 | 1/3 | 0 | 1 | 0 | 2/3 |
| S₅ | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 3 |

In the x_1 - equation of the current optimal tableau, we have:

$$x_1 - (1/3)S_1 + (2/3)S_2 = 10/3$$

Thus the new constraint expressed in terms of the current non basic variables becomes:

$$(10/3) + (1/3) S_1 - (2/3)S_2 = 3 - S_5$$

$$\Rightarrow (10/3) + (1/3)S_1 - (2/3)S_2 + S_5 = 3$$

$\therefore (1/3)S_1 - (2/3)S_2 + S_5 = (-1/3)$ We add this equation in the optimal tableau

| B.V | X ₁ | X ₂ | S ₁ | S ₂ | S ₃ | S ₄ | S ₅ | R.H.S |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|-------|
| Z | 0 | 0 | 1/3 | 4/3 | 0 | 0 | 0 | 38/3 |
| X ₂ | 0 | 1 | 2/3 | -1/3 | 0 | 0 | 0 | 4/3 |
| X ₁ | 1 | 0 | -1/3 | 2/3 | 0 | 0 | 0 | 10/3 |
| S ₃ | 0 | 0 | -1 | 1 | 1 | 0 | 0 | 3 |
| S ₄ | 0 | 0 | -2/3 | 1/3 | 0 | 1 | 0 | 2/3 |
| S ₅ | 0 | 0 | 1/3 | -2/3 | 0 | 0 | 1 | -1/3 |

By using dual simplex method the optimal tableau is:

| B.V | X ₁ | X ₂ | S ₁ | S ₂ | S ₃ | S ₄ | S ₅ | R.H.S |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|-------|
| Z | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 12 |
| X ₂ | 0 | 1 | 1/2 | 0 | 0 | 0 | -1/2 | 3/2 |
| X ₁ | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 3 |
| S ₃ | 0 | 0 | -1/2 | 0 | 1 | 0 | 3/2 | 5/2 |
| S ₄ | 0 | 0 | -1/2 | 0 | 0 | 1 | 1/2 | 1/2 |
| S ₂ | 0 | 0 | -1/2 | 1 | 0 | 0 | -3/2 | 1/2 |

Lecture 9

(B) Change Affecting optimality:

The current solution will be not optimal only if the coefficients of the objective equation violate the optimality condition.

(1) Changes in the objective function:

1-If the changes in the objective function involve the coefficients of a current basic variable, determine the new dual values and then use them to recompute the new Z-equation coefficients.

2-If the changes involve non basic variables only, use the current dual values (directly from the current tableau) and recomputed the Z-equation coefficients of the involved non basic variables only. No other changes will occur in the tableau.

Example(3): Suppose that the objective function in example (1) is changed from

$$Z=3x_1 + 2x_2 \text{ to } Z=5x_1 + 4x_2$$

the primal model:

$$\begin{aligned} \text{Max } Z &= 5x_1 + 4x_2 \\ \text{s.t } & x_1 + 2x_2 \leq 6 \\ & 2x_1 + 2x_2 \leq 8 \\ & -x_1 + x_2 \leq 1 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The dual model:

$$\begin{aligned} \text{Min } W &= 6y_1 + 8y_2 + y_3 + 2y_4 \\ \text{Sub to } & y_1 + 2y_2 - y_3 \geq 5 \\ & 2y_1 + y_2 + y_3 + y_4 \geq 4 \\ & y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

The changes involve both x_1 and x_2 which happen to be basic in the current solution. Thus we must determine the new dual values. Notice that the order of the basic variables in the current tableau is (x_2, x_1, S_3, S_4)

$$(y_1, y_2, y_3, y_4) = (4, 5, 0, 0)^* \begin{pmatrix} 2/3 & -1/3 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 1/3 & 0 & 1 \end{pmatrix} = (1, 2, 0, 0)$$

The next step is to recompute the Z-equation coefficients by taking the difference between the left and right sides of the dual constraints.

$$\begin{aligned} x_1 - \text{coefficient} &= y_1 + 2y_2 - y_3 - 5 \\ &= 1(1) + 2(2) - 0 - 5 = 0 \end{aligned}$$

$$\begin{aligned} x_2 - \text{coefficient} &= 2y_1 + y_2 + y_3 + y_4 - 4 \\ &= 2(1) + 2 + 0 + 0 - 4 = 0 \end{aligned}$$

$$s_1 - \text{coefficient} = y_1 - 0 = 1 - 0 = 1$$

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$$s_2 - \text{coefficient} = y_2 - 0 = 2 - 0 = 2$$

$$s_3 - \text{coefficient} = y_3 - 0 = 0 - 0 = 0$$

$$s_4 - \text{coefficient} = y_4 - 0 = 0 - 0 = 0$$

$\therefore Z - C_j \geq 0$, the changes indicated in the objective function will not change the optimum variables or their values. The only change is the value of $Z = 5 * (10/3) + 4 * (4/3) = 22$, where $(x_1 = 10/3, x_2 = 4/3)$ in the primal optimal tableau in example (1).

Example(4): Suppose that the objective function change from:

$$Z = 3x_1 + 2x_2 \quad \text{to} \quad Z = 4x_1 + x_2$$

$$\therefore (y_1, y_2, y_3, y_4) = (1, 4, 0, 0) * \begin{pmatrix} 2/3 & -1/3 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 1/3 & 0 & 1 \end{pmatrix}$$

$$= (-2/3, 7/3, 0, 0)$$

Since (S_1) has a negative coefficient, S_1 must enter the solution and optimality is recovered by applying the regular simplex method.

| B.V | X₁ | X₂ | S₁ | S₂ | S₃ | S₄ | R.H.S |
|------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|--------------|
| Z | 0 | 0 | -2/3 | 7/3 | 0 | 0 | 44/3 |

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| | | | | | | | |
|----------------------|---|---|------|------|---|---|------|
| X₂ | 0 | 1 | 2/3 | -1/3 | 0 | 0 | 4/3 |
| X₁ | 1 | 0 | -1/3 | 2/3 | 0 | 0 | 10/3 |
| S₃ | 0 | 0 | -1 | 1 | 1 | 0 | 3 |
| S₄ | 0 | 0 | -2/3 | 1/3 | 0 | 1 | 2/3 |

| B.V | X₁ | X₂ | S₁ | S₂ | S₃ | S₄ | R.H.S |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|--------------|
| Z | 0 | 1 | 0 | 2 | 0 | 0 | 16 |
| S₁ | 0 | 3/2 | 1 | -1/2 | 0 | 0 | 2 |
| X₁ | 1 | 1/2 | 0 | 1/2 | 0 | 0 | 4 |
| S₃ | 0 | 3/2 | 0 | 1/2 | 1 | 0 | 5 |
| S₄ | 0 | 1 | 0 | 0 | 0 | 1 | 2 |

Since $Z - C_j \geq 0$, (R.H.S +v), the optimal solution is : $Z=16, x_1=4, x_2=0$

And the dual $(y_1, y_2, y_3, y_4) = (0, 2, 0, 0)$ (row of objective function)

(2) Changes in activity's usage of Resources (a_{ij}):

A change in an activity's usage of resources (a_{ij}) can affect only the optimality of the solution; since it affects the left side of its dual constraint .However we must restrict this statement to activities that are currently non basic. A change in the constraint coefficients of the basic activities will affect the inverse and could lead to complications in the computations

Example: Suppose that the activity (x₂) of the first and second constraints is

changed from $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ in the following problem:

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Primal
 Max $Z=4x_1+x_2$
 s.t $x_1+2x_2\leq 6$
 $2x_1+x_2\leq 8$
 $-x_1+x_2\leq 1$
 $x_2\leq 2$
 $x_1,x_2\geq 0$

New primal
 Max $Z=4x_1+x_2$
 s.t $x_1+4x_2\leq 6$
 $2x_1+3x_2\leq 8$
 $-x_1+x_2\leq 1$
 $x_2\leq 2$
 $x_1,x_2\geq 0$

Dual
 Min $W=6y_1+8y_2+y_3+2y_4$
 Sub to $y_1+2y_2-y_3\geq 4$
 $4y_1+3y_2+y_3+y_4\geq 1$
 $y_1,y_2,y_3,y_4\geq 0$

The associated dual constraint is $4y_1 + 3y_2 + y_3 + y_4 \geq 1$ since the objective function remains unchanged, the dual values in the primal optimal tableau in example(1) is:

$(0, 2, 0, 0) = (y_1, y_2, y_3, y_4)$

New x_2 –coefficient = $4(0) + 3(2) + 1(0) + 1(0) - 1 = 5 \geq 0$

Hence the proposed change does not affect the optimum solution

(3) Addition of a new Activity (x_i):

We can think of adding a new activity as a non basic activity that started originally in the model with all zero coefficients in the objective and constraints.

