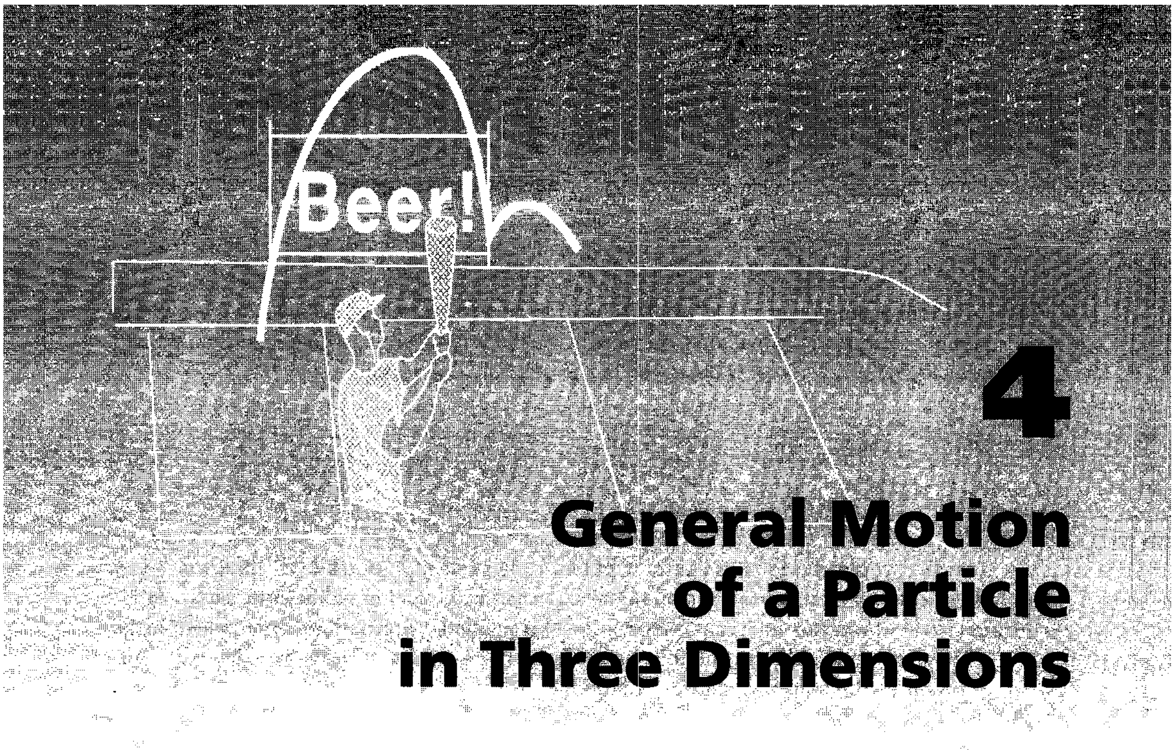

4 General Motion of a Particle in Three Dimensions

Sir Isaac Newton, and his followers, have also a very odd opinion concerning the work of God. According to their doctrine, God almighty needs to wind up his watch from time to time; otherwise it would cease to move. He had not, it seems, sufficient foresight to make it a perpetual motion.

Nay, the machine of God's making, is so imperfect, according to these gentlemen, that He is obliged to clean it now and then by an extraordinary concourse, and even to mend it, as a clockmaker mends his work; who must consequently be so much the more unskillful a workman, as He is often obliged to mend his work and set it right. According to my opinion, the same force and vigour [energy] remains always in the world, and only passes from one part to another, agreeable to the laws of nature, and the beautiful pre-established order—

Gottfried Wilhelm Leibniz—*Letter to Caroline, Princess of Wales*, 1715; The Leibniz-Clarke Correspondence, Manchester, Manchester Univ. Press, 1956

4.1 | Introduction: General Principles

We now examine the general case of the motion of a particle in three dimensions. The vector form of the equation of motion for such a particle is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{4.1.1}$$

in which $\mathbf{p} = m\mathbf{v}$ is the linear momentum of the particle. This vector equation is equivalent to three scalar equations in Cartesian coordinates.

$$\begin{aligned} F_x &= m\ddot{x} \\ F_y &= m\ddot{y} \\ F_z &= m\ddot{z} \end{aligned} \tag{4.1.2}$$

The three force components may be explicit or implicit functions of the coordinates, their time and spatial derivatives, and possibly time itself. There is no general method for obtaining an analytic solution to the above equations of motion. In problems of even the mildest complexity, we might have to resort to the use of applied numerical techniques; however, there are many problems that can be solved using relatively simple analytical methods. It may be true that such problems are sometimes overly simplistic in their representation of reality. However, they ultimately serve as the basis of models of real physical systems, and so it is well worth the effort that we take here to develop the analytical skills necessary to solve such idealistic problems. Even these may prove capable of taxing our analytic ability.

It is rare that one knows the explicit way in which $\mathbf{F}$ depends on time; therefore, we do not worry about this situation but instead focus on the more normal situation in which $\mathbf{F}$ is known as an explicit function of spatial coordinates and their derivatives. The simplest situation is one in which $\mathbf{F}$ is known to be a function of spatial coordinates only. We devote most of our effort to solving such problems. There are many only slightly more complex situations, in which $\mathbf{F}$ is a known function of coordinate derivatives as well. Such cases include projectile motion with air resistance and the motion of a charged particle in a static electromagnetic field. We will solve problems such as these, too. Finally, $\mathbf{F}$ may be an implicit function of time, as in situations where the coordinate and coordinate derivative dependency is nonstatic. A prime example of such a situation involves the motion of a charged particle in a time-varying electromagnetic field. We will not solve problems such as these. For now, we begin our study of three-dimensional motion with a development of several powerful analytical techniques that can be applied when $\mathbf{F}$ is a known function of $\mathbf{r}$ and/or $\dot{\mathbf{r}}$.

The Work Principle

Work done on a particle causes it to gain or lose kinetic energy. The work concept was introduced in Chapter 2 for the case of motion of a particle in one dimension. We would like to generalize the results obtained there to the case of three-dimensional motion. To do so, we first take the dot product of both sides of Equation 4.1.1 with the velocity $\mathbf{v}$:

$$\mathbf{F}\cdot\mathbf{v} = \frac{d\mathbf{p}}{dt}\cdot\mathbf{v} = \frac{d(m\mathbf{v})}{dt}\cdot\mathbf{v} \tag{4.1.3}$$

Because $d(\mathbf{v}\cdot\mathbf{v})/dt = 2\mathbf{v}\cdot\dot{\mathbf{v}}$, and assuming that the mass is constant, independent of the velocity of the particle, we may write Equation 4.1.3 as

$$\mathbf{F}\cdot\mathbf{v} = \frac{d}{dt}\left(\tfrac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) = \frac{dT}{dt} \tag{4.1.4}$$

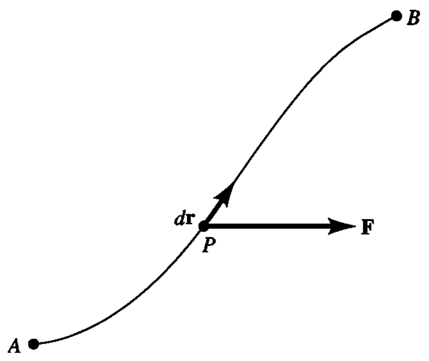

Figure 4.1.1 The work done by a force $\mathbf{F}$ is the line integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$.

in which T is the kinetic energy, $mv^2/2$. Because $\mathbf{v} = d\mathbf{r}/dt$, we can rewrite Equation 4.1.4 and then integrate the result to obtain

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{dT}{dt} \tag{4.1.5a}$$

$$\therefore \int \mathbf{F} \cdot d\mathbf{r} = \int dT = T_f - T_i = \Delta T \tag{4.1.5b}$$

The left-hand side of this equation is a *line integral*, or the integral of $F_r\, dr$, the component of $\mathbf{F}$ parallel to the particle's displacement vector $d\mathbf{r}$. The integral is carried out along the trajectory of the particle from some initial point in space A to some final point B. This situation is pictured in Figure 4.1.1. The line integral represents the work done on the particle by the force $\mathbf{F}$ as the particle moves along its trajectory from A to B. The right-hand side of the equation is the net change in the kinetic energy of the particle. $\mathbf{F}$ is the net sum of all vector forces acting on the particle; hence, the equation states that the work done on a particle by the net force acting on it, in moving from one position in space to another, is equal to the difference in the kinetic energy of the particle at those two positions.

Conservative Forces and Force Fields

In Chapter 2 we introduced the concept of potential energy. We stated there that if the force acting on a particle were *conservative*, it could be derived as the derivative of a scalar potential energy function, $F_x = -dV(x)/dx$. This condition led us to the notion that the work done by such a force in moving a particle from point A to point B along the x-axis was $\int F_x dx = -\Delta V = V(A) - V(B)$, or equal to minus the change in the potential energy of the particle. Thus, we no longer required a detailed knowledge of the motion of the particle from A to B to calculate the work done on it by a conservative force. We needed to know only that it started at point A and ended up at point B. The work done depended only upon the potential energy function evaluated at the endpoints of the motion. Moreover, because the work done was also equal to the change in kinetic energy of the particle, $\Delta T = T(B) - T(A)$, we were able to establish a general conservation of total energy principle, namely, $E_{tot} = V(A) + T(A) = V(B) + T(B) =$ constant throughout the motion of the particle.

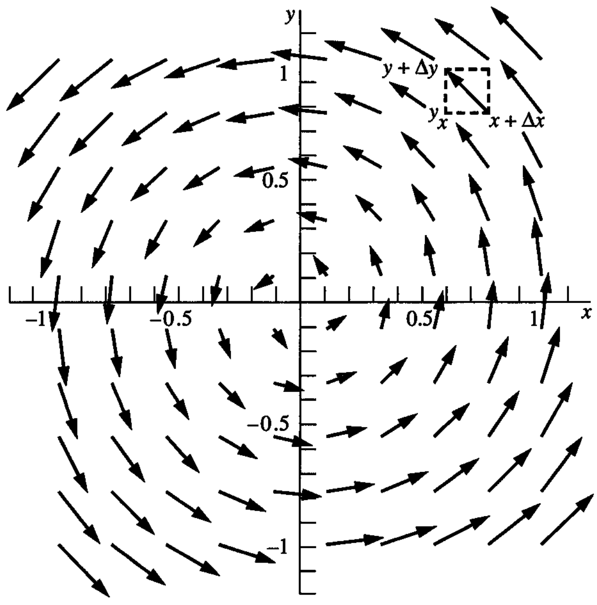

Figure 4.1.2 A nonconservative force field whose force components are $F_x = -by$ and $F_y = +bx$.

This principle was based on the condition that the force acting on the particle was conservative. Indeed, the very name implies that something is being conserved as the particle moves under the action of such a force. We would like to generalize this concept for a particle moving in three dimensions, and, more importantly, we would like to define just what is meant by the word *conservative*. Clearly, we would like to have some prescription that tells us whether or not a particular force is conservative and, thus, whether or not a potential energy function exists for the particle. Then we could invoke the powerful conservation of energy principle in solving the motion of a particle.

In searching for such a prescription, we first describe an example of a nonconservative force that, in fact, is a well-defined function of position but cannot be derived from a potential energy function. This should give us a hint of the critical characteristic that a force must have if it is to be conservative. Consider the two-dimensional force field depicted in Figure 4.1.2. The term *force field* simply means that if a small test particle[1] were to be placed at any point (x_1, y_1) on the xy plane, it would experience a force **F**. Thus, we can think of the xy plane as permeated, or "mapped out," with the potential for generating a force.

This situation can be mathematically described by assigning a vector **F** to every point in the xy plane. The field is, therefore, a vector field, represented by the function $\mathbf{F}(x, y)$. Its components are $F_x = -by$ and $F_y = +bx$, where b is some constant. The arrows

[1] A test particle is one whose mass is small enough that its presence does not alter its environment. Conceptually, we might imagine it placed at some point in space to serve as a "test probe" for the suspected presence of forces. The forces are "sensed" by observing any resultant acceleration of the test particle. We further imagine that its presence does not disturb the sources of those forces.

in the figure represent the vector $\mathbf{F} = -\mathbf{i}by + \mathbf{j}bx$ evaluated at each point on which the center of the arrow is located. You can see by looking at the figure that there seems to be a general counterclockwise "circulation" of the force vectors around the origin. The magnitude of the vectors increases with increasing distance from the origin. If we were to turn a small test particle loose in such a "field," the particle would tend to circulate counterclockwise, gaining kinetic energy all the while.

