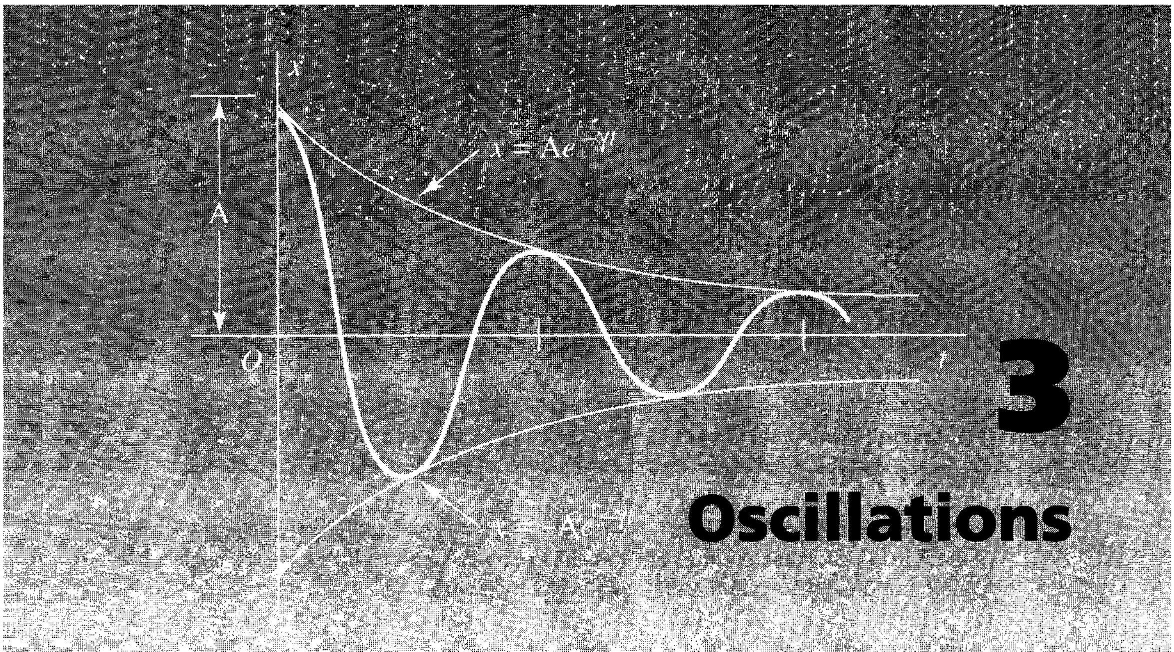

3 Oscillations

"These are the Phenomena of Springs and springy bodies, which as they have not hitherto been by any that I know reduced to Rules—It is very evident that the Rule or Law of Nature in every springing body is, that the force or power thereof to restore itself to its natural position is always proportionate to the distance or space it is removed therefrom—"

Robert Hooke—*De Potentia Restitutiva,* 1678

3.1 | Introduction

The solar system was the most fascinating and intensively studied mechanical system known to early humans. It is a marvelous example of periodic motion. It is not clear how long people would have toiled in mechanical ignorance were it not for this periodicity or had our planet been the singular observable member of the solar system. Everywhere around us we see systems engaged in a periodic dance: the small oscillations of a pendulum clock, a child playing on a swing, the rise and fall of the tides, the swaying of a tree in the wind, the vibrations of the strings on a violin. Even things that we cannot see march to the tune of a periodic beat: the vibrations of the air molecules in the woodwind instruments of a symphony, the hum of the electrons in the wires of our modern civilization, the vibrations of the atoms and molecules that make up our bodies. It is fitting that we cannot even say the word *vibration* properly without the tip of the tongue oscillating.

The essential feature that all these phenomena have in common is *periodicity,* a pattern of movement or displacement that repeats itself over and over again. The pattern may be simple or it may be complex. For example, Figure 3.1.1(a) shows a record of the horizontal displacement of a supine human body resting on a nearly frictionless surface, such as a thin layer of air. The body oscillates horizontally back and forth due to the mechanical action of the heart, pumping blood through and around the aortic

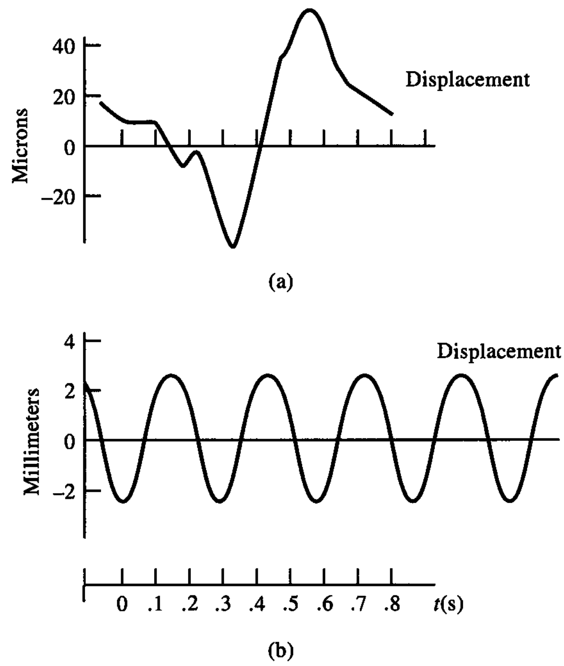

Figure 3.1.1 (a) Recoil vibrations of a human subject resting on a frictionless surface (in response to the pumping action of the heart). (b) Horizontal displacement of a simple pendulum about equilibrium.

arch. Such a recording is called a *ballistocardiogram.*[1] Figure 3.1.1(b) shows the almost perfect sine curve representing the horizontal displacement of a simple pendulum executing small oscillations about its equilibrium position. In both cases, the horizontal axis represents the steady advance of time. The period of the motion is readily identified as the time required for one complete cycle of the motion to occur.

It is with the hope of being able to describe all the complicated forms of periodic motion Mother Nature exhibits, such as that shown in Figure 3.1.1(a), that we undertake an analysis of her simplest form—*simple harmonic motion* (exemplified in Fig. 3.1.1(b)).

Simple harmonic motion exhibits two essential characteristics. (1) It is described by a second-order, linear differential equation with constant coefficients. Thus, the *superposition principle* holds; that is, if two particular solutions are found, their sum is also a solution. We will see evidence of this in the examples to come. (2) The period of the motion, or the time required for a particular configuration (not only position, but velocity as well) to repeat itself, is independent of the maximum displacement from equilibrium. We have already remarked that Galileo was the first to exploit this essential feature of the pendulum by using it as a clock. These features are true only if the displacements from equilibrium are "small." "Large" displacements result in the appearance of nonlinear terms in the differential equations of motion, and the resulting oscillatory solutions no longer obey the principle of superposition or exhibit amplitude-independent periods. We briefly consider this situation toward the end of this chapter.

[1] George B. Benedek and Felix M. H. Villars, *Physics—with Illustrative Examples from Medicine and Biology,* Addison-Wesley, New York, 1974.

3.2 | Linear Restoring Force: Harmonic Motion

One of the simplest models of a system executing simple harmonic motion is a mass on a frictionless surface attached to a wall by means of a spring. Such a system is shown in Figure 3.2.1. If X_e is the unstretched length of the spring, the mass will sit at that position, undisturbed, if initially placed there at rest. This position represents the equilibrium configuration of the mass, that is, the one in which its potential energy is a minimum or, equivalently, where the net force on it vanishes. If the mass is pushed or pulled away from this position, the spring will be either compressed or stretched. It will then exert a force on the mass, which will always attempt to restore it to its equilibrium configuration.

We need an expression for this restoring force if we are to calculate the motion of the mass. We can estimate the mathematical form of this force by appealing to arguments based on the presumed nature of the potential energy of this system. Recall from Example 2.3.3 that the Morse potential—the potential energy function of the diatomic hydrogen molecule, a bound system of two particles—has the shape of a well or a cup. Mathematically, it was given by the following expression:

$$V(x) = V_0(1 - \exp(-x/\delta))^2 - V_0 \tag{3.2.1}$$

We showed that this function exhibited quadratic behavior near its minimum and that the resulting force between the two atoms was linear, always acting to restore them to their equilibrium configuration. In general, any potential energy function can be described approximately by a polynomial function of the displacement x for displacements not too far from equilibrium

$$V(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{3.2.2a}$$

Furthermore, because only *differences* in potential energies are relevant for the behavior of physical systems, the constant term in each of the above expressions may be taken to be zero; this amounts to a simple reassignment of the value of the potential energy at some reference point. We also argue that the linear term in the above expression must be identically zero. This condition follows from the fact that the first derivative of any function must vanish at its minimum, presuming that the function and its derivatives are continuous, as they must be if the function is to describe the behavior of a real, physical system. Thus, the approximating polynomial takes the form

$$V(x) = a_2 x^2 + a_3 x^3 + \cdots \tag{3.2.2b}$$

For example, Figure 3.2.2(a) is a plot of the Morse potential along with an approximating eighth-order polynomial "best fit." The width δ of the potential and its depth

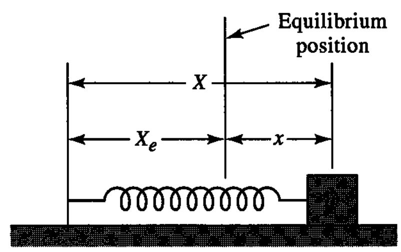

Figure 3.2.1 A model of the simple harmonic oscillator.

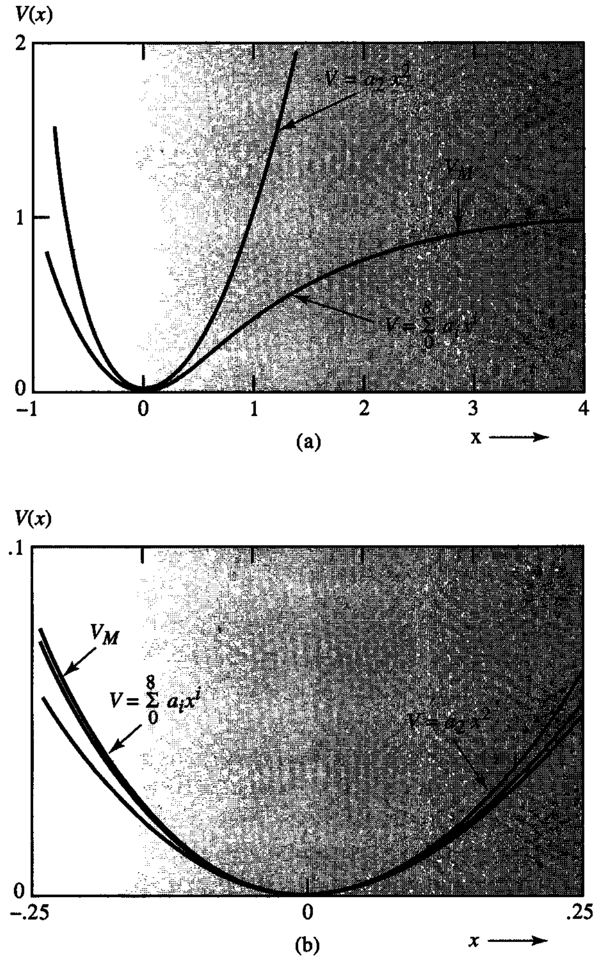

Figure 3.2.2 (a) The Morse potential, its eighth-order approximating polynomial and the quadratic term only. (b) Same as (a) but magnified in scale around $x = 0$.

(the V_0 coefficient) were both set equal to 1.0 (the bare constant V_0 was set equal to 0). The fit was made over the rather sizable range $\Delta x = [-1, 4] = 5\delta$. The result is

$$V(x) = \sum_{i=0}^{8} a_i x^i \tag{3.2.2c}$$

$$\begin{array}{lll} a_0 = \ 1.015 \cdot 10^{-4} & a_1 = \ 0.007 & a_2 = \ 0.995 \\ a_3 = -1.025 & a_4 = \ 0.611 & a_5 = -0.243 \\ a_6 = \ 0.061 & a_7 = -0.009 & a_8 = \ 5.249 \cdot 10^{-4} \end{array}$$

The polynomial function fits the Morse potential quite well throughout the quoted displacement range. If one examines closely the coefficients of the eighth-order fit, one sees that the first two terms are essentially zero, as we have argued they should be. Therefore, we also show plotted only the quadratic term $V(x) \approx a_2 \cdot x^2$. It seems as though this term does not agree very well with the Morse potential. However, if we "explode" the plot around $x = 0$ (see Fig. 3.2.2(b)), we see that for small displacements—say, $-0.1\delta \le x \le +0.1\delta$—there is virtually no difference among the purely quadratic term, the eighth-order polynomial fit, and the actual Morse potential. For such small displacements, the potential function is, indeed, purely quadratic. One might argue that this example was contrived; however, it is fairly representative of many physical systems.

The potential energy function for the system of spring and mass must exhibit similar behavior near the equilibrium position at X_e, dominated by a purely quadratic term. The spring's restoring force is thus given by the familiar Hooke's law,

$$F(x) = -\frac{dV(x)}{dx} = -(2a_2)x = -kx \tag{3.2.3}$$

where $k = 2a_2$ is the *spring constant*. In fact, this is how we define small displacements from equilibrium, that is, those for which Hooke's law is valid or the restoring force is linear. That the derived force must be a restoring one is a consequence of the fact that the derivative of the potential energy function must be negative for positive displacements from equilibrium and vice versa for negative ones. Newton's second law of motion for the mass can now be written as

$$m\ddot{x} + kx = 0 \tag{3.2.4a}$$

$$\ddot{x} + \frac{k}{m}x = 0 \tag{3.2.4b}$$

Equation 3.2.4b can be solved in a wide variety of ways. It is a second-order, linear differential equation with constant coefficients. As previously stated, the principle of superposition holds for its solutions. Before solving the equation here, we point out those characteristics we expect the solution to exhibit. First, the motion is both periodic and bounded. The mass vibrates back and forth between two limiting positions. Suppose we pull the mass out to some position x_{m1} and then release it from rest. The restoring force, initially equal to $-kx_{m1}$, pulls the mass toward the left in Figure 3.2.1, where it vanishes at $x = 0$, the equilibrium position. The mass now finds itself moving to the left with some velocity v, and so it passes on through equilibrium. Then the restoring force begins to build up strength as the spring compresses, but now directed toward the right. It slows the mass down until it stops, just for an instant, at some position, $-x_{m2}$. The spring, now fully compressed, starts to shove the mass back toward the right. But again momentum carries it through the equilibrium position until the now-stretching spring finally manages to stop it—we might guess—at x_{m1}, the initial configuration of the system. This completes one cycle of the motion—a cycle that repeats itself, apparently forever! Clearly, the resultant functional dependence of x upon t must be represented by a periodic and bounded function. Sine and/or cosine functions come to mind, because they exhibit the sort of behavior we are describing here. In fact, sines and cosines are the real solutions of Equation 3.2.4b. Later on, we show that other functions, imaginary exponentials, are actually equivalent to sines and cosines and are easier to use in describing the more complicated systems soon to be discussed.

A solution is given by

$$x = A\sin(\omega_0 t + \phi_0) \tag{3.2.5}$$

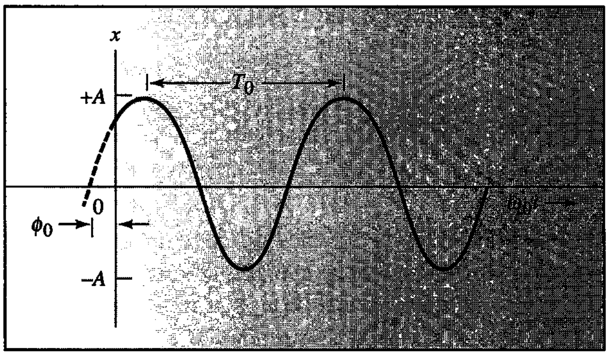

Figure 3.2.3 Displacement versus $\omega_0 t$ for the simple harmonic oscillator.

which can be verified by substituting it into Equation 3.2.4b

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{3.2.6}$$

is the *angular frequency* of the system. The motion represented by Equation 3.2.5 is a sinusoidal oscillation about equilibrium. A graph of the displacement x versus $\omega_0 t$ is shown in Figure 3.2.3. The motion exhibits the following features. (1) It is characterized by a single angular frequency ω_0. The motion repeats itself after the angular argument of the sine function ($\omega_0 t + \phi_0$) advances by 2π or after one cycle has occurred (hence, the name *angular frequency* for ω_0). The time required for a phase advance of 2π is given by

$$\begin{aligned} \omega_0(t+T_0)+\phi_0 &= \omega_0 t+\phi_0+2\pi \\ \therefore T_0 &= \frac{2\pi}{\omega_0} \end{aligned} \tag{3.2.7}$$

T_0 is called the period of the motion. (2) The motion is bounded; that is, it is confined within the limits $-A \le x \le +A$. A, the maximum displacement from equilibrium, is called the amplitude of the motion. It is independent of the angular frequency ω_0. (3) The phase angle ϕ_0 is the initial value of the angular argument of the sine function. It determines the value of the displacement x at time $t = 0$. For example, at $t = 0$ we have

$$x(t=0) = A\sin(\phi_0) \tag{3.2.8}$$

The maximum displacement from equilibrium occurs at a time t_m given by the condition that the angular argument of the sine function is equal to $\pi/2$, or

$$\omega_0 t_m = \frac{\pi}{2} - \phi_0 \tag{3.2.9}$$

One commonly uses the term *frequency* to refer to the reciprocal of the period of the oscillation or

$$f_0 = \frac{1}{T_0} \tag{3.2.10}$$

where f_0 is the number of cycles of vibration per unit time. It is related to the angular frequency ω_0 by

$$2\pi f_0 = \omega_0 \tag{3.2.11a}$$

$$f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{3.2.11b}$$

The unit of frequency (cycles per second, or s^{-1}) is called the *hertz* (Hz) in honor of Heinrich Hertz, who is credited with the discovery of radio waves. Note that 1 Hz = 1 s^{-1}. The word *frequency* is used sloppily sometimes to mean either cycles per second or radians per second (angular frequency). The meaning is usually clear from the context.

Constants of the Motion and Initial Conditions

Equation 3.2.5, the solution for simple harmonic motion, contains two arbitrary constants, A and ϕ_0. The value of each constant can be determined from knowledge of the initial conditions of the specific problem at hand. As an example of the simplest and most commonly described initial condition, consider a mass initially displaced from equilibrium to a position x_m, where it is then released from rest. The displacement at $t = 0$ is a maximum. Therefore, $A = x_m$ and $\phi_0 = \pi/2$.

As an example of another simple situation, suppose the oscillator is at rest at $x = 0$, and at time $t = 0$ it receives a sharp blow that imparts to it an initial velocity v_0 in the positive x direction. In such a case the initial phase is given by $\phi_0 = 0$. This automatically ensures that the solution yields $x = 0$ at $t = 0$. The amplitude can be found by differentiating x to get the velocity of the oscillator as a function of time and then demanding that the velocity equal v_0 at $t = 0$. Thus,

$$v(t) = \dot{x}(t) = \omega_0 A\cos(\omega_0 t + \phi_0) \tag{3.2.12a}$$

$$v(0) = v_0 = \omega_0 A \tag{3.2.12b}$$

$$\therefore\ A = \frac{v_0}{\omega_0} \tag{3.2.12c}$$

For a more general scenario, consider a mass initially displaced to some position x_0 and given an initial velocity v_0. The constants can then be determined as follows:

$$x(0) = A\sin\phi_0 = x_0 \tag{3.2.13a}$$

$$\dot{x}(0) = \omega_0 A\cos\phi_0 = v_0 \tag{3.2.13b}$$

$$\therefore\ \tan\phi_0 = \frac{\omega_0 x_0}{v_0} \tag{3.2.13c}$$

$$A^2 = x_0^2 + \frac{v_0^2}{\omega_0^2} \tag{3.2.13d}$$

This more general solution reduces to either of those described above, as can easily be seen by setting v_0 or x_0 equal to zero.