Example: suppose the original problem:

Max $Z=3x_1+2x_2$
 s.t $x_1+2x_2\leq 6$
 $2x_1+x_2\leq 8$
 $-x_1+x_2\leq 1$
 $x_2\leq 2$
 $x_1,x_2\geq 0$

And the primal optimal tableau is:

| B.V | X1 | X2 | S1 | S2 | S3 | S4 | R.H.S |
|-----|----|----|------|------|----|----|-------|
| Z | 0 | 0 | 1/3 | 4/3 | 0 | 0 | 38/3 |
| X2 | 0 | 1 | 2/3 | -1/3 | 0 | 0 | 4/3 |
| X1 | 1 | 0 | -1/3 | 2/3 | 0 | 0 | 10/3 |
| S3 | 0 | 0 | -1 | 1 | 1 | 0 | 3 |
| S4 | 0 | 0 | -2/3 | 1/3 | 0 | 1 | 2/3 |

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Let we add the new activity x_3 to the objective function and the constraints as follows:

$$\text{Max } Z=3x_1+2x_2+(3/2)x_3$$

$$\text{s.t } x_1+2x_2+(3/4)x_3 \leq 6$$

$$2x_1+x_2+(3/4)x_3 \leq 8$$

$$-x_1+x_2-x_3 \leq 1$$

$$x_2 \leq 2$$

$$x_1, x_2, x_3 \geq 0$$

The addition of a new activity is equivalent to combining the analysis of making changes in the objective and the resource usages the first thing to do is to check the corresponding dual constraint:

$$(3/4)y_1 + (3/4)y_2 - y_3 \geq 3/2$$

Since x_3 is regarded as a non basic variable in the original tableau, the dual values remain unchanged, thus the coefficients of x_3 in the current optimal tableau is:

$$(3/4)(1/3)+(3/4)(4/3)-1(0)-3/2=-1/4 \quad \text{coefficient of } x_3 \text{ in Z-equation}$$

Where $(y_1, y_2, y_3)=(1/3, 4/3, 0)$ in the optimal tableau in example (1)

Then we find the coefficients of x_3 in the rows of tableau (in constraints) as follows:

$$\begin{pmatrix} 3/2 & -1/3 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -2/3 & 1/3 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 3/4 \\ 3/4 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \\ -1 \\ -1/4 \end{pmatrix}$$

Since the coefficient of x_3 is negative $(-1/4)$, we use simplex tableau to find the optimal solution as follows:

| B.V. | X₁ | X₂ | X₃ | S₁ | S₂ | S₃ | S₄ | R.H.S |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|--------------|
| Z | 0 | 0 | -1/4 | 1/3 | 4/3 | 0 | 0 | 38/3 |
| X₂ | 0 | 1 | 1/4 | 2/3 | -1/3 | 0 | 0 | 4/3 |
| X₁ | 1 | 0 | 1/4 | -1/3 | 2/3 | 0 | 0 | 10/3 |
| S₃ | 0 | 0 | -1 | -1 | 1 | 1 | 0 | 3 |
| S₄ | 0 | 0 | -1/4 | -2/3 | 1/3 | 0 | 1 | 2/3 |

| B.V. | X₁ | X₂ | X₃ | S₁ | S₂ | S₃ | S₄ | R.H.S |
|-------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|--------------|
| Z | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 14 |

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| | | | | | | | | |
|-------|---|----|---|-------|--------|---|---|--------|
| X_3 | 0 | 4 | 1 | $8/3$ | $-4/3$ | 0 | 0 | $16/3$ |
| X_1 | 1 | -1 | 0 | -1 | 1 | 0 | 0 | 2 |
| S_3 | 0 | 4 | 0 | $5/3$ | $-1/3$ | 1 | 0 | $25/3$ |
| S_4 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 2 |

The new optimal solution is: $Z = 14$, $x_1 = 2$, $x_2 = 0$, $x_3 = 16/3$

Lecture 10

Transportation model

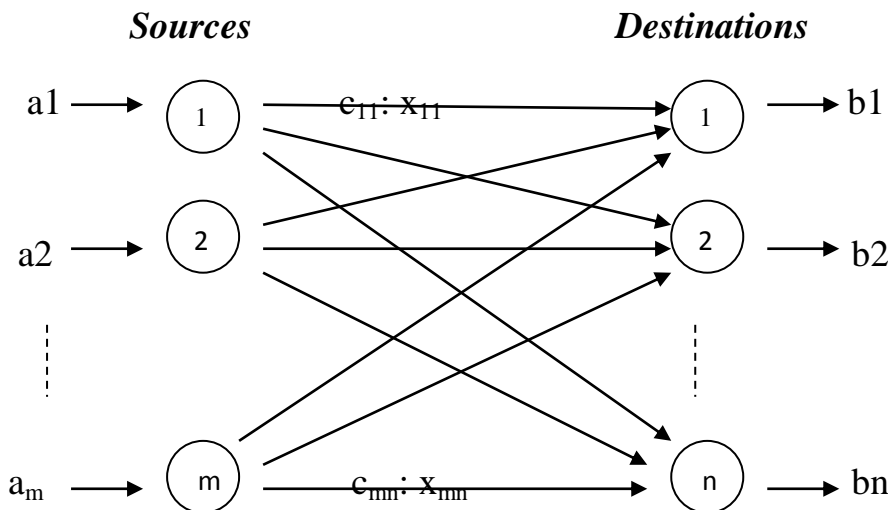
The transportation model is basically a linear program that can be solved by the regular simplex method. However, its special structure allows the development of a solution procedure, called the transportation technique that is computationally more efficient. It can be extended to cover a number of important applications, including the assignment model, the transshipment model, the transportation problem and its extensions are also special cases of network model.

Definition:

The transportation model seeks the determination of a transportation plan of a single commodity from a number of sources to a number of destinations the data of the model include:

1. Level of supply at each source and amount of demand at each destination.
2. The unit transportation cost of the commodity from each source to each destination.

Since there is only one commodity, a destination can receive its demand from one or more sources .The objective of the model is to determine the amount to be shipped from each source to each destination such that total transportation cost is minimized.



The figure above depicts the transportation model (T.M) as a network with m sources and n destinations. A source or a destination is represented by a nod. The arc joining a source and a destination represents the rote through which the commodity is transported. The amount of supply at source i is a_i and the demand at destination j is b_j . The unit transportation cost between source i and destination j is c_{ij} .

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Let x_{ij} represent the amount transported from source i to destination j , then the LP model representing the transportation problem is given generally as:

$$\text{Min } Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

Sub.to

$$\sum_{j=1}^n x_{ij} \leq a_i \quad i = 1, 2, \dots, m$$

$$\sum_{i=1}^m x_{ij} \geq b_j \quad j = 1, 2, \dots, n$$

$$x_{ij} \geq 0 \quad \text{for all } i, j$$

The first set of constraints stipulates that the sum of the shipments from a source cannot exceed its supply; similarly, the second set requires that the sum of the shipments to a destination must satisfy its demand.

The model described above implies that the total supply $\sum_{i=0}^m a_i$ must least equal total demand $\sum_{j=0}^n b_j$.

If $\sum_{i=0}^m a_i = \sum_{j=0}^n b_j$, the resulting formulation is called *balanced* transportation model.

It differs from the model above only in the fact that all constraints are equations; that is:

$$\sum_{j=1}^n x_{ij} = a_i \quad i=1, 2, \dots, m$$

$$\sum_{i=1}^m x_{ij} = b_j \quad j=1, 2, \dots, n$$

In real life it is not necessarily true that supply equal demand or, for that matter, exceed it, however, a T.M can always be balanced. The balancing, in addition to its usefulness in modeling certain practical situations, is important for the development of a solution method that fully exploits the special structure of the T.M.

Example: (standard T.M)

The MG Auto Company has plants in Los Angeles, Detroit, and New Orleans. Its major distribution centers are located in Denver and Miami .The capacities of the three plants

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during the next quarter are 1000, 1500 and 1200 cars. The quarterly demands at the two distribution centers are 2300 and 1400 cars. The train transportation cost per car per mile is approximately 8 cents. The mileage chart between the plants and distribution centers is as follows:

| | <i>Denver</i> | <i>Miami</i> |
|--------------------|---------------|--------------|
| <i>Los Angeles</i> | 1000 | 2690 |
| <i>Detroit</i> | 1250 | 1350 |
| <i>New Orleans</i> | 1275 | 850 |

The mileage chart can be translated to cost per car at the rate of 8 cents per mile. This yields the following costs which represent c_{ij} in the general model:

| | <i>Denver</i> | <i>Miami</i> |
|--------------------|---------------|--------------|
| <i>Los Angeles</i> | 80 | 215 |
| <i>Detroit</i> | 100 | 108 |
| <i>New Orleans</i> | 102 | 68 |

Using numeric codes to represent the plants and distribution centers, we let x_{ij} represent the number of cars transported from source i to destination j .