This situation, at first glance, does not appear to be so unusual. After all, when you drop a ball in a gravitational force field, it falls and gains kinetic energy, with an accompanying loss of an equal amount of potential energy. The question here is, can we even define a potential energy function for this circulating particle such that it would lose an amount of "potential energy" equal to the kinetic energy it gained, thus preserving its overall energy, as it travels from one point to another? That is not the case here. If we were to calculate the work done on this particle in tracing out some path that came back on itself (such as the rectangular path indicated by the dashed line in Figure 4.1.2), we would obtain a nonzero result! In traversing such a loop over and over again, the particle would continue to gain kinetic energy equal to the nonzero value of work done per loop. But if the particle could be assigned a potential energy dependent only upon its (x, y) position, then its change in potential energy upon traversing the closed loop would be zero. It should be clear that there is no way in which we could assign a unique value of potential energy for this particle at any particular point on the xy plane. Any value assigned would depend on the previous history of the particle. For example, how many loops has the particle already made before arriving at its current position?

We can further expose the nonuniqueness of any proposed potential energy function by examining the work done on the particle as it travels between two points A and B but along two different paths. First, we let the particle move from (x, y) to $(x + \Delta x, y + \Delta y)$ by traveling in the $+x$ direction to $(x + \Delta x, y)$ and then in the $+y$ direction to $(x + \Delta x, y + \Delta y)$. Then we let the particle travel first along the $+y$ direction from (x,y) to $(x,y + \Delta y)$ and then along the $+x$ direction to $(x + \Delta x, y + \Delta y)$. We see that a different amount of work is done depending upon which path we let the particle take. If this is true, then the work done cannot be set equal to the difference between the values of some scalar potential energy function evaluated at the two endpoints of the motion, because such a difference would give a unique, *path-independent* result. The difference in work done along these two paths is equal to $2b\Delta x\Delta y$ (see Equation 4.1.6). This difference is just equal to the value of the closed-loop work integral; therefore, the statement that the work done in going from one point to another in this force field is path-dependent is equivalent to the statement that the closed-loop work integral is nonzero. The particular force field represented in Figure 4.1.2 demands that we know the complete history of the particle to calculate the work done and, therefore, its kinetic energy gain. The potential energy concept, from which the force could presumably be derived, is rendered meaningless in this particular context.

The only way in which we could assign a unique value to the potential energy would be if the closed-loop work integral vanished. In such cases, the work done along a path from A to B would be path-independent and would equal both the potential energy loss and the kinetic energy gain. The total energy of the particle would be a constant, independent of its location in such a force field! We, therefore, must find the constraint that a particular force must obey if its closed-loop work integral is to vanish.

To find the desired constraint, let us calculate the work done in taking a test particle counterclockwise around the rectangular loop of area $\Delta x \Delta y$ from the point (x,y) and back again, as indicated in Figure 4.1.2. We get the following result:

$$\begin{aligned} W &= \oint \mathbf{F} \cdot d\mathbf{r} \\ &= \int_x^{x+\Delta x} F_x(y)\,dx + \int_y^{y+\Delta y} F_y(x+\Delta x)\,dy \\ &\quad + \int_{x+\Delta x}^{x} F_x(y+\Delta y)\,dx + \int_{y+\Delta y}^{y} F_y(x)\,dy \\ &= \int_y^{y+\Delta y} (F_y(x+\Delta x) - F_y(x))\,dy \\ &\quad + \int_x^{x+\Delta x} (F_x(y) - F_x(y+\Delta y))\,dx \\ &= (b(x+\Delta x) - bx)\Delta y + (b(y+\Delta y) - by)\Delta x \\ &= 2b\Delta x \Delta y \end{aligned} \tag{4.1.6}$$

The work done is nonzero and is proportional to the area of the loop, $\Delta A = \Delta x \cdot \Delta y$, which was chosen in an arbitrary fashion. If we divide the work done by the area of the loop and take limits as $\Delta A \to 0$, we obtain the value $2b$. The result is dependent on the precise nature of this particular nonconservative force field.

If we reverse the direction of one of the force components—say, let $F_x = +by$ (thus "destroying" the circulation of the force field but everywhere preserving its magnitude)—then the work done per unit area in traversing the closed loop vanishes. The resulting force field is conservative and is shown in Figure 4.1.3. Clearly, the value of the closed-loop

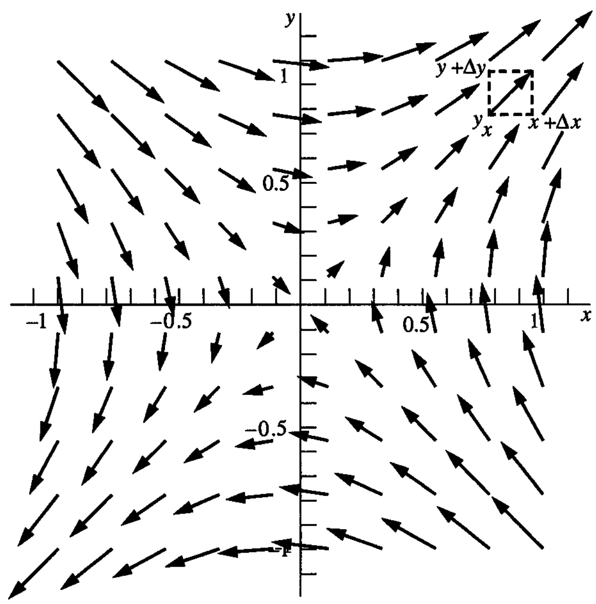

Figure 4.1.3 A conservative force field whose components are $F_x = by$ and $F_y = bx$.

integral depends upon the precise way in which the vector force **F** changes its direction as well as its magnitude as we move around on the xy plane.

There is obviously some sort of constraint that **F** must obey if the closed-loop work integral is to vanish. We can derive this condition of constraint by evaluating the forces at $x + \Delta x$ and $y + \Delta y$ using a Taylor expansion and then inserting the resultant expansion into the closed-loop work integral of Equation 4.1.6. The result follows:

$$\begin{aligned} F_x(y+\Delta y) &= F_x(y) + \frac{\partial F_x}{\partial y}\Delta y \\ F_y(x+\Delta x) &= F_y(x) + \frac{\partial F_y}{\partial x}\Delta x \end{aligned} \tag{4.1.7}$$

$$\begin{aligned} \oint \mathbf{F}\cdot d\mathbf{r} &= \int_y^{y+\Delta y}\left(\frac{\partial F_y}{\partial x}\Delta x\right)dy - \int_x^{x+\Delta x}\left(\frac{\partial F_x}{\partial y}\Delta y\right)dx \\ &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\Delta x\Delta y = 2b\Delta x\Delta y \end{aligned} \tag{4.1.8}$$

This last equation contains the term $(\partial F_y/\partial x - \partial F_x/\partial y)$, whose zero or nonzero value represents the test we are looking for. If this term were identically equal to zero instead of $2b$, then the closed-loop work integral would vanish, which would ensure the existence of a potential energy function from which the force could be derived.

This condition is a rather simplified version of a very general mathematical theorem called *Stokes' theorem.*[2] It is written as

$$\begin{aligned} \oint \mathbf{F}\cdot d\mathbf{r} &= \int_S \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}}\,da \\ \text{curl}\,\mathbf{F} &= \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \end{aligned} \tag{4.1.9}$$

The theorem states that the closed-loop line integral of *any* vector function **F** is equal to curl **F** · **n** da integrated over a surface S surrounded by the closed loop. The vector **n** is a unit vector normal to the surface-area integration element da. Its direction is that of the advance of a right-hand screw turned in the same rotational sense as the direction of traversal around the closed loop. In Figure 4.1.2, **n** would be directed out of the paper. The surface would be the rectangular area enclosed by the dashed rectangular loop. Thus, a vanishing curl **F** ensures that the line integral of **F** around a closed path is zero and, thus, that **F** is a conservative force.

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[2] See any advanced calculus textbook (e.g., S. I. Grossman and W. R. Derrick, *Advanced Engineering Mathematics*, Harper Collins, New York, 1988) or any advanced electricity and magnetism textbook (e.g., J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory*, Addison-Wesley, New York, 1992).

4.2 | The Potential Energy Function in Three-Dimensional Motion: The Del Operator

Assume that we have a test particle subject to some force whose curl vanishes. Then all the components of curl **F** in Equation 4.1.9 vanish. We can make certain that the curl vanishes if we derive **F** from a potential energy function $V(x,y,z)$ according to

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z} \tag{4.2.1}$$

For example, the z component of curl **F** becomes

$$\frac{\partial F_x}{\partial y} = -\frac{\partial^2 V}{\partial y \partial x} \qquad \frac{\partial F_y}{\partial x} = -\frac{\partial^2 V}{\partial x \partial y} = -\frac{\partial^2 V}{\partial y \partial x} \qquad \therefore \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0 \tag{4.2.2}$$

This last step follows if we assume that V is everywhere continuous and differentiable. We reach the same conclusion for the other components of curl **F**. One might wonder whether there are other reasons why curl **F** might vanish, besides its being derivable from a potential energy function. However, curl **F** = 0 is a necessary and sufficient condition for the existence of $V(x,y,z)$ such that Equation 4.2.1 holds.[3]

We can now express a conservative force **F** vectorially as

$$\mathbf{F} = -\mathbf{i}\frac{\partial V}{\partial x} - \mathbf{j}\frac{\partial V}{\partial y} - \mathbf{k}\frac{\partial V}{\partial z} \tag{4.2.3}$$

This equation can be written more succinctly as

$$\mathbf{F} = -\nabla V \tag{4.2.4}$$

where we have introduced the vector operator *del*:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{4.2.5}$$

The expression ∇V is also called the *gradient of* V and is sometimes written grad V. Mathematically, the gradient of a function is a vector that represents the maximum spatial derivative of the function in direction and magnitude. Physically, the negative gradient of the potential energy function gives the direction and magnitude of the force that acts on a particle located in a field created by other particles. The meaning of the negative sign is that the particle is urged to move in the direction of *decreasing* potential energy rather than in the opposite direction. This is illustrated in Figure 4.2.1. Here the potential energy function is plotted out in the form of contour lines representing the curves of constant potential energy. The force at any point is always normal to the equipotential curve or surface passing through the point in question.

[3] See, for example, S. I. Grossman, op cit. Also, Feng presents an interesting discussion of conservancy criteria when the force field contains singularities in *Amer. J. Phys.* 37, 616 (1969).

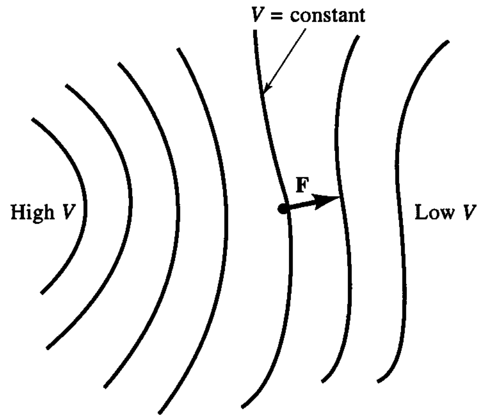

Figure 4.2.1 A force field represented by equipotential contour curves.

We can express curl **F** using the del operator. Look at the components of curl **F** in Equation 4.1.9. They are the components of the vector $\nabla \times \mathbf{F}$. Thus, $\nabla \times \mathbf{F} = \text{curl } \mathbf{F}$. The condition that a force be conservative can be written compactly as

$$\nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0 \qquad (4.2.6)$$

Furthermore, if $\nabla \times \mathbf{F} = 0$, then **F** can be derived from a scalar function V by the operation $\mathbf{F} = -\nabla V$, since $\nabla \times \nabla V \equiv 0$, or the curl of any gradient is identically 0.

