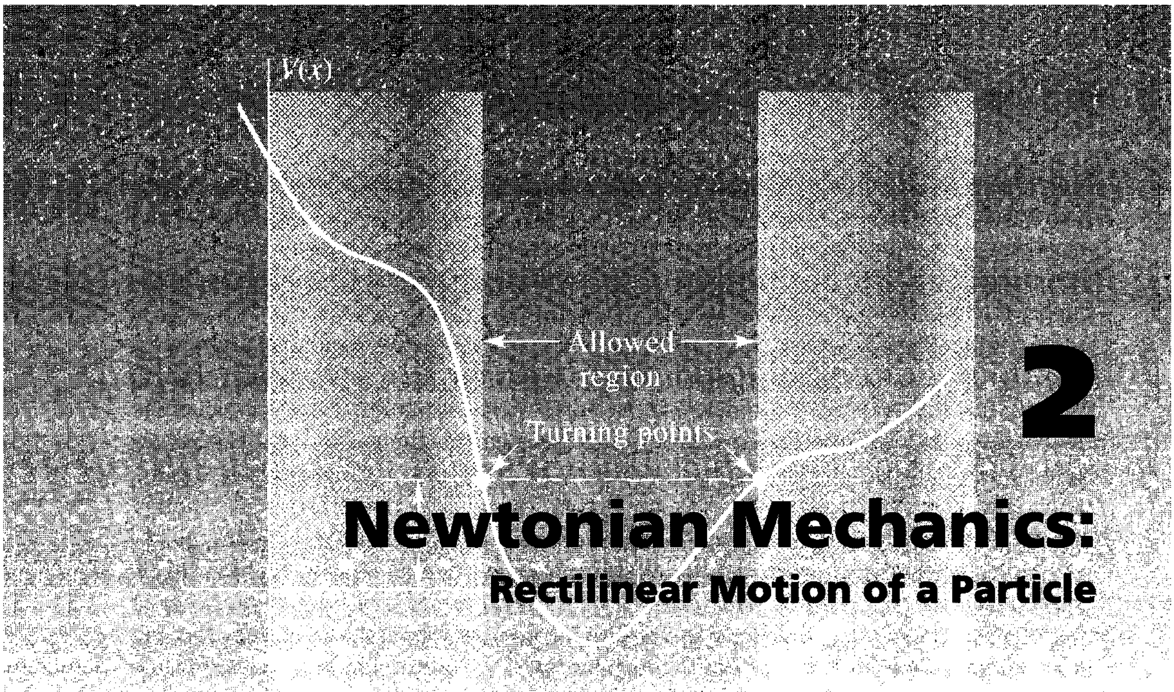

2 Newtonian Mechanics: Rectilinear Motion of a Particle

"Salviati: But if this is true, and if a large stone moves with a speed of, say, eight while a smaller one moves with a speed of four, then when they are united, the system will move with a speed less than eight; but the two stones when tied together make a stone larger than that which before moved with a speed of eight. Hence the heavier body moves with less speed than the lighter; an effect which is contrary to your supposition. Thus you see how, from your supposition that the heavier body moves more rapidly than the lighter one, I infer that the heavier body moves more slowly."

Galileo—*Dialogues Concerning Two New Sciences*

2.1 | Newton's Laws of Motion: Historical Introduction

In his *Principia* of 1687, Isaac Newton laid down three fundamental laws of motion, which would forever change mankind's perception of the world:

I. Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

II. The change of motion is proportional to the motive force impressed and is made in the direction of the line in which that force is impressed.

III. To every action there is always imposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.

These three laws of motion are now known collectively as Newton's laws of motion or, more simply, as Newton's laws. It is arguable whether or not these are indeed all his laws. However, no one before Newton stated them quite so precisely, and certainly no one before him had such a clear understanding of the overall implication and power of these laws. The behavior of natural phenomena that they imply seems to fly in the face of common experience. As any beginning student of physics soon discovers, Newton's laws become "reasonable" only with the expenditure of great effort in attempting to understand thoroughly the apparent vagaries of physical systems.

Aristotle (384–322 B.C.E.) had frozen the notion of the way the world works for almost 20 centuries by invoking powerfully logical arguments that led to a physics in which all moving, earthbound objects ultimately acquired a state of rest unless acted upon by some motive force. In his view, a force was required to keep earthly things moving, even at constant speed—a law in distinct contradiction with Newton's first and second laws. On the other hand, heavenly bodies dwelt in a more perfect realm where perpetual circular motion was the norm and no forces were required to keep this celestial clockwork ticking.

Modern scientists heap scorn upon Aristotle for burdening us with such obviously flawed doctrine. He is particularly criticized for his failure to carry out even the most modest experiment that would have shown him the error of his ways. At that time, though, it was a commonly held belief that experiment was not a suitable enterprise for any self-respecting philosopher, and thus Aristotle, raised with that belief, failed to acquire a true picture of nature. This viewpoint is a bit misleading, however. Although he did no experiments in natural philosophy, Aristotle was a keen observer of nature, one of the first. If he was guilty of anything, it was less a failure to observe nature than a failure to follow through with a process of abstraction based upon observation. Indeed, bodies falling through air accelerate initially, but ultimately they attain a nearly constant velocity of fall. Heavy objects, in general, fall faster than lighter ones. It takes a sizable force to haul a ship through water, and the greater the force, the greater the ship's speed. A spear thrown vertically upward from a moving chariot will land behind the charioteer, not on top of him. And the motion of heavenly bodies does go on and on, apparently following a curved path forever without any visible motive means. Of course, nowadays we can understand these things if we pay close attention to all the variables that affect the motion of objects and then apply Newton's laws correctly.