Hence the following Lp model presenting the problem has all equality constraints:

$$\text{Min } Z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

$$\text{Sub to } \quad x_{11} + x_{12} \quad \quad \quad = 1000$$

$$\quad \quad \quad + x_{21} + x_{22} \quad \quad \quad = 1500$$

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$$\begin{aligned}
 & +X_{31} + X_{32} = 1200 \\
 X_{11} & + X_{21} + X_{31} = 2300 \\
 X_{12} & + X_{22} + X_{32} = 1400 \\
 & x_{ij} \geq 0 \quad \text{for all } i, j
 \end{aligned}$$

and the transportation table is as follows:

| | <i>Denver</i> | <i>Miami</i> | supply |
|--------------------|-----------------|-----------------|---------------|
| <i>Los Angeles</i> | 80 X_{11} | 215 X_{21} | 1000 |
| <i>Detroit</i> | 100 X_{21} | 108 X_{22} | 1500 |
| <i>New Orleans</i> | 102 X_{31} | 68 X_{32} | 1200 |
| Demand | 2300 | 1400 | |

Since the total supply (1000+1500+1200=3700) is equal the total demand (2300+1400=3700), the resulting T.M is balanced.

Example (2):-

Suppose that the Detroit plant capacity is 1300 cars (instead of 1500).the situation is said to be unbalanced because the total supply=3500 dose not equal the total demand=3700.

| | <i>Denver</i> | <i>Miami</i> | supply |
|--------------------|-----------------|-----------------|---------------|
| <i>Los Angeles</i> | 80 X_{11} | 215 X_{21} | 1000 |
| <i>Detroit</i> | 100 X_{21} | 108 X_{22} | 1300 |
| <i>New Orleans</i> | 102 X_{31} | 68 X_{32} | 1200 |

Demand 2300 1400

Stated differently, this unbalanced situation means that it will not be possible to fill all the demand at the distribution centers.

Our objective is to reformulate the T.M in a manner that will distribute the shortage quantity (3700-3500=200 cars) optimally among the distribution centers. Since demand exceeds supply, a dummy source can be added with its capacity equal to 200 Cars. The following table summarizes the balanced model under the new Capacity restriction of the Detroit Plant.

| | <i>Denver</i> | <i>Miami</i> | supply |
|--------------------|---------------|--------------|---------------|
| <i>Los Angeles</i> | X_{11} 80 | X_{21} 215 | 1000 |
| <i>Detroit</i> | X_{21} 100 | X_{22} 108 | 1300 |
| <i>New Orleans</i> | X_{31} 102 | X_{32} 68 | 1200 |
| <i>Dummy plant</i> | X_{41} 0 | X_{42} 0 | 200 |
| Demand | 2300 | 1400 | |

In a similar manner, if the supply exceeds the demand, we can add a dummy destination that will absorb the difference. For example suppose in the above example that the demand at Denver drops to (1900) Cars. The following table summarizes the model with the dummy distribution center.

| | <i>Denver</i> | <i>Miami</i> | <i>dummy</i> | supply |
|--------------------|---------------|--------------|--------------|---------------|
| <i>Los Angeles</i> | X_{11} 80 | X_{21} 215 | X_{31} 0 | 1000 |
| <i>Detroit</i> | X_{21} 100 | X_{22} 108 | X_{32} 0 | 1500 |

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| | | | | |
|--------------------|-------------------------|------------------------|-----------------------|-------------|
| <i>New Orleans</i> | X_{31} ¹⁰² | X_{32} ⁶⁸ | X_{33} ⁰ | 1200 |
|--------------------|-------------------------|------------------------|-----------------------|-------------|

Demand **1900** **1400** **400**

Lecture 11

-Solution of the Transportation Problem

In this section we introduce the details for solving the T.M the method uses the steps of the simplex method directly and differs only in the details of implementing the optimality and feasibility conditions.

-Transportation Technique

The basic steps of the Transportation Technique are:

Step (1): Determine a starting feasible solution.

Step (2): Determine an entering variable from among the non basic variables. If all such variables satisfy the optimality condition (of the simplex method), hence obtain the optimal solution .otherwise go to step3.

Step (3): Determine a leaving variable (using the feasibility condition) form among the variables of the current basic solution, and then find the new basic solution. Return to step2.

-Methods to finding the basic feasible solution:

- 1) Northwest- Corner method.
- 2) Last Cost method.
- 3) Vogel's approximation method.

1-Northwest-Corner Method:

Example:

| | A | B | C | D | Supply |
|---|----------------|---------------|----------------|----------------|--------|
| I | 10 X_{11} | 0 X_{12} | 20 X_{13} | 11 X_{14} | 15 |

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| | | | | | |
|---------------|----------|-----------|-----------|-----------|-----------|
| II | 12 | 7 | 9 | 20 | 25 |
| | X_{21} | X_{22} | X_{23} | X_{24} | |
| III | 0 | 14 | 16 | 18 | 5 |
| | X_{31} | X_{32} | X_{33} | X_{34} | |
| Demand | 5 | 15 | 15 | 10 | |

The northwest-corner method starts by allocating the maximum amount allowable by the supply and demand to the variable x_{11} (the northwest-corner of the tableau). The satisfied column(row) is then crossed out, indicating that the remaining variables in the crossed-out column (row) equal zero. If a column and a row are satisfied simultaneously, only one (either one) may be crossed out. The process is completed when exactly one row or one column is left uncrossed out. The procedure described above is now applied to the above example as follows:-

A B C D Supply

| | | | | | |
|---------------|--------------|---------------|---------------|---------------|--|
| | 10 | 0 | 20 | 11 | |
| I | 5 | 10 | | | 15 10 0 |
| II | 12 | 7 | 9 | 20 | 25 20 5 0 |
| III | 0 | 14 | 16 | 18 | 5 0 |
| Demand | 5 | 15 | 15 | 10 | |

And the associated transportation cost is:

$$\text{Total cost} = 5 \times 10 + 5 \times 7 + 15 \times 9 + 5 \times 20 + 5 \times 18 = 410\$$$

2-The Least-Cost Method

The procedure is as follows. Assign as much as possible to the variable with the smallest unit cost in the entire tableau. Cross out the satisfied row or column. (As in the northwest-corner method, if both a column and row are satisfied simultaneously, only one may be crossed out).

After adjusting the supply and demand for all uncrossed-out rows and columns, repeat the process by assigning as much as possible to the variable with the smallest uncrossed-out unit cost. The procedure is complete when exactly one row or one column is left uncrossed-out.

Example:-

A B C D Supply

| | | | | | |
|--------|--------------|---------------|---------------|---------------|-------------------------------|
| | 10 | 0 | 20 | 11 | |
| I | 0 | 15 | | 0 | 15 0 |
| II | 12 | 7 | 9 | 20 | 25 10 0 |
| III | 0 | 14 | 16 | 18 | 5 0 |
| Demand | 5 | 15 | 15 | 10 | |

The total cost = $15 \times 0 + 15 \times 9 + 10 \times 20 + 5 \times 0 = 335\$$

3-Vogel's Approximation Method (VAM)

This method is a heuristic and usually provides a better starting solution than the two methods described above. In fact, VAM generally yields an optimum, or close to optimum, starting solution.

-The steps of the VAM:

Step (1): Evaluate a penalty for each row (column) by subtracting the smallest cost element in the row (column) from the next smallest cost element in the same row (column).

Step (2): Indent the row or column with the largest penalty, breaking ties arbitrarily. Allocate as much as possible to the variable with the least cost in the selected row or column. Adjust the supply and demand and cross out the satisfied row or column. If

a row and a column are satisfied simultaneously, only one of them is crossed out and the remaining row(column) is assigned a Zero supply (demand).

Any row or column with zero supply or demand should not be used in computing future penalties (in step 3).

Step (3):

(a) If exactly one row or one column remains uncrossed out, stop.

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(b) If only one row (column) with positive supply (demand) remains uncrossed out, determine the basic variable in the row (column) by the least-cost method.

(c) If all uncrossed out rows and columns have (assigned) zero supply and demand, determine the zero basic variables by the least-cost method .stop.

(d) Otherwise, recomputed the penalties for the uncrossed-out rows and columns then go to (step 2). (Notice that the rows and columns with assigned zero supply and demand should not be used in computing these penalties).