We are now able to generalize the conservation of energy principle to three dimensions. The work done by a conservative force in moving a particle from point A to point B can be written as

$$\begin{aligned} \int_A^B \mathbf{F}\cdot d\mathbf{r} &= -\int_A^B \nabla\, V(\mathbf{r})\cdot d\mathbf{r} = -\int_{A_x}^{B_x} \frac{\partial V}{\partial x}dx - \int_{A_y}^{B_y} \frac{\partial V}{\partial y}dy - \int_{A_z}^{B_z} \frac{\partial V}{\partial z}dz \\ &= -\int_A^B dV(\mathbf{r}) = -\Delta V = V(A) - V(B) \end{aligned} \qquad (4.2.7)$$

The last step illustrates the fact that $\nabla V \cdot d\mathbf{r}$ is an *exact* differential equal to dV. The work done by any net force is always equal to the change in kinetic energy, so

$$\begin{aligned} &\int_A^B \mathbf{F}\cdot d\mathbf{r} = \Delta T = -\Delta V \\ &\therefore \Delta(T+V) = 0 \\ &\therefore T(A) + V(A) = T(B) + V(B) = E = \text{constant} \end{aligned} \qquad (4.2.8)$$

and we have arrived at our desired law of conservation of total energy.

If $\mathbf{F}'$ is a nonconservative force, it cannot be set equal to $-\nabla V$. The work increment $\mathbf{F}' \cdot d\mathbf{r}$ is not an exact differential and cannot be equated to $-dV$. In those cases where both conservative forces **F** and nonconservative forces $\mathbf{F}'$ are present, the total work increment is $(\mathbf{F}+\mathbf{F}')\cdot d\mathbf{r} = -dV + \mathbf{F}'\cdot d\mathbf{r} = dT$, and the generalized form of the work energy theorem becomes

$$\int_A^B \mathbf{F}'\cdot d\mathbf{r} = \Delta(T+V) = \Delta E \tag{4.2.9}$$

The total energy E does not remain a constant throughout the motion of the particle but increases or decreases depending upon the nature of the nonconservative force $\mathbf{F}'$. In the case of dissipative forces such as friction and air resistance, the direction of $\mathbf{F}'$ is always *opposite* the motion; hence, $\mathbf{F}'\cdot d\mathbf{r}$ is negative, and the total energy of the particle decreases as it moves through space.

EXAMPLE 4.2.1

Given the two-dimensional potential energy function

$$V(\mathbf{r}) = V_0 - \tfrac{1}{2}k\delta^2 e^{-r^2/\delta^2}$$

where $\mathbf{r} = \mathbf{i}x + \mathbf{j}y$ and V_0, k, and δ are constants, find the force function.

Solution:

We first write the potential energy function as a function of x and y,

$$V(x,y) = V_0 - \tfrac{1}{2}k\delta^2 e^{-(x^2+y^2)/\delta^2}$$

and then apply the gradient operator:

$$\begin{aligned}\mathbf{F} &= -\nabla V = -\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right)V(x,y)\\ &= -k(\mathbf{i}x + \mathbf{j}y)e^{-(x^2+y^2)/\delta^2}\\ &= -k\mathbf{r}e^{-r^2/\delta^2}\end{aligned}$$

Notice that the constant V_0 does not appear in the force function; its value is arbitrary. It simply raises or lowers the value of the potential energy function by a constant everywhere on the x, y plane and, thus, has no effect on the resulting force function.

We have plotted the potential energy function in Figure 4.2.2(a) and the resulting force function in Figure 4.2.2(b). The constants were taken to be $V_0 = 1$, $\delta^2 = 1/3$, and $k = 6$. The "hole" in the potential energy surface reaches greatest depth at the origin, which is obviously the location of a source of attraction. The concentric circles around the center of the hole are *equipotentials*—lines of constant potential energy. The radial lines are lines of steepest descent that depict the gradient of the potential energy surface. The slope of a radial line at any point on the plane is proportional to the force that a particle would experience there. The force field in Figure 4.2.2(b) shows the force vectors pointing towards the origin. They weaken both far from and near to the origin, where the slope of the potential energy function approaches zero.

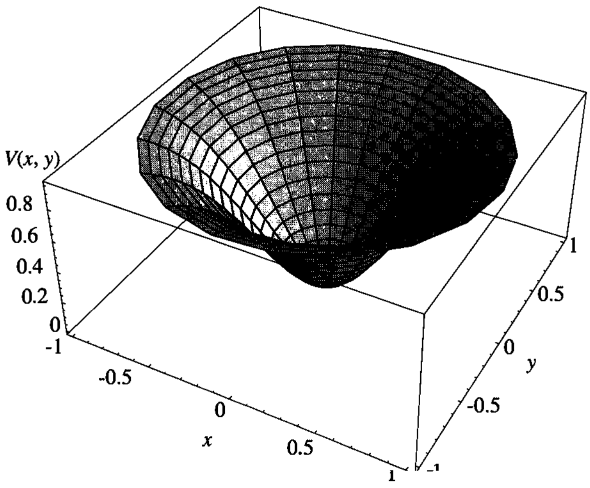

Figure 4.2.2a The potential energy function $V(x, y) = V_0 - \frac{1}{2}k\delta^2 e^{-(x^2+y^2)/\delta^2}$.

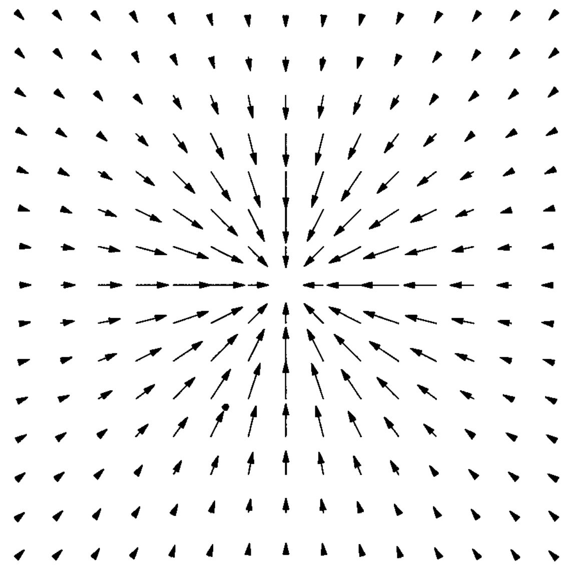

Figure 4.2.2b Force field gradient of potential energy function in Figure 4.2.2(a); $\mathbf{F} = -\Delta V - k(\mathbf{i}x + \mathbf{j}y)e^{-(x^2+y^2)/\delta^2}$.

EXAMPLE 4.2.2

Suppose a particle of mass m is moving in the above force field, and at time $t = 0$ the particle passes through the origin with speed v_0. What will the speed of the particle be at some small distance away from the origin given by $\mathbf{r} = \mathbf{e}_r\Delta$, where $\Delta \ll \delta$?

Solution:

The force is conservative, because a potential energy function exists. Thus, the total energy $E = T + V = \text{constant}$,

$$E = \frac{1}{2}mv^2 + V(\mathbf{r}) = \frac{1}{2}mv_0^2 + V(0)$$

and solving for v, we obtain

$$\begin{aligned}
v^2 &= v_0^2 + \frac{2}{m}[V(0) - V(\mathbf{r})] \\
&= v_0^2 + \frac{2}{m}\left[\left(V_0 - \tfrac{1}{2}k\delta^2\right) - \left(V_0 - \tfrac{1}{2}k\delta^2 e^{-\Delta^2/\delta^2}\right)\right] \\
&= v_0^2 - \frac{k\delta^2}{m}[1 - e^{-\Delta^2/\delta^2}] \\
&\approx v_0^2 - \frac{k\delta^2}{m}[1 - (1 - \Delta^2/\delta^2)] \\
&= v_0^2 - \frac{k}{m}\Delta^2
\end{aligned}$$

The potential energy is a quadratic function of the displacement Δ from the origin for small displacements, so this solution reduces to the conservation of energy for the simple harmonic oscillator

EXAMPLE 4.2.3

Is the force field $\mathbf{F} = \mathbf{i}xy + \mathbf{j}xz + \mathbf{k}yz$ conservative? The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xz & yz \end{vmatrix} = \mathbf{i}(z - x) + \mathbf{j}0 + \mathbf{k}(z - x)$$

The final expression is not zero for all values of the coordinates; hence, the field is *not* conservative.

EXAMPLE 4.2.4

For what values of the constants a, b, and c is the force $\mathbf{F} = \mathbf{i}(ax + by^2) + \mathbf{j}cxy$ conservative? Taking the curl, we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ax + by^2 & cxy & 0 \end{vmatrix} = \mathbf{k}(c - 2b)y$$

This shows that the force is conservative, provided $c = 2b$. The value of a is immaterial.

EXAMPLE 4.2.5

Show that the inverse-square law of force in three dimensions $\mathbf{F} = (-k/r^2)\mathbf{e}_r$ is conservative by the use of the curl. Use spherical coordinates. The curl is given in Appendix F as

$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta r & \mathbf{e}_\phi r \sin\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & rF_\phi \sin\theta \end{vmatrix}$$

We have $F_r = -k/r^2$, $F_\theta = 0$, $F_\phi = 0$. The curl then reduces to

$$\nabla \times \mathbf{F} = \frac{\mathbf{e}_\theta}{r\sin\theta} \frac{\partial}{\partial \phi}\left(\frac{-k}{r^2}\right) - \frac{\mathbf{e}_\phi}{r} \frac{\partial}{\partial \theta}\left(\frac{-k}{r^2}\right) = 0$$

which, of course, vanishes because both partial derivatives are zero. Thus, the force in question is conservative.

4.3 Forces of the Separable Type: Projectile Motion

A Cartesian coordinate system can be frequently chosen such that the components of a force field involve the respective coordinates alone, that is,

$$\mathbf{F} = \mathbf{i}F_x(x) + \mathbf{j}F_y(y) + \mathbf{k}F_z(z) \tag{4.3.1}$$

Forces of this type are *separable*. The curl of such a force is identically zero:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x(x) & F_y(y) & F_z(z) \end{vmatrix} = 0 \tag{4.3.2}$$

The x component is $\partial F_z(z)/\partial y - \partial F_y(y)/\partial z$ and a similar expression holds for the other components; therefore, the field is conservative because each partial derivative is of the mixed type and vanishes identically, because the coordinates x, y, and z are independent variables. The integration of the differential equations of motion is then very simple because each component equation is of the type $m\ddot{x} = F_x(x)$. In this case the equations can be solved by the methods described under rectilinear motion in Chapter 2.

In the event that the force components involve the time and the time derivatives of the respective coordinates, then it is no longer true that the force is necessarily conservative. Nevertheless, if the force is separable, then the component equations of motion are of the form $m\ddot{x} = F_x(x,\dot{x},t)$ and may be solved by the methods used in Chapter 2. Some examples of separable forces, both conservative and nonconservative, are discussed here and in the sections to follow.

Motion of a Projectile in a Uniform Gravitational Field

While a professor at Padua, Italy, during the years 1602–1608, Galileo spent much of his time projecting balls horizontally into space by rolling them down an inclined plane at the bottom of which he had attached a curved deflector. He hoped to demonstrate that the horizontal motion of objects would persist in the absence of frictional forces. If this were true, then the horizontal motion of heavy projectiles should not be affected much by air resistance and should occur at a constant speed. Galileo had already demonstrated that balls rolling down inclined planes attained a speed that was proportional to their time of roll, and so he could vary the speed of a horizontally projected ball in a controlled way. He observed that the horizontal distance traveled by a projectile increased in direct proportion to its speed of projection from the plane, thus, experimentally demonstrating his conviction. During these investigations, he was stunned to find that the paths these projectiles followed were parabolas! In 1609, already knowing the answer (affirming as gospel what every modern, problem-solving student of physics knows from experience), Galileo was able to prove mathematically that the parabolic trajectory of projectiles was a natural consequence of horizontal motion that was unaccelerated—and an independent vertical motion that was. Indeed, he understood the consequences of this motion as well. Before finally publishing his work in 1638 in *Discourse of Two New Sciences*, he wrote the following in a letter to one of his many scientific correspondents, Giovanni Baliani:

> . . . I treat also of the motion of projectiles, demonstrating various properties, among which is the proof that the projectile thrown by the projector, as would be the ball shot by firing artillery, makes its maximum flight and falls at the greatest distance when the piece is elevated at half a right angle, that is at 45°; and moreover, that other shots made at greater or less elevation come out equal when the piece is elevated an equal number of degrees above and below the said 45°.[4]

Not unlike the funding situation that science and technology finds itself in today, fundamental problems in the fledgling science of Galileo's time, which piqued the interest of the interested few, stood a good chance of being addressed if they related in some way to the military enterprise. Indeed, solving the motion of a projectile is one of the most famous problems in classical mechanics, and it is no accident that Galileo made the discovery partially supported by funds ultimately derived from wealthy patrons attempting to gain some military advantage over their enemies.