Simple Harmonic Motion as the Projection of a Rotating Vector

Imagine a vector **A** rotating at a constant angular velocity ω_0. Let this vector denote the position of a point P in uniform circular motion. The projection of the vector onto a line (which we call the x-axis) in the same plane as the circle traces out simple harmonic motion. Suppose the vector **A** makes an angle θ with the x-axis at some time t, as shown in Figure 3.2.4. Because $\dot{\theta} = \omega_0$, the angle θ increases with time according to

$$\theta = \omega_0 t + \theta_0 \tag{3.2.14}$$

where θ_0 is the value of θ at $t = 0$. The projection of P onto the x-axis is given by

$$x = A\cos\theta = A\cos(\omega_0 t + \theta_0) \tag{3.2.15}$$

This point oscillates in simple harmonic motion as P goes around the circle in uniform angular motion.

Our picture describes x as a cosine function of t. We can show the equivalence of this expression to the sine function given by Equation 3.2.5 by measuring angles to the vector **A** from the y-axis, instead of the x-axis as shown in Figure 3.2.4. If we do this, the projection of **A** onto the x-axis is given by

$$x = A\sin(\omega_0 t + \phi_0) \tag{3.2.16}$$

We can see this equivalence in another way. We set the phase difference between ϕ_0 and θ_0 to $\pi/2$ and then substitute into the above equation, obtaining

$$\phi_0 - \theta_0 = \frac{\pi}{2} \tag{3.2.17a}$$

$$\begin{aligned}\cos(\omega_0 t + \theta_0) &= \cos\left(\omega_0 t + \phi_0 - \frac{\pi}{2}\right)\\ &= \sin(\omega_0 t + \phi_0)\end{aligned} \tag{3.2.17b}$$

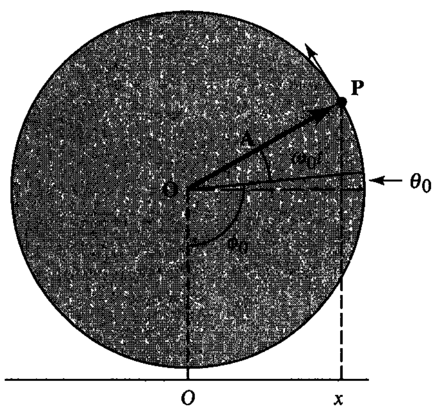

Figure 3.2.4 Simple harmonic motion as a projection of uniform circular motion.

We now see that simple harmonic motion can be described equally well by a sine function or a cosine function. The one we choose is largely a matter of taste; it depends upon our choice of initial phase angle to within an arbitrary constant.

You might guess from the above commentary that we could use a sum of sine and cosine functions to represent the general solution for harmonic motion. For example, we can convert the sine solution of Equation 3.2.5 directly to such a form, using the trigonometric identity for the sine of a sum of angles:

$$\begin{aligned} x(t) &= A\ \sin(\omega_0 t + \phi_0) = A\ \sin\phi_0\ \cos\omega_0 t + A\ \cos\phi_0\ \sin\omega_0 t \\ &= C\ \cos\omega_0 t + D\ \sin\omega_0 t \end{aligned} \tag{3.2.18}$$

Neither A nor ϕ_0 appears explicitly in the solution. They are there implicitly; that is,

$$\tan\phi_0 = \frac{C}{D} \qquad A^2 = C^2 + D^2 \tag{3.2.19}$$

There are occasions when this form may be the preferred one.

Effect of a Constant External Force on a Harmonic Oscillator

Suppose the same spring shown in Figure 3.2.1 is held in a vertical position, supporting the same mass m (Fig. 3.2.5). The total force acting is now given by adding the weight mg to the restoring force,

$$F = -k(X - X_e) + mg \tag{3.2.20}$$

where the positive direction is down. This equation could be written $F = -kx + mg$ by defining x to be $X - X_e$, as previously. However, it is more convenient to define the variable x in a different way, namely, as the displacement from the *new* equilibrium position

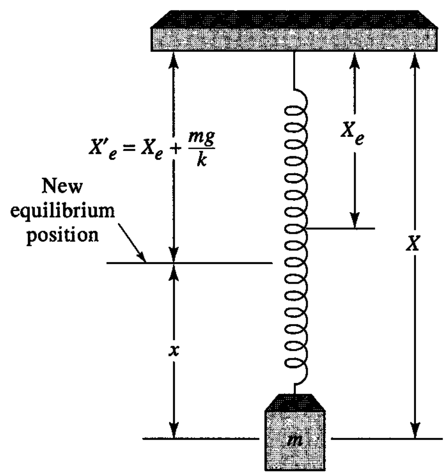

Figure 3.2.5 The vertical case for the harmonic oscillator.

X_e' obtained by setting $F = 0$ in Equation 3.2.20: $0 = -k(X_e' - X_e) + mg$, which gives $X_e' = X_e + mg/k$. We now define the displacement as

$$x = X - X_e' = X - X_e - \frac{mg}{k} \tag{3.2.21}$$

Putting this into Equation 3.2.20 gives, after a very little algebra,

$$F = -kx \tag{3.2.22}$$

so the differential equation of motion is again

$$m\ddot{x} + kx = 0 \tag{3.2.23}$$

and our solution in terms of our newly defined x is identical to that of the horizontal case. It should now be evident that *any* constant external force applied to a harmonic oscillator merely shifts the equilibrium position. The equation of motion remains unchanged if we measure the displacement x from the new equilibrium position.

EXAMPLE 3.2.1

When a light spring supports a block of mass m in a vertical position, the spring is found to stretch by an amount D_1 over its unstretched length. If the block is furthermore pulled downward a distance D_2 from the equilibrium position and released—say, at time $t = 0$—find (a) the resulting motion, (b) the velocity of the block when it passes back upward through the equilibrium position, and (c) the acceleration of the block at the top of its oscillatory motion.

Solution:

First, for the equilibrium position we have

$$F_x = 0 = -kD_1 + mg$$

where x is chosen positive downward. This gives us the value of the stiffness constant:

$$k = \frac{mg}{D_1}$$

From this we can find the angular frequency of oscillation:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{D_1}}$$

We shall express the motion in the form $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$. Then

$$\dot{x} = -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t.$$

From the initial conditions we find

$$x_0 = D_2 = A \qquad \dot{x}_0 = 0 = B\omega_0 \qquad B = 0$$

The motion is, therefore, given by

(a) $$x(t) = D_2 \cos\left(\sqrt{\frac{g}{D_1}}t\right)$$

in terms of the given quantities. Note that the mass m does not appear in the final expression. The velocity is then

$$\dot{x}(t) = -D_2\sqrt{\frac{g}{D_1}} \sin\left(\sqrt{\frac{g}{D_1}}t\right)$$

and the acceleration

$$\ddot{x}(t) = -D_2 \frac{g}{D_1} \cos\left(\sqrt{\frac{g}{D_1}}t\right)$$

As the block passes upward through the equilibrium position, the argument of the sine term is $\pi/2$ (one-quarter period), so

(b) $$\dot{x} = -D_2\sqrt{\frac{g}{D_1}} \qquad (center)$$

At the top of the swing the argument of the cosine term is π (one-half period), which gives

(c) $$\ddot{x} = D_2 \frac{g}{D_1} \qquad (top)$$

In the case $D_1 = D_2$, the downward acceleration at the top of the swing is just g. This means that the block, at that particular instant, is in *free fall*; that is, the spring is exerting zero force on the block.

EXAMPLE 3.2.2

The Simple Pendulum

The so-called simple pendulum consists of a small plumb bob of mass m swinging at the end of a light, inextensible string of length l, Figure 3.2.6. The motion is along a circular arc defined by the angle θ, as shown. The restoring force is the component of the weight mg acting in the direction of increasing θ along the path of motion: $F_s = -mg \sin\theta$. If we treat the bob as a particle, the differential equation of motion is, therefore,

$$m\ddot{s} = -mg \sin\theta$$

Now $s = l\theta$, and, for small θ, $\sin\theta = \theta$ to a fair approximation. So, after canceling the m's and rearranging terms, we can write the differential equation of motion in terms of either θ or s as follows:

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad \ddot{s} + \frac{g}{l}s = 0$$

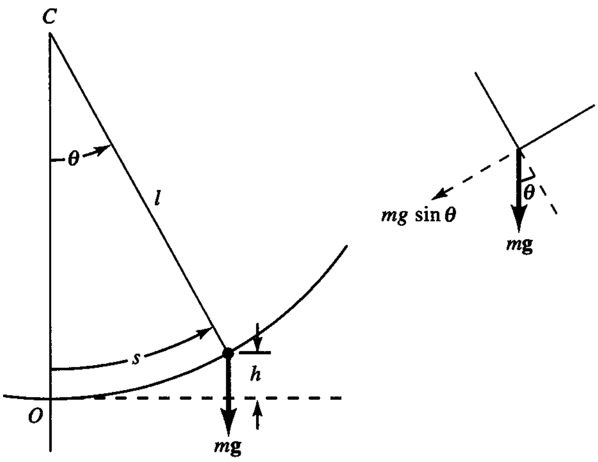

Figure 3.2.6 The simple pendulum.

Although the motion is along a curved path rather than a straight line, the differential equation is mathematically identical to that of the linear harmonic oscillator, Equation 3.2.4b, with the quantity g/l replacing k/m. Thus, to the extent that the approximation $\sin\theta = \theta$ is valid, we can conclude that the motion is simple harmonic with angular frequency

$$\omega_0 = \sqrt{\frac{g}{l}}$$

and period

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$$

This formula gives a period of very nearly 2 s, or a half-period of 1 s, when the length l is 1 m. More accurately, for a half-period of 1 s, known as the "seconds pendulum," the precise length is obtained by setting $T_0 = 2$ s and solving for l. This gives $l = g/\pi^2$ numerically, when g is expressed in m/s^2. At sea level at a latitude of 45°, the value of the acceleration of gravity is $g = 9.8062$ m/s^2. Accordingly, the length of a seconds pendulum at that location is 9.8062/9.8696 = 0.9936 m.

3.3 | Energy Considerations in Harmonic Motion

Consider a particle under the action of a linear restoring force $F_x = -kx$. Let us calculate the work done by an external force F_{ext} in moving the particle from the equilibrium position ($x = 0$) to some position x. Assume that we move the particle very slowly so that it does not gain any kinetic energy; that is, the applied external force is barely greater in magnitude than the restoring force $-kx$; hence, $F_{ext} = -F_x = kx$, so

$$W = \int_0^x F_{ext}\, dx = \int_0^x kx\, dx = \frac{k}{2}x^2 \tag{3.3.1}$$

In the case of a spring obeying Hooke's law, the work is stored in the spring as potential energy: $W = V(x)$, where

$$V(x) = \tfrac{1}{2}kx^2 \tag{3.3.2}$$

Thus, $F_x = -dV/dx = -kx$, as required by the definition of V. The total energy, when the particle is undergoing harmonic motion, is given by the sum of the kinetic and potential energies, namely,

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 \tag{3.3.3}$$

This equation epitomizes the harmonic oscillator in a rather fundamental way: The kinetic energy is quadratic in the velocity variable, and the potential energy is quadratic in the displacement variable. The total energy is constant if there are no other forces except the restoring force acting on the particle.

The motion of the particle can be found by starting with the energy equation (3.3.3). Solving for the velocity gives

$$\dot{x} = \pm\left(\frac{2E}{m} - \frac{kx^2}{m}\right)^{1/2} \tag{3.3.4}$$

which can be integrated to give t as a function of x as follows:

$$t = \int \frac{dx}{\pm[(2E/m) - (k/m)x^2]^{1/2}} = \mp(m/k)^{1/2} \cos^{-1}(x/A) + C \tag{3.3.5}$$

in which C is a constant of integration and A is the amplitude given by

$$A = \left(\frac{2E}{k}\right)^{1/2} \tag{3.3.6}$$

Upon solving the integrated equation for x as a function of t, we find the same relationship as in the preceding section, with the addition that we now have an explicit value for the amplitude. We can also obtain the amplitude directly from the energy equation (3.3.3) by finding the turning points of the motion where $\dot{x} = 0$: The value of x must lie between $\pm A$ in order for $\dot{x}$ to be real. This is illustrated in Figure 3.3.1.

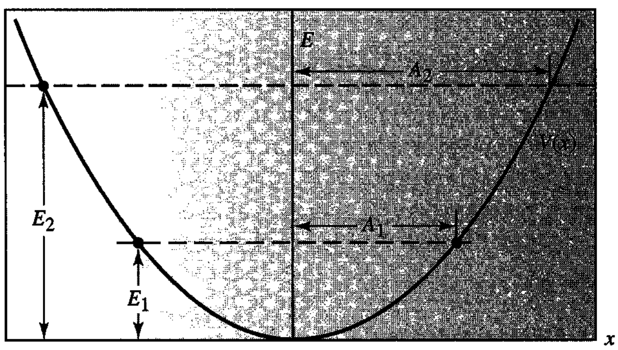

Figure 3.3.1 Graph of the parabolic potential energy function of the harmonic oscillator. The turning points defining the amplitude are indicated for two different values of the total energy.

We also see from the energy equation that the maximum value of the speed, which we call v_{max}, occurs at $x = 0$. Accordingly, we can write

$$E = \tfrac{1}{2}mv_{max}^2 = \tfrac{1}{2}kA^2 \tag{3.3.7}$$

As the particle oscillates, the kinetic and potential energies continually change. The constant total energy is entirely in the form of kinetic energy at the center, where $x = 0$ and $\dot{x} = \pm v_{max}$, and it is all potential energy at the extrema, where $\dot{x} = 0$ and $x = \pm A$.

EXAMPLE 3.3.1

The Energy Function of the Simple Pendulum

The potential energy of the simple pendulum (Fig. 3.2.6) is given by the expression

$$V = mgh$$

where h is the vertical distance from the reference level (which we choose to be the level of the equilibrium position). For a displacement through an angle θ (Fig. 3.2.6), we see that $h = l - l\cos\theta$, so

$$V(\theta) = mgl(1 - \cos\theta)$$

Now the series expansion for the cosine is $\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \cdots$, so for small θ we have approximately $\cos\theta = 1 - \theta^2/2$. This gives

$$V(\theta) = \tfrac{1}{2}mgl\,\theta^2$$

or, equivalently, because $s = l\theta$,

$$V(s) = \tfrac{1}{2}\frac{mg}{l}s^2$$

Thus, to a first approximation, the potential energy function is quadratic in the displacement variable. In terms of s, the total energy is given by

$$E = \tfrac{1}{2}m\dot{s}^2 + \tfrac{1}{2}\frac{mg}{l}s^2$$

in accordance with the general statement concerning the energy of the harmonic oscillator discussed above.

EXAMPLE 3.3.2

Calculate the average kinetic, potential, and total energies of the harmonic oscillator. (Here we use the symbol K for kinetic energy and T_0 for the period of the motion.)

Solution:

$$\langle K\rangle = \frac{1}{T_0}\int_0^{T_0} K(t)\,dt = \frac{1}{T_0}\int_0^{T_0} \tfrac{1}{2}m\dot{x}^2\,dt$$

but

$$x = A\sin(\omega_0 t + \phi_0)$$
$$\dot{x} = \omega_0 A\cos(\omega_0 t + \phi_0)$$

Setting $\phi_0 = 0$ and letting $u = \omega_0 t = (2\pi/T_0)\cdot t$, we obtain

$$\langle K\rangle = \frac{1}{T_0}\left[\tfrac{1}{2}m\omega_0^2 A^2 \int_0^{T_0}\cos^2(\omega_0 t)\,dt\right]$$
$$= \frac{1}{2\pi}\left[\tfrac{1}{2}m\omega_0^2 A^2 \int_0^{2\pi}\cos^2 u\,du\right]$$

We can make use of the fact that

$$\frac{1}{2\pi}\int_0^{2\pi}(\sin^2 u + \cos^2 u)\,du = \frac{1}{2\pi}\int_0^{2\pi}du = 1$$

to obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2 u\,du = \tfrac{1}{2}$$

because the areas under the $\cos^2$ and $\sin^2$ terms throughout one cycle are identical. Thus,

$$\langle K\rangle = \tfrac{1}{4}m\omega_0^2 A^2$$

The calculation of the average potential energy proceeds along similar lines.

$$V = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2\sin^2\omega_0 t$$
$$\langle V\rangle = \tfrac{1}{2}kA^2\frac{1}{T_0}\int_0^{T_0}\sin^2\omega_0 t\,dt$$
$$= \tfrac{1}{2}kA^2\frac{1}{2\pi}\int_0^{2\pi}\sin^2 u\,du$$
$$= \tfrac{1}{4}kA^2$$

Now, because $k/m = \omega_0^2$ or $k = m\omega_0^2$, we obtain

$$\langle V\rangle = \tfrac{1}{4}kA^2 = \tfrac{1}{4}m\omega_0^2 A^2 = \langle K\rangle$$
$$\langle E\rangle = \langle K\rangle + \langle V\rangle = \tfrac{1}{2}m\omega_0^2 A^2 = \tfrac{1}{2}kA^2 = E$$

The average kinetic energies and potential energies are equal; therefore, the average energy of the oscillator is equal to its total instantaneous energy.

3.4 | Damped Harmonic Motion

The foregoing analysis of the harmonic oscillator is somewhat idealized in that we have failed to take into account frictional forces. These are always present in a mechanical system to some extent. Analogously, there is always a certain amount of resistance in an electrical circuit. For a specific model, let us consider an object of mass m that is supported by

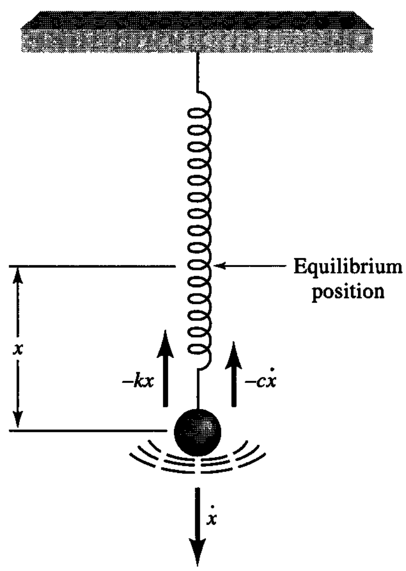

Figure 3.4.1 A model for the damped harmonic oscillator.

a light spring of stiffness k. We assume that there is a viscous retarding force that is a linear function of the velocity, such as is produced by air drag at low speeds.[2] The forces are indicated in Figure 3.4.1.

If x is the displacement from equilibrium, then the restoring force is $-kx$, and the retarding force is $-c\dot{x}$, where c is a constant of proportionality. The differential equation of motion is, therefore, $m\ddot{x} = -kx - c\dot{x}$, or

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{3.4.1}$$

As with the undamped case, we divide Equation 3.4.1 by m to obtain

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{3.4.2}$$

If we substitute the *damping factor* γ, defined as

$$\gamma \equiv \frac{c}{2m} \tag{3.4.3}$$

and ω_0^2 $(=k/m)$ into Equation 3.4.2, it assumes the simpler form

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{3.4.4}$$

The presence of the velocity-dependent term $2\gamma\dot{x}$ complicates the problem; simple sine or cosine solutions do not work, as can be verified by trying them. We introduce a method of solution that works rather well for second-order differential equations with constant

[2]Nonlinear drag is more realistic in many situations; however, the equations of motion are much more difficult to solve and are not treated here.

coefficients. Let D be the differential operator d/dt. We "operate" on x with a quadratic function of D chosen in such a way that we generate Equation 3.4.4:

$$\left[D^2 + 2\gamma D + \omega_0^2\right]x = 0 \tag{3.4.5a}$$

We interpret this equation as an "operation" by the term in brackets on x. The operation by D^2 means first operate on x with D and then operate on the result of that operation with D again. This procedure yields $\ddot{x}$, the first term in Equation 3.4.4. The operator equation (Equation 3.4.5a) is, therefore, equivalent to the differential equation (Equation 3.4.4). The simplification that we get by writing the equation this way arises when we factor the operator term, using the binomial theorem, to obtain

$$\left[D + \gamma - \sqrt{\gamma^2 - \omega_0^2}\right]\left[D + \gamma + \sqrt{\gamma^2 - \omega_0^2}\right]x = 0 \tag{3.4.5b}$$

The operation in Equation 3.4.5b is identical to that in Equation 3.4.5a, but we have reduced the operation from second-order to a product of two first-order ones. Because the order of operation is arbitrary, the general solution is a sum of solutions obtained by setting the result of each first-order operation on x equal to zero. Thus, we obtain

$$x(t) = A_1 e^{-(\gamma - q)t} + A_2 e^{-(\gamma + q)t} \tag{3.4.6}$$

where

$$q = \sqrt{\gamma^2 - \omega_0^2} \tag{3.4.7}$$

The student can verify that this is a solution by direct substitution into Equation 3.4.4. A problem that we soon encounter, though, is that the above exponents may be real or complex, because the factor q could be imaginary. We see what this means in just a minute.