Example:- Solve the following (T.M) using VAM .

| | A | B | C | supply |
|---------------|----------------|----------------|-----------------|---------------|
| I | 90 X_{11} | 80 X_{12} | 100 X_{13} | 1000 |
| II | 20 X_{21} | 40 X_{22} | 50 X_{23} | 1900 |
| III | 40 X_{31} | 90 X_{32} | 60 X_{33} | 1600 |
| Demand | 700 | 2000 | 1800 | 4500 |
| | | | | 4500 |

Solution:

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| | A | B | C | Supply | panalty1 | panalty2 | panalty3 |
|--------|------------|-------------|-------------|-------------|----------|----------|----------|
| I | X 90 | 100 80 | 900 100 | 1000 | 10 | 10 | 20 |
| II | X 20 | 1900 40 | X 50 | 1900 | 20 | - | - |
| III | 700 40 | X 90 | 900 60 | 1600 | 20 | 20 | 30 |
| demand | 700 | 2000 | 1800 | | | | |
| | 20 | (40) | (10) | | | | |
| | (50) | 10 | 40 | | | | |
| | - | 10 | (40) | | | | |

The total cost is= $100*80+900*100+1900*40+700*40+900*60= (256000)$

Lecture 12

-Finding the optimal solution of (T.M)

1-Determination of entering variable (method of multiplier)

The entering variable is determined by using the optimality condition of simplex method. Another method, called the stepping-stone procedure is also a variable for determining the entering variable. In the method of multipliers we associate the multiplier u_i and v_j with row i and column j of the transportation tableau. For each basic variables x_{ij} in the current solution, the multipliers u_i and v_j must satisfy the following equation:

$$u_i + v_j = c_{ij} \quad \text{for each basic variable } x_{ij}$$

These equations yield $(m+n-1)$ equations (because there are only $(m+n-1)$ basic variable) in $(m+n)$ unknowns.

The values of multipliers can be determined from these equation by assuming an arbitrary values for any one of the multipliers ($u_1=zero$) and the solving the $(m+n-1)$ equations in the remaining $(m+n-1)$ unknown multipliers.

The non basic variable x_{pq} is given by:

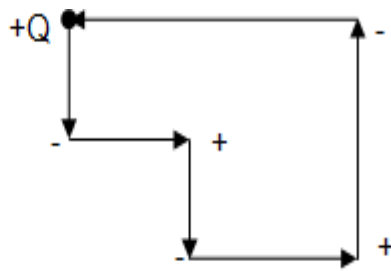
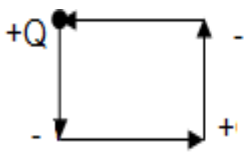
$$\bar{C}_{pq} = u_p + v_q - c_{pq} \quad \text{for each non basic variable } x_{pq}.$$

Then the entering variable is then selected as the non basic variable with the (most positive \bar{C}_{pq}) compare with the minimum optimality condition of simplex method.

2-Determination of leaving variable (Loop construction).

This step is equivalent to applying the feasibility condition in simplex method. For the purpose of determining the minimum ratio, we construct a closed Loop for the current entering variable. The Loop starts and ends at the designated nonbasic variable. It consists of successive horizontal and vertical (connected) segments whose end points must be basic variables, except for the end points that are associated with the entering variable. This means that every corner element of the loop must be a cell containing a basic variable. For example:

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Example: Find the optimal solution to the following T.M :

| | | Destination | | | | supply |
|--------|-----|-------------|----|----|----|--------|
| | I | 10 | 0 | 20 | 11 | 15 |
| | II | 12 | 7 | 9 | 20 | 25 |
| | III | 0 | 14 | 16 | 18 | 5 |
| Demand | | 5 | 15 | 15 | 10 | |

Solution:

By applying North-west corner method the starting solution is as follows:

| | | | | |
|-------------------|------------------------------------|--------------------|------------------------------------|---|
| 5 / 10 | 10 / 0 | X / 20 | X / 11 | 15 10 0 |
| X / 12 | 5 / 7 | 15 / 9 | 5 / 20 | 25 20 50 |
| X / 0 | X / 14 | X / 16 | 5 / 18 | 5 0 |
| 5 0 | 15 5 0 | 15 0 | 10 5 0 | |

Now we find the optimal solution:

1) For each basic variable x_{ij} : $c_{ij} = u_i + v_j$

Let $u_1=0$

$$\rightarrow c_{11} = u_1 + v_1 = 10 \rightarrow v_1 = 10$$

$$c_{12} = u_1 + v_2 = 0 \rightarrow v_2 = 0$$

$$c_{22} = u_2 + v_2 = 7 \rightarrow u_2 = 7$$

$$c_{23} = u_2 + v_3 = 9 \rightarrow v_3 = 2$$

$$c_{24} = u_2 + v_4 = 20 \rightarrow v_4 = 13$$

$$c_{34} = u_3 + v_4 = 18 \rightarrow u_3 = 5$$

2) For each non basic variable: $\bar{C}_{pq} = u_p + v_q - c_{pq}$

$$\rightarrow \bar{C}_{13} = u_1 + v_3 - c_{13} = 0 + 2 - 20 = -18$$

$$\bar{C}_{14} = u_1 + v_4 - c_{14} = 0 + 13 - 11 = 2$$

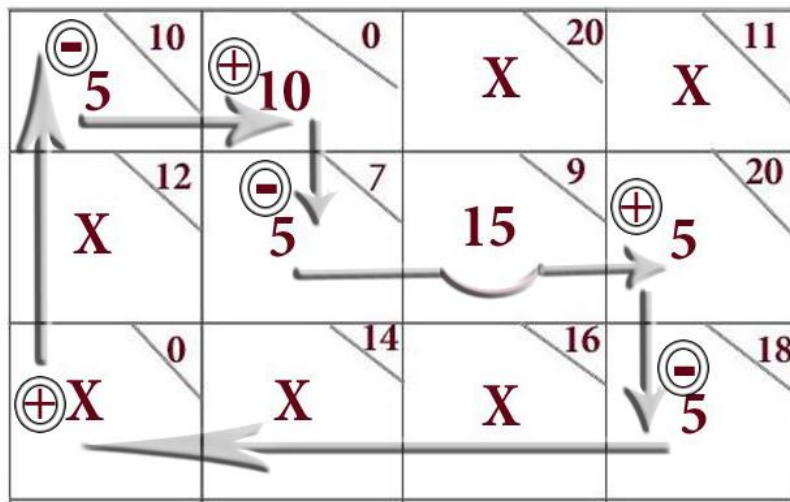
$$\bar{C}_{21} = u_2 + v_1 - c_{21} = 7 + 10 - 12 = 5$$

$$\bar{C}_{31} = u_3 + v_1 - c_{31} = 5 + 10 - 0 = 15$$

$$\bar{C}_{32} = u_3 + v_2 - c_{32} = 5 + 0 - 14 = -9$$

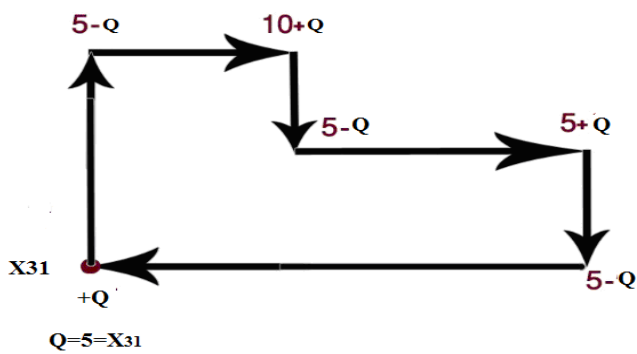
$$\bar{C}_{33} = u_3 + v_3 - c_{33} = 5 + 2 - 16 = -9$$

Since x_{31} has the most positive ($\bar{C}_{31}=15$), x_{31} it is selected as the entering variable.



This loop may be defined in terms of basic variables as:

$X_{31} \rightarrow X_{11} \rightarrow X_{12} \rightarrow X_{22} \rightarrow X_{24} \rightarrow X_{34} \rightarrow X_{31}$. It is immaterial whether the loop is traced in a clockwise or counterclockwise direction.



Hence the new solution is as follows:

| | | | | |
|---|----|----|----|----|
| 0 | 15 | X | X | 15 |
| X | 0 | 15 | 10 | 25 |
| 5 | X | X | X | 5 |
| 5 | 15 | 15 | 10 | |

Now the new basic solution in the above table is degenerate, since the basic variables x_{11} and x_{22} are zero. Now the zero basic variables are treated as any other positive basic variables.

H.W: check for optimality by computing the new multipliers, recomputed the steps (1) and (2) and stop when all the values of step (2) are zero or negative (optimal solution).

Lecture 13

Network Analysis

Reference Books:

- 1-Anderson, Sweeney, and Williams, AN INTRODUCTION TO MANAGEMENT SCIENCE, QUANTITATIVE APPROACHES TO DECISION MAKING, 7th edition, West Publishing Company, 1994
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- 3-Daellenbach, George, McNickle, INTRODUCTION TO OPERATIONS RESEARCH TECHNIQUES, 2nd edition, Allyn and Bacon. Inc, 1983
- 4-Lawrence Lapin, QUANTITATIVE METHODS for Business Decisions with Cases, 4th edition Harcourt Brace Jovanovich, Inc., 1988
- 5-T. A. Burley and G O'sullivan, OPERATIONAL RESEARCH, MacMillan Education Ltd., 1990

- Introduction

A **project** defines a combination of interrelated activities that must be executed in a certain order before the entire task can be completed. An **activity** in a project is usually viewed as a job requiring time and resources for its completion.