In 1597, Galileo had entered into a 10-year collaboration with a toolmaker, Marc' antonio Mazzoleni. In Galileo's day, the use of cannons to pound away at castle walls was more art than science. The Marquis del Monte in Florence and General del Monte in Padua, with whom Galileo had worked earlier, wondered if it were possible to devise a lightweight military "compass" that could be used to gauge the distance and height of a target,

[4]See for example, S. Drake and J. MacLachlan, Galileo's Discovery of the Parabolic Trajectory, *Scienti. Amer.* 232, 102–110, (March, 1975). Also see S. Drake, *Galileo at Work*, Mineola, NY, Dover, 1978.

to measure the angle of elevation of the cannon and to track the path of its projectile. Galileo solved the problem and developed the military compass, which his toolmaker produced in quantity in his workshop. There was a ready market for these devices, and they sold well. However, Galileo gained most of the support that enabled him to carry out his own investigation of motion by instructing students in the use of the compass and charging them 120 *lire* for the privilege. Though, like many professors today with which many readers of this text are likely familiar, Galileo more than resented any labor that prevented him from pursuing his own interests. "I'm always at the service of this or that person. I have to consume many hours of the day—often the best ones—in the service of others." Fortunately, he found enough time to carry out his experiments with rolling balls, which led to his discovery of the parabolic trajectory and ultimately helped lead Newton to the discovery of the classical laws of motion.

In 1611, Galileo informed Antonio de'Medici of his work on projectiles, which no doubt the powerful de'Medici family of Florence put to good use . . . and no doubt, went a long way towards helping Galileo secure their undying gratitude and unending patronage.

So with undying gratitude to Galileo and his successor, Newton, here we take only a few minutes—and not years—to solve the projectile problem.

No Air Resistance

For simplicity, we first consider the case of a projectile moving with no air resistance. Only one force, gravity, acts on the projectile, and, consistent with Galileo's observations as we shall see, it affects only its vertical motion. Choosing the z-axis to be vertical, we have the following equation of motion:

$$m\frac{d^2\mathbf{r}}{dt^2} = -\mathbf{k}mg \tag{4.3.3}$$

In the case of projectiles that don't rise too high or travel too far, we can take the acceleration of gravity, g, to be constant. Then the force function is conservative and of the separable type, because it is a special case of Equation 4.3.1. v_0 is the initial speed of the projectile, and the origin of the coordinate system is its initial position. Furthermore, there is no loss of generality if we orient the coordinate system so that the x-axis lies along the projection of the initial velocity onto the xy horizontal plane. Because there are no horizontally directed forces acting on the projectile, the motion occurs solely in the xz vertical plane. Thus, the position of the projectile at any time is (see Figure 4.3.1)

$$\mathbf{r} = \mathbf{i}x + \mathbf{k}z \tag{4.3.4}$$

The speed of the projectile can be calculated as a function of its height, z, using the energy equation (Equation 4.2.8)

$$\tfrac{1}{2}m(\dot{x}^2 + \dot{z}^2) + mgz = \tfrac{1}{2}mv_0^2 \tag{4.3.5a}$$

or equivalently,

$$v^2 = v_0^2 - 2gz \tag{4.3.5b}$$

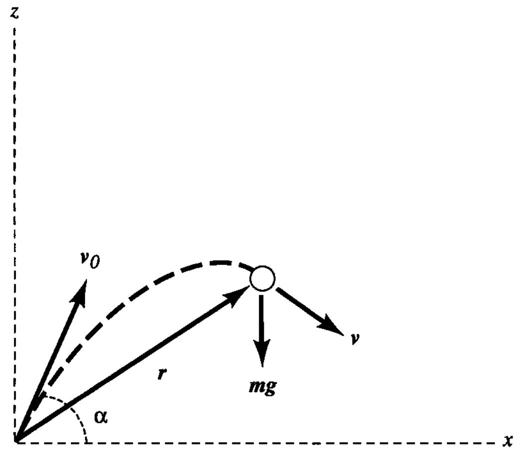

Figure 4.3.1 The parabolic path of a projectile.

We can calculate the velocity of the projectile at any instant of time by integrating Equation 4.3.3

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -\mathbf{k}gt + \mathbf{v}_0 \tag{4.3.6a}$$

The constant of integration is the initial velocity $\mathbf{v}_0$. In terms of unit vectors, the velocity is

$$\mathbf{v} = \mathbf{i}v_0 \cos\alpha + \mathbf{k}(v_0 \sin\alpha - gt) \tag{4.3.6b}$$

Integrating once more yields the position vector

$$\mathbf{r} = -\mathbf{k}\tfrac{1}{2}gt^2 + \mathbf{v}_0 t + \mathbf{r}_0 \tag{4.3.7a}$$

The constant of integration is the initial position of the projectile, $\mathbf{r}_0$, which is equal to zero; therefore, in terms of unit vectors, Equation 4.3.7a becomes

$$\mathbf{r} = \mathbf{i}(v_0 \cos\alpha)t + \mathbf{k}\left((v_0 \sin\alpha)t - \tfrac{1}{2}gt^2\right) \tag{4.3.7b}$$

In terms of components, the position of the projectile at any instant of time is

$$\begin{aligned} x &= \dot{x}_0 t = (v_0 \cos\alpha)t \\ y &= \dot{y}_0 t \equiv 0 \\ z &= \dot{z}_0 t - \tfrac{1}{2}gt^2 = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2 \end{aligned} \tag{4.3.7c}$$

$\dot{x}_0 = v_0 \cos\alpha$, $\dot{y}_0 = 0$, and $\dot{z}_0 = v_0 \sin\alpha$ are the components of the initial velocity $\mathbf{v}_0$.

We can now show, as Galileo did in 1609, that the path of the projectile is a parabola. We find $z(x)$ by using the first of Equations 4.3.7c to solve for t as a function of x and then substitute the resulting expression in the third of Equations 4.3.7c

$$t = \frac{x}{v_0 \cos\alpha} \tag{4.3.8}$$

$$z = (\tan\alpha)x - \left(\frac{g}{2v_0^2 \cos^2\alpha}\right)x^2 \tag{4.3.9}$$

Equation 4.3.9 is the equation of a parabola and is shown in Figure 4.3.1.

Like Galileo, we calculate several properties of projectile motion: (1) the maximum height, z_{max}, of the projectile, (2) the time, t_{max}, it takes to reach maximum height, (3) the time of flight, T, of the projectile, and (4) the range, R, and maximum range, R_{max}, of the projectile.

- First, we calculate the maximum height obtained by the projectile by using Equation 4.3.5b and noting that at maximum height the vertical component of the velocity of the projectile is zero so that its velocity is in the horizontal direction and equal to the constant horizontal component, $v_0 \cos \alpha$. Thus

$$v_0^2 \cos^2\alpha = v_0^2 - 2gz_{max} \tag{4.3.10}$$

We solve this to obtain

$$z_{max} = \frac{v_0^2 \sin^2\alpha}{2g} \tag{4.3.11}$$

- The time it takes to reach maximum height can be obtained from Equation 4.3.6b where we again make use of the fact that at maximum height, the vertical component of the velocity vanishes, so

$$v_0 \sin\alpha - gt_{max} = 0$$

or

$$t_{max} = \frac{v_0 \sin\alpha}{g} \tag{4.3.12}$$

- We can obtain the total time of flight T of the projectile by setting $z = 0$ in the last of Equations 4.3.7c, which yields

$$T = \frac{2v_0 \sin\alpha}{g} \tag{4.3.13}$$

This is twice the time it takes the projectile to reach maximum height. This indicates that the upward flight of the projectile to the apex of its trajectory is symmetrical to its downward flight away from it.

- Finally, we calculate the range of the projectile by substituting the total time of flight, T, into the first of Equations 4.3.7c, obtaining

$$R = x = \frac{v_0^2 \sin^2 2\alpha}{g} \tag{4.3.14}$$

R has its maximum value $R_{max} = v_0^2/g$ at $\alpha = 45°$.

Linear Air Resistance

We now consider the motion of a projectile subject to the force of air resistance. In this case, the motion does not conserve total energy, which continually diminishes during the flight of the projectile. To solve the problem analytically, we assume that the resisting force varies linearly with the velocity. To simplify the resulting equation of motions, we take the constant of proportionality to be $m\gamma$ where m is the mass of the projectile. The equation of motion is then

$$m\frac{d^2\mathbf{r}}{dt^2} = -m\gamma\mathbf{v} - \mathbf{k}mg \tag{4.3.15}$$

Upon canceling m's, the equation simplifies to

$$\frac{d^2\mathbf{r}}{dt^2} = -\gamma\mathbf{v} - \mathbf{k}g \tag{4.3.16}$$

Before integrating, we write Equation 4.3.16 in component form

$$\begin{aligned} \ddot{x} &= -\gamma\dot{x} \\ \ddot{y} &= -\gamma\dot{y} \\ \ddot{z} &= -\gamma\dot{z} - g \end{aligned} \tag{4.3.17}$$

We see that the equations are separated; therefore, each can be solved individually by the methods of Chapter 2. Using the results from Example 2.4.1, we can write down the solutions immediately, noting that here $\gamma = c_1/m$, c_1 being the linear drag coefficient. The results are

$$\begin{aligned} \dot{x} &= \dot{x}_0 e^{-\gamma t} \\ \dot{y} &= \dot{y}_0 e^{-\gamma t} \\ \dot{z} &= \dot{z}_0 e^{-\gamma t} - \frac{g}{\gamma}(1 - e^{-\gamma t}) \end{aligned} \tag{4.3.18}$$

for the velocity components. As before, we orient the coordinate system such that the x-axis lies along the projection of the initial velocity onto the xy horizontal plane. Then $\dot{y} = \dot{y}_0 = 0$

and the motion is confined to the xz vertical plane. Integrating once more, we obtain the position coordinates

$$
\begin{aligned}
x &= \frac{\dot{x}_0}{\gamma}(1 - e^{-\gamma t}) \\
z &= \left(\frac{\dot{z}_0}{\gamma} + \frac{g}{\gamma^2}\right)(1 - e^{-\gamma t}) - \frac{g}{\gamma}t
\end{aligned}
\tag{4.3.19}
$$

We have taken the initial position of the projectile to be zero, the origin of the coordinate system. This solution can be written vectorially as

$$
\mathbf{r} = \left(\frac{\mathbf{v}_0}{\gamma} + \frac{\mathbf{k}g}{\gamma^2}\right)(1 - e^{-\gamma t}) - \mathbf{k}\frac{gt}{\gamma} \tag{4.3.20}
$$

which can be verified by differentiation.

Contrary to the case of zero air resistance the path of the projectile is not a parabola, but rather a curve that lies below the corresponding parabolic trajectory. This is illustrated in Figure 4.3.2. Inspection of the x equation shows that, for large t, the value of x approaches the limiting value

$$
x \rightarrow \frac{\dot{x}_0}{\gamma} \tag{4.3.21}
$$

This means that the complete trajectory of the projectile, if it did not hit anything, would have a vertical asymptote as shown in Figure 4.3.2.

In the actual motion of a projectile through the atmosphere, the law of resistance is by no means linear; it is a very complicated function of the velocity. An accurate calculation of the trajectory can be done by means of numerical integration methods. (See the reference cited in Example 2.4.3.)

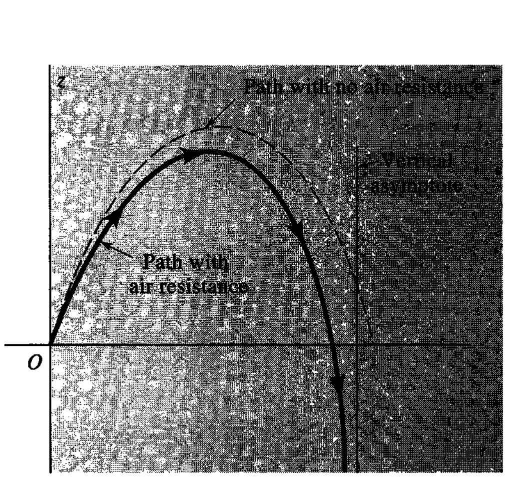

Figure 4.3.2 Comparison of the paths of a projectile with and without air resistance.