There are three possible scenarios:

I. q real > 0	*Overdamping*
II. q real $= 0$	*Critical damping*
III. q imaginary	*Underdamping*

I. *Overdamped.* Both exponents in Equation 3.4.6 are real. The constants A_1 and A_2 are determined by the initial conditions. The motion is an exponential decay with two different decay constants, $(\gamma - q)$ and $(\gamma + q)$. A mass, given some initial displacement and released from rest, returns slowly to equilibrium, prevented from oscillating by the strong damping force. This situation is depicted in Figure 3.4.2.

II. *Critical damping.* Here $q = 0$. The two exponents in Equation 3.4.6 are each equal to γ. The two constants A_1 and A_2 are no longer independent. Their sum forms a single constant A. The solution degenerates to a single exponential decay function. A completely general solution requires two different functions and independent constants to satisfy the boundary conditions specified by an initial position and velocity. To find a solution with two independent constants, we return to Equation 3.4.5b:

$$(D + \gamma)(D + \gamma)x = 0 \tag{3.4.8a}$$

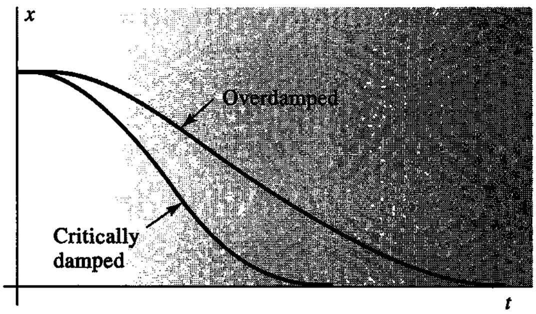

Figure 3.4.2 Displacement versus time for critically damped and overdamped oscillators released from rest after an initial displacement.

Switching the order of operation does not work here, because the operators are the same. We have to carry out the entire operation on x before setting the result to zero. To do this, we make the substitution $u = (D + \gamma)x$, which gives

$$\begin{aligned}(D+\gamma)u &= 0 \\ u &= Ae^{-\gamma t}\end{aligned} \tag{3.4.8b}$$

Equating this to $(D + \gamma)x$, the final solution is obtained as follows:

$$\begin{aligned}Ae^{-\gamma t} &= (D+\gamma)x \\ A &= e^{\gamma t}(D+\gamma)x = D(xe^{\gamma t}) \\ \therefore\ xe^{\gamma t} &= At + B \\ x(t) &= Ate^{-\gamma t} + Be^{-\gamma t}\end{aligned} \tag{3.4.9}$$

The solution consists of two different functions, $te^{-\gamma t}$ and $e^{-\gamma t}$, and two constants of integration, A and B, as required. As in case I, if a mass is released from rest after an initial displacement, the motion is nonoscillatory, returning asymptotically to equilibrium. This case is also shown in Figure 3.4.2. Critical damping is highly desirable in many systems, such as the mechanical suspension systems of motor vehicles.

III. *Underdamping*. If the constant γ is small enough that $\gamma^2 - \omega_0^2 < 0$, the factor q in Equation 3.4.7 is imaginary. A mass initially displaced and then released from rest oscillates, not unlike the situation described earlier for no damping force at all. The only difference is the presence of the real factor $-\gamma$ in the exponent of the solution that leads to the ultimate death of the oscillatory motion. Let us now reverse the factors under the square root sign in Equation 3.4.7 and write q as $i\omega_d$. Thus,

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \tag{3.4.10}$$

where ω_0 and ω_d are the angular frequencies of the undamped and underdamped harmonic oscillators, respectively. We now rewrite the general solution represented by Equation 3.4.6 in terms of the factors described here,

$$\begin{aligned}x(t) &= C_+e^{-(\gamma - i\omega_d)t} + C_-e^{-(\gamma + i\omega_d)t} \\ &= e^{-\gamma t}(C_+e^{i\omega_d t} + C_-e^{-i\omega_d t})\end{aligned} \tag{3.4.11}$$

where the constants of integration are C_+ and C_-. The solution contains a sum of imaginary exponentials. But the solution must be real—it is supposed to describe the real world! This reality demands that C_+ and C_- be complex conjugates of each other, a condition that ultimately allows us to express the solution in terms of sines and/or cosines. Thus, taking the complex conjugate of Equation 3.4.11,

$$x^*(t) = e^{-\gamma t}(C_+^* e^{-i\omega_d t} + C_-^* e^{+i\omega_d t}) = x(t) \tag{3.4.12a}$$

Because $x(t)$ is real, $x^*(t) = x(t)$, and, therefore,

$$\begin{aligned} \therefore\ C_+^* &= C_- = C \\ C_-^* &= C_+ = C^* \\ \therefore\ x(t) &= e^{-\gamma t}(C^* e^{+i\omega_d t} + Ce^{-i\omega_d t}) \end{aligned} \tag{3.4.12b}$$

It looks as though we have a solution that now has only a single constant of integration. In fact, C is a complex number. It is composed of two constants. We can express C and C^* in terms of two real constants, A and θ_0, in the following way.

$$\begin{aligned} C_- &= C = \frac{A}{2}e^{-i\theta_0} \\ C_+ &= C^* = \frac{A}{2}e^{+i\theta_0} \end{aligned} \tag{3.4.13}$$

We soon see that A is the maximum displacement and θ_0 is the initial phase angle of the motion. Thus, Equation 3.4.12b becomes

$$x(t) = e^{-\gamma t}\left(\frac{A}{2}e^{+i(\omega_d t+\theta_0)} + \frac{A}{2}e^{-i(\omega_d t+\theta_0)}\right) \tag{3.4.14}$$

We now apply Euler's identity[3] to the above expressions, thus obtaining

$$\begin{aligned} \frac{A}{2}e^{+i(\omega_d t+\theta_0)} &= \frac{A}{2}\cos(\omega_d t+\theta_0) + i\frac{A}{2}\sin(\omega_d t+\theta_0) \\ \frac{A}{2}e^{-i(\omega_d t+\theta_0)} &= \frac{A}{2}\cos(\omega_d t+\theta_0) - i\frac{A}{2}\sin(\omega_d t+\theta_0) \\ \therefore\ x(t) &= e^{-\gamma t}(A\cos(\omega_d t+\theta_0)) \end{aligned} \tag{3.4.15}$$

Following our discussion in Section 3.2 concerning the rotating vector construct, we see that we can express the solution equally well as a sine function:

$$x(t) = e^{-\gamma t}\,(A\sin(\omega_d t + \phi_0)) \tag{3.4.16}$$

The constants A, θ_0, and ϕ_0 have the same interpretation as those of Section 3.2. In fact, we see that the solution for the underdamped oscillator is nearly identical to that of the undamped oscillator. There are two differences: (1) The presence of the real exponential factor $e^{-\gamma t}$ leads to a gradual death of the oscillations, and (2) the underdamped oscillator's angular frequency is ω_d, not ω_0, because of the presence of the damping force.

[3] Euler's identity relates imaginary exponentials to sines and cosines. It is given by the expression $e^{iu} = \cos u + i\sin u$. This equality is demonstrated in Appendix D.

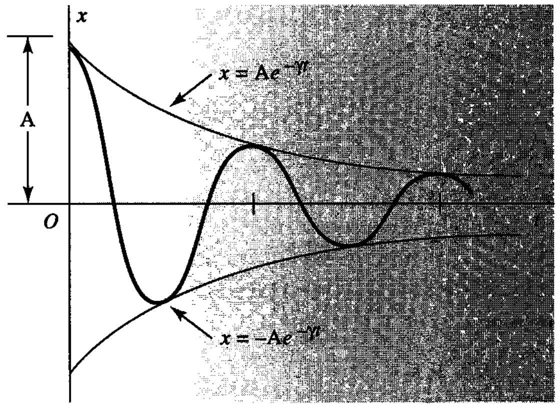

Figure 3.4.3 Graph of displacement versus time for the underdamped harmonic oscillator.

The underdamped oscillator vibrates a little more slowly than does the undamped oscillator. The period of the underdamped oscillator is given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\left(\omega_0^2 - \gamma^2\right)^{1/2}} \tag{3.4.17}$$

Figure 3.4.3 is a plot of the motion. Equation 3.4.15a shows that the two curves given by $x = Ae^{-\gamma t}$ and $x = -Ae^{-\gamma t}$ form an envelope of the curve of motion because the cosine factor takes on values between +1 and −1, including +1 and −1, at which points the curve of motion touches the envelope. Accordingly, the points of contact are separated by a time interval of one-half period, $T_d/2$. These points, however, are not quite the maxima and minima of the displacement. It is left to the student to show that the actual maxima and minima are also separated in time by the same amount. In one complete period the amplitude diminishes by a factor $e^{-\gamma T_d}$; also, in a time $\gamma^{-1} = 2m/c$ the amplitude decays by a factor $e^{-1} = 0.3679$.

In summary, our analysis of the freely running harmonic oscillator has shown that the presence of damping of the linear type causes the oscillator, given an initial motion, to eventually return to a state of rest at the equilibrium position. The return to equilibrium is either oscillatory or not, depending on the amount of damping. The critical condition, given by $\gamma = \omega_0$, characterizes the limiting case of the nonoscillatory mode of return.

Energy Considerations

The total energy of the damped harmonic oscillator is given by the sum of the kinetic and potential energies:

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 \tag{3.4.18}$$

This is constant for the undamped oscillator, as stated previously. Let us differentiate the above expression with respect to t:

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x} \tag{3.4.19}$$

Now the differential equation of motion is $m\ddot{x} + c\dot{x} + kx = 0$, or $m\ddot{x} + kx = -c\dot{x}$. Thus, we can write

$$\frac{dE}{dt} = -c\dot{x}^2 \tag{3.4.20}$$

for the time rate of change of total energy. We see that it is given by the product of the damping force and the velocity. Because this is always either zero or negative, the total energy continually decreases and, like the amplitude, eventually becomes negligibly small. The energy is dissipated as frictional heat by virtue of the viscous resistance to the motion.

Quality Factor

The rate of energy loss of a weakly damped harmonic oscillator is best characterized by a single parameter Q, called the *quality factor* of the oscillator. It is defined to be 2π times the energy stored in the oscillator divided by the energy lost in a single period of oscillation T_d. If the oscillator is weakly damped, the energy lost per cycle is small and Q is, therefore, large. We calculate Q in terms of parameters already derived and show that this is true.

The average rate of energy dissipation for the damped oscillator is given by Equation 3.4.20, $\dot{E} = -c\dot{x}^2$, so we need to calculate $\dot{x}$. Equation 3.4.16 gives $x(t)$:

$$x = Ae^{-\gamma t}\sin(\omega_d t + \phi_0) \tag{3.4.21a}$$

Differentiating it, we obtain

$$\dot{x} = -Ae^{-\gamma t}(\gamma\sin(\omega_d t + \phi_0) - \omega_d\cos(\omega_d t + \phi_0)) \tag{3.4.21b}$$

The energy lost during a single cycle of period $T_d = 2\pi/\omega_d$ is

$$\Delta E = \int_0^{T_d} \dot{E}\,dt \tag{3.4.22a}$$

If we change the variable of integration to $\theta = \omega_d t + \phi_0$, then $dt = d\theta/\omega_d$ and the integral over the period T_d transforms to an integral, from ϕ_0 to $\phi_0 + 2\pi$. The value of the integral over a full cycle doesn't depend on the initial phase ϕ_0 of the motion, so, for the sake of simplicity, we drop it from the limits of integration:

$$\begin{aligned}\Delta E &= \frac{1}{\omega_d}\int_0^{2\pi}\dot{E}\,d\theta \\ &= -\frac{cA^2}{\omega_d}\int_0^{2\pi} e^{-2\gamma t}\left[\gamma^2\sin^2\theta - 2\gamma\omega_d\sin\theta\cos\theta + \omega_d^2\cos^2\theta\right]d\theta\end{aligned} \tag{3.4.22b}$$

Now we can extract the exponential factor $e^{-2\gamma t}$ from inside the integral, because in the case of weak damping ($\gamma \ll \omega_d$) its value does not change very much during a single cycle of oscillation:

$$\Delta E = \frac{-cA^2}{\omega_d}e^{-2\gamma t}\int_0^{2\pi}\left(\gamma^2\sin^2\theta - 2\gamma\omega_d\sin\theta\cos\theta + \omega_d^2\cos^2\theta\right)d\theta \tag{3.4.22c}$$

The integral of both $\sin^2\theta$ and $\cos^2\theta$ over one cycle is π, while the integral of the $\sin\theta$ $\cos\theta$ product vanishes. Thus, we have

$$\begin{aligned}\Delta E &= \frac{-cA^2}{\omega_d}\pi e^{-2\gamma t}\left(\gamma^2+\omega_d^2\right) = -cA^2 e^{-2\gamma t}\omega_0^2\left(\frac{\pi}{\omega_d}\right)\\ &= -\gamma m\omega_0^2 A^2 e^{-2\gamma t}T_d\end{aligned} \tag{3.4.22d}$$

where we have made use of the relations $\omega_0^2 = \omega_d^2 + \gamma^2$ and $\gamma = c/2m$. Now, if we identify the damping factor γ with a time constant τ, such that $\gamma = (2\tau)^{-1}$, we obtain for the *magnitude* of the energy loss in one cycle

$$\begin{aligned}\Delta E &= \left(\tfrac{1}{2}mA^2\omega_0^2 e^{-t/\tau}\right)\frac{T_d}{\tau}\\ \frac{\Delta E}{E} &= \frac{T_d}{\tau}\end{aligned} \tag{3.4.22e}$$

where the energy stored in the oscillator (see Example 3.3.2) at any time t is

$$E(t) = \tfrac{1}{2}m\omega_0^2 A^2 e^{-t/\tau} \tag{3.4.23}$$

Clearly, the energy remaining in the oscillator during any cycle dies away exponentially with time constant τ. We, therefore, see that the quality factor Q is just 2π times the inverse of the ratios given in the expression above, or

$$Q = \frac{2\pi}{(T_d/\tau)} = \frac{2\pi\tau}{(2\pi/\omega_d)} = \omega_d\tau = \frac{\omega_d}{2\gamma} \tag{3.4.24}$$

For weak damping, the period of oscillation T_d is much less than the time constant τ, which characterizes the energy loss rate of the oscillator. Q is large under such circumstances. Table 3.4.1 gives some values of Q for several different kinds of oscillators.

TABLE 3.4.1 Values of Q for Several Physical Systems

Earth (for earthquake)	250–1400
Piano string	3000
Crystal in digital watch	10^4
Microwave cavity	10^4
Excited atom	10^7
Neutron star	10^{12}
Excited Fe^{57} nucleus	3×10^{12}

EXAMPLE 3.4.1

An automobile suspension system is critically damped, and its period of free oscillation with no damping is 1 s. If the system is initially displaced by an amount x_0 and released with zero initial velocity, find the displacement at $t = 1$ s.

Solution:

For critical damping we have $\gamma = c/2m = (k/m)^{1/2} = \omega_0 = 2\pi/T_0$. Hence, $\gamma = 2\pi\ \mathrm{s}^{-1}$ in our case, because $T_0 = 1$ s. Now the general expression for the displacement in the critically damped case (Equation 3.4.9) is $x(t) = (At + B)e^{-\gamma t}$, so, for $t = 0$, $x_0 = B$. Differentiating, we have $\dot{x}(t) = (A - \gamma B - \gamma At)e^{-\gamma t}$, which gives $\dot{x}_0 = A - \gamma B = 0$, so $A = \gamma B = \gamma x_0$ in our problem. Accordingly,

$$x(t) = x_0(1 + \gamma t)e^{-\gamma t} = x_0(1 + 2\pi t)e^{-2\pi t}$$

is the displacement as a function of time. For $t = 1$ s, we obtain

$$x_0(1 + 2\pi)e^{-2\pi} = x_0(7.28)e^{-6.28} = 0.0136\, x_0$$

The system has practically returned to equilibrium.

EXAMPLE 3.4.2

The frequency of a damped harmonic oscillator is one-half the frequency of the same oscillator with no damping. Find the ratio of the maxima of successive oscillations.

Solution:

We have $\omega_d = \frac{1}{2}\omega_0 = (\omega_0^2 - \gamma^2)^{1/2}$, which gives $\omega_0^2/4 = \omega_0^2 - \gamma^2$, so $\gamma = \omega_0(3/4)^{1/2}$. Consequently,

$$\gamma T_d = \omega_0(3/4)^{1/2}\,[2\pi/(\omega_0/2)] = 10.88$$

Thus, the amplitude ratio is

$$e^{-\gamma T_d} = e^{-10.88} = 0.00002$$

This is a *highly damped* oscillator.

EXAMPLE 3.4.3

Given: The terminal speed of a baseball in free fall is 30 m/s. Assuming a linear air drag, calculate the effect of air resistance on a simple pendulum, using a baseball as the plumb bob.

Solution:

In Chapter 2 we found the terminal speed for the case of linear air drag to be given by $v_t = mg/c_1$, where c_1 is the linear drag coefficient. This gives

$$\gamma = \frac{c_1}{2m} = \frac{(mg/v_t)}{2m} = \frac{g}{2v_t} = \frac{9.8\ \text{ms}^{-2}}{60\ \text{ms}^{-1}} = 0.163\ \text{s}^{-1}$$

for the exponential damping constant. Consequently, the baseball pendulum's amplitude drops off by a factor e^{-1} in a time $\gamma^{-1} = 6.13$ s. This is independent of the length of the pendulum. Earlier, in Example 3.2.2, we showed that the angular frequency of oscillation of the simple pendulum of length l is given by $\omega_0 = (g/l)^{1/2}$ for *small* amplitude. Therefore, from Equation 3.4.17, the period of our pendulum is

$$T_d = 2\pi\left(\omega_0^2 - \gamma^2\right)^{-1/2} = 2\pi\left(\frac{g}{l} - 0.0265\ \text{s}^{-2}\right)^{-1/2}$$

In particular, for a baseball "seconds pendulum" for which the half-period is 1 s in the absence of damping, we have $g/l = \pi^2$, so the half-period with damping in our case is

$$\frac{T_d}{2} = \pi(\pi^2 - 0.0265)^{-1/2}\ \text{s} = 1.00134\ \text{s}$$

Our solution somewhat exaggerates the effect of air resistance, because the drag function for a baseball is more nearly quadratic than linear in the velocity except at very low velocities, as discussed in Section 2.4.

EXAMPLE 3.4.4

A spherical ball of radius 0.00265 m and mass 5×10^{-4} kg is attached to a spring of force constant $k = .05$ N/m underwater. The mass is set to oscillate under the action of the spring. The coefficient of viscosity η for water is 10^{-3} Ns/m^2. (a) Find the number of oscillations that the ball will execute in the time it takes for the amplitude of the oscillation to drop by a factor of 2 from its initial value. (b) Calculate the Q of the oscillator.