Project management has evolved as a field with the development of two analytical techniques for planning, scheduling, and controlling of projects. These are the **project evaluation and review technique (PERT)** and the **critical path method (CPM)**. These techniques were developed by two groups almost simultaneously. CPM was developed by E. I. Du Pont de Nemours & Company as an application to construction projects and was later extended to a more advanced status by Mauchly Associates. PERT was developed by the U.S. Navy by a consulting firm for scheduling the research and development activities for the Polaris missile program.

Although PERT and CPM were developed independently, they are similar in principle. Today, PERT and CPM actually comprise one technique and the differences, if any, are only historical. Consequently, both techniques are referred to as “project scheduling” techniques.

Project scheduling by PERT-CPM consists of three basic phases:

Planning

- Breaking down the project into distinct activities.
- Determining the time estimates for these activities.
- Constructing a network diagram with each arc representing the activity.

Scheduling

- constructing a time chart showing the start and the finish times for each activity as well as its relationship to other activities in the project;
- pinpointing the critical (in view of time) activities that require special attention if the project is to be completed on time.

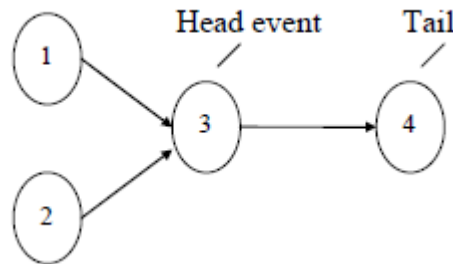
- Showing the amount of slack (or float) times for the non-critical activities;

Controlling

- Using the network diagram and the time chart for making periodic progress reports;
- updating the network

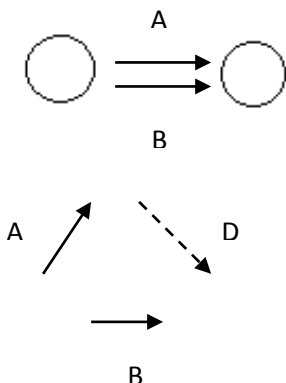
2. Network Diagram Representations and Network Construction.

The network diagram represents the interdependencies and precedence relationships among the activities of the project. An **arrow** is commonly used to represent an activity, with its head indicating the direction of progress in the project. An **event** represents a point in time that signifies the completion of some activities and the beginning of new ones. The following diagram shows an example, where activities (1,3) and (2,3) must be completed before activity (3,4) can start.



Rules for constructing a network diagram:

- 1- Each activity is represented by one and only one arrow in the network;
- 2- No two activities can be identified by the same head and tail events (a **dummy** activity is introduced in such situations);



In this case, D is the dummy activity.

Example 1:

Subject: Operation Research

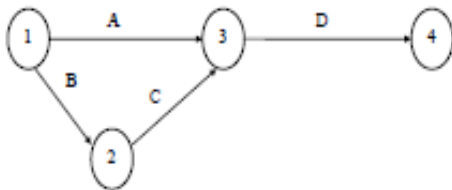
The Galaxy plc is to buy a small business, Tiny Ltd. The whole procedure involves four activities:

- A. Develop a list of sources for financing.
- B. Analyses the financial records of Tiny Ltd.
- C. Develop a business plan (sales projections, cash flow projections,
- D. Submit a proposal to a lending institution.

The precedence relationship of these four activities is described as in the table below. Construct the network diagram.

| Activity | Immediate Predecessor |
|----------|-----------------------|
| A | - |
| B | - |
| C | B |
| D | A,C |

Solution:



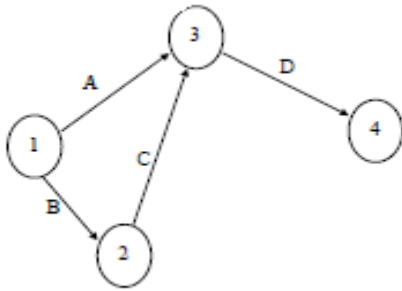
Example 2:

Construct the network based on the Table of information

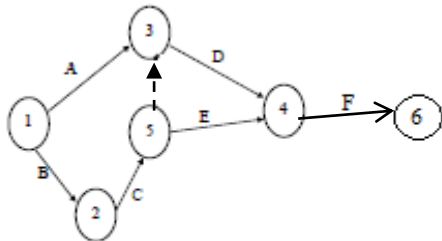
| Activity | Immediate Predecessor |
|----------|-----------------------|
| A | - |
| B | - |
| C | B |
| D | A, C |
| E | C |
| F | D,E |

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For activities A,B,C,D, the network portion is as follows:



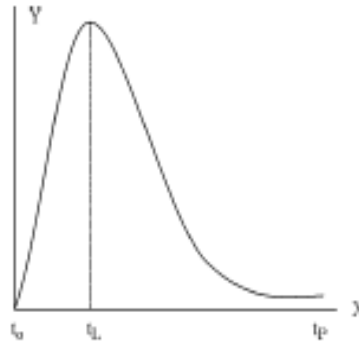
When activity E, which has C as its immediate predecessor, is to be added, we come across a problem because activities A and C both end at node 3. If activity E is to happen after C, it has to be after A as well in this arrangement, which is not true according to the specification. The solution is to add a dummy activity between C and node 3 in order to add E correctly. This is shown below.



Lecture 14

PERT activities are probabilistic in nature. The time required to complete the PERT activity cannot be specified correctly. Because of uncertainties in carrying out the activity, the time cannot be specified correctly. Say, for example, if you ask a contractor how much time it takes to construct the house, he may answer you that it may take 5 to 6 months. This is because of his expectation of uncertainty in carrying out each one of the activities in the construction of the house. Another example is if somebody asks you how much time you require to reach railway station from your house, you may say that

it may take 1 to 1½ hours. This is because you may think that you may not get a transport facility in time. Or on the way to station, you may come across certain work, which may cause delay in your journey from house to station. Hence PERT network is used when the activity times are probabilistic.



There are three time estimates in PERT, they are:

a- optimistic time: Optimistic time is represented by t_0 . Here the estimator thinks that everything goes on well and he will not come across any sort of uncertainties and estimates lowest time as far as possible. He is optimistic in his thinking.

b- Pessimistic time: This is represented by t_p . Here estimator thinks that everything goes wrong and expects all sorts of uncertainties and estimates highest possible time. He is pessimistic in his thinking.

c- Likely time: This is represented by t_L . This time is in between optimistic and pessimistic times. Here the estimator expects he may come across some sort of uncertainties and many a time the things will go right.

So while estimating the time for a PERT activity, the estimator will give the three times estimates. When these three estimates are plotted on a graph, the probability distribution that we get is closely associated with *Beta Distribution curve*. For a Beta distribution curve as shown in the figure above, the characteristics are:

Standard deviation (σ) = $(t_p - t_0)/6$, $t_p - t_0$ is known as range.

Variance (σ^2) = $[(t_p - t_0)/6]^2$

Expected Time or Average Time (t_E) = $(t_0 + 4t_L + t_p) / 6$

These equations are very important in the calculation of PERT times.

Example:

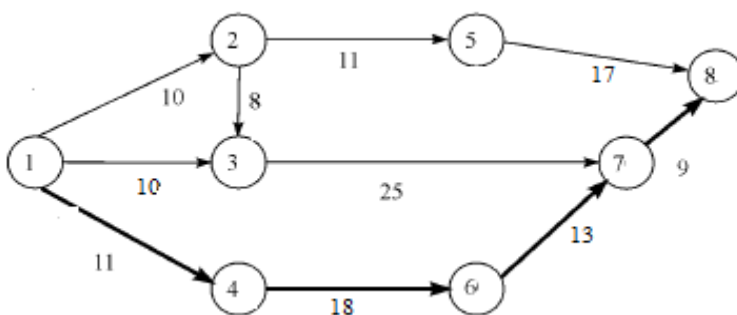
A project consists of 9 activities and the three time estimates are given below. Write the network for the given project and find the completion time (TE).