Horizontal Range

The horizontal range of a projectile with linear air drag is found by setting $z = 0$ in the second of Equations 4.3.19 and then eliminating t among the two equations. From the first of Equations 4.3.19, we have $1 - \gamma x/\dot{x}_0 = e^{-\gamma t}$, so $t = -\gamma^{-1}\ln(1-\gamma x/\dot{x}_0)$. Thus, the horizontal range x_{max} is given by the implicit expression

$$\left(\frac{\dot{z}_0}{\gamma}+\frac{g}{\gamma^2}\right)\frac{\gamma x_{max}}{\dot{x}_0}+\frac{g}{\gamma^2}\ln\left(1-\frac{\gamma x_{max}}{\dot{x}_0}\right)=0 \tag{4.3.22}$$

This is a transcendental equation and must be solved by some approximation method to find x_h. We can expand the logarithmic term by use of the series

$$\ln(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \cdots \tag{4.3.23}$$

which is valid for $|u| < 1$. With $u = \gamma x_{max}/\dot{x}_0$, it is left as a problem to show that this leads to the following expression for the horizontal range:

$$x_{max} = \frac{2\dot{x}_0\dot{z}_0}{g} - \frac{8\dot{x}_0\dot{z}_0^2}{3g^2}\gamma + \cdots \tag{4.3.24a}$$

If the projectile is fired at angle of elevation α with initial speed v_0, then $\dot{x}_0 = v_0 \cos\alpha$, $\dot{z}_0 = v_0 \sin\alpha$, and $2\dot{x}_0\dot{z}_0 = 2v_0^2 \sin\alpha \cos\alpha = v_0^2 \sin 2\alpha$. An equivalent expression is then

$$x_{max} = \frac{v_0^2 \sin 2\alpha}{g} - \frac{4v_0^3 \sin 2\alpha \sin\alpha}{3g^2}\gamma + \cdots \tag{4.3.24b}$$

The first term on the right is the range in the absence of air resistance. The remainder is the decrease due to air resistance.

EXAMPLE 4.3.1

Horizontal Range of a Golf Ball

For objects of baseball or golf-ball size traveling at normal speeds, the air drag is more nearly quadratic in v, rather than linear, as pointed out in Section 2.4. However, the approximate expression found above can be used to find the range for flat trajectories by "linearizing" the force function given by Equation 2.4.3, which may be written in three dimensions as

$$\mathbf{F}(\mathbf{v}) = -\mathbf{v}(c_1 + c_2|\mathbf{v}|)$$

To linearize it, we set $|\mathbf{v}|$ equal to the initial speed v_0, and so the constant γ is given by

$$\gamma = \frac{c_1 + c_2 v_0}{m}$$

(A better approximation would be to take the average speed, but that is not a given quantity.) Although this method exaggerates the effect of air drag, it allows a quick ballpark estimate to be found easily.

For a golf ball of diameter $D = 0.042$ m and mass $m = 0.046$ kg, we find that c_1 is negligible and so

$$\gamma = \frac{c_2 v_0}{m} = \frac{0.22 D^2 v_0}{m} = \frac{0.22(0.042)^2 v_0}{0.046} = 0.0084 v_0$$

numerically, where v_0 is in ms^{-1}. For a chip shot with, say, $v_0 = 20\ \mathrm{ms}^{-1}$, we find $\gamma = 0.0084 \times 20 = 0.17\ \mathrm{s}^{-1}$. The horizontal range is then, for $\alpha = 30°$,

$$x_{max} = \frac{(20)^2 \sin 60°}{9.8}\ \mathrm{m} - \frac{4(20)^3 \sin 60° \sin 30° \times 0.17}{3(9.8)^2}\ \mathrm{m} = 35.3\ \mathrm{m} - 8.2\ \mathrm{m} = 27.1\ \mathrm{m}$$

Our estimate, thus, gives a reduction of about one-fourth due to air drag on the ball.

EXAMPLE 4.3.2

A "Tape Measure" Home Run

Here we calculate what is required of a baseball player to hit a *tape measure home run,* or one that travels a distance in excess of 500 feet. In Section 2.4, we mentioned that the force of air drag on a baseball is essentially proportional to the square of its speed that is, $\mathbf{F}_D(v) = -c_2 |v|\, \mathbf{v}$. The actual air drag force on a baseball is more complicated than that. For example, the "constant" of proportionality c_2 varies somewhat with the speed of the baseball, and the air drag depends, among other things, on its spin and the way its cover is stitched on. We assume, however, for our purposes here that the above equation describes the situation adequately enough with the caveat that we take $c_2 = 0.15$ instead of the value 0.22 that we used previously. This value "normalizes" the air drag factor of a baseball traveling at speeds near 100 mph to that used by Robert Adair in *The Physics of Baseball.*[4]

Trajectories of bodies subject to an air drag force that depends upon the square of its speed cannot not be calculated analytically, so we use *Mathematica,* a computer software tool (see Appendix I), to find a numerical solution for the trajectory of a baseball in fight. Our goal is to find the minimum velocity and optimum angle of launch that a baseball batter must achieve to propel a baseball to maximum range. The situation we analyze concerns the longest home run ever hit in a regular-season, major league baseball game according to the *Guinness Book of Sports Records,* namely, a ball struck by

[4] R. K. Adair, *The Physics of Baseball,* 2nd ed., New York, Harper Collins.

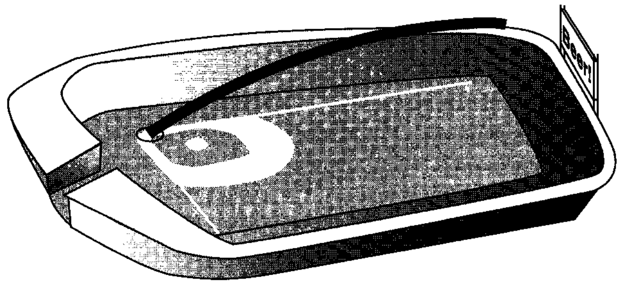

Figure 4.3.3a Trajectory of Mickey Mantle's home run on April 17, 1953, in Griffith Stadium, Washington, D.C.

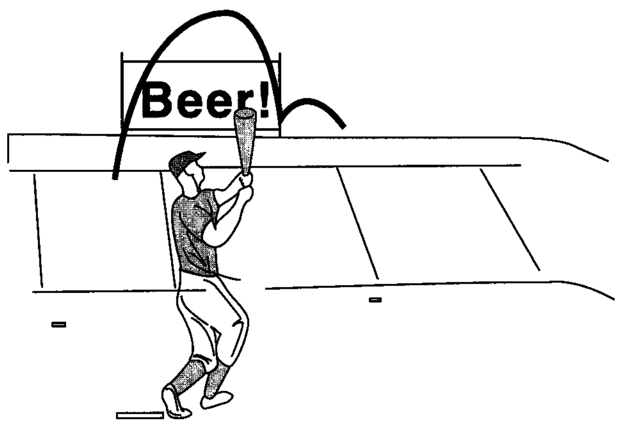

Figure 4.3.3b Trajectory of Mantle's home run as seen from the batter's perspective. (Mantle, a switch hitter, was actually batting right-handed against the left-handed Stobbs. This photo, showing him batting left-handed, is for the sake of illustration only.)

Mickey Mantle in 1953 that is claimed to have traveled 565 feet over the left field bleachers in old Griffith Stadium in Washington, D.C. The following is an account of that historic home run,[5] which one of your authors (GLC) was privileged to see while watching the baseball game as a bright-eyed young boy from those very left field bleachers for which he paid an entrance fee of 25¢ (oh, how times have changed).

> The Yankees were playing the Senators at Griffith Stadium in Washington, D.C. (The Washington Senators baseball club and Griffith Stadium no longer exist.) The stadium was a little sandbox of a ballpark but, as Mickey Mantle said, "It wasn't that easy to hit a home run there. There was a 90-foot wall in centerfield and there always seemed to be a breeze blowing in."
>
> Lefty Chuck Stobbs was on the mound. A light wind was blowing out from home plate for a change. It was two years to the day since Mickey's first major league game. Mickey stepped up to the plate. Stobbs fired a fast ball just below the letters, right where the Mick liked them, and he connected full-on with it. The ball took off toward the 391-foot sign in left-centerfield. It soared past the fence, over the bleachers and was headed out of the park when it ricocheted off a beer sign on the auxiliary football scoreboard (see Figures 4.3.3a and b). Although, slightly impeded, it continued its flight over neighboring

[5]This account of Mantle's Guinness Book of Sports Records home run can be found at the website, http://www.themick.com/10homers.html.

Fifth Street and landed in the backyard of 434 Oakdale Street, several houses up the block.

Billy Martin was on third when Mickey connected and, as a joke, he pretended to tag up like it was just a long fly ball. Mickey didn't notice Billy's shenanigans ("I used to keep my head down as I rounded the bases after a home run. I didn't want to show up the pitcher. I figured he felt bad enough already") and almost ran into Billy! If not for third base coach, Frank Crosetti, he would have. Had Mickey touched Billy he would have automatically been declared out and would have been credited with only a double.

Meanwhile, up in the press box, Yankees PR director, Red Patterson, cried out, "That one's got to be measured!" He raced out of the park and around to the far side of the park where he found 10-year-old Donald Dunaway with the ball. Dunaway showed Red the ball's impact in the yard and Red paced off the distance to the outside wall of Griffith Stadium. Contrary to popular myth, he did not use a tape measure, although he and Mickey were photographed together with a giant tape measure shortly after the historic blast. Using the dimensions of the park, its walls, and the distance he paced off, Patterson calculated the ball traveled 565 feet. However, sports writer Joe Trimble, when adding together the distances, failed to account for the three-foot width of the wall and came up with the 562-foot figure often cited. However, 565 feet is the correct number.

This was the first ball to ever go over Griffith Stadium's leftfield bleachers. Most believe the ball would have gone even further had it not hit the scoreboard (see Figure 4.3.3b). At any rate, it became one of the most famous home runs ever. It was headline news in a number of newspapers and a major story across the country. From that date forward, long home runs were referred to as "tape measure home runs."

So, did Mickey Mantle really hit a 565 foot home run, and, if so, at what angle did he strike the ball and what initial velocity did he impart to it? The equation of motion of a baseball subject to quadratic air drag is

$$m\ddot{\mathbf{r}} = -c_2\,|v|\,\mathbf{v} - mg\mathbf{k}$$

This separates into two component equations

$$m\ddot{x} = -c_2\,|v|\,\dot{x}$$
$$m\ddot{z} = -c_2\,|v|\,\dot{z} - mg$$

Letting $\gamma = c_2/m$, we obtain

$$\ddot{x} = -\gamma(\dot{x}^2 + \dot{z}^2)^{1/2}\dot{x}$$
$$\ddot{z} = -\gamma(\dot{x}^2 + \dot{z}^2)^{1/2}\dot{z} - g$$

Understandably, the game of baseball being the great American pastime, the weight (5.125 oz) and diameter (2.86 in) of the baseball are given in English units. In metric units, they are $m = 0.145$ kg and $D = 0.0728$ m respectively, so

$$\gamma = \frac{c_2}{m} = \frac{0.15D^2}{m} = \frac{0.15(0.0728)^2}{0.145}\ \text{meters}^{-1} = 0.0055\ \text{meters}^{-1}$$

The numerical solution to these second-order, coupled nonlinear differential equations can be generated by using the discussion of Mathematica given in Appendix I. Here we

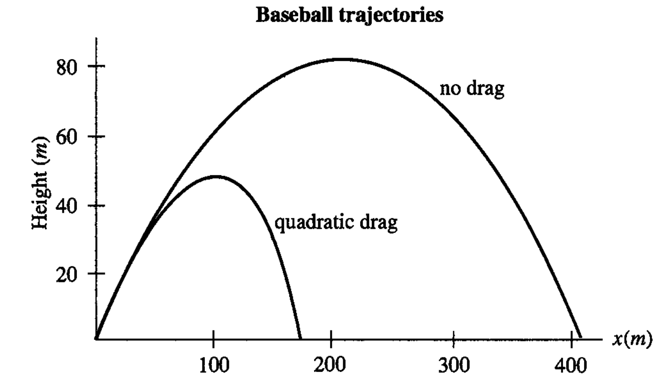

Figure 4.3.4 The calculated range of a baseball with quadratic air drag and without air drag. The range of the baseball is 172.2 m (565 ft) for an initial speed of 143.2 mph and elevation angle of 39 degrees.

simply outline the solution process, which involves an iterative procedure.