Solution:

Stokes' law for objects moving in a viscous medium can be used to find c, the constant of proportionality of the $\dot{x}$ term, in the equation of motion (Equation 3.4.1) for the damped oscillator. The relationship is

$$c = 6\pi\eta r = 5 \cdot 10^{-5}\ \text{Ns/m}$$

The energy of the oscillator dies away exponentially with time constant τ, and the amplitude dies away as $A = A_0 e^{-t/2\tau}$. Thus,

$$\frac{A}{A_0} = \frac{1}{2} = e^{-t/2\tau}$$
$$\therefore t = 2\tau \ln 2$$

Consequently, the number of oscillations during this time is

$$\begin{aligned} n &= \omega_d t/2\pi \\ &= \omega_d \tau (\ln 2)/\pi \\ &= Q(\ln 2)/\pi \end{aligned}$$

Because $\omega_0^2 = k/m = 100\ \text{s}^{-2}$, $\tau = m/c = 10$ s, and $\gamma = 1/2\tau = 0.05\ \text{s}^{-1}$, we obtain

$$Q = \left(\omega_0^2 - \gamma^2\right)^{1/2}\tau = (100 - 0.0025)^{1/2}\ 10 = 100$$
$$n = Q(\ln 2)/\pi = 22$$

If we had asked how many oscillations would occur in the time it takes for the amplitude to drop to $e^{-1/2}$, or about 0.606 times its initial value, the answer would have been $Q/2\pi$. Clearly Q is a measure of the rate at which an oscillator loses energy.

*3.5 | Phase Space

A physical system in motion that does not dissipate energy remains in motion. One that dissipates energy eventually comes to rest. An oscillating or rotating system that does not dissipate energy repeats its configuration each cycle. One that dissipates energy never does. The evolution of such a physical system can be graphically illustrated by examining its motion in a special space called *phase space,* rather than real space. The phase space for a single particle whose motion is restricted to lie along a single spatial coordinate consists of all the possible points in a "plane" whose horizontal coordinate is its position x and whose vertical coordinate is its velocity $\dot{x}$. Thus, the "position" of a particle on the phase-space plane is given by its "coordinates" $(x, \dot{x})$.[4] The future state of motion of such a particle is completely specified if its position and velocity are known simultaneously—say, its initial conditions $x(t_0)$ and $\dot{x}(t_0)$. We can, thus, picture the evolution of the motion of the particle from that point on by plotting its coordinates in phase space. Each point in such a plot can be thought of as a precursor for the next point. The trajectory of these points in phase space represents the complete time history of the particle.

Simple Harmonic Oscillator: No Damping Force

The simple harmonic oscillator that we discuss in this section is an example of a particle whose motion is restricted to a single dimension. Let's examine the phase-space motion of a simple harmonic oscillator that is not subject to any damping force. The solutions for its position and velocity as functions of time were given previously by Equations 3.2.5 and 3.2.12a:

$$x(t) = A \sin(\omega_0 t + \phi_0) \tag{3.5.1a}$$

$$\dot{x}(t) = A\omega_0 \cos(\omega_0 t + \phi_0) \tag{3.5.1b}$$

Letting $y = \dot{x}$ we eliminate t from these two parametric equations to find the equation of the trajectory of the oscillator in phase space:

$$\begin{aligned} x^2(t) + \frac{y^2(t)}{\omega_0^2} &= A^2(\sin^2(\omega_0 t + \phi_0) + \cos^2(\omega_0 t + \phi_0)) = A^2 \\ \therefore \frac{x^2}{A^2} + \frac{y^2}{A^2\omega_0^2} &= 1 \end{aligned} \tag{3.5.2}$$

Equation 3.5.2 is the equation of an ellipse whose semimajor axis is A and whose semiminor axis is $\omega_0 A$. Shown in Figure 3.5.1 are several phase-space trajectories for the harmonic oscillator. The trajectories differ only in the amplitude A of the oscillation.

Note that the phase-path trajectories never intersect. The existence of a point common to two different trajectories would imply that two different future motions could evolve

*Again, as noted in Chapter 2, sections in the text marked with an asterisk may be skipped with impunity.

[4]Strictly speaking, phase space is defined as the ensemble of points (x, p) where x and p are the position and momentum of the particle. Because momentum is directly proportional to velocity, the space defined here is essentially a phase space.

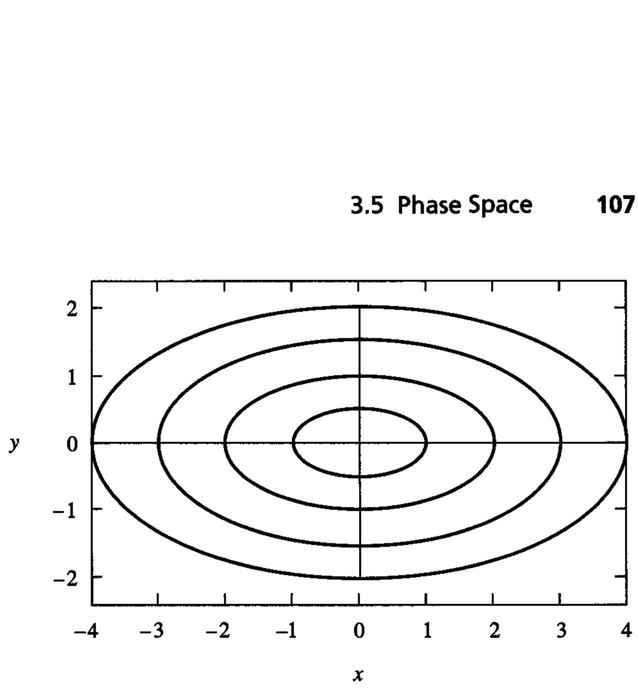

Figure 3.5.1 Phase-space plot for the simple harmonic oscillator ($\omega_0 = 0.5\ s^{-1}$). No damping force ($\gamma = 0\ s^{-1}$).

from a single set of conditions ($x(t_i)$, $\dot{x}(t_i)$) at some time t_i. This cannot happen because, starting with specific values of $x(t_i)$ and $\dot{x}(t_i)$, Newton's laws of motion completely determine a unique future state of motion for the system.

Also note that the trajectories in this case form closed paths. In other words, the motion repeats itself, a consequence of the conservation of the total energy of the harmonic oscillator. In fact, the equation of the phase-space trajectory (Equation 3.5.2) is nothing more than a statement that the total energy is conserved. We can show this by substituting $E = \frac{1}{2}kA^2$ and $\omega_0^2 = k/m$ into Equation 3.5.2, obtaining

$$\frac{x^2}{2E/k} + \frac{y^2}{2E/m} = 1 \tag{3.5.3a}$$

which is equivalent to (replacing y with $\dot{x}$)

$$\tfrac{1}{2}kx^2 + \tfrac{1}{2}m\dot{x}^2 = V + T = E \tag{3.5.3b}$$

the energy equation (Equation 3.3.3) for the harmonic oscillator. Each closed phase-space trajectory, thus, corresponds to some definite, conserved total energy.

EXAMPLE 3.5.1

Consider a particle of mass m subject to a force of strength $+kx$, where x is the displacement of the particle from equilibrium. Calculate the phase space trajectories of the particle.

Solution:

The equation of motion of the particle is $m\ddot{x} = kx$. Letting $\omega^2 = k/m$ we have $\ddot{x} - \omega^2 x = 0$. Letting $y = \dot{x}$ and $y' = dy/dx$ we have $\dot{y} = \dot{x}y' = yy' = \omega^2 x$ or $ydy = \omega^2 xdx$. The solution is $y^2 - \omega^2 x^2 = C$ in which C is a constant of integration. The phase space trajectories are branches of a hyperbola whose asymptotes are $y = \pm\omega x$. The resulting phase space plot is shown in Figure 3.5.2. The trajectories are open ended, radiating away from the origin, which is an unstable equilibrium point.

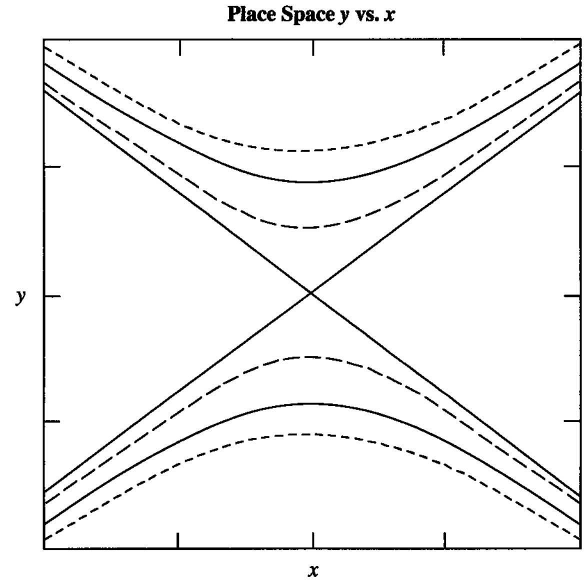

Figure 3.5.2 Phase space plot for $\ddot{x} - \omega^2 x = 0$.

The Underdamped Harmonic Oscillator

The phase-space trajectories for the harmonic oscillator subject to a weak damping force can be calculated in the same way as before. We anticipate, though, that the trajectories will not be closed. The motion does not repeat itself, because energy is constantly being dissipated. For the sake of illustration, we assume that the oscillator is started from rest at position x_0. The solutions for x and $\dot{x}$ are given by Equations 3.4.21a and b:

$$x = Ae^{-\gamma t}\sin(\omega_d t + \phi_0) \tag{3.5.4a}$$

$$\dot{x} = -Ae^{-\gamma t}(\gamma \sin(\omega_d t + \phi_0) - \omega_d \cos(\omega_d t + \phi_0)) \tag{3.5.4b}$$

Remember that because the initial phase angle ϕ_0 is given by the condition that $\dot{x}_0 = 0$, its value for the damped oscillator is not $\pi/2$ but $\phi_0 = \tan^{-1}\omega_d/\gamma$. It is difficult to eliminate t by brute force in the above parametric equations. Instead, we can illuminate the motion in phase space by applying a sequence of substitutions and linear transformations of the phase-space coordinates that simplifies the above expressions, leading to the form we've already discussed for the harmonic oscillator. First, substitute $\rho = Ae^{-\gamma t}$ and

$$\theta = \omega_d t + \phi_0$$

into the above equations, obtaining

$$x = \rho \sin\theta \tag{3.5.4c}$$

$$\dot{x} = -\rho(\gamma \sin\theta - \omega_d \cos\theta) \tag{3.5.4d}$$

Next, we apply the linear transformation $y = \dot{x} + \gamma x$ to Equation 3.5.4d, obtaining

$$y = \omega_d \rho \cos\theta \tag{3.5.5}$$

We then square this equation and carry out some algebra to obtain

$$\begin{aligned} y^2 &= \omega_d^2\rho^2(1-\sin^2\theta) \\ y^2 &= \omega_d^2(\rho^2 - x^2) \\ \frac{x^2}{\rho^2} + \frac{y^2}{\omega_d^2\rho^2} &= 1 \end{aligned} \tag{3.5.6}$$

Voila! Equation 3.5.6 is identical *in form* to Equation 3.5.2. But here the variable y is a linear combination of x and $\dot{x}$ so the ensemble of points (x, y) represents a *modified* phase space. The trajectory of the oscillator in this space is an ellipse whose major and minor axes, characterized by ρ and $\omega_d\rho$, decrease exponentially with time. The trajectory starts off with a maximum value of $x_0(=A\sin\phi_0)$ and then spirals inward toward the origin. The result is shown in Figure 3.5.3(a). The behavior of the trajectory in the x–$\dot{x}$ plane is similar and is shown in Figure 3.5.3(b). Two trajectories are shown in the plots for the cases of strong and weak damping. Which is which should be obvious.

As before, Equation 3.5.6 is none other than the energy equation for the damped harmonic oscillator. We can compare it to the results we obtained in our discussion in Section 3.4 for the rate of energy dissipation in the weakly damped oscillator. In the case of weak damping, the damping factor γ is small compared to ω_0, the undamped oscillator angular frequency (see Equation 3.4.10), and, thus, we have

$$\omega_d \approx \omega_0 \qquad y \approx \dot{x} \tag{3.5.7}$$

Hence, Equation 3.5.6 becomes

$$\frac{x^2}{\rho^2} + \frac{\dot{x}^2}{\rho^2\omega_0^2} = 1 \tag{3.5.8}$$

Note that this equation is identical in form to Equation 3.5.6, and consequently the trajectory seen in the x–$\dot{x}$ plane of Figure 3.5.3(b) for the case of weak damping is virtually identical to the modified phase-space trajectory of the weakly damped oscillator shown in Figure 3.5.3(a). Finally, upon substituting k/m for ω_0^2 and $A^2e^{-2\gamma t}$ for ρ^2, we obtain

$$\begin{aligned} \tfrac{1}{2}kx^2 + \tfrac{1}{2}m\dot{x}^2 &= \tfrac{1}{2}kA^2e^{-2\gamma t} \\ &= \tfrac{1}{2}m\omega_0^2A^2e^{-2\gamma t} \end{aligned} \tag{3.5.9}$$

If we compare this result with Equation 3.4.23, we see that it represents the total energy remaining in the oscillator at any subsequent time t:

$$V(t) + T(t) = E(t) \tag{3.5.10}$$

The energy of the weakly damped harmonic oscillator dies away exponentially with a time constant $\tau = (2\gamma)^{-1}$. The spiral nature of its phase-space trajectory reflects this fact.

The Critically Damped Harmonic Oscillator

Equation 3.4.9 gave the solution for the critically damped oscillator:

$$x = (At + B)e^{-\gamma t} \tag{3.5.11}$$

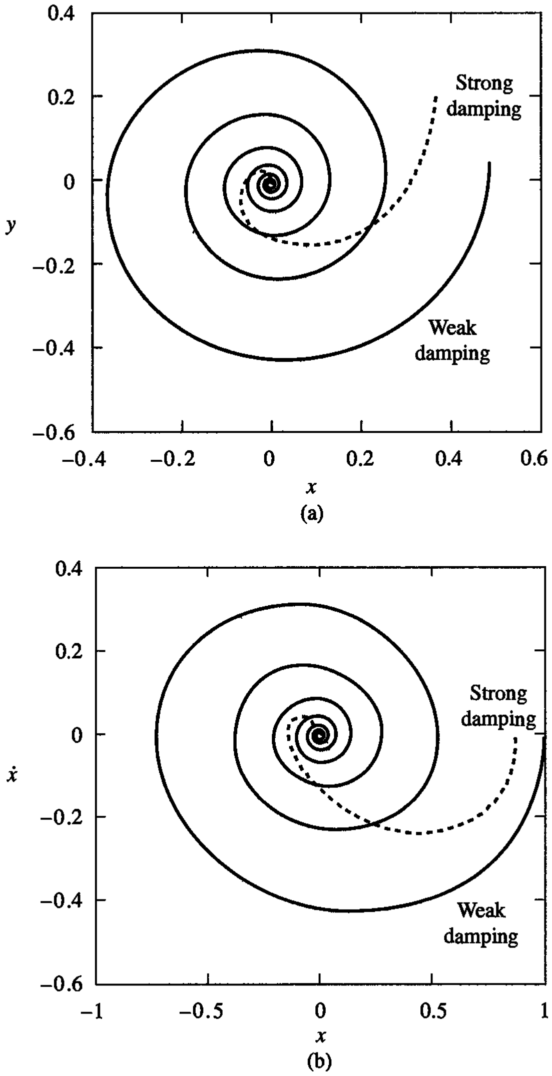

Figure 3.5.3 (a) Modified phase-space plot (see text) for the simple harmonic oscillator. (b) Phase-space plot ($\omega_0 = 0.5\ s^{-1}$). Underdamped case: (1) weak damping ($\gamma = 0.05\ s^{-1}$) and (2) strong damping ($\gamma = 0.25\ s^{-1}$).

Taking the derivative of this equation, we obtain

$$\dot{x} = -\gamma(At + B)e^{-\gamma t} + Ae^{-\gamma t} \tag{3.5.12}$$

or

$$\dot{x} + \gamma x = Ae^{-\gamma t} \tag{3.5.13}$$

This last equation indicates that the phase-space trajectory should approach a straight line whose intercept is zero and whose slope is equal to $-\gamma$. The phase-space plot is shown in Figure 3.5.4 for motion starting off with the conditions $(x_0, \dot{x}_0) = (1, 0)$.

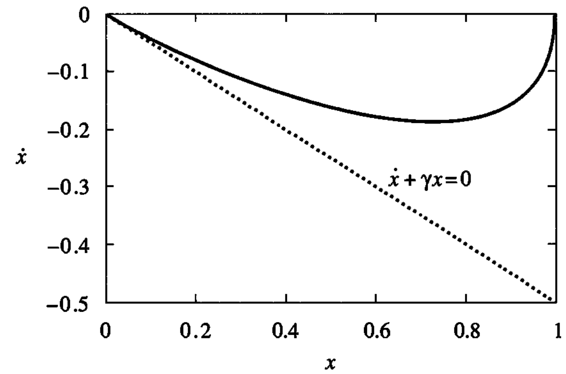

Figure 3.5.4 Phase-space plot for the simple harmonic oscillator ($\omega_0 = 0.5\ s^{-1}$). Critical damping ($\gamma = 0.5\ s^{-1}$).

The Overdamped Oscillator

Overdamping occurs when the damping parameter γ is larger than the angular frequency ω_0. Equation 3.4.6 then gives the solution for the motion:

$$x(t) = A_1 e^{-(\gamma-q)t} + A_2 e^{-(\gamma+q)t} \tag{3.5.14}$$

in which all the exponents are real. Taking the derivative of this equation, we find

$$\dot{x}(t) = -\gamma x + q e^{-\gamma t}(A_1 e^{qt} - A_2 e^{-qt}) \tag{3.5.15}$$

As in the case of critical damping, the phase path approaches zero along a straight line. However, approaches along two different lines are possible. To see what they are, it is convenient to let the motion start from rest at some displacement x_0. Given these conditions, a little algebra yields the following values for A_1 and A_2:

$$A_1 = \frac{(\gamma+q)}{2q} x_0 \qquad A_2 = -\frac{(\gamma-q)}{2q} x_0 \tag{3.5.16}$$

Some more algebra yields the following for two different linear combinations of x and $\dot{x}$:

$$\dot{x} + (\gamma - q)x = (\gamma - q)x_0 e^{-(\gamma+q)t} \tag{3.5.17a}$$

$$\dot{x} + (\gamma + q)x = (\gamma + q)x_0 e^{-(\gamma-q)t} \tag{3.5.17b}$$

The term on the right-hand side of each of the above equations dies out with time, and, thus, the phase-space asymptotes are given by the pairs of straight lines:

$$\dot{x} = -(\gamma - q)x \tag{3.5.18a}$$

$$\dot{x} = -(\gamma + q)x \tag{3.5.18b}$$

Except for special cases, phase-space paths of the motion always approach zero along the asymptote whose slope is $-(\gamma - q)$. That asymptote invariably "springs into existence" much faster than the other, because its exponential decay factor is $(\gamma + q)$ (Equations 3.5.17), the larger of the two.

Figure 3.5.5 shows the phase-space plot for an overdamped oscillator whose motion starts off with the values $(x_0, \dot{x}_0) = (1,0)$, along with the asymptote whose slope is $-(\gamma - q)$. Note how rapidly the trajectory locks in on the asymptote, unlike the case of critical damping, where it reaches the asymptote only toward the end of its motion. Obviously, overdamping is the most efficient way to knock the oscillation out of oscillatory motion!

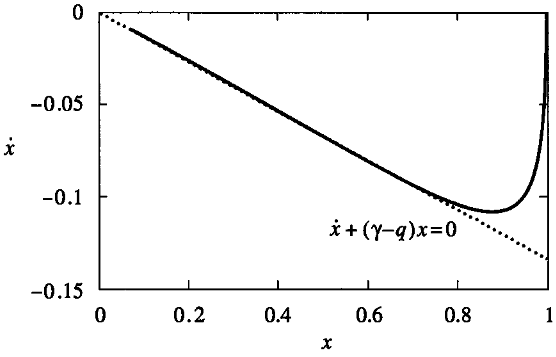

Figure 3.5.5 Phase-space plot for the simple harmonic oscillator ($\omega_0 = 0.5\ s^{-1}$). Overdamping ($\gamma = 1\ s^{-1}$).