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| Activities | | Days | | |
|------------|----------|-------|-------|-------|
| <i>i</i> | <i>j</i> | T_O | T_L | T_P |
| 1 | 2 | 5 | 12 | 17 |
| 1 | 3 | 8 | 10 | 13 |
| 1 | 4 | 9 | 11 | 12 |
| 2 | 3 | 5 | 8 | 9 |
| 2 | 5 | 9 | 11 | 13 |
| 4 | 6 | 14 | 18 | 22 |
| 3 | 7 | 21 | 25 | 30 |
| 6 | 7 | 8 | 13 | 17 |
| 6 | 8 | 14 | 17 | 21 |
| 7 | 8 | 6 | 9 | 12 |

Solution:

| Predecessor event | Successor event | Time in days | | | $T_E = (t_O + 4t_L + t_P)/6$ | Range $t_P - t_O$ | S.D (σ) $(t_P - t_O) / 6$ | Variance σ^2 |
|-------------------|-----------------|--------------|-------|-------|------------------------------|-------------------|------------------------------------|---------------------|
| | | t_O | t_L | t_P | | | | |
| 1 | 2 | 5 | 12 | 17 | 9.66 (10) | 12 | 2 | 4 |
| 1 | 3 | 8 | 10 | 13 | 10.17 (10) | 5 | 0.83 | 0.69 |
| 1 | 4 | 9 | 11 | 12 | 10.83 (11) | 3 | 0.5 | 0.25 |
| 2 | 3 | 5 | 8 | 9 | 7.67 (8) | 4 | 0.66 | 0.44 |
| 2 | 5 | 9 | 11 | 13 | 11.00 (11) | 4 | 0.66 | 0.44 |
| 4 | 6 | 14 | 18 | 22 | 18.00 (18) | 8 | 1.33 | 1.78 |
| 3 | 7 | 21 | 25 | 30 | 25.18 (25) | 9 | 1.5 | 2.25 |
| 6 | 7 | 8 | 13 | 17 | 12.83 (13) | 9 | 1.5 | 2.25 |
| 5 | 8 | 14 | 17 | 21 | 17.17 (17) | 7 | 1.16 | 1.36 |
| 7 | 8 | 6 | 9 | 12 | 9.00 (9) | 6 | 1.0 | 1.0 |



H.W:

A project consists of 7 activities and the three time estimates are given below.

| Activities | | Time in weeks | | |
|------------|----------|---------------|-------|-------|
| <i>i</i> | <i>j</i> | t_o | t_l | t_p |
| 1 | 2 | 1 | 1 | 7 |
| 1 | 3 | 1 | 4 | 7 |
| 1 | 4 | 2 | 2 | 8 |
| 2 | 5 | 1 | 1 | 1 |
| 3 | 5 | 2 | 5 | 14 |
| 4 | 6 | 2 | 5 | 8 |
| 5 | 6 | 3 | 6 | 15 |

- 1- Write the network for the given projects.
- 2- Find the completion time (T_E).
- 3- Calculated the expected variances (σ^2) for each.

Lecture 15

-Determination of the Critical Path

An activity is said to be **critical** if a delay in its start will cause a delay in the completion date of the entire project. A **non-critical** activity is an activity that has time to spare (known as slack or float time) within the entire project. A **critical path is a sequence of connected critical activities that leads from the source node to the sink node.**

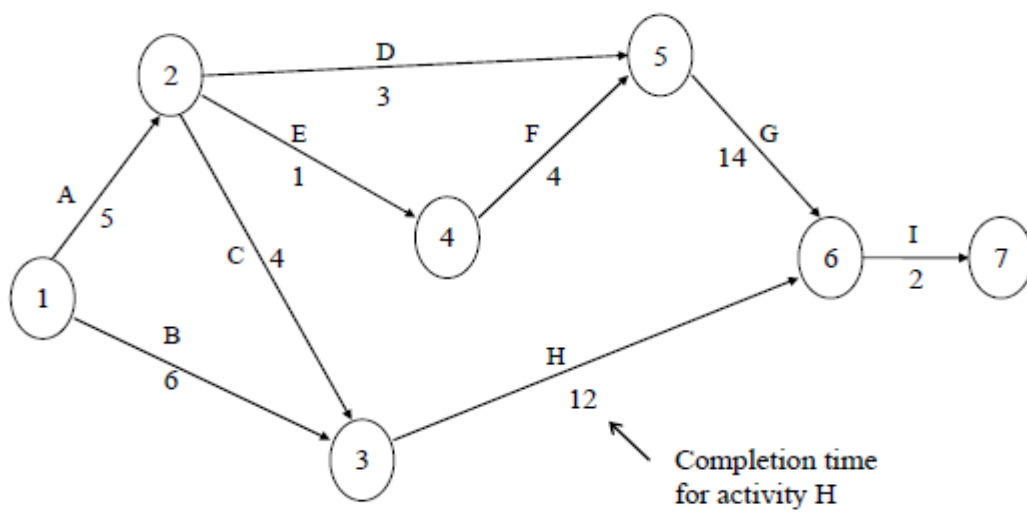
We will discuss the determination of the critical path through the following example.

Example:

A project consists of 9 activities and completion time (T_E) are given below.

| activity | Immediate predecessor | Completion time(weeks) |
|----------|-----------------------|------------------------|
| A | - | 5 |
| B | - | 6 |
| C | A | 4 |
| D | A | 3 |
| E | A | 1 |
| F | E | 4 |
| G | D,F | 14 |
| H | B,C | 12 |
| I | G,H | 2 |

Solution:



Starting at the network's source node (node 1) we will have to compute the **earliest start time** for each activity in the network. Let's (n) the number of nodes then:

ES_i = earliest start time for a particular activity ($ES_1=0$, $i=1,2,3,\dots,n$)

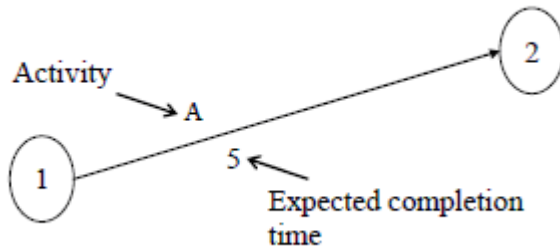
t_{ij} = expected completion time between the activities i and j . ($j=1,2,\dots,n$),

The earliest start time can be calculated by the following expression for a given activity:

$$ES_{i+1} = \max (ES_i + t_{ij})$$

For example, for activity an $ES_1 = 0$ and $t_{12} = 5$; thus $ES_2 = 0+5 = 5$.

We will write ES directly on the network. Using activity an example, we have



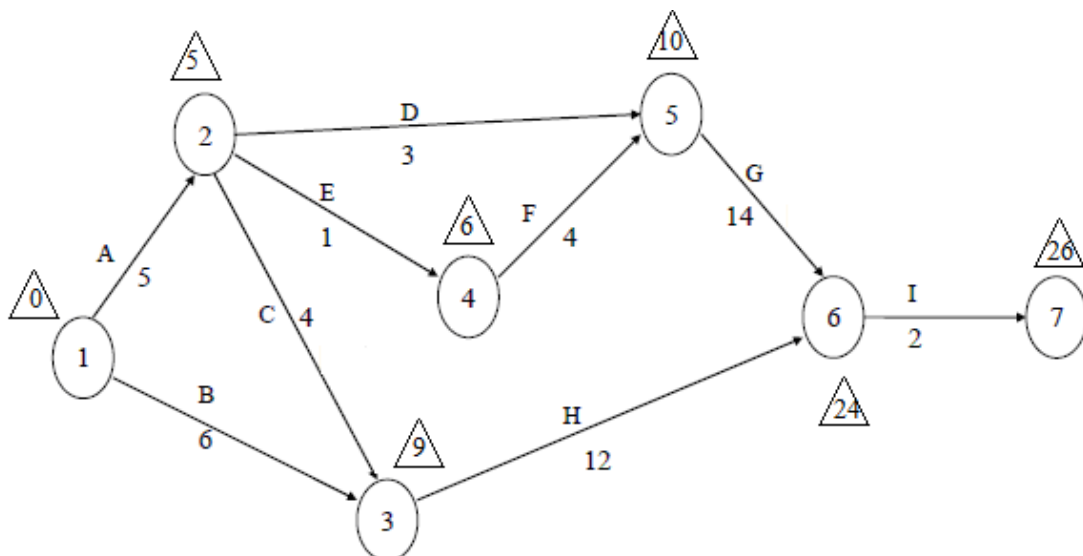
$$ES_1=0, ES_2=5$$

Since activities leaving a node cannot be started until all immediate preceding activities have been completed, the following rule determines the earliest start time for activities.

Earliest Start Time Rule (ES):

The earliest start time for an activity leaving a particular node is equal to the largest of the earliest finish time for all activities entering the node.

Using this rule, the earliest start for each activity is written onto the network, which now looks as follows:



As has demonstrated, proceeding in a *forward pass* through the network, we can establish the earliest start time for each activity.