- First, we make reasonable guesses for the initial velocity (v_0) and angle (θ_0) of the baseball and then solve the coupled differential equations using these values.
- Plot the trajectory and find the x-axis intercept (the range)
- Hold v_0 fixed, and repeat the above using different values of θ_0 until we find the value of θ_0 that yields the maximum range
- Hold θ_0 fixed at the value found above (that yields the maximum range), and repeat the procedure again, but varying v_0 until we find the value that yields the required range of Mickey's tape measure home run, 565 feet (172.2 m)

The resultant trajectory is shown in Figure 4.3.4 along with the parameters that generated that trajectory. For comparison, we also show the trajectory of a similarly struck baseball in the absence of air resistance. We find that Mickey had to strike the ball at an elevation angle of $\theta_0 = 39°$ with an initial velocity of $v_0 = 143.2$ mph. Are these values reasonable? We would guess that the initial angle ought to be a bit less than the 45° one finds for the case of no air resistance. With resistance, a smaller launch angle (rather than one greater than 45°) corresponds to less time spent in flight during which air resistance can effectively act. What about the initial speed? Chuck Stobbs threw a baseball not much faster than 90 mph. Mantle could swing a bat such that its speed when striking the ball was approximately 90 mph. The coefficient of restitution (see Chapter 7) of baseballs is such that the resultant velocity imparted to the batted ball would be about 130 mph, so the value we've estimated is somewhat high but not outrageously so. If the ball Mantle hit in Griffith Stadium was assisted by a moderate tailwind, his Herculean swat seems possible. Wouldn't it have been spectacular to have seen Mantle hit one like that—in a vacuum?

4.4 The Harmonic Oscillator in Two and Three Dimensions

Consider the motion of a particle subject to a linear restoring force that is always directed toward a fixed point, the origin of our coordinate system. Such a force can be represented by the expression

$$\mathbf{F} = -k\mathbf{r} \tag{4.4.1}$$

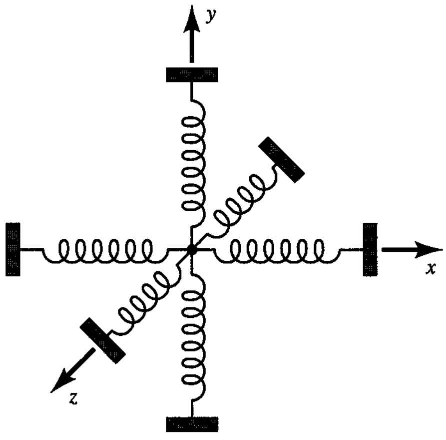

Figure 4.4.1 A model of a three-dimensional harmonic oscillator.

Accordingly, the differential equation of motion is simply expressed as

$$m\frac{d^2\mathbf{r}}{dt^2} = -k\mathbf{r} \tag{4.4.2}$$

The situation can be represented approximately by a particle attached to a set of elastic springs as shown in Figure 4.4.1. This is the three-dimensional generalization of the linear oscillator studied earlier. Equation 4.4.2 is the differential equation of the *linear isotropic oscillator.*

The Two-Dimensional Isotropic Oscillator

In the case of motion in a single plane, Equation 4.4.2 is equivalent to the two component equations

$$\begin{aligned} m\ddot{x} &= -kx \\ m\ddot{y} &= -ky \end{aligned} \tag{4.4.3}$$

These are separated, and we can immediately write down the solutions in the form

$$x = A\cos(\omega t + \alpha) \qquad y = B\cos(\omega t + \beta) \tag{4.4.4}$$

in which

$$\omega = \left(\frac{k}{m}\right)^{1/2} \tag{4.4.5}$$

The constants of integration A, B, α, and β are determined from the initial conditions in any given case.

To find the equation of the path, we eliminate the time t between the two equations. To do this, let us write the second equation in the form

$$y = B\cos(\omega t + \alpha + \Delta) \tag{4.4.6}$$

where

$$\Delta = \beta - \alpha \tag{4.4.7}$$

Then

$$y = B[\cos(\omega t + \alpha)\cos\Delta - \sin(\omega t + \alpha)\sin\Delta] \tag{4.4.8}$$

Combining the above with the first of Equations 4.4.4, we then have

$$\frac{y}{B} = \frac{x}{A}\cos\Delta - \left(1 - \frac{x^2}{A^2}\right)^{1/2}\sin\Delta \tag{4.4.9}$$

and upon transposing and squaring terms, we obtain

$$\frac{x^2}{A^2} - xy\frac{2\cos\Delta}{AB} + \frac{y^2}{B^2} = \sin^2\Delta \tag{4.4.10}$$

which is a quadratic equation in x and y. Now the general quadratic

$$ax^2 + bxy + cy^2 + dx + ey = f \tag{4.4.11}$$

represents an ellipse, a parabola, or a hyperbola, depending on whether the discriminant

$$b^2 - 4ac \tag{4.4.12}$$

is negative, zero, or positive, respectively. In our case the discriminant is equal to $-(2\sin\Delta/AB)^2$, which is negative, so the path is an ellipse as shown in Figure 4.4.2.

In particular, if the phase difference Δ is equal to $\pi/2$, then the equation of the path reduces to the equation

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \tag{4.4.13}$$

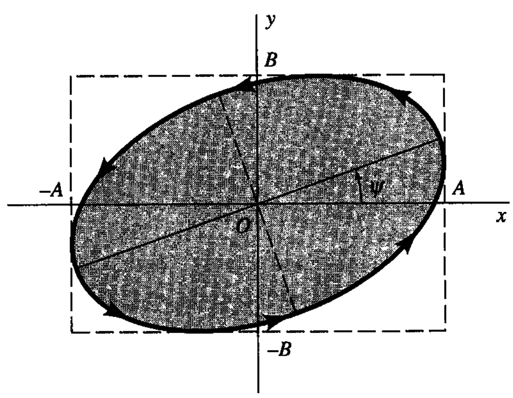

Figure 4.4.2 The elliptical path of a two-dimensional isotropic oscillator.

which is the equation of an ellipse whose axes coincide with the coordinate axes. On the other hand, if the phase difference is 0 or π, then the equation of the path reduces to that of a straight line, namely,

$$y = \pm \frac{B}{A} x \tag{4.4.14}$$

The positive sign is taken if $\Delta = 0$, and the negative sign, if $\Delta = \pi$. In the general case it is possible to show that the axis of the elliptical path is inclined to the x-axis by the angle ψ, where

$$\tan 2\psi = \frac{2AB \cos \Delta}{A^2 - B^2} \tag{4.4.15}$$

The derivation is left as an exercise.

The Three-Dimensional Isotropic Harmonic Oscillator

In the case of three-dimensional motion, the differential equation of motion is equivalent to the three equations

$$m\ddot{x} = -kx \qquad m\ddot{y} = -ky \qquad m\ddot{z} = -kz \tag{4.4.16}$$

which are separated. Hence, the solutions may be written in the form of Equations 4.4.4, or, alternatively, we may write

$$\begin{aligned} x &= A_1 \sin \omega t + B_1 \cos \omega t \\ y &= A_2 \sin \omega t + B_2 \cos \omega t \\ z &= A_3 \sin \omega t + B_3 \cos \omega t \end{aligned} \tag{4.4.17a}$$

The six constants of integration are determined from the initial position and velocity of the particle. Now Equations 4.4.16 can be expressed vectorially as

$$\mathbf{r} = \mathbf{A} \sin \omega t + \mathbf{B} \cos \omega t \tag{4.4.17b}$$

in which the components of $\mathbf{A}$ are A_1, A_2, and A_3, and similarly for $\mathbf{B}$. It is clear that the motion takes place entirely in a single plane, which is common to the two constant vectors $\mathbf{A}$ and $\mathbf{B}$, and that the path of the particle in that plane is an ellipse, as in the two-dimensional case. Hence, the analysis concerning the shape of the elliptical path under the two-dimensional case also applies to the three-dimensional case.

Nonisotropic Oscillator

The previous discussion considered the motion of the isotropic oscillator, wherein the restoring force is independent of the direction of the displacement. If the magnitudes of the components of the restoring force depend on the direction of the displacement, we have the case of the *nonisotropic oscillator*. For a suitable choice of axes, the differential equations for the nonisotropic case can be written

$$\begin{aligned} m\ddot{x} &= -k_1 x \\ m\ddot{y} &= -k_2 y \\ m\ddot{z} &= -k_3 z \end{aligned} \tag{4.4.18}$$

Here we have a case of *three* different frequencies of oscillation, $\omega_1 = \sqrt{k_1/m}$, $\omega_2 = \sqrt{k_2/m}$, and $\omega_3 = \sqrt{k_3/m}$, and the motion is given by the solutions

$$\begin{aligned} x &= A\cos(\omega_1 t + \alpha) \\ y &= B\cos(\omega_2 t + \beta) \\ z &= C\cos(\omega_3 t + \gamma) \end{aligned} \tag{4.4.19}$$

Again, the six constants of integration in the above equations are determined from the initial conditions. The resulting oscillation of the particle lies entirely within a rectangular box (whose sides are $2A$, $2B$, and $2C$) centered on the origin. In the event that ω_1, ω_2, and ω_3 are commensurate—that is, if

$$\frac{\omega_1}{n_1} = \frac{\omega_2}{n_2} = \frac{\omega_3}{n_3} \tag{4.4.20}$$

where n_1, n_2, and n_3 are integers—the path, called a *Lissajous* figure, is closed, because after a time $2\pi n_1/\omega_1 = 2\pi n_2/\omega_2 = 2\pi n_3/\omega_3$ the particle returns to its initial position and the motion is repeated. (In Equation 4.4.20 we assume that any common integral factor is canceled out.) On the other hand, if the ω's are *not* commensurate, the path is not closed. In this case the path may be said to completely fill the rectangular box mentioned above, at least in the sense that if we wait long enough, the particle comes arbitrarily close to any given point.

The net restoring force exerted on a given atom in a solid crystalline substance is approximately linear in the displacement in many cases. The resulting frequencies of oscillation usually lie in the infrared region of the spectrum: 10^{12} to 10^{14} vibrations per second.

Energy Considerations

In the preceding chapter we showed that the potential energy function of the one-dimensional harmonic oscillator is quadratic in the displacement, $V(x) = \frac{1}{2}kx^2$. For the general three-dimensional case, it is easy to verify that

$$V(x,y,z) = \tfrac{1}{2}k_1x^2 + \tfrac{1}{2}k_2y^2 + \tfrac{1}{2}k_3z^2 \tag{4.4.21}$$

because $F_x = -\partial V/\partial x = -k_1 x$, and similarly for F_y and F_z. If $k_1 = k_2 = k_3 = k$, we have the isotropic case, and

$$V(x,y,z) = \tfrac{1}{2}k(x^2 + y^2 + z^2) = \tfrac{1}{2}kr^2 \tag{4.4.22}$$

The total energy in the isotropic case is then given by the simple expression

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kr^2 = E \tag{4.4.23}$$

which is similar to that of the one-dimensional case discussed in the previous chapter.

EXAMPLE 4.4.1

A particle of mass m moves in two dimensions under the following potential energy function:

$$V(\mathbf{r}) = \tfrac{1}{2}k(x^2 + 4y^2)$$

Find the resulting motion, given the initial condition at $t = 0$: $x = a$, $y = 0$, $\dot{x} = 0$, $\dot{y} = v_0$.