EXAMPLE 3.5.2

A particle of unit mass is subject to a damping force $-\dot{x}$ and a force that depends on its displacement x from the origin that varies as $+x - x^3$. (a) Find the points of equilibrium of the particle and specify whether or not they are stable or unstable. (b) Use *Mathcad* to plot phase-space trajectories for the particle for three sets of starting conditions: $(x, y) =$ (i) $(-1, 1.40)$ (ii) $(-1, 1.45)$ (iii) $(0.01, 0)$ and describe the resulting motion.

Solution:

(a) The equation of motion is

$$\ddot{x}+\dot{x}-x+x^3=0$$

Let $y=\dot{x}$. Then

$$\dot{y}=-y+x-x^3$$

At equilibrium, both $y = 0$ and $\dot{y} = 0$. This is satisfied if

$$x - x^3 = x(1 - x^2) = x(1 - x)(1 + x) = 0$$

Thus, there are three equilibrium points $x = 0$ and $x = \pm 1$.

We can determine whether or not they are stable by *linearizing* the equation of motion for small excursions away from those points. Let u represent a small excursion of the particle away from an equilibrium point, which we designated by x_0. Thus, $x = x_0 + u$ and the equation of motion becomes

$$y=\dot{u} \qquad \text{and} \qquad \dot{y}=-y+(x_0+u)-(x_0+u)^3$$

Carrying out the expansion and dropping all terms non-linear in u, we get

$$\dot{y}=-y+\left(1-3x_0^2\right)u+x_0\left(1-x_0^2\right)$$

The last term is zero, so

$$\dot{y}=-y+\left(1-3x_0^2\right)u$$

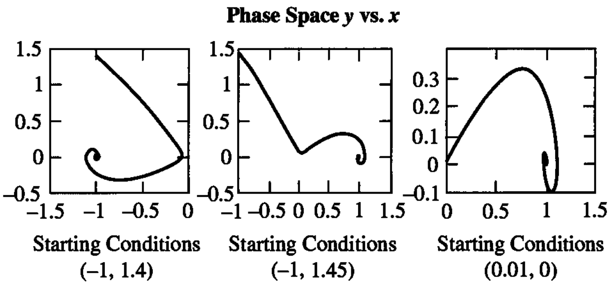

Figure 3.5.6 Phase space plots for $\ddot{x}+\dot{x}-x+x^3=0$.

If $(1-3x_0^2)<0$ the motion is a stable, damped oscillation that eventually ceases at $x=x_0$. If $(1-3x_0^2)>0$ the particle moves away from x_0 and the equilibrium is unstable. Thus, $x=\pm1$ are points of stable equilibrium and x_0 is an unstable point.

(b) The three graphs in Figure 3.5.6 were generated by using *Mathcad's rkfixed* equation solver to solve the complete nonlinear equation of motion numerically. In all cases, no matter how the motion is started, the particle veers away from $x=0$ and ultimately terminates at $x=\pm1$. The motion for the third set of starting conditions is particularly illuminating. The particle is started at rest near, but not precisely at, $x=0$. The particle is repelled away from that point, goes into damped oscillation about $x=1$, and eventually comes to rest there. The points $x=\pm1$ are called *attractors* and the point $x=0$ is called a *repellor*.

3.6 | Forced Harmonic Motion: Resonance

In this section we study the motion of a damped harmonic oscillator that is subjected to a periodic driving force by an external agent. Suppose a force of the form $F_0\cos\omega t$ is exerted upon such an oscillator. The equation of motion is

$$m\ddot{x}=-kx-c\dot{x}+F_0\cos\omega t \tag{3.6.1}$$

The most striking feature of such an oscillator is the way in which it responds as a function of the driving frequency even when the driving force is of fixed amplitude. A remarkable phenomenon occurs when the driving frequency is close in value to the natural frequency ω_0 of the oscillator. It is called *resonance*. Anyone who has ever pushed a child on a swing knows that the amplitude of oscillation can be made quite large if even the smallest push is made at just the right time. Small, periodic forces exerted on oscillators at frequencies well above or below the natural frequency are much less effective; the amplitude remains small. We initiate our discussion of forced harmonic motion with a qualitative description of the behavior that we might expect. Then we carry out a detailed analysis of the equation of motion (Equation 3.6.1), with our eyes peeled for the appearance of the phenomenon of resonance.

We already know that the undamped harmonic oscillator, subjected to any sort of disturbance that displaces it from its equilibrium position, oscillates at its natural frequency, $\omega_0=\sqrt{(k/m)}$. The dissipative forces inevitably present in any real system changes the frequency of the oscillator slightly, from ω_0 to ω_d, and cause the free oscillation to die out. This motion is represented by a solution to the *homogeneous* differential equation (Equation 3.4.1, which is Equation 3.6.1 without the driving force present). A periodic driving force does two things to the oscillator: (1) It initiates a "free" oscillation at its natural frequency, and (2) it forces the oscillator to vibrate eventually at the driving frequency ω. For a short time the actual motion is a linear superposition of oscillations at these two frequencies, but with one dying away and the other persisting. The motion that dies away is called the *transient*. The final surviving motion, an oscillation at the driving frequency, is called the *steady-state* motion. It represents a solution to the inhomogeneous equation (Equation 3.6.1). Here we focus only upon the steady-state motion, whose anticipated features we describe below. To aid in the descriptive process, we assume for the moment that the damping term $-c\dot{x}$ is vanishingly small. Unfortunately, this approximation leads to the physical absurdity that the transient term never dies out—a rather paradoxical situation for a phenomenon described by the word *transient*! We just ignore this difficulty and focus totally upon the steady-state description, in hopes that the simplicity gained by this approximation gives us insight that helps when we finally solve the problem of the driven, damped oscillator.

In the absence of damping, Equation 3.6.1 can be written as

$$m\ddot{x} + kx = F_0 \cos \omega t \tag{3.6.2}$$

The most dramatic feature of the resulting motion of this driven, undamped oscillator is a catastrophically large response at $\omega = \omega_0$. This we shall soon see, but what response might we anticipate at both extremely low ($\omega \ll \omega_0$) and high ($\omega \gg \omega_0$) frequencies? At low frequencies, we might expect the inertial term $m\ddot{x}$ to be negligible compared to the spring force $-kx$. The spring should appear to be quite stiff, compressing and relaxing very slowly, with the oscillator moving pretty much in phase with the driving force. Thus, we might guess that

$$x \approx A \cos \omega t$$

$$A = \frac{F_0}{k}$$

At high frequencies the acceleration should be large, so we might guess that $m\ddot{x}$ should dominate the spring force $-kx$. The response, in this case, is controlled by the mass of the oscillator. Its displacement should be small and 180° out of phase with the driving force, because the acceleration of a harmonic oscillator is 180° out of phase with the displacement. The veracity of these preliminary considerations emerge during the process of obtaining an actual solution.

First, let us solve Equation 3.6.2, representing the driven, undamped oscillator. In keeping with our previous descriptions of harmonic motion, we try a solution of the form

$$x(t) = A \cos(\omega t - \phi)$$

Thus, we assume that the steady-state motion is harmonic and that in the steady state it ought to respond at the driving frequency ω. We note, though, that its response might differ in phase from that of the driving force by an amount ϕ. ϕ is not the result of some initial condition! (It does not make any sense to talk about initial conditions for a steady-state solution.) To see if this assumed solution works, we substitute it into Equation 3.6.2, obtaining

$$-m\omega^2 A \cos(\omega t - \phi) + kA \cos(\omega t - \phi) = F_0 \cos \omega t$$

This works if ϕ can take on only two values, 0 and π. Let us see what is implied by this requirement. Solving the above equation for $\phi = 0$ and π, respectively, yields

$$A = \frac{F_0/m}{\left(\omega_0^2 - \omega^2\right)} \quad \phi = 0 \quad \omega < \omega_0$$
$$= \frac{F_0/m}{\left(\omega^2 - \omega_0^2\right)} \quad \phi = \pi \quad \omega > \omega_0$$

We plot the amplitude A and phase angle ϕ as functions of ω in Figure 3.6.1. Indeed, as can be seen from the plots, as ω passes through ω_0, the amplitude becomes catastrophically large, and, perhaps even more surprisingly, the displacement shifts discontinuously from being in phase with the driving force to being 180° out of phase. True, these results are not physically possible. However, they are idealizations of real situations. As we shall soon see, if we throw in just a little damping, at ω close to ω_0 the amplitude becomes large but finite. The phase shift "smooths out"; it is no longer discontinuous, although the shift is still quite abrupt.

(**Note:** *The behavior of the system mimics our description of the low-frequency and high-frequency limits.*)

The 0° and 180° phase differences between the displacement and driving force can be simply and vividly demonstrated. Hold the lighter end of a pencil or a pair of scissors (closed) or a spoon delicately between forefinger and thumb, squeezing just hard enough that it does not drop. To demonstrate the 0° phase difference, slowly move your hand back and forth horizontally in a direction parallel to the line formed between your forefinger and thumb. The bottom of this makeshift pendulum swings back and forth in phase with the hand motion and with a larger amplitude than the hand motion. To see the 180°

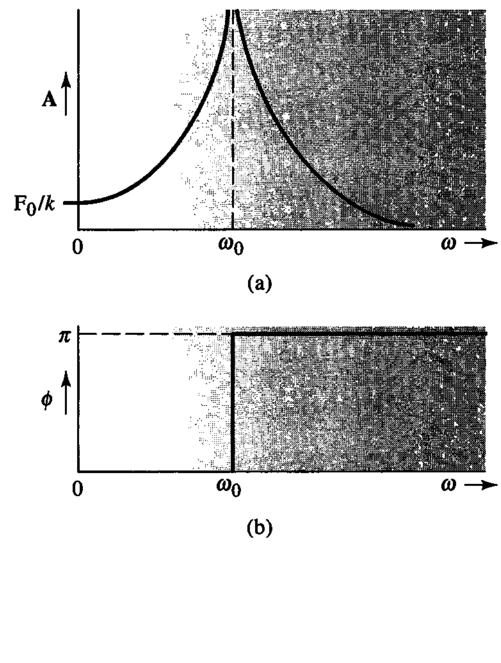

Figure 3.6.1 (a) The amplitude of a driven oscillator versus ω with no damping. (b) The phase lag of the displacement relative to the driving force versus ω.

phase shift, move your hand back and forth rather rapidly (high frequency). The bottom of the pendulum hardly moves at all, but what little motion it does undergo is 180° out of phase with the hand motion.

The Driven, Damped Harmonic Oscillator

We now seek the steady-state solution to Equation 3.6.1, representing the driven, damped harmonic oscillator. It is fairly straightforward to solve this equation directly, but it is algebraically simpler to use complex exponentials instead of sines and/or cosines. First, we represent the driving force as

$$F = F_0 e^{i\omega t} \tag{3.6.3}$$

so that Equation 3.6.1 becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t} \tag{3.6.4}$$

The variable x is now complex, as is the applied force F. Remember, though, that by Euler's identity the real part of F is $F_0 \cos \omega t$.[5] If we solve Equation 3.6.4 for x, its real part will be a solution to Equation 3.6.1. In fact, when we find a solution to the above complex equation (Equation 3.6.4), we can be sure that the real parts of both sides are equal (as are the imaginary parts). It is the real parts that are equivalent to Equation 3.6.1 and, thus, the real, physical situation.

For the steady-state solution, let us, therefore, try the complex exponential

$$x(t) = Ae^{i(\omega t - \phi)} \tag{3.6.5}$$

where the amplitude A and phase difference ϕ are constants to be determined. If this "guess" is correct, we must have

$$m\frac{d^2}{dt^2}Ae^{i(\omega t-\phi)} + c\frac{d}{dt}Ae^{i(\omega t-\phi)} + kAe^{i(\omega t-\phi)} = F_0 e^{i\omega t} \tag{3.6.6a}$$

be true for all values of t. Upon performing the indicated operations and canceling the common factor $e^{i\omega t}$, we find

$$-m\omega^2 A + i\omega c A + kA = F_0 e^{i\phi} = F_0(\cos\phi + i\sin\phi) \tag{3.6.6b}$$

Equating the real and imaginary parts yields the two equations

$$\begin{aligned} A(k - m\omega^2) &= F_0 \cos\phi \\ c\omega A &= F_0 \sin\phi \end{aligned} \tag{3.6.7a}$$

Upon dividing the second by the first and using the identity $\tan\phi = \sin\phi/\cos\phi$, we obtain the following relation for the phase angle:

$$\tan\phi = \frac{c\omega}{k - m\omega^2} \tag{3.6.7b}$$

[5]For a proof of Euler's identity, see Appendix D.

By squaring both sides of Equations 3.6.7a and adding and employing the identity $\sin^2 \phi + \cos^2 \phi = 1$, we find

$$A^2(k - m\omega^2)^2 + c^2\omega^2 A^2 = F_0^2 \tag{3.6.7c}$$

We can then solve for A, the amplitude of the steady-state oscillation, as a function of the driving frequency:

$$A(\omega) = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}} \tag{3.6.7d}$$

In terms of our previous abbreviations $\omega_0^2 = k/m$ and $\gamma = c/2m$, we can write the expressions in another form, as follows:

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2} \tag{3.6.8}$$

$$A(\omega) = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\right]^{1/2}} \tag{3.6.9}$$

A plot of the above amplitude A and phase difference ϕ versus driving frequency ω (Fig. 3.6.2) reveals a fetching similarity to the plots of Figure 3.6.1 for the case of the undamped oscillator. As can be seen from the plots, as the damping term approaches 0, the resonant peak gets larger and narrower, and the phase shift sharpens up, ultimately approaching infinity and discontinuity, respectively, at ω_0. What is not so obvious from these plots is that the amplitude resonant frequency is not ω_0 when damping is present (although the phase shift always passes through $\pi/2$ at ω_0)! Amplitude resonance occurs at some other value ω_r, which can be calculated by differentiating $A(\omega)$ and setting the result equal to zero. Upon solving the resultant equation for ω, we obtain

$$\omega_r^2 = \omega_0^2 - 2\gamma^2 \tag{3.6.10}$$

ω_r approaches ω_0 as γ, the damping term, goes to zero. Because the angular frequency of the freely running damped oscillator is given by $\omega_d = (\omega_0^2 - \gamma^2)^{1/2}$, we have

$$\omega_r^2 = \omega_d^2 - \gamma^2 \tag{3.6.11}$$

When the damping is weak, and only under this condition, the resonant frequency ω_r, the freely running, damped oscillator frequency ω_d, and the natural frequency ω_0 of the undamped oscillator are essentially identical.

At the extreme of strong damping, no amplitude resonance occurs if $\gamma > \omega_0/\sqrt{2}$, because the amplitude then becomes a monotonically decreasing function of ω. To see this, consider the limiting case $\gamma^2 = \omega_0^2/2$. Equation 3.6.9 then gives

$$A(\omega) = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 2\omega_0^2\omega^2\right]^{1/2}} = \frac{F_0/m}{\left(\omega_0^4 + \omega^4\right)^{1/2}} \tag{3.6.12}$$

which clearly decreases with increasing values of ω, starting with $\omega = 0$.

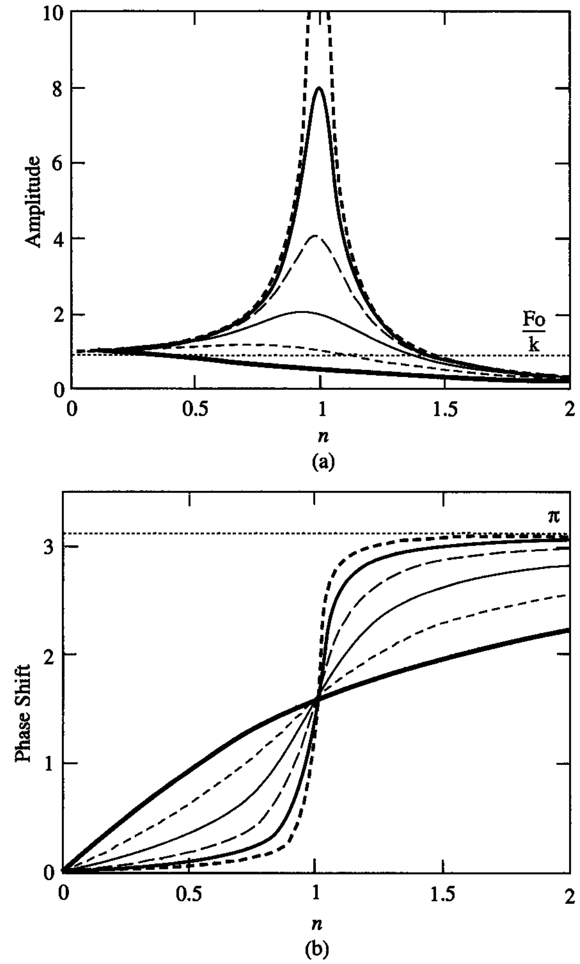

Figure 3.6.2 (a) Amplitude $A/(F_0/k)$ and (b) phase shift ϕ vs. driving frequency ($n = \omega/\omega_0$) for values of the damping constant γ given by $\gamma = 2^{-i}\,\omega_0$ ($i = 0, 1 \ldots 5$). Larger values of A and more abrupt phase shifts correspond to decreasing values of γ.

EXAMPLE 3.6.1

A seismograph may be modeled as a mass suspended by springs and a *dashpot* from a platform attached to the Earth (Figure 3.6.3). Oscillations of the Earth are passed through the platform to the suspended mass, which has a "pointer" to record its displacement relative to the platform. The dashpot provides a damping force. Ideally, the displacement A of the mass relative to the platform should closely mimic the displacement of the Earth D. Find the equation of motion of the mass m and choose parameters ω_0 and γ to insure that A lies within 10% of D. Assume during a ground tremor that the Earth oscillates with simple harmonic motion at $f = 10$ Hz.

Solution:

First we calculate the equation of motion of the mass m. Suppose the platform moves downward a distance z relative to its initial position and that m moves downward to a position y relative to the platform. The plunger in the dashpot is moving downward with

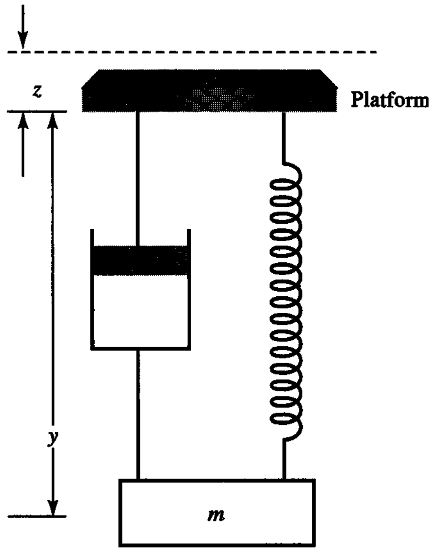

Figure 3.6.3 Seismograph model.

speed $\dot{z}$ while the pot containing the damping fluid is moving downward with speed $\dot{y}+\dot{z}$; therefore, the retarding, damping force is given by $c\dot{y}$. If l is the natural length of the spring, then

$$F = mg - c\dot{y} - k(y-l) = m(\ddot{y}+\ddot{z})$$

We let $y = x + mg/k + l$, so that x is the displacement of the mass from its equilibrium position (see Figure 3.2.5), and, in terms of x, the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{z}$$

During the tremor, as the platform oscillates with simple harmonic motion of amplitude D and angular frequency $\omega = 2\pi f$, we have $z = De^{i\omega t}$. Thus,

$$m\ddot{x} + c\dot{x} + kx = mD\omega^2 e^{i\omega t}$$

Comparing with Equation 3.6.4, and associating F_0/m with $D\omega^2$, the solution for the amplitude of oscillation given by Equation 3.6.9 can be expressed here as

$$A = D\omega^2\left[\left(\omega_0^2-\omega^2\right)^2 + 4\gamma^2\omega^2\right]^{-1/2}$$

Dividing numerator and denominator by ω^2 we obtain

$$A = D\left[\left(\frac{\omega_0^2}{\omega^2}-1\right)^2 + 4\frac{\gamma^2}{\omega^2}\right]^{-1/2}$$

Expanding the term in the denominator gives

$$A = D\left[1+\frac{\omega_0^4}{\omega^4}+\frac{2}{\omega^2}\left(2\gamma^2-\omega_0^2\right)\right]^{-1/2}$$

We can insure that $A \approx D$ for reasonable values of ω by setting $2\gamma^2 - \omega_0^2 = 0$ and $\omega_0/\omega < 0$. For example, for a fractional difference between A and D of 10%, we require that

$$\frac{D-A}{D} = 1 - \left(1 + \frac{\omega_0^4}{\omega^4}\right)^{-1/2} \approx \frac{1}{2}\frac{\omega_0^4}{\omega^4} < \frac{1}{10} \quad \text{or} \quad \omega_0 < 0.84\omega$$

This means that the free-running frequency of the oscillator is

$$f_0 = \omega_0/2\pi \leq 8\,\text{Hz}$$

The damping parameter should be

$$\gamma = \omega_0/\sqrt{2} = 36.$$

Typically, this requires the use of "soft" springs and a heavy mass.