We now continue the algorithm for finding the critical path by making a *backward pass* calculation. Starting at the sink node (node 7) and using a latest start time of 26 weeks for activity I, we trace back through the network, to computing a **latest finish time (Lf)** for each activity. Let

LS_j = latest finish time for a particular activity (assume that $Lf_n = ES_n$)

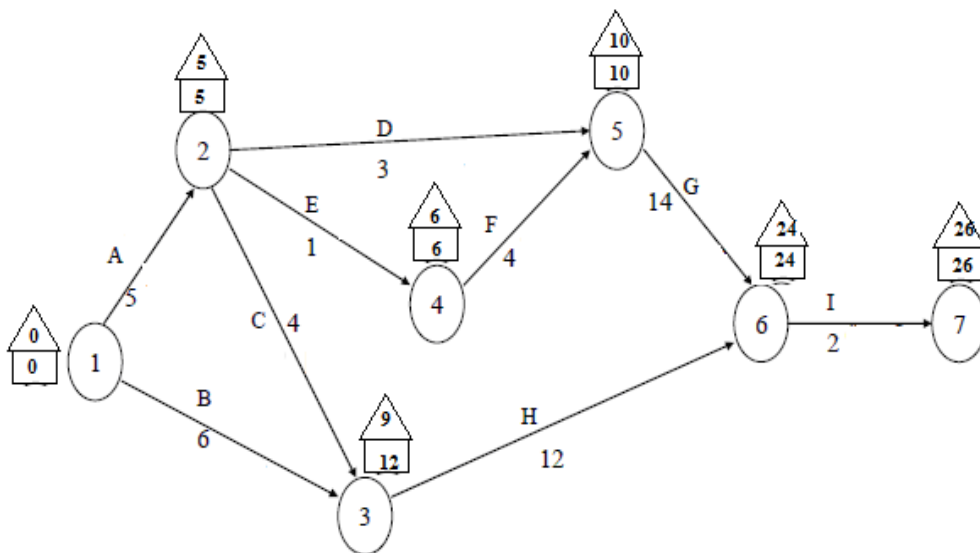
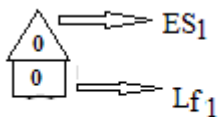
The latest finish time is given by the following expression:

$Lf_{n-1} = \min(Lf_n - t_{ij})$, where (n is the number of nodes)

Latest finish time Rule (Lf) :

The latest time finish for an activity entering a particular node is equal to the smallest of the latest finish times for all activities leaving the node.

The PERT and CPM network with both ES,Lf for the example is shown below.



To find the critical path, we find

1) earliest start time as follows:

Let $ES_1 = 0$

$ES_2 = (ES_1 + t_{12}) = 0 + 5 = 5$

$ES_3 = \text{Max}(ES_1 + t_{13}, ES_2 + t_{23}) = \text{Max}(0 + 6, 5 + 4) = 9$

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$$ES_4=(ES_2+t_{24})=5+1=6$$

$$ES_5=Max(ES_4+t_{45},ES_2+t_{25})=Max(6+4,5+3)=10$$

$$ES_6=(ES_5+t_{56})=(10+14)=24$$

$$ES_7=(ES_6+t_{67})=24+2=26$$

2) latest finish time:

$$\text{Let } Lf_7=ES_7=26$$

$$Lf_6=(Lf_7-t_{67})=26-2=24$$

$$Lf_5=(Lf_6-t_{56})=24-14=10$$

$$Lf_4=(Lf_5-t_{45})=10-4=6$$

$$Lf_3=(Lf_6-t_{36})=24-12=12$$

$$Lf_2=Min(Lf_4-t_{24}, Lf_5-t_{25}, Lf_3-t_{23})=Min(6-1,10-3,12-4)=5$$

$$Lf_1=Min(Lf_3-t_{13}, Lf_2-t_{12})=Min(12-6,5-5)=0$$

From the above diagram, we find the amount of slack or free time associated with each of the activities.

Slack time is defined as the length of time an activity can be delayed without affecting the total time required to complete the project. The amount of slack is computed as follows:

Slack = $Lf_j - ES_i$, Activities with zero slack are the critical path activities.

According to the finished PERT/CPM network, we arrive at the following table of information (the project schedule) for the shopping centre project.

| Event (i) | ES_i | Lf_j | Slack(Lf_j-ES_i) | Critical path |
|-----------|--------|--------|----------------------|---------------|
| 1 | 0 | 0 | 0 | yes |
| 2 | 5 | 5 | 0 | yes |
| 3 | 9 | 12 | 3 | no |
| 4 | 6 | 6 | 0 | yes |
| 5 | 10 | 10 | 0 | yes |
| 6 | 24 | 24 | 0 | yes |
| 7 | 26 | 26 | 0 | yes |

Thus the critical path of the above network is (1 - 2 - 4 - 5 - 6 - 7)

The PERT/CPM critical path procedure:

Step 1: Develop a list of activities that make up the project.

Step 2: Determine the immediate predecessor activities for each activity listed in the project.

Step 3: Estimate the completion time for each activity.

Step 4: Draw a network depicting the activities and immediate predecessors listed in Steps 1&2.

Step 5 :Using the network and the activity time estimates, determine the earliest start times for each activity by making a forward pass through the network.

Step 6: Using the project completion time identified in Step 5 as the latest finish time for the last activity, make a backward pass through the network to identify the latest finish for each activity.

Step 7: Use the difference between the latest finish time and the earliest start time for each activity to identify the slack time available for the activity.

Step 8: Find the activities with zero slack; these are the critical path activities.

Lecture 16

-Game theory

Introduction

It frequently happens that you are faced with having to make a decision or choose a best strategy from several possible choices. For instance, you might need to decide whether to invest in stocks or bonds, or you might need to choose an offensive play to use in a football game. In both of these examples, the result depends on something you cannot control. In the first case, your success partly depends on the future behavior of the economy. In the second case, it depends on the defensive strategy chosen by the opposing team.

We can model situations like this using **game theory**. We represent the various options and payoffs in a matrix and can then calculate the best single strategy or combination of strategies using matrix algebra and techniques from linear programming. Game theory is yet another illustration of the power of matrix algebra and linear programming.

Game theory is very new compared with most of the mathematics you learn. It was invented in the 1920's by the noted mathematicians Émile Borel (1871–1956) and John

von Neumann (1903–1957). The connection with linear programming was discovered even more recently, in 1947, by von Neumann.

Dominating strategy

A strategy dominates another strategy of a player if it always gives a better payoff to that player, regardless of what the other players are doing. It weakly dominates the other strategy if it is always at least as good.

Extensive game

An extensive game (or extensive form game) describes with a tree how a game is played. It depicts the order in which players make moves, and the information each player has at each decision point.

Game

A game is a formal description of a strategic situation.

Game theory

Game theory is the formal study of decision-making where several players must make choices that potentially affect the interests of the other players.

Mixed strategy

A mixed strategy is an active randomization, with given a probability that determines the player's decision. As a special case, a mixed strategy can be the deterministic choice of one of the given pure strategies

Player

A player is an agent who makes decisions in a game.

Strategic form

A game in strategic form, also called normal form, is a compact representation of a game in which players simultaneously choose their strategies. The resulting payoffs are presented in a table with a cell for each strategy combination.

Strategy

In a game in strategic form, a strategy is one of the given possible actions of a player. In an extensive game, a strategy is a complete plan of choices, one for each decision point of the player.

Zero-sum game

A game is said to be zero-sum if for any outcome, the sum of the payoffs to all players is zero. In a two-player zero-sum game, one player’s gain is the other player’s loss, so their interests are diametrically opposed.

The Two-Person, Zero-Sum Game

Consider a competitive situation with two players and arbitrarily assume that player **I** is a woman and player **II** is a man. The game is specified by the sets of strategies available to the two players and the payoff matrix. The set of strategies for player **I** is indexed from 1 through m . The set of strategies for player **II** is indexed from 1 through n . The payoff matrix (see Table 1) specifies the gain or profit to player **I** for every strategy pair (i, j) .

Table (1) Payoff Matrix

| | | Player II | | | |
|-----------------|-----|------------------|----------|-----|----------|
| | | 1 | 2 | ... | n |
| Player I | 1 | p_{11} | p_{12} | ... | p_{1n} |
| | 2 | p_{21} | p_{22} | ... | p_{2n} |
| | . | . | . | . | . |
| | . | . | . | . | . |
| | m | p_{m1} | p_{m2} | ... | p_{mn} |

The two players select their strategies simultaneously, and when player **I** uses strategy i and player **II** uses strategy j , player **I** receives the payoff p_{ij} from player **II**. A positive number is a gain for player **I** and a negative number is a loss (a gain for player **II**). A gain to one player is a loss to the other, thus providing the zero-sum feature. The payoff obtained when the two players select their strategies is the *value* of the game.

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Each player knows all strategies available to the other, and they both agree on the payoff matrix.

Solutions are either pure or mixed strategies. With a pure strategy a player chooses only one strategy in a play of the game. In a mixed strategy, a player chooses one of several strategies according some probability distribution.

| | B1 | B2 | min |
|-----|----------|----|----------|
| A1 | -3 | 3 | -3 |
| A2 | -2 | 4 | -2 |
| A3 | 2 | 3 | <u>2</u> |
| max | <u>2</u> | 4 | |

Lecture 17

Rules for game theory

a) Looking for pure strategy (saddle point).