Solution:

This is a nonisotropic oscillator potential. The force function is

$$\mathbf{F} = -\nabla V = -\mathbf{i}kx - \mathbf{j}4ky = m\ddot{\mathbf{r}}$$

The component differential equations of motion are then

$$m\ddot{x} + kx = 0 \qquad m\ddot{y} + 4ky = 0$$

The x-motion has angular frequency $\omega = (k/m)^{1/2}$, while the y-motion has angular frequency just twice that, namely, $\omega_y = (4k/m)^{1/2} = 2\omega$. We shall write the general solution in the form

$$x = A_1 \cos\omega t + B_1 \sin\omega t$$
$$y = A_2 \cos 2\omega t + B_2 \sin 2\omega$$

To use the initial condition we must first differentiate with respect to t to find the general expression for the velocity components:

$$\dot{x} = -A_1\omega \sin\omega t + B_1\omega \cos\omega t$$
$$\dot{y} = -2A_2\omega \sin 2\omega t + 2B_2\omega \cos 2\omega t$$

Thus, at $t = 0$, we see that the above equations for the components of position and velocity reduce to

$$a = A_1 \qquad 0 = A_2 \qquad 0 = B_1\omega \qquad v_0 = 2B_2\omega$$

These equations give directly the values of the amplitude coefficients, $A_1 = a$, $A_2 = B_1 = 0$, and $B_2 = v_0/2\omega$, so the final equations for the motion are

$$x = a \cos\omega t$$
$$y = \frac{v_0}{2\omega} \sin 2\omega t$$

The path is a Lissajous figure having the shape of a figure-eight as shown in Figure 4.4.3.

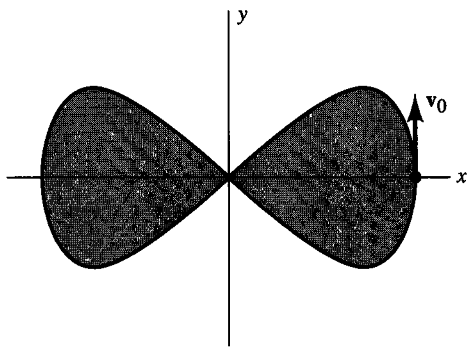

Figure 4.4.3 A Lissajous figure.

4.5 Motion of Charged Particles in Electric and Magnetic Fields

When an electrically charged particle is in the vicinity of other electric charges, it experiences a force. This force **F** is said to be caused by the electric field **E**, which arises from these other charges. We write

$$\mathbf{F} = q\mathbf{E} \tag{4.5.1}$$

where q is the electric charge carried by the particle in question.[6] The equation of motion of the particle is then

$$m\frac{d^2\mathbf{r}}{dt^2} = q\mathbf{E} \tag{4.5.2a}$$

or, in component form,

$$\begin{aligned} m\ddot{x} &= qE_x \\ m\ddot{y} &= qE_y \\ m\ddot{z} &= qE_z \end{aligned} \tag{4.5.2b}$$

The field components are, in general, functions of the position coordinates x, y, and z. In the case of time-varying fields (that is, if the charges producing **E** are moving), the components also involve t.

Let us consider a simple case, namely, that of a uniform constant electric field. We can choose one of the axes—say, the z-axis—to be in the direction of the field. Then $E_x = E_y = 0$, and $E = E_z$. The differential equations of motion of a particle of charge q moving in this field are then

$$\ddot{x} = 0 \qquad \ddot{y} = 0 \qquad \ddot{z} = \frac{qE}{m} = \text{constant} \tag{4.5.3}$$

[6] In SI units, F is in newtons, q in coulombs, and E in volts per meter.

These are of exactly the same form as those for a projectile in a uniform gravitational field. The path is, therefore, a parabola, if $\dot{x}$ and $\dot{y}$ are not both zero initially. Otherwise, the path is a straight line, as with a body falling vertically.

Textbooks dealing with electromagnetic theory[7] show that

$$\nabla \times \mathbf{E} = 0 \tag{4.5.4}$$

if **E** is due to static charges. This means that motion in such a field is conservative, and that there exists a potential function Φ such that $\mathbf{E} = -\nabla\Phi$. The potential energy of a particle of charge q in such a field is then $q\Phi$, and the total energy is constant and is equal to $\frac{1}{2}mv^2 + q\Phi$.

In the presence of a static magnetic field **B** (called the magnetic induction), the force acting on a moving particle is conveniently expressed by means of the cross product, namely,

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) \tag{4.5.5}$$

where **v** is the velocity and q is the charge.[8] The differential equation of motion of a particle moving in a purely magnetic field is then

$$m\frac{d^2\mathbf{r}}{dt^2} = q(\mathbf{v} \times \mathbf{B}) \tag{4.5.6}$$

Equation 4.5.6 states that the acceleration of the particle is always at right angles to the direction of motion. This means that the tangential component of the acceleration ($\dot{v}$) is zero, and so the particle moves with constant speed. This is true even if **B** is a varying function of the position **r**, as long as it does not vary with time.

EXAMPLE 4.5.1

Let us examine the motion of a charged particle in a uniform constant magnetic field. Suppose we choose the z-axis to be in the direction of the field; that is, we write

$$\mathbf{B} = \mathbf{k}B$$

The differential equation of motion now reads

$$m\frac{d^2\mathbf{r}}{dt^2} = q(\mathbf{v} \times \mathbf{k}B) = qB\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & 1 \end{vmatrix}$$

$$m(\mathbf{i}\ddot{x} + \mathbf{j}\ddot{y} + \mathbf{k}\ddot{z}) = qB(\mathbf{i}\dot{y} - \mathbf{j}\dot{x})$$

[7] For example, Reitz, Milford, and Christy, op cit.

[8] Equation 4.5.5 is valid for SI units: F is in newtons, q in coulombs, v in meters per second, and B in webers per square meter.

Equating components, we have

$$\begin{aligned} m\ddot{x} &= qB\dot{y} \\ m\ddot{y} &= -qB\dot{x} \\ \ddot{z} &= 0 \end{aligned} \tag{4.5.7}$$

Here, for the first time we meet a set of differential equations of motion that are *not* of the separated type. The solution is relatively simple, however, for we can integrate at once with respect to t, to obtain

$$\begin{aligned} m\dot{x} &= qBy + c_1 \\ m\dot{y} &= -qBx + c_2 \\ \dot{z} &= \text{constant} = \dot{z}_0 \end{aligned}$$

or

$$\dot{x} = \omega y + C_1 \qquad \dot{y} = -\omega x + C_2 \qquad \dot{z} = \dot{z}_0 \tag{4.5.8}$$

where we have used the abbreviation $\omega = qB/m$. The c's are constants of integration, and $C_1 = c_1/m$, $C_2 = c_2/m$. Upon inserting the expression for $\dot{y}$ from the second part of Equation 4.5.8 into the first part of Equation 4.5.7, we obtain the following separated equation for x:

$$\ddot{x} + \omega^2 x = \omega^2 a \tag{4.5.9}$$

where $a = C_2/\omega$. The solution is

$$x = a + A\cos(\omega t + \theta_0) \tag{4.5.10}$$

where A and θ_0 are constants of integration. Now, if we differentiate with respect to t, we have

$$\dot{x} = -A\omega\sin(\omega t + \theta_0) \tag{4.5.11}$$

The above expression for $\dot{x}$ may be substituted for the left-hand side of the first of Equations 4.5.8 and the resulting equation solved for y. The result is

$$y = b - A\sin(\omega t + \theta_0) \tag{4.5.12}$$

where $b = -C_1/\omega$. To find the form of the path of motion, we eliminate t between Equation 4.5.10 and Equation 4.5.12 to get

$$(x-a)^2 + (y-b)^2 = A^2 \tag{4.5.13}$$

Thus, the projection of the path of motion on the xy plane is a circle of radius A centered at the point (a, b). Because, from the third of Equations 4.5.8, the speed in the z direction is constant, we conclude that the path is a *helix*. The axis of the winding path is in the direction of the magnetic field, as shown in Figure 4.5.1. From Equation 4.5.12 we have

$$\dot{y} = -A\omega\cos(\omega t + \theta_0) \tag{4.5.14}$$

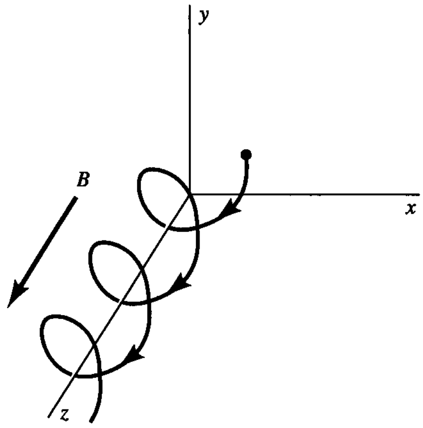

Figure 4.5.1 The helical path of a particle moving in a magnetic field.

Upon eliminating t between Equation 4.5.11 and Equation 4.5.14, we find

$$\dot{x}^2 + \dot{y}^2 = A^2\omega^2 = A^2\left(\frac{qB}{m}\right)^2 \tag{4.5.15}$$

Letting $v_1 = (\dot{x}^2 + \dot{y}^2)^{1/2}$, we see that the radius A of the helix is given by

$$A = \frac{v_1}{\omega} = v_1\frac{m}{qB} \tag{4.5.16}$$

If there is no component of the velocity in the z direction, the path is a circle of radius A. It is evident that A is directly proportional to the speed v_1 and that the angular frequency ω of motion in the circular path is independent of the speed. The angular frequency ω is known as the cyclotron frequency. The cyclotron, invented by Ernest Lawrence, depends for its operation on the fact that ω is independent of the speed of the charged particle.

4.6 Constrained Motion of a Particle

When a moving particle is restricted geometrically in the sense that it must stay on a certain definite surface or curve, the motion is said to be *constrained*. A piece of ice sliding around a bowl and a bead sliding on a wire are examples of constrained motion. The constraint may be complete, as with the bead, or it may be one-sided, as with the ice in the bowl. Constraints may be fixed, or they may be moving. In this chapter we study only fixed constraints.

The Energy Equation for Smooth Constraints

The total force acting on a particle moving under constraint can be expressed as the vector sum of the net external force **F** and the force of constraint **R**. The latter force is the reaction of the constraining agent upon the particle. The equation of motion may, therefore, be written

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{R} \tag{4.6.1}$$

If we take the dot product with the velocity **v**, we have

$$m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v} + \mathbf{R}\cdot\mathbf{v} \tag{4.6.2}$$

Now in the case of a *smooth* constraint—for example, a frictionless surface—the reaction **R** is normal to the surface or curve while the velocity **v** is tangent to the surface. Hence, **R** is perpendicular to **v**, and the dot product **R** · **v** vanishes. Equation 4.6.2 then reduces to

$$\frac{d}{dt}\left(\tfrac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) = \mathbf{F}\cdot\mathbf{v} \tag{4.6.3}$$

Consequently, if **F** is conservative, we can integrate as in Section 4.2, and we find that, even though the particle is constrained to move along the surface or curve, its total energy remains constant, namely,

$$\tfrac{1}{2}mv^2 + V(x,y,z) = \text{constant} = E \tag{4.6.4}$$

We might, of course, have expected this to be the case for frictionless constraints.

EXAMPLE 4.6.1

A particle is placed on top of a smooth sphere of radius a. If the particle is slightly disturbed, at what point will it leave the sphere?

Solution:

The forces acting on the particle are the downward force of gravity and the reaction **R** of the spherical surface. The equation of motion is

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} + \mathbf{R}$$

Let us choose coordinate axes as shown in Figure 4.6.1. The potential energy is then mgz, and the energy equation reads

$$\tfrac{1}{2}mv^2 + mgz = E$$

From the initial conditions ($v = 0$ for $z = a$) we have $E = mga$, so, as the particle slides down, its speed is given by the equation

$$v^2 = 2g(a - z)$$

Now, if we take radial components of the equation of motion, we can write the force equation as

$$-\frac{mv^2}{a} = -mg\cos\theta + R = -mg\frac{z}{a} + R$$

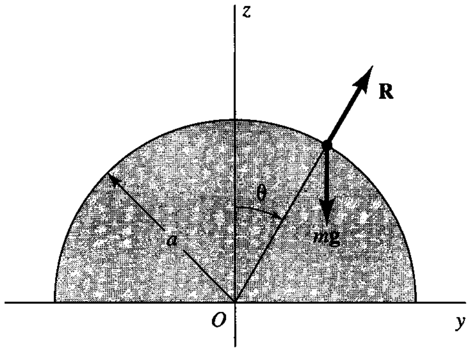

Figure 4.6.1 A particle sliding on a smooth sphere.