Amplitude of Oscillation at the Resonance Peak

The steady-state amplitude at the resonant frequency, which we call A_{max}, is obtained from Equations 3.6.9 and 3.6.10. The result is

$$A_{max} = \frac{F_0/m}{2\gamma\sqrt{\omega_0^2 - \gamma^2}} \tag{3.6.13a}$$

In the case of weak damping, we can neglect γ^2 and write

$$A_{max} \simeq \frac{F_0}{2\gamma m\omega_0} \tag{3.6.13b}$$

Thus, the amplitude of the induced oscillation at the resonant condition becomes very large if the damping factor γ is very small, and conversely. In mechanical systems large resonant amplitudes may or may not be desirable. In the case of electric motors, for example, rubber or spring mounts are used to minimize the transmission of vibration. The stiffness of these mounts is chosen so as to ensure that the resulting resonant frequency is far from the running frequency of the motor.

Sharpness of the Resonance: Quality Factor

The sharpness of the resonance peak is frequently of interest. Let us consider the case of weak damping $\gamma \ll \omega_0$. Then, in the expression for steady-state amplitude (Equation 3.6.9), we can make the following substitutions:

$$\begin{aligned} \omega_0^2 - \omega^2 &= (\omega_0 + \omega)(\omega_0 - \omega) \\ &\simeq 2\omega_0(\omega_0 - \omega) \end{aligned} \tag{3.6.14a}$$

$$4\gamma^2\omega^2 \simeq 4\gamma^2\omega_0^2 \tag{3.6.14b}$$

These, together with the expression for A_{max}, allow us to write the amplitude equation in the following approximate form:

$$A(\omega) \approx \frac{A_{max}\gamma}{\sqrt{(\omega_0 - \omega)^2 + \gamma^2}} \tag{3.6.15}$$

The above equation shows that when $|\omega_0 - \omega| = \gamma$ or, equivalently, if

$$\omega = \omega_0 \pm \gamma \tag{3.6.16}$$

then

$$A^2 = \tfrac{1}{2} A_{max}^2 \tag{3.6.17}$$

This means that γ is a measure of the width of the resonance curve. Thus, 2γ is the frequency difference between the points for which the energy is down by a factor of $\frac{1}{2}$ from the energy at resonance, because the energy is proportional to A^2.

The quality factor Q defined in Equation 3.4.24, which characterizes the rate of energy loss in the undriven, damped harmonic oscillator, also characterizes the sharpness of the resonance peak for the driven oscillator. In the case of weak damping, Q can be expressed as

$$Q = \frac{\omega_d}{2\gamma} \approx \frac{\omega_0}{2\gamma} \tag{3.6.18}$$

Thus, the total width $\Delta\omega$ at the half-energy points is approximately

$$\Delta\omega = 2\gamma \approx \frac{\omega_0}{Q} \tag{3.6.19a}$$

or, because $\omega = 2\pi f$,

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta f}{f_0} \approx \frac{1}{Q} \tag{3.6.19b}$$

giving the fractional width of the resonance peak.

This last expression for Q, so innocuous-looking, represents a key feature of feedback and control in electrical systems. Many electrical systems require the existence of a well-defined and precisely maintained frequency. High Q (of order 10^5) quartz oscillators, vibrating at their resonant frequency, are commonly employed as the control element in feedback circuits to provide frequency stability. A high Q results in a sharp resonance. If the frequency of the circuit under control by the quartz oscillator starts to wander or drift by some amount δf away from the resonance peak, feedback circuitry, exploiting the sharpness of the resonance, drives the circuit vigorously back toward the resonant frequency. The higher the Q of the oscillator and, thus, the narrower δf, the more stable the output of the frequency of the circuit.

The Phase Difference ϕ

Equation 3.6.8 gives the difference in phase ϕ between the applied driving force and the steady-state response:

$$\phi = \tan^{-1}\left[\frac{2\gamma\omega}{\left(\omega_0^2 - \omega^2\right)}\right] \tag{3.6.20}$$

The phase difference is plotted in Figure 3.6.2(b). We saw that for the driven, undamped oscillator, ϕ was 0° for $\omega < \omega_0$ and 180° for $\omega > \omega_0$. These values are the low- and high-frequency limits of the real motion. Furthermore, ϕ changed discontinuously at $\omega = \omega_0$. This, too, is an idealization of the real motion where the transition between the two limits is smooth, although for very small damping it is quite abrupt, changing essentially from one limit to the other as ω passes through a region within $\pm\gamma$ about ω_0.

At low driving frequencies $\omega \ll \omega_0$, we see that $\phi \to 0$ and the response is nearly in phase with the driving force. That this is reasonable can be seen upon examination of the amplitude of the oscillation (Equation 3.6.9). In the low-frequency limit, it becomes

$$A(\omega \to 0) \approx \frac{F_0/m}{\omega_0^2} = \frac{F_0/m}{k/m} = \frac{F_0}{k} \tag{3.6.21}$$

In other words, just as we claimed during our preliminary discussion of the driven oscillator, the spring, and not the mass or the friction, controls the response; the mass is slowly pushed back and forth by a force acting against the retarding force of the spring.

At resonance the response can be enormous. Physically, how can this be? Perhaps some insight can be gained by thinking about pushing a child on a swing. How is it done? Clearly, anyone who has experience pushing a swing does not stand behind the child and push when the swing is on the backswing. One pushes in the same direction the swing is *moving*, essentially in phase with its velocity, regardless of its position. To push a small child, we usually stand somewhat to the side and give a very small shove forward as the swing passes through the equilibrium position, when its speed is a maximum and the displacement is zero! In fact, this is the optimum way to achieve a resonance condition; a rather gentle force, judiciously applied, can lead to a large amplitude of oscillation. The maximum amplitude at resonance is given by Equation 3.6.13a and, in the case of weak damping, by Equation 3.6.13b, $A_{max} \approx F_0/2\gamma m\omega_0$. But from the expression above for the amplitude as $\omega \to 0$, we have $A(\omega \to 0) \approx F_0/m\omega_0^2$. Hence, the ratio is

$$\frac{A_{max}}{A(\omega \to 0)} = \frac{F_0/(2\gamma m\omega_0)}{F_0/\left(m\omega_0^2\right)} = \frac{\omega_0}{2\gamma} = \omega_0\tau = Q \tag{3.6.22}$$

The result is simply the Q of the oscillator. Imagine what would happen to the child on the swing if there were no frictional losses! We would continue to pump little bits of energy into the swing on a cycle-by-cycle basis, and with no energy loss per cycle, the amplitude would soon grow to a catastrophic dimension.

Now let us look at the phase difference. At $\omega = \omega_0$, $\phi = \pi/2$. Hence, the displacement "lags," or is behind, the driving force by 90°. In view of the foregoing discussion, this should make sense. The optimum time to dump energy into the oscillator is when it swings through zero at maximum velocity, that is, when the power input $\mathbf{F} \cdot \mathbf{v}$ is a maximum. For example, the real part of Equation 3.6.5 gives the displacement of the oscillator:

$$x(t) = A(\omega)Re(e^{i(\omega t-\phi)}) = A(\omega)\cos(\omega t - \phi) \tag{3.6.23}$$

and at resonance, for small damping, this becomes

$$\begin{aligned} x(t) &= A(\omega_0)\cos(\omega_0 t - \pi/2) \\ &= A(\omega_0)\sin\omega_0 t \end{aligned} \tag{3.6.24}$$

The velocity, in general, is

$$\dot{x}(t) = -\omega A(\omega)\sin(\omega t - \phi) \tag{3.6.25}$$

which at resonance becomes

$$\dot{x}(t) = \omega_0 A(\omega_0)\cos\omega_0 t \tag{3.6.26}$$

Because the driving force at resonance is given by

$$F = F_0\, Re(e^{i\omega_0 t}) = F_0 \cos\omega_0 t \tag{3.6.27}$$

we can see that the driving force is indeed in phase with the velocity of the oscillator, or 90° ahead of the displacement.

Finally, for large values of ω, $\omega \gg \omega_0$, $\phi \to \pi$, and the displacement is 180° out of phase with the driving force. The amplitude of the displacement becomes

$$A(\omega \gg \omega_0) \approx \frac{F_0}{m\omega^2} \tag{3.6.28}$$

In this case, the amplitude falls off as $1/\omega^2$. The mass responds essentially like a free object, being rapidly shaken back and forth by the applied force. The main effect of the spring is to cause the displacement to lag behind the driving force by 180°.

Electrical–Mechanical Analogs

When an electric current flows in a circuit comprising inductive, capacitative, and resistive elements, there is a precise analogy with a moving mechanical system of masses and springs with frictional forces of the type studied previously. Thus, if a current $i = dq/dt$ (q being the charge) flows through an inductance L, the potential difference across the inductance is $L\ddot{q}$, and the stored energy is $\frac{1}{2}L\dot{q}^2$. Hence, inductance and charge are analogous to mass and displacement, respectively, and potential difference is analogous to force. Similarly, if a capacitance C carries a charge q, the potential difference is $C^{-1}q$, and the stored energy is $\frac{1}{2}C^{-1}q^2$. Consequently, we see that the reciprocal of C is analogous to the stiffness constant of a spring. Finally, for an electric current i flowing through a resistance R, the potential difference is $iR = \dot{q}R$, and the rate of energy dissipation is $i^2R = \dot{q}^2R$ in analogy with the quantity $c\dot{x}^2$ for a mechanical system. Table 3.6.1 summarizes the situation.

TABLE 3.6.1 Electrical-Mechanical Analogs

	Mechanical		Electrical
x	Displacement	q	Charge
$\dot{x}$	Velocity	$\dot{q} = i$	Current
m	Mass	L	Inductance
k	Stiffness	C^{-1}	Reciprocal of capacitance
c	Damping resistance	R	Resistance
F	Force	V	Potential difference

EXAMPLE 3.6.2

The exponential damping factor γ of a spring suspension system is one-tenth the critical value. If the undamped frequency is ω_0, find (a) the resonant frequency, (b) the quality factor, (c) the phase angle ϕ when the system is driven at a frequency $\omega = \omega_0/2$, and (d) the steady-state amplitude at this frequency.

Solution:

(a) We have $\gamma = \gamma_{crit}/10 = \omega_0/10$, from Equation 3.4.7, so from Equation 3.6.10,

$$\omega_r = \left[\omega_0^2 - 2(\omega_0/10)^2\right]^{1/2} = \omega_0(0.98)^{1/2} = 0.99\omega_0$$

(b) The system can be regarded as weakly damped, so, from Equation 3.6.18,

$$Q \simeq \frac{\omega_0}{2\gamma} = \frac{\omega_0}{2(\omega_0/10)} = 5$$

(c) From Equation 3.6.8 we have

$$\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right) = \tan^{-1}\left[\frac{2(\omega_0/10)(\omega_0/2)}{\omega_0^2 - (\omega_0/2)^2}\right]$$
$$= \tan^{-1} 0.133 = 7.6°$$

(d) From Equation 3.6.9 we first calculate the value of the resonance denominator:

$$D(\omega = \omega_0/2) = \left[\left(\omega_0^2 - \omega_0^2/4\right)^2 + 4(\omega_0/10)^2(\omega_0/2)^2\right]^{1/2}$$
$$= [(9/16) + (1/100)]^{1/2}\omega_0^2 = 0.7566\omega_0^2$$

From this, the amplitude is

$$A(\omega = \omega_0/2) = \frac{F_0/m}{0.7566\omega_0^2} = 1.322\frac{F_0}{m\omega_0^2}$$

Notice that the factor $(F_0/m\omega_0^2) = F_0/k$ is the steady-state amplitude for zero driving frequency.

*3.7 The Nonlinear Oscillator: Method of Successive Approximations

When a system is displaced from its equilibrium position, the restoring force may vary in a manner other than in direct proportion to the displacement. For example, a spring may not obey Hooke's law exactly; also, in many physical cases the restoring force function is inherently nonlinear, as is the case with the simple pendulum discussed in the example to follow.

In the nonlinear case the restoring force can be expressed as

$$F(x) = -kx + \epsilon(x) \tag{3.7.1}$$

in which the function $\epsilon(x)$ represents the departure from linearity. It is necessarily quadratic, or higher order, in the displacement variable x. The differential equation of motion under such a force, assuming no external forces are acting, can be written in the form

$$m\ddot{x} + kx = \epsilon(x) = \epsilon_2 x^2 + \epsilon_3 x^3 + \cdots \tag{3.7.2}$$

Here we have expanded $\epsilon(x)$ as a power series.

Solving the above type of equation usually requires some method of approximation. To illustrate one method, we take a particular case in which only the cubic term in $\epsilon(x)$ is of importance. Then we have

$$m\ddot{x} + kx = \epsilon_3 x^3 \tag{3.7.3}$$

Upon division by m and introduction of the abbreviations $\omega_0^2 = k/m$ and $\epsilon_3/m = \lambda$, we can write

$$\ddot{x} + \omega_0^2 x = \lambda x^3 \tag{3.7.4}$$

We find the solution by the *method of successive approximations*.

Now we know that for $\lambda = 0$ a solution is $x = A\cos\omega_0 t$. Suppose we try a *first* approximation of the same form,

$$x = A\cos\omega t \tag{3.7.5}$$

where, as we see, ω is not quite equal to ω_0. Inserting our trial solution into the differential equation gives

$$-A\omega^2 \cos\omega t + A\omega_0^2 \cos\omega t = \lambda A^3 \cos^3\omega t = \lambda A^3\left(\tfrac{3}{4}\cos\omega t + \tfrac{1}{4}\cos 3\omega t\right) \tag{3.7.6a}$$

In the last step we have used the trigonometric identity $\cos^3 u = \frac{3}{4}\cos u + \frac{1}{4}\cos 3u$, which is easily derived by use of the relation $\cos^3 u = [(e^{iu} + e^{-iu})/2]^3$. Upon transposing and collecting terms, we get

$$\left(-\omega^2 + \omega_0^2 - \tfrac{3}{4}\lambda A^2\right)A\cos\omega t - \tfrac{1}{4}\lambda A^3 \cos 3\omega t = 0 \tag{3.7.6b}$$

Excluding the trivial case $A = 0$, we see that our trial solution does not exactly satisfy the differential equation. However, an approximation to the value of ω, *which is valid for small*

λ, is obtained by setting the quantity in parentheses equal to zero. This yields

$$\omega^2 = \omega_0^2 - \tfrac{3}{4}\lambda A^2 \tag{3.7.7a}$$

$$\omega = \omega_0\left(1 - \frac{3\lambda A^2}{4\omega_0^2}\right)^{1/2} \tag{3.7.7b}$$

for the frequency of our freely running nonlinear oscillator. As we can see, it is a function of the amplitude A.

To obtain a better solution, we must take into account the dangling term in Equation 3.7.6b involving the third harmonic, $\cos 3\omega t$. Accordingly, we take a *second* trial solution of the form

$$x = A\cos\omega t + B\cos 3\omega t \tag{3.7.8}$$

Putting this into the differential equation, we find, after collecting terms,

$$\begin{gathered}\left(-\omega^2 + \omega_0^2 - \tfrac{3}{4}\lambda A^2\right)A\cos\omega t + \left(-9B\omega^2 + \omega_0^2 B - \tfrac{1}{4}\lambda A^3\right)\cos 3\omega t \\ + (\text{terms involving } B\lambda \text{ and higher multiples of } \omega t) = 0\end{gathered} \tag{3.7.9a}$$

Setting the first quantity in parentheses equal to zero gives the same value for ω found in Equations 3.7.7. Equating the second to zero gives a value for the coefficient B, namely,

$$B = \frac{\frac{1}{4}\lambda A^3}{-9\omega^2 + \omega_0^2} = \frac{\lambda A^3}{-32\omega_0^2 + 27\lambda A^2} \simeq -\frac{\lambda A^3}{32\omega_0^2} \tag{3.7.9b}$$

where we have assumed that the term in the denominator involving λA^2 is small enough to neglect. Our second approximation can be expressed as

$$x = A\cos\omega t - \frac{\lambda A^3}{32\omega_0^2}\cos 3\omega t \tag{3.7.10}$$

We stop at this point, but the process could be repeated to find yet a third approximation, and so on.

The above analysis, although it is admittedly very crude, brings out two essential features of free oscillation under a nonlinear restoring force; that is, the period of oscillation is a function of the amplitude of vibration, and the oscillation is not strictly sinusoidal but can be considered as the superposition of a mixture of harmonics. The vibration of a nonlinear system driven by a purely sinusoidal driving force is also distorted; that is, it contains harmonics. The loudspeaker of a stereo system, for example, may introduce distortion (harmonics) over and above that introduced by the electronic amplifying system.

EXAMPLE 3.7.1

The Simple Pendulum as a Nonlinear Oscillator

In Example 3.2.2 we treated the simple pendulum as a linear harmonic oscillator by using the approximation $\sin\theta \simeq \theta$. Actually, the sine can be expanded as a power series,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

so the differential equation for the simple pendulum, $\ddot{\theta}+(g/l)\sin\theta = 0$, may be written in the form of Equation 3.7.2, and, by retaining only the linear and the cubic terms in the expansion for the sine, the differential equation becomes

$$\ddot{\theta}+\omega_0^2\theta = \frac{\omega_0^2}{3!}\theta^3$$

in which $\omega_0^2 = g/l$. This is mathematically identical to Equation 3.7.4 with the constant $\lambda = \omega_0^2/3! = \omega_0^2/6$. The improved expression for the angular frequency, Equation 3.7.7b, then gives

$$\omega = \omega_0\left[1-\frac{3(\omega_0^2/6)A^2}{4\omega_0^2}\right]^{1/2} = \omega_0\left(1-\frac{A^2}{8}\right)^{1/2}$$

and

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}\left(1-\frac{A^2}{8}\right)^{-1/2} = T_0\left(1-\frac{A^2}{8}\right)^{-1/2}$$

for the period of the simple pendulum. Here A is the amplitude of oscillation expressed in radians. Our method of approximation shows that the period for nonzero amplitude is longer by the factor $(1-A^2/8)^{-1/2}$ than that calculated earlier, assuming $\sin\theta = \theta$. For instance, if the pendulum is swinging with an amplitude of $90° = \pi/2$ radians (a fairly large amplitude), the factor is $(1-\pi^2/32)^{-1/2} = 1.2025$, so the period is about 20% longer than the period for small amplitude. This is considerably greater than the increase due to damping of the baseball pendulum, treated in Example 3.4.3.