Example: suppose we have two players A, B with the following payoff matrix:

Since $\max.\min = \min.\max$ then there exist a saddle point =2 ,the saddle point gives the value of the game ,then the optimal strategy for A is A3 and the optimal strategy for B is B1

To find a saddle point

- 1- Find the smallest element in each row (the largest element in the rows=max.min)
- 2- Find the largest element in each column (the smallest element in the columns=min.max)
- 3- If max.min=min.max, then there exist a saddle point (value of the game).
- 4- If max.min≠min.max, then the value of the game lies between them.
- 5- Maybe there is more than one saddle point; hence there exist more than one optimal solution.

b) Dominated Strategies

A dominated strategy is a strategy that yields a payoff for one of the players that is less than or equal to the payoff for some other strategy for all actions of the opponent. For player I, strategy *i* is dominated by strategy *k* if

$$p_{ij} \leq p_{kj} \quad \text{for } j = 1, \dots, n \quad \dots (1)$$

In other words *i* is dominated by *k* if every element of row *i* in the payoff matrix is less than or equal to every corresponding element of row *k*.

For player II, a strategy is dominated if every element of a column is greater than or equal to every corresponding element of some other column.

Strategy *j* is dominated by strategy *k* if

$$p_{ij} \geq p_{ik} \quad \text{for } i = 1, \dots, m \quad \dots (2)$$

It should be clear that dominated strategies will not be used in the solution of the game. If a dominated strategy is used, a better solution is always obtained by replacing it with the dominating strategy. The first step in the solution process is to find and eliminate the dominated strategies from the game.

Example: solve the following payoff matrix:

| | | | |
|------------|----------|-----------|------------|
| B | | | |
| A | I | II | III |
| I | 2 | 4 | 5 |
| II | 3 | 1 | 6 |
| III | 2 | 4 | 3 |

Solution:

$\therefore (2,4,3) \leq (2,4,5) \Rightarrow A_3 \leq A_1$

Then in the rows the strategy I is dominated by strategy III, therefore the new matrix is:

| | | | |
|-----------------|----------|-----------|------------|
| B A \ | I | II | III |
| I | 2 | 4 | 5 |
| II | 3 | 1 | 6 |

Now in the column a strategy is dominated if every element of a column is greater than or equal to every corresponding element of some other column.

$\therefore (5,6) \geq (2,3) \Rightarrow B_3 \geq B_1$

Then in the columns the strategy I is dominated by strategy III, therefore the new matrix is:

| | | |
|-----------------|----------|-----------|
| B A \ | I | II |
| I | 2 | 4 |
| II | 3 | 1 |

When only one row (or column) remains, it is always possible to reduce the number of columns (or rows) to one by dominance. Then we find the solution to the game with both players using dominate rule.

| | | | |
|-----------------|----------|-----------|------------|
| B A \ | I | II | Min |
| I | 4 | 4 | 2 |
| II | 5 | 1 | 1 |
| max | 3 | 4 | |

Since $\max.\min \neq \min.\max$, there is no saddle point ($2 \neq 3$). therefore we solve the above matrix using methods of mixed strategy.

Lecture 18

-Solved for a mixed strategy.

1- Solved a mixed strategy (2x2) by Arithmetic method.

Example: find the best strategy and the value of the following payoff matrix:

| | | |
|-----------|-----------|-----------|
| | B1 | B2 |
| A1 | 2 | -1 |
| A2 | -1 | 0 |

Solution:

| | | | |
|------------|-----------|-----------|------------|
| | B1 | B2 | Min |
| A1 | 2 | -1 | -1 |
| A2 | -1 | 0 | -1 |
| max | 2 | 0 | |

Since $\min.\max \neq \max.\min$.their is no saddle point.

| | | | |
|-----------|-----|---------|----------|
| A1 | G11 | G12 | G21-G22 |
| A2 | G21 | G22 | G11- G12 |
| | | G12-G22 | G11- G21 |

 \Rightarrow

| | | | |
|-----------|-----------|-----------|---|
| | B1 | B2 | |
| A1 | 2 | -1 | 1 |
| A2 | -1 | 0 | 3 |
| | 1 | 3 | |

The best strategy for A :(1/4, 3/4)

The best strategy for B :(1/4, 3/4)

The game value =

$$G11 \cdot \frac{|G21 - G22|}{|G21 - G22| + |G11 - G12|} + G21 \cdot \frac{|G11 - G12|}{|G21 - G22| + |G11 - G12|} = 2 \cdot \frac{1}{4} + (-1) \cdot \frac{3}{4} = \frac{-1}{4}$$

2- Solved a mixed strategy (2x2) by Algebraic method.

Example:

| | | |
|-----------|-----------|-----------|
| | B1 | B2 |
| A1 | 0 | -3 |
| A2 | -1 | 0 |

Solution:

Suppose (x) is the time probability of player A is must play in the strategy A1.

Suppose (1- x) is the time probability of player A is must play in the strategy A2.

$$0 \cdot x + (-1) \cdot (1-x) = -3x + 0 \cdot (1-x)$$

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$$-1+x = -3x \implies 4x = 1 \implies x = 1/4 \text{ the time in the strategy A1.}$$

$$\implies 1-x = 1 - (1/4) = 3/4 \text{ the time of strategy A2.}$$

Suppose (y) is the time probability of player B is must play in the strategy B1.

Suppose (1-y) is the time probability of player B is must play in the strategy B2.

$$0.y - 3(1-y) = -y + 0.(1-y)$$

$$-3+3y = -y \implies y+3y = 3 \implies y = 3/4 \implies 1-y = 1/4$$

The best strategy for A=(1/4,3/4)

The best strategy for B=(3/4,1/4)

The value of game = $0.(1/4) + (-1).3/4 = -3/4$ or $= -3.(1/4) + 0.(3/4) = -3/4$.

H.W: Find the best strategy and the value of the following game matrix:

| | B1 | B2 | B2 |
|-----------|-----------|-----------|-----------|
| A1 | 5 | 50 | 50 |
| A2 | 1 | 1 | 0.1 |
| A3 | 10 | 1 | 10 |

Lecture 19

3- Solved (mx2) or (2xn) matrix by Graphical method.

Example: using graphical method to solve the following matrix game:

| | B1 | B2 | B2 |
|----|----|----|----|
| A1 | 1 | 3 | 10 |
| A2 | 8 | 5 | 2 |

Solution:

Since $\max.\min \neq \min.\max$ ($2 \neq 5$), therefore there is no saddle point.

Suppose (x_1) is the probability of choose player A the strategy A1.

Suppose (x_2) is the probability of choose player A the strategy A2.

Where: $x_2 = 1 - x_1$, $x_1 + x_2 = 1$

Suppose Z is the value of the game between the players A and B.

If the player B chooses the first column, then the result chose of the player A is:

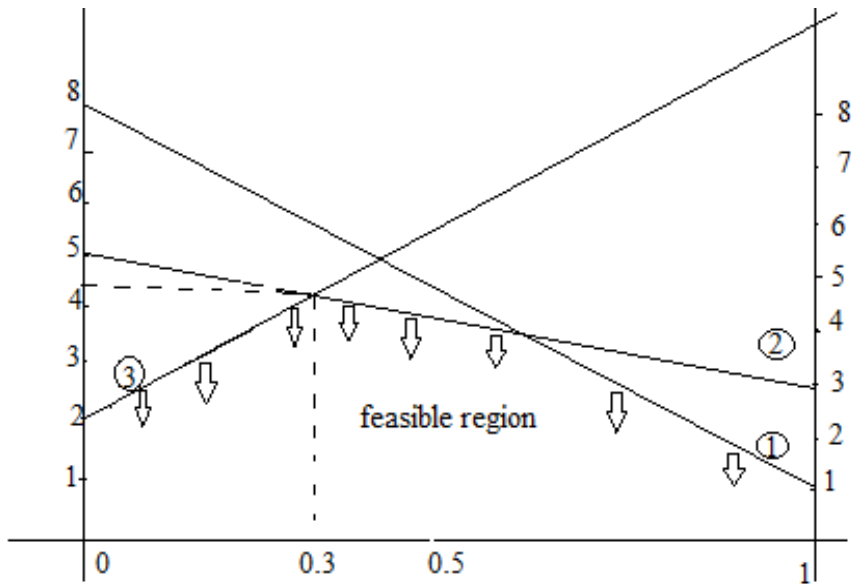
$$x_1 + 8x_2 \geq Z$$

Since $x_2 = 1 - x_1$

$$\Rightarrow x_1 + 8(1 - x_1) \geq Z$$

$$\Rightarrow Z + 7x_1 \leq 8 \quad \dots \text{constraint (1)}$$

And if the player B chooses the second column, then the result chose of the player A is:



Then from intersection the constraint (2),(3) we find the value of the game (Z) as follows :

$$Z+2x_1=-5$$

$$\underline{Z-8x_1= 2}$$

$-10x_1=-3 \Rightarrow x_1=3/10$ substitution it in eq.(2) $\Rightarrow Z=5-2(3/10)=4.4$ the value of game.

$$x_2=1-x_1=1-0.3=0.7 .$$

the best strategy of A is(0.3,0.7)