That Aristotle failed to extract Newton's laws from such observations of the real world is a consequence only of the fact that he observed the world and interpreted its workings in a rather superficial way. He was basically unaware of the then-subtle effects of air resistance, friction, and the like. It was only with the advent of the ability and motivation to carry out precise experiments followed by a process of abstraction that led to the revolutionary point of view of nature represented by the Newtonian paradigm. Even today, the workings of that paradigm are most easily visualized in the artificial realm of our own minds, emptied of the real world's imperfections of friction and air resistance (look at any elementary physics book and see how often one encounters the phrase "neglecting friction"). Aristotle's physics, much more than Newton's, reflects the workings of a nature quite coincident with the common misconception of modern people in general (including the typical college student who chooses a curriculum curiously devoid of courses in physics).

There is no question that the first law, the so-called law of inertia, had already been set forth prior to the time of Newton. This law, commonly attributed to Galileo (1564–1642), was actually first formulated by René Descartes (1596–1650). According to Descartes, "inertia" made bodies persist in motion forever, not in perfect

Aristotelian circles but in a straight line. Descartes came to this conclusion not by experiment but by pure thought. In contrast to belief in traditional authority (which at that time meant belief in the teachings of Aristotle), Descartes believed that only one's own thinking could be trusted. It was his intent to "explain effects by their causes, and not causes by their effects." For Descartes, pure reasoning served as the sole basis of certainty. Such a paradigm would aid the transition from an Aristotelian worldview to a Newtonian one, but it contained within itself the seeds of its own destruction.

It was not too surprising that Descartes failed to grasp the implication of his law of inertia regarding planetary motion. Planets certainly did not move in straight lines. Descartes, more ruthless in his methods of thought than any of his predecessors, reasoned that some physical thing had to "drive" the planets along in their curved paths. Descartes rebelled in horror at the notion that the required physical force was some invisible entity reaching out across the void to grab the planets and hold them in their orbits. Moreover, having no knowledge of the second law, Descartes never realized that the required force was not a "driving" force but a force that had to be directed "inward" toward the Sun. He, along with many others of that era, was certain that the planets had to be pushed along in their paths around the Sun (or Earth). Thus, he concocted the notion of an all-pervading, ether-like fluid made of untold numbers of unseen particles, rotating in vortices, within which the planets were driven round and round—an erroneous conclusion that arose from the fancies of a mind engaged only in pure thought, minimally constrained by experimental or observational data.

Galileo, on the other hand, mainly by clear argument based on actual experimental results, had gradually commandeered a fairly clear understanding of what would come to be the first of Newton's laws, as well as the second. A necessary prelude to the final synthesis of a correct system of mechanics was his observation that a pendulum undergoing small oscillations was isochronous; that is, its period of oscillation was independent of its amplitude. This discovery led to the first clocks capable of making accurate measurements of small time intervals, a capability that Aristotle did not have. Galileo would soon exploit this capability in carrying out experiments of unprecedented precision with objects either freely falling or sliding down inclined planes. Generalizing from the results of his experiments, Galileo came very close to formulating Newton's first two laws.

For example, concerning the first law, Galileo noted, as had Aristotle, that an object sliding along a level surface indeed came to rest. But here Galileo made a wonderful mental leap that took him far past the dialectics of Aristotle. He imagined a second surface, more slippery than the first. An object given a push along the second surface would travel farther before stopping than it would if given a similar push along the first surface. Carrying this process of abstraction to its ultimate conclusion, Galileo reasoned that an object given a push along a surface of "infinite slipperiness" (i.e., "neglecting friction") would, in fact, go on forever, never coming to rest. Thus, contrary to Aristotle's physics, he reasoned that a force is not required merely to keep an object in motion. In fact, some force must be applied to stop it. This is very close to Newton's law of inertia but, astonishingly, Galileo did not argue that motion, in the absence of forces, would continue forever in a straight line!

For Galileo and his contemporaries, the world was not an impersonal one ruled by mechanical laws. Instead, it was a cosmos that marched to the tune of an infinitely intelligent craftsman. Following the Aristotelian tradition, Galileo saw a world ordered according to the perfect figure, the circle. Rectilinear motion implied disorder. Objects that found themselves in such a state of affairs would not continue to fly in a straight line forever but would ultimately lapse into their more natural state of perfect circular motion. The experiments necessary to discriminate between straight-line motion forever and straight-line motion ultimately evolving to pure circular motion obviously could not be performed in practice, but only within the confines of one's own mind, and only if that mind had been properly freed from the conditioning of centuries of ill-founded dogma. Galileo, brilliant though he was, still did battle with the ghosts of the past and had not yet reached that required state of mind.

Galileo's experiments with falling bodies led him to the brink of Newton's second law. Again, as Aristotle had known, Galileo saw that heavy objects, such as stones, did fall faster than lighter ones, such as feathers. However, by carefully timing similarly shaped objects, albeit of different weights, Galileo discovered that such objects accelerated as they fell and all reached the ground at more or less the same time! Indeed, very heavy objects, even though themselves differing greatly in weight, fell at almost identical rates, with a speed that increased about 10 m/s each second. (Incidentally, the famous experiment of dropping cannonballs from the Leaning Tower of Pisa might not have been carried out by Galileo but by one of his chief Aristotelian antagonists at Pisa, Giorgio Coressio, and in hopes not of refuting but of confirming the Aristotelian view that larger bodies must fall more quickly than small ones!)[1] It was again through a process of brilliant abstraction that Galileo realized that if the effects of air resistance could be eliminated, all objects would fall with the same acceleration, regardless of weight or shape. Thus, even more of Aristotle's edifice was torn apart; a heavier weight does not fall faster than a light one, and a force