*The Self-Limiting Oscillator: Numerical Solution

Certain nonlinear oscillators exhibit an effect that cannot be generated by any linear oscillator—the limit cycle, that is, its oscillations are self-limiting. Examples of nonlinear oscillators that exhibit self-limiting behavior are the van der Pol oscillator, intensively studied by van der Pol[6] in his investigation of vacuum tube circuits, and the simple mechanical oscillator subject to dry friction (see Computer Problem 3.5), studied by Lord Rayleigh in his investigation of the vibrations of violin strings driven by bow strings.[7] Here we discuss a variant of the van der Pol equation of motion that describes a nonlinear oscillator, exhibiting self-limiting behavior whose limit cycle we can calculate explicitly rather than numerically. Consider an oscillator subject to a nonlinear damping force, whose overall equation of motion is

$$\ddot{x}-\gamma\left(A^2-x^2-\frac{\dot{x}^2}{\beta^2}\right)\dot{x}+\omega_0^2x = 0 \qquad (3.7.11)$$

[6] B. van der Pol, *Phil. Mag.* 2, 978 (1926). Also see T. L. Chow, *Classical Mechanics*, New York, NY Wiley, 1995.

[7] P. Smith and R. Smith, *Mechanics*, Chichester, England Wiley, 1990.

Van der Pol's equation is identical to Equation 3.7.11 without the third term in parentheses, the velocity-dependent damping factor, $\dot{x}^2/\beta^2$ (see Computer Problem 3.3). The limit cycle becomes apparent with a slight rearrangement of the above terms and a substitution of the phase-space variable y for $\dot{x}$:

$$\dot{y} - \gamma A^2\left[1 - \left(\frac{x^2}{A^2} + \frac{y^2}{A^2\beta^2}\right)\right]y + \omega_0^2 x = 0 \tag{3.7.12}$$

The nonlinear damping term is *negative* for all points (x, y) inside the ellipse given by

$$\frac{x^2}{A^2} + \frac{y^2}{A^2\beta^2} = 1 \tag{3.7.13}$$

It is zero for points on the ellipse and positive for points outside the ellipse. Therefore, no matter the state of the oscillator (described by its current position in phase space), it is driven toward states whose phase-space points lie along the ellipse. In other words, no matter how the motion is started, the oscillator ultimately vibrates with simple harmonic motion of amplitude A; its behavior is said to be "self-limiting," and this ellipse in phase space is called its limit cycle. The van der Pol oscillator behaves this way, but its limit cycle cannot be seen quite so transparently.

A complete solution can only be carried out numerically. We have used *Mathcad* to do this. For ease of calculation, we have set the factors A, β, and ω_0 equal to one. This amounts to transforming the elliptical limit cycle into a circular one of unit radius and scaling angular frequencies of vibration to ω_0. Thus, Equation 3.7.12 takes on the simple form

$$\dot{y} - \gamma(1 - x^2 - y^2)y + x = 0 \tag{3.7.14}$$

A classic way to solve a single second-order differential equation is to turn it into an equivalent system of first-order ones and then use Runge–Kutta or some equivalent technique to solve them (see Appendix I). With the substitution of y for $\dot{x}$, we obtain the following two first-order differential equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \gamma(1 - x^2 - y^2)y \end{aligned} \tag{3.7.15}$$

In fact, these equations do not have to be solved numerically. One can easily verify that they have analytic solutions $x = \cos t$ and $y = -\sin t$, which represent the final limiting motion on the unit circle $x^2 + y^2 = 1$. It is captivating, however, to let the motion start from arbitrary values that lie both within and without the limit cycle, and watch the system evolve toward its limit cycle. This behavior can be observed only by solving the equations numerically—for example, using *Mathcad*.

As in the preceding chapter, we use the *Mathcad* equation solver, *rkfixed*, which employs the fourth-order Runge–Kutta technique to numerically solve first-order differential equations. We represent the variables x and y in *Mathcad* as x_1 and x_2, the components of a two-dimensional vector $\mathbf{x} = (x_1, x_2)$.

Mathcad Procedure

- Define a two-dimensional vector $\mathbf{x} = (x_1, x_2)$ containing initial values (x_0, y_0); that is,

$$\mathbf{x} = \begin{pmatrix} -0.5 \\ 0 \end{pmatrix}$$

(This starts motion off at $(x_0, y_0) = (-0.5, 0)$.)

- Define a vector-valued function $D(t, \mathbf{x})$ containing the first derivatives of the unknown functions $x(t)$ and $y(t)$ (Equations 3.7.15):

$$D(t, \mathbf{x}) = \begin{pmatrix} x_2 \\ -x_1 + \gamma\left(1 - x_1^2 - x_2^2\right)x_2 \end{pmatrix}$$

- Decide on time interval [0, T] and the number of points, *npts*, within that interval where solutions are to be evaluated.
- Pass this information to the function *rkfixed* (or *Rkadapt* if the motion changes too rapidly within small time intervals somewhere within the time interval [0, T] that you have selected); that is,

$$Z = rkfixed(\mathbf{x}, 0, T, npts, D)$$

or

$$Z = Rkadapt(\mathbf{x}, 0, T, npts, D)$$

The function *rkfixed* (or *Rkadapt*) returns a matrix Z (in this case, two rows and three columns) whose first column contains the times t_i where the solution was evaluated and whose remaining two columns contain the values of $x(t_i)$ and $y(t_i)$. *Mathcad's* graphing feature can then be used to generate the resulting phase-space plot, a two-dimensional scatter plot of $y(t_i)$ versus $x(t_i)$.

Figure 3.7.1 shows the result of a numerical solution to the above equation of motion. Indeed, as advertised, the system either spirals in or spirals out, finally settling on the limit cycle in which the damping force disappears. Once the oscillator "locks in" on its limit cycle, its motion is simply that of the simple harmonic oscillator, repetitive and completely predictable.

*3.8 | The Nonlinear Oscillator: Chaotic Motion

When do nonlinear oscillations occur in nature? We answer that question with a tautology: They occur when the equations of motion are nonlinear. This means that if there are two (or more) solutions, $x_1(t)$ and $x_2(t)$, to a nonlinear equation of motion, any *arbitrary* linear combination of them, $\alpha x_1(t) + \beta x_2(t)$ is, in general, not linear. We can illustrate this with a simple example. The first nonlinear oscillator discussed in Section 3.7 was described by Equation 3.7.4:

$$\ddot{x} + \omega_0^2 x = \lambda x^3 \tag{3.8.1}$$

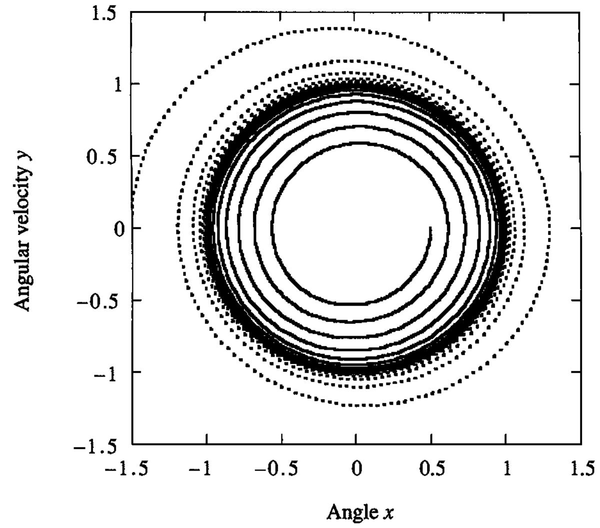

Figure 3.7.1 Phase-space plot for the self-limiting oscillator, with damping factor $\gamma = 0.1$ and starting values (x_0, y_0) of (a) (0.5,0), solid curve, and (b) (−1.5,0), dashed curve.

Assume that x_1 and x_2 individually satisfy the above equation. First, substitute their linear combination into the left side,

$$\begin{aligned}\alpha\ddot{x}_1 + \beta\ddot{x}_2 + \omega_0^2(\alpha x_1 + \beta x_2) &= \alpha\left(\ddot{x}_1 + \omega_0^2 x_1\right) + \beta\left(\ddot{x}_2 + \omega_0^2 x_2\right) \\ &= \alpha\left(\lambda x_1^3\right) + \beta\left(\lambda x_2^3\right)\end{aligned} \tag{3.8.2}$$

where the last step follows from the fact that x_1 and x_2 are assumed to be solutions to Equation 3.8.1. Now substitute the linear combination into the right side of Equation 3.8.1 and equate it to the result of Equation 3.8.2:

$$(\alpha x_1 + \beta x_2)^3 = \left(\alpha x_1^3 + \beta x_2^3\right) \tag{3.8.3a}$$

With a little algebra, Equation 3.8.3a can be rewritten as

$$\alpha(\alpha^2 - 1)x_1^3 + 3\alpha^2\beta x_1^2 x_2 + 3\alpha\beta^2 x_1 x_2^2 + \beta(\beta^2 - 1)x_2^3 = 0 \tag{3.8.3b}$$

x_1 and x_2 are solutions to the equation of motion that vary with time t. Thus, the only way Equation 3.8.3b can be satisfied at all times is if α and β are identically zero, which violates the postulate that they are arbitrary factors. Clearly, if x_1 and x_2 are solutions to the nonlinear equation of motion, any linear combination of them is not. It is this nonlinearity that gives rise to the fascinating behavior of chaotic motion.

The essence of the chaotic motion of a nonlinear system is erratic and unpredictable behavior. It occurs in simple mechanical oscillators, such as pendula or vibrating objects, that are "overdriven" beyond their linear regime where their potential energy function is a quadratic function of distance from equilibrium (see Section 3.2). It occurs in the weather, in the convective motion of heated fluids, in the motion of objects bound to our solar system, in laser cavities, in electronic circuits, and even in some chemical reactions. Chaotic oscillation in such systems manifests itself as nonrepetitive behavior. The oscillation is bounded, but each "cycle" of oscillation is like none in the past or future. The oscillation seems to exhibit all the vagaries of purely random motion. Do not be confused

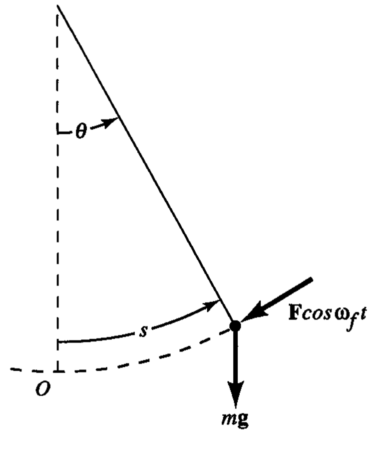

Figure 3.8.1 A simple pendulum driven in a resistive medium by a sinusoidally varying force, $F \cos \omega_f t$. The force is applied tangential to the arc path of the pendulum.

by this statement. "Chaotic" behavior of classical systems does not mean that they do not obey deterministic laws of nature. They do. Given initial conditions and the forces to which they are subject, classical systems do evolve in time in a way that is completely determined. We just may not be able to calculate that evolution with any degree of certainty.

We do not treat chaotic motion in great detail. Such treatment is beyond the mission of this text. The reader who wishes to remedy this deficiency is referred to many fine treatments of chaotic motion elsewhere.[8] Here we are content to introduce the phenomenon of chaos with an analysis of the damped simple pendulum that also can be driven into a chaotic state. We show that slight changes in the driving parameter can lead to wide divergences in the resulting motion, thus rendering prediction of its long-term evolution virtually impossible.

The Driven, Damped Harmonic Oscillator

We developed the equation of motion for the simple pendulum in Example 3.2.2. With the addition of a damping term and a forcing term, it becomes

$$m\ddot{s} = -c\dot{s} - mg \sin\theta + F \cos \omega_f t \tag{3.8.4}$$

where we have assumed that the driving force, $F \cos \omega_f t$, is applied tangent to the path of the pendulum whose arc distance from equilibrium is s (see Fig. 3.8.1).[9]

Let $s = l\theta$, $\gamma = c/m$, $\omega_r^2 = g/l$, and $\alpha = F/ml$, and apply a little algebra to obtain

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2 \sin\theta = \alpha \cos \omega_f t \tag{3.8.5}$$

---

[8] J. B. Marion and S. T. Thornton, *Classical Dynamics*, 5th ed., Brooks/Cole—Thomson Learning, Belmont, CA, 2004.

[9] The equation of motion of the simple pendulum in terms of the angular variable θ can be derived most directly using the notion of applied torques and resulting rates of change of angular momentum. These concepts are not fully developed until Chapter 7.

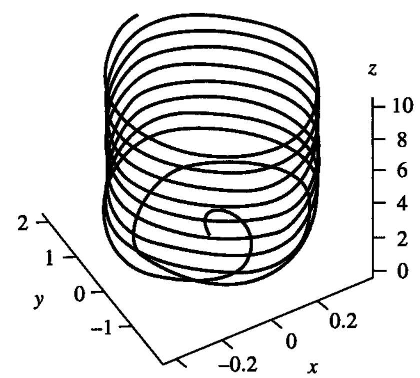

Figure 3.8.2 Three-dimensional phase-space plot of a driven, damped simple pendulum. The driving parameter is $\alpha = 0.9$. The driving angular frequency ω and damping parameter γ are $\frac{2}{3}$ and $\frac{1}{2}$ respectively. Coordinates plotted are $x = \theta/2\pi$, $y = \dot{\theta}$, $z = \omega t/2\pi$.

In our earlier discussion of the simple pendulum, we restricted the analysis of its motion to the regime of small oscillations where the approximation $\sin\theta \approx \theta$ could be used. We do not do that here. It is precisely when the pendulum is driven out of the small-angle regime that the nonlinear effect of the $\sin\theta$ term manifests itself, sometimes in the form of chaotic motion.

We simplify our analysis by scaling angular frequencies in units of ω_0 (in essence, let $\omega_0 = 1$), and we simplify notation by letting $x = \theta$ and $\omega = \omega_f$. The above equation becomes

$$\ddot{x} + \gamma\dot{x} + \sin x = \alpha \cos \omega t \tag{3.8.6}$$

Exactly as before, we transform this second-order differential equation into three first-order ones by letting $y = \dot{x}$ and $z = \omega t$:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x - \gamma y + \alpha \cos z \\ \dot{z} &= \omega \end{aligned} \tag{3.8.7}$$

Remember, these equations are dimensionless, and the driving angular frequency ω is a multiple of ω_0.

We use *Mathcad* as in the preceding example to solve these equations under a variety of conditions. For the descriptions that follow, we vary the driving "force" α and hold fixed both the driving frequency ω and the damping parameter γ at $\frac{2}{3}$ and $\frac{1}{2}$, respectively. The starting coordinates (x_0, y_0, z_0) of the motion are (0,0,0) unless otherwise noted.

- *Driving parameter:* $\alpha = 0.9$
  These conditions lead to periodic motion. The future behavior of the pendulum is predictable. We have allowed the motion to evolve for a duration T equivalent to 10 driving cycles.[10] We show in Figure 3.8.2 a three-dimensional phase-space trajectory of the motion. The vertical axis represents the z-coordinate, or the flow of time, while the horizontal axes represent the two phase-space coordinates x and y. The trajectory starts at coordinates (0,0,0) and spirals outward and upward in corkscrew-like fashion with the flow of time. There are 10 spirals corresponding to

[10]The duration of one driving cycle is $\tau = 2\pi/\omega$.

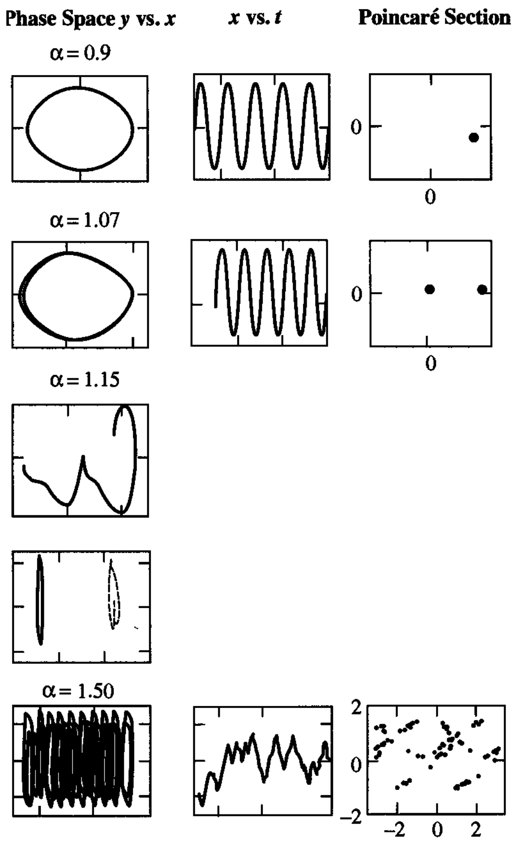

Figure 3.8.3 Damped, driven pendulum for different driving force parameters α. (i) Phase-space plots on the left (ii) angle vs. time in the center (iii) Poincaré sections on the right. Only phase-space plots of first two and last two cycles shown for $\alpha = 1.15$. Each plot represents two sets of starting conditions in which the initial angular velocities differ by only 1 part in 10^5.

the evolution of the motion over 10 driving cycle periods. The transient behavior dies out after the first few cycles as the pendulum attains a state of stable, steady-state oscillation. This is evident upon examination of the first of the top row of graphs in Figure 3.8.3. It is a two-dimensional projection of the three-dimensional, phase-space plot during the last 5 of the 10 total driving cycles. The resulting closed curve actually consists of five superimposed projected curves. The perfect superposition and closure demonstrates the stability and exact repeatability of the oscillation.

The second graph in the first row of Figure 3.8.3 is a plot of the angular position of the pendulum x as a function of the number of elapsed driving cycles n $(=\omega t/2\pi)$. The repeatability of the oscillation, cycle after cycle, is evident here as well.

The third graph in the top row is a *Poincaré section* plot of the motion. Think of it as a stroboscopic snapshot of the three-dimensional, phase-space trajectory taken every drive cycle period. The times at which the snapshots are taken can be envisioned as a series of two-dimensional planes parallel to the x–y plane separated by a single drive cycle period. The intersection of the trajectory with any of these horizontal planes, or "slices," is a single point whose (x, y) phase-space coordinates represent the current state of the motion. The single point shown in the plot for this driving cycle parameter is actually five different superimposed

Poincaré sections taken during the last five cycles of the motion. This means that after the initial transient effects die out, the (x, y) phase-space coordinates repeat exactly every subsequent drive cycle period. In other words, the pendulum is oscillating at a single frequency, the drive cycle frequency, as one might expect.

- *Driving parameter:* $\alpha = 1.07$
  This value leads to an interesting effect shown in the second row of graphs in Figure 3.8.3, known as *period doubling*, in which the motion repeats itself exactly every other drive cycle. Close examination of the phase-space plot reveals two closed loops, one for each drive cycle. You might need glasses to see the effect in the second plot (angle vs. time), but close scrutiny reveals that there is a slight vertical displacement between adjacent cycles and that every other cycle is identical. The Poincaré section shows the effect best: two discrete points can be seen indicating that the motion consists of two different but repetitive oscillations.
- *Driving parameter:* $\alpha = 1.15$
  starting coordinates: $(-0.9, 0.54660, 0)$ and $(-0.9, 0.54661, 0)$
  This particular value of driving parameter leads to *chaotic motion* and allows us to graphically illustrate one of its defining characteristics: unpredictable behavior. Take a look at the two phase-space plots defined by $\alpha = 1.15$ in the third and fourth row of Figure 3.8.3. The first one is a phase-space plot of the first two cycles of the motion, for two different trials, each one started with the two slightly different set of starting coordinates given above. In each trial, the pendulum was started from the position, $x = -0.9$ at time $t = 0$, but with slightly different angular velocities y that differed by only 1 part in 10^5. The trajectory shown in the first plot is thus two trajectories, one for each trial. The two trajectories are identical indicating that the motion for the two trials during the first two cycles are virtually indistinguishable. Note that in each case, the pendulum has moved one cycle to the left in the phase-space coordinate x when it reaches its maximum speed y in the negative direction. This tells us that the pendulum swung through one complete revolution in the clockwise direction.