Hence,

$$R = mg\frac{z}{a} - \frac{mv^2}{a} = mg\frac{z}{a} - \frac{m}{a}2g(a-z)$$
$$= \frac{mg}{a}(3z - 2a)$$

Thus, R vanishes when $z = \frac{2}{3}a$, at which point the particle leaves the sphere. This may be argued from the fact that the sign of R changes from positive to negative there.

EXAMPLE 4.6.2

Constrained Motion on a Cycloid

Consider a particle sliding under gravity in a smooth cycloidal trough, Figure 4.6.2, represented by the parametric equations

$$x = A(2\phi + \sin 2\phi)$$
$$z = A(1 - \cos 2\phi)$$

where ϕ is the parameter. Now the energy equation for the motion, assuming no y-motion, is

$$E = \frac{m}{2}v^2 + V(z) = \frac{m}{2}(\dot{x}^2 + \dot{z}^2) + mgz$$

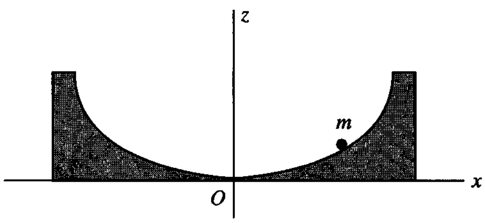

Figure 4.6.2 A particle sliding in a smooth cycloidal trough.

Because $\dot{x} = 2A\dot{\phi}(1 + \cos 2\phi)$ and $\dot{z} = 2A\dot{\phi}\sin 2\phi$, we find the following expression for the energy in terms of ϕ:

$$E = 4mA^2\dot{\phi}^2(1 + \cos 2\phi) + mgA(1 - \cos 2\phi)$$

or, by use of the identities $1 + \cos 2\phi = 2\cos^2\phi$ and $1 - \cos 2\phi = 2\sin^2\phi$,

$$E = 8mA^2\dot{\phi}^2 \cos^2\phi + 2mgA\sin^2\phi$$

Let us introduce the variable s defined by $s = 4A\sin\phi$. The energy equation can then be written

$$E = \frac{m}{2}\dot{s}^2 + \frac{1}{2}\left(\frac{mg}{4A}\right)s^2$$

This is just the energy equation for harmonic motion in the single variable s. Thus, the particle undergoes periodic motion whose frequency is independent of the amplitude of oscillation, unlike the simple pendulum for which the frequency depends on the amplitude. The periodic motion in the present case is said to be *isochronous*. (The linear harmonic oscillator under Hooke's law is, of course, isochronous.)

The Dutch physicist and mathematician Christiaan Huygens discovered the above fact in connection with attempts to improve the accuracy of pendulum clocks. He also discovered the theory of evolutes and found that the evolute of a cycloid is also a cycloid. Hence, by providing cycloidal "cheeks" for a pendulum, the motion of the bob must follow a cycloidal path, and the period is, thus, independent of the amplitude. Though ingenious, the invention never found extensive practical use.

Problems

4.1 Find the force for each of the following potential energy functions:
(a) $V = cxyz + C$
(b) $V = \alpha x^2 + \beta y^2 + \gamma z^2 + C$
(c) $V = ce^{-(\alpha x + \beta y + \gamma z)}$
(d) $V = cr^n$ in spherical coordinates

4.2 By finding the curl, determine which of the following forces are conservative:
(a) $\mathbf{F} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$
(b) $\mathbf{F} = \mathbf{i}y - \mathbf{j}x + \mathbf{k}z^2$
(c) $\mathbf{F} = \mathbf{i}y + \mathbf{j}x + \mathbf{k}z^3$
(d) $\mathbf{F} = -kr^{-n}\mathbf{e}_r$ in spherical coordinates

4.3 Find the value of the constant c such that each of the following forces is conservative:
(a) $\mathbf{F} = \mathbf{i}xy + \mathbf{j}cx^2 + \mathbf{k}z^3$
(b) $\mathbf{F} = \mathbf{i}(z/y) + c\mathbf{j}(xz/y^2) + \mathbf{k}(x/y)$

4.4 A particle of mass m moving in three dimensions under the potential energy function $V(x, y, z) = \alpha x + \beta y^2 + \gamma z^3$ has speed v_0 when it passes through the origin.
(a) What will its speed be if and when it passes through the point $(1, 1, 1)$?
(b) If the point $(1, 1, 1)$ is a turning point in the motion ($v = 0$), what is v_0?
(c) What are the component differential equations of motion of the particle?
(*Note*: It is *not* necessary to solve the differential equations of motion in this problem.)

4.5 Consider the two force functions
(a) $\mathbf{F} = \mathbf{i}x + \mathbf{j}y$
(b) $\mathbf{F} = \mathbf{i}y - \mathbf{j}x$
Verify that (a) is conservative and that (b) is nonconservative by showing that the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is independent of the path of integration for (a), but not for (b), by taking two paths in which the starting point is the origin (0, 0), and the endpoint is (1, 1). For one path take the line $x = y$. For the other path take the x-axis out to the point (1, 0) and then the line $x = 1$ up to the point (1, 1).

4.6 Show that the variation of gravity with height can be accounted for approximately by the following potential energy function:

$$V = mgz\left(1 - \frac{z}{r_e}\right)$$

in which r_e is the radius of the Earth. Find the force given by the above potential function. From this find the component differential equations of motion of a projectile under such a force. If the vertical component of the initial velocity is v_{0z}, how high does the projectile go? (Compare with Example 2.3.2.)

4.7 Particles of mud are thrown from the rim of a rolling wheel. If the forward speed of the wheel is v_0, and the radius of the wheel is b, show that the greatest height above the ground that the mud can go is

$$b + \frac{v_0^2}{2g} + \frac{gb^2}{2v_0^2}$$

At what point on the rolling wheel does this mud leave?
(*Note:* It is necessary to assume that $v_0^2 \geq bg$.)

4.8 A gun is located at the bottom of a hill of constant slope ϕ. Show that the range of the gun measured up the slope of the hill is

$$\frac{2v_0^2 \cos\alpha \sin(\alpha - \phi)}{g \cos^2\phi}$$

where α is the angle of elevation of the gun, and that the maximum value of the slope range is

$$\frac{v_0^2}{g(1 + \sin\phi)}$$

4.9 A cannon that is capable of firing a shell at speed V_0 is mounted on a vertical tower of height h that overlooks a level plain below.
(a) Show that the elevation angle α at which the cannon must be set to achieve maximum range is given by the expression

$$\csc^2\alpha = 2\left(1 + \frac{gh}{V_0^2}\right)$$

(b) What is the maximum range R of the cannon?

4.10 A movable cannon is positioned somewhere on the level plain below the cannon mounted on the tower of Problem 4.9. How close must it be positioned from the tower to fire a shell that can hit the cannon in that tower? Assume the two cannons have identical muzzle velocities V_0.

4.11 While playing in Yankee, Stadium, Mickey Mantle hits a baseball that attains a maximum height of 69 ft and strikes the ground 328 ft away from home plate unless it is caught by an outfielder. Assume that the outfielder can catch the ball sometime before it strikes the ground—only if it is less than 9.8 ft above the ground. Assume that Mantle hit the ball when it was 3.28 ft above the ground, and assume no air resistance. Within what horizontal distance can the fielder catch the ball?

4.12 A baseball pitcher can throw a ball more easily horizontally than vertically. Assume that the pitcher's throwing speed varies with elevation angle approximately as $v_0 \cos \frac{1}{2}\theta_0$ m/s, where θ_0 is the initial elevation angle and v_0 is the initial velocity when the ball is thrown horizontally. Find the angle θ_0 at which the ball must be thrown to achieve maximum (**a**) height and (**b**) range.
Find the values of the maximum (**c**) height and (**d**) range. Assume no air resistance and let $v_0 = 25$ m/s.

4.13 A gun can fire an artillery shell with a speed V_0 in any direction. Show that a shell can strike any target within the surface given by

$$g^2r^2 = V_0^4 - 2gV_0^2 z$$

where z is the height of the target and r is its horizontal distance from the gun. Assume no air resistance.

4.14 Write down the component form of the differential equations of motion of a projectile if the air resistance is proportional to the square of the speed. Are the equations separated? Show that the x component of the velocity is given by

$$\dot{x} = \dot{x}_0 e^{-\gamma s}$$

where s is the distance the projectile has traveled along the path of motion, and $\gamma = c_2/m$.

4.15 Fill in the steps leading to Equations 4.3.24a and b, giving the horizontal range of a projectile that is subject to linear air drag.

4.16 The initial conditions for a two-dimensional isotropic oscillator are as follows: $t = 0$, $x = A$, $y = 4A$, $\dot{x} = 0$, $\dot{y} = 3\omega A$ where ω is the angular frequency. Find x and y as functions of t. Show that the motion takes place entirely within a rectangle of dimensions $2A$ and $10A$. Find the inclination ψ of the elliptical path relative to the x-axis. Make a sketch of the path.

4.17 A small lead ball of mass m is suspended by means of six light springs as shown in Figure 4.4.1. The stiffness constants are in the ratio 1:4:9, so that the potential energy function can be expressed as

$$V = \frac{k}{2}(x^2 + 4y^2 + 9z^2)$$

At time $t = 0$ the ball receives a push in the (1, 1, 1) direction that imparts to it a speed v_0 at the origin. If $k = \pi^2 m$, numerically find x, y, and z as functions of the time t. Does the ball ever retrace its path? If so, for what value of t does it first return to the origin with the same velocity that it had at $t = 0$?

4.18 Complete the derivation of Equation 4.4.15.

4.19 An atom is situated in a simple cubic crystal lattice. If the potential energy of interaction between any two atoms is of the form $cr^{-\alpha}$, where c and α are constants and r is the distance between the two atoms, show that the total energy of interaction of a given atom with its six nearest neighbors is approximately that of the three-dimensional harmonic oscillator potential

$$V \simeq A + B(x^2 + y^2 + z^2)$$

where A and B are constants.

[*Note:* Assume that the six neighboring atoms are fixed and are located at the points $(\pm d, 0, 0), (0, \pm d, 0), (0, 0, \pm d)$, and that the displacement (x, y, z) of the given atom from the equilibrium position $(0, 0, 0)$ is small compared to d. Then $V = \Sigma c r_i^{-\alpha}$ where

$$r_1 = [(d - x)^2 + y^2 + z^2]^{1/2},$$

with similar expressions for $r_2, r_3, \ldots, r_6$. See the approximation formulas in Appendix D.]

4.20 An electron moves in a force field due to a uniform electric field $\mathbf{E}$ and a uniform magnetic field $\mathbf{B}$ that is at right angles to $\mathbf{E}$. Let $\mathbf{E} = \mathbf{j}E$ and $\mathbf{B} = \mathbf{k}B$. Take the initial position of the electron at the origin with initial velocity $\mathbf{v}_0 = \mathbf{i}v_0$ in the x direction. Find the resulting motion of the particle. Show that the path of motion is a cycloid:

$$x = a \sin \omega t + bt$$
$$y = a(1 - \cos \omega t)$$
$$z = 0$$

Cycloidal motion of electrons is used in an electronic tube called a magnetron to produce the microwaves in a microwave oven.

4.21 A particle is placed on a smooth sphere of radius b at a distance $b/2$ above the central plane. As the particle slides down the side of the sphere, at what point will it leave?

4.22 A bead slides on a smooth rigid wire bent into the form of a circular loop of radius b. If the plane of the loop is vertical, and if the bead starts from rest at a point that is level with the center of the loop, find the speed of the bead at the bottom and the reaction of the wire on the bead at that point.

4.23 Show that the period of the particle sliding in the cycloidal trough of Example 4.6.2 is $4\pi(A/g)^{1/2}$.

Computer Problems

C 4.1 A bomber plane, about to drop a bomb, suffers a malfunction of its targeting computer. The pilot notes that there is a strong horizontal wind, so she decides to release the bomb anyway, directly over the visually sighted target, as the plane flies over it directly into the wind. She calculates the required *ground speed* of the aircraft for her flying altitude of 50,000 feet and realizes that there is no problem flying her craft at that speed. She is perfectly confident that the wind speed will offset the plane's speed and blow the bomb "backwards" onto the intended target. She adjusts the ground speed of the aircraft accordingly and informs the bombardier to release the bomb at the precise instant that the target