  However, the second graph, where we have plotted the two phase-space trajectories for the 99th and 100th cycles, shows that the motion of the pendulum has diverged dramatically between the two trials. The trajectory for the first trial is centered on $x = -2$ on the left-hand side of the graph, indicating that after 98 drive cycle periods have elapsed, the pendulum has made two more complete 2π clockwise revolutions than it did counterclockwise. The trajectory for the second trial is centered about $x = 6$ on the right-hand side of the graph, indicating that its slightly different starting angular velocity resulted in the pendulum making six more counterclockwise revolutions than it did clockwise. Furthermore, the phase-space trajectories for the two trials now have dramatically different shapes, indicating that the oscillation of the pendulum is quite different at this point in the two trials. An effect such as this invariably occurs if the parameters α, γ, and ω are set for chaotic motion. In the case here, if we were trying to predict the future motion of the pendulum by integrating the equations of motion numerically with a precision no better than 10^{-5}, we would fail miserably. Because any numerical solution has some precisional limit, even a completely deterministic system, such as we have in Newtonian dynamics, ultimately behaves in an unpredictable fashion—in other words, in a chaotic way.

- *Driving parameter:* $\alpha = 1.5$
  This value for the driving parameter also leads to chaotic motion. The three graphs in the last row of Figure 3.8.3 illustrate a second defining characteristic of chaotic motion, namely, nonrepeatability. Two hundred drive cycles have been plotted and during no single cycle is the motion identical to that of any other. If we had plotted y vs. x *modulo* 2π, as is done in many treatments of chaotic motion (thus, discounting all full revolutions by restricting the angular variable to the interval $[-\pi, \pi]$), the entire allowed area on the phase-space plot would be filled up, a clear signature of chaotic motion. The signature is still obvious in the Poincaré section plot, which actually consists of 200 distinct points, indicating that the motion never repeats itself during any drive cycle.

Finally, the richness of the motion of the driven, damped pendulum discussed here was elicited by simply varying the driving parameter within the interval [0.9, 1.5]. We saw that one value led to periodic behavior, one led to period doubling and two led to chaotic motion. Apparently, when one deals with driven, nonlinear oscillators, chaotic motion lurks just around the corner from the rather mundane periodic behavior that we and our predecessors have beat into the ground in textbooks throughout the past several hundred years. We urge each student to investigate these motions for him- or herself using a computer. It is remarkable how the slightest change in the parameters governing the equations of motion either leads to or terminates chaotic behavior, but, of course, that's what chaos is all about.

*3.9 | Nonsinusoidal Driving Force: Fourier Series

To determine the motion of a harmonic oscillator that is driven by an external periodic force that is *other* than "pure" sinusoidal, it is necessary to employ a somewhat more involved method than that of the previous sections. In this more general case it is convenient to use the *principle of superposition*. The principle is applicable to any system governed by a linear differential equation. In our application, the principle states that if the external driving force acting on a damped harmonic oscillator is given by a superposition of force functions

$$F_{ext} = \sum_n F_n(t) \tag{3.9.1}$$

such that the differential equation

$$m\ddot{x}_n + c\dot{x}_n + kx_n = F_n(t) \tag{3.9.2}$$

is individually satisfied by the functions $x_n(t)$, then the solution of the differential equation of motion

$$m\ddot{x} + c\dot{x} + kx = F_{ext} \tag{3.9.3}$$

is given by the superposition

$$x(t) = \sum_n x_n(t) \tag{3.9.4}$$

The validity of the principle is easily verified by substitution:

$$m\ddot{x}+c\dot{x}+kx=\sum_n(m\ddot{x}_n+c\dot{x}_n+kx_n)=\sum_n F_n(t)=F_{ext} \tag{3.9.5}$$

In particular, when the driving force is periodic—that is, if for any value of the time t

$$F_{ext}(t)=F_{ext}(t+T) \tag{3.9.6}$$

where T is the period—then the force function can be expressed as a superposition of harmonic terms according to *Fourier's theorem*. This theorem states that any periodic function $f(t)$ can be expanded as a sum as follows:

$$f(t)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}[a_n\cos(n\omega t)+b_n\sin(n\omega t)] \tag{3.9.7}$$

The coefficients are given by the following formulas (derived in Appendix G):

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos(n\omega t)\,dt \qquad n=0,1,2,\ldots \tag{3.9.8a}$$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin(n\omega t)\,dt \qquad n=1,2,\ldots \tag{3.9.8b}$$

Here T is the period and $\omega=2\pi/T$ is the fundamental frequency. If the function $f(t)$ is an *even* function—that is, if $f(t)=f(-t)$—then the coefficients $b_n=0$ for all n. The series expansion is then known as a *Fourier cosine series*. Similarly, if we have an *odd* function so that $f(t)=-f(-t)$, then the a_n vanish, and the series is called a *Fourier sine series*. By use of the relation $e^{iu}=\cos u+i\sin u$, it is straightforward to verify that Equations 3.9.7 and 3.9.8a and b may also be expressed in complex exponential form as follows:

$$f(t)=\sum_n c_n e^{in\omega t} \qquad n=0,\pm1,\pm2,\ldots \tag{3.9.9}$$

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2}f(t)e^{-in\omega t}\,dt \tag{3.9.10}$$

Thus, to find the steady-state motion of our harmonic oscillator subject to a given periodic driving force, we express the force as a Fourier series of the form of Equation 3.9.7 or 3.9.9, using Equations 3.9.8a and b or 3.9.10 to determine the Fourier coefficients a_n and b_n, or c_n. For each value of n, corresponding to a given harmonic $n\omega$ of the fundamental driving frequency ω, there is a response function $x_n(t)$. This function is the steady-state solution of the driven oscillator treated in Section 3.6. The superposition of all the $x_n(t)$ gives the actual motion. In the event that one of the harmonics of the driving frequency coincides, or nearly coincides, with the resonance frequency ω_r, then the response at that harmonic dominates the motion. As a result, if the damping constant γ is very small, the resulting oscillation may be very nearly sinusoidal even if a highly nonsinusoidal driving force is applied.

EXAMPLE 3.9.1

Periodic Pulse

To illustrate the above theory, we analyze the motion of a harmonic oscillator that is driven by an external force consisting of a succession of rectangular pulses:

$$F_{ext}(t) = F_0 \qquad NT - \tfrac{1}{2}\Delta T \le t \le NT + \tfrac{1}{2}\Delta T$$
$$F_{ext}(t) = 0 \qquad \text{Otherwise}$$

where $N = 0, \pm 1, \pm 2, \ldots$, T is the time from one pulse to the next, and ΔT is the width of each pulse as shown in Figure 3.9.1. In this case, $F_{ext}(t)$ is an even function of t, so it can be expressed as a Fourier cosine series. Equation 3.9.8a gives the coefficients a_n,

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-\Delta T/2}^{+\Delta T/2} F_0 \cos(n\omega t)\,dt \\ &= \frac{2}{T}F_0\left[\frac{\sin(n\omega t)}{n\omega}\right]_{-\Delta T/2}^{+\Delta T/2} \\ &= F_0\,\frac{2\sin(n\pi\Delta T/T)}{n\pi}. \end{aligned} \tag{3.9.11a}$$

where in the last step we use the fact that $\omega = 2\pi/T$. We see also that

$$a_0 = \frac{2}{T}\int_{-\Delta T/2}^{+\Delta T/2} F_0\,dt = F_0\,\frac{2\Delta T}{T} \tag{3.9.11b}$$

Thus, for our periodic pulse force we can write

$$\begin{aligned} F_{ext}(t) = F_0\Bigg[&\frac{\Delta T}{T} + \frac{2}{\pi}\sin\left(\pi\frac{\Delta T}{T}\right)\cos(\omega t) + \frac{2}{2\pi}\sin\left(2\pi\frac{\Delta T}{T}\right)\cos(2\omega t) \\ &+ \frac{2}{3\pi}\sin\left(3\pi\frac{\Delta T}{T}\right)\cos(3\omega t) + \cdots\Bigg] \end{aligned} \tag{3.9.12}$$

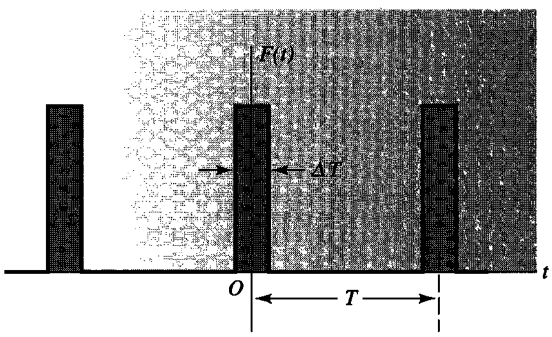

Figure 3.9.1 Rectangular-pulse driving force.

The first term in the above series expansion is just the *average* value of the external force: $F_{avg} = F_0(\Delta T/T)$. The second term is the Fourier component at the fundamental frequency ω. The remaining terms are harmonics of the fundamental: 2ω, 3ω, and so on.

Referring to Equations 3.6.5 and 3.9.4, we can now write the final expression for the motion of our pulse-driven oscillator. It is given by the superposition principle,

$$x(t) = \sum_n x_n(t) = \sum_n A_n \cos(n\omega t - \phi_n) \tag{3.9.13}$$

in which the respective amplitudes are (Equation 3.6.9)

$$A_n = \frac{a_n/m}{D_n(\omega)} = \frac{(F_0/m)(2/n\pi)\sin(n\pi\Delta T/T)}{\left[\left(\omega_0^2 - n^2\omega^2\right)^2 + 4\gamma^2 n^2\omega^2\right]^{1/2}} \tag{3.9.14}$$

and the phase angles (Equation 3.6.8)

$$\phi_n = \tan^{-1}\left(\frac{2\gamma n\omega}{\omega_0^2 - n^2\omega^2}\right) \tag{3.9.15}$$

Here m is the mass, γ is the decay constant, and ω_0 is the frequency of the freely running oscillator with no damping.

As a specific numerical example, let us consider the spring suspension system of Example 3.6.1 under the action of a periodic pulse for which the pulse width is one tenth the pulse period: $\Delta T/T = 0.1$. As before, we shall take the damping constant to be one-tenth critical, $\gamma = 0.1\,\omega_0$, and the pulse frequency to be one-half the undamped frequency of the system: $\omega = \omega_0/2$. The Fourier series for the driving force (Equation 3.9.12) is then

$$\begin{aligned} F_{ext}(t) &= F_0\left[0.1 + \frac{2}{\pi}\sin(0.1\pi)\cos(\omega t) + \frac{2}{2\pi}\sin(0.2\pi)\cos(2\omega t)\right. \\ &\quad \left. + \frac{2}{3\pi}\sin(0.3\pi)\cos(3\omega t) + \cdots\right] \\ &= F_0[0.1 + 0.197\cos(\omega t) + 0.187\cos(2\omega t) + 0.172\cos(3\omega t) + \cdots] \end{aligned}$$

The resonance denominators in Equation 3.9.14 are given by

$$D_n = \left[\left(\omega_0^2 - n^2\frac{\omega_0^2}{4}\right)^2 + 4(0.1)^2\omega_0^2 n^2\frac{\omega_0^2}{4}\right]^{1/2} = \left[\left(1 - \frac{n^2}{4}\right)^2 + 0.01n^2\right]^{1/2}\omega_0^2$$

Thus,

$$D_0 = \omega_0^2 \qquad D_1 = 0.757\omega_0^2 \qquad D_2 = 0.2\omega_0^2 \qquad D_3 = 1.285\,\omega_0^2$$

The phase angles (Equation 3.9.15) are

$$\phi_n = \tan^{-1}\left(\frac{0.2n\omega_0^2/2}{\omega_0^2 - n^2\omega_0^2/4}\right) = \tan^{-1}\left(\frac{0.4n}{4 - n^2}\right)$$

which gives

$$\phi_0 = 0 \qquad \phi_1 = \tan^{-1}(0.133) = 0.132$$

$$\phi_2 = \tan^{-1}\infty = \pi/2 \qquad \phi_3 = \tan^{-1}(-0.24) = -0.236$$

The steady-state motion of the system is, therefore, given by the following series (Equation 3.9.13):

$$x(t) = \frac{F_0}{m\omega_0^2}[0.1 + 0.26\cos(\omega t - 0.132) + 0.935\sin(2\omega t) + 0.134\cos(3\omega t + 0.236) + \cdots]$$

The dominant term is the one involving the second harmonic $2\omega = \omega_0$, because ω_0 is close to the resonant frequency. Note also the phase of this term:

$$\cos(2\omega t - \pi/2) = \sin(2\omega t).$$

Problems

3.1 A guitar string vibrates harmonically with a frequency of 512 Hz (one octave above middle C on the musical scale). If the amplitude of oscillation of the centerpoint of the string is 0.002 m (2 mm), what are the maximum speed and the maximum acceleration at that point?

3.2 A piston executes simple harmonic motion with an amplitude of 0.1 m. If it passes through the center of its motion with a speed of 0.5 m/s, what is the period of oscillation?

3.3 A particle undergoes simple harmonic motion with a frequency of 10 Hz. Find the displacement x at any time t for the following initial condition:

$$t = 0 \qquad x = 0.25\,\text{m} \qquad \dot{x} = 0.1\,\text{m/s}$$

3.4 Verify the relations among the four quantities C, D, ϕ_0, and A given by Equation 3.2.19.

3.5 A particle undergoing simple harmonic motion has a velocity $\dot{x}_1$ when the displacement is x_1 and a velocity $\dot{x}_2$ when the displacement is x_2. Find the angular frequency and the amplitude of the motion in terms of the given quantities.

3.6 On the surface of the moon, the acceleration of gravity is about one-sixth that on the Earth. What is the half-period of a simple pendulum of length 1 m on the moon?

3.7 Two springs having stiffness k_1 and k_2, respectively, are used in a vertical position to support a single object of mass m. Show that the angular frequency of oscillation is $[(k_1 + k_2)/m]^{1/2}$ if the springs are tied in parallel, and $[k_1k_2/(k_1 + k_2)m]^{1/2}$ if the springs are tied in series.

3.8 A spring of stiffness k supports a box of mass M in which is placed a block of mass m. If the system is pulled downward a distance d from the equilibrium position and then released, find the force of reaction between the block and the bottom of the box as a function of time. For what value of d does the block just begin to leave the bottom of the box at the top of the vertical oscillations? Neglect any air resistance.

3.9 Show that the ratio of two successive maxima in the displacement of a damped harmonic oscillator is constant. (*Note:* The maxima do not occur at the points of contact of the displacement curve with the curve $Ae^{-\gamma t}$.)

3.10 A damped harmonic oscillator with $m = 10$ kg, $k = 250$ N/m, and $c = 60$ kg/s is subject to a driving force given by $F_0 \cos \omega t$, where $F_0 = 48$ N.

(a) What value of ω results in steady-state oscillations with maximum amplitude? Under this condition:

(b) What is the maximum amplitude?
(c) What is the phase shift?

3.11 A mass m moves along the x-axis subject to an attractive force given by $17\beta^2 mx/2$ and a retarding force given by $3\beta m\dot{x}$, where x is its distance from the origin and β is a constant. A driving force given by $mA\cos\omega t$, where A is a constant, is applied to the particle along the x-axis.
(a) What value of ω results in steady-state oscillations about the origin with maximum amplitude?
(b) What is the maximum amplitude?

3.12 The frequency f_d of a damped harmonic oscillator is 100 Hz, and the ratio of the amplitude of two successive maxima is one half.
(a) What is the undamped frequency f_0 of this oscillator?
(b) What is the resonant frequency f_r?

3.13 Given: The amplitude of a damped harmonic oscillator drops to $1/e$ of its initial value after n complete cycles. Show that the ratio of period of the oscillation to the period of the same oscillator with no damping is given by

$$\frac{T_d}{T_0} = \left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2} \approx 1 + \frac{1}{8\pi^2 n^2}$$

where the approximation in the last expression is valid if n is large. (See the approximation formulas in Appendix D.)

3.14 Work all parts of Example 3.6.2 for the case in which the exponential damping factor γ is one-half the critical value and the driving frequency is equal to $2\omega_0$.

3.15 For a lightly damped harmonic oscillator $\gamma \ll \omega_0$, show that the driving frequency for which the steady-state amplitude is one-half the steady-state amplitude at the resonant frequency is given by $\omega \simeq \omega_0 \pm \gamma\sqrt{3}$.

3.16 If a series LCR circuit is connected across the terminals of an electric generator that produces a voltage $V = V_0 e^{i\omega t}$, the flow of electrical charge q through the circuit is given by the following second-order differential equation:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0 e^{i\omega t}$$

(a) Verify the correspondence shown in Table 3.6.1 between the parameters of a driven mechanical oscillator and the above driven electrical oscillator.
(b) Calculate the Q of the electrical circuit in terms of the coefficients of the above differential equation.
(c) Show that, in the case of small damping, Q can be written as $Q = R_0/R$, where $R_0 = \sqrt{L/C}$ is the *characteristic impedance* of the circuit.

3.17 A damped harmonic oscillator is driven by an external force of the form

$$F_{ext} = F_0 \sin\omega t$$

Show that the steady-state solution is given by

$$x(t) = A(\omega)\sin(\omega t - \phi)$$

where $A(\omega)$ and ϕ are identical to the expressions given by Equations 3.6.9 and 3.6.8.

3.18 Solve the differential equation of motion of the damped harmonic oscillator driven by a damped harmonic force:

$$F_{ext}(t) = F_0 e^{-\alpha t} \cos \omega t$$

(Hint: $e^{-\alpha t} \cos \omega t = Re(e^{-\alpha t + i\omega t}) = Re(e^{\beta t})$, *where* $\beta = -\alpha + i\omega$. *Assume a solution of the form* $Ae^{\beta t - i\phi}$.*)*

3.19 A simple pendulum of length l oscillates with an amplitude of 45°.
(a) What is the period?
(b) If this pendulum is used as a laboratory experiment to determine the value of g, find the error included in the use of the elementary formula $T_0 = 2\pi(l/g)^{1/2}$.
(c) Find the approximate amount of third-harmonic content in the oscillation of the pendulum.

3.20 Verify Equations 3.9.9 and 3.9.10 in the text.

3.21 Show that the Fourier series for a periodic square wave is

$$f(t) = \frac{4}{\pi}\left[\sin(\omega t) + \tfrac{1}{3}\sin(3\omega t) + \tfrac{1}{5}\sin(5\omega t) + \cdots\right]$$

where

$$f(t) = +1 \qquad \text{for } 0 < \omega t < \pi,\ 2\pi < \omega t < 3\pi, \text{ and so on}$$
$$f(t) = -1 \qquad \text{for } \pi < \omega t < 2\pi,\ 3\pi < \omega t < 4\pi, \text{ and so on}$$

3.22 Use the above result to find the steady-state motion of a damped harmonic oscillator that is driven by a periodic square-wave force of amplitude F_0. In particular, find the relative amplitudes of the first three terms, A_1, A_3, and A_5 of the response function $x(t)$ in the case that the third harmonic 3ω of the driving frequency coincides with the frequency ω_0 of the undamped oscillator. Let the quality factor $Q = 100$.

3.23 **(a)** Derive the first-order differential equation, dy/dx, describing the phase-space trajectory of the simple harmonic oscillator.
(b) Solve the equation, proving that the trajectory is an ellipse.

3.24 Let a particle of unit mass be subject to a force $x - x^3$ where x is its displacement from the coordinate origin.
(a) Find the equilibrium points, and tell whether they are stable or unstable.
(b) Calculate the total energy of the particle, and show that it is a conserved quantity.
(c) Calculate the trajectories of the particle in phase space.

3.25 A simple pendulum whose length $l = 9.8$ m satisfies the equation

$$\ddot{\theta} + \sin\theta = 0$$

(a) If Θ_0 is the amplitude of oscillation, show that its period T is given by

$$T = 4\int_0^{\pi/2} \frac{d\phi}{(1 - \alpha \sin^2\phi)^{1/2}} \qquad \text{where } \alpha = \sin^2 \tfrac{1}{2}\Theta_0$$

(b) Expand the integrand in powers of α, integrate term by term, and find the period T as a power series in α. Keep terms up to and including $O(\alpha^2)$.
(c) Expand α in a power series of Θ_0, insert the result into the power series found in (b), and find the period T as a power series in Θ_0. Keep terms up to and including $O(\Theta_0^2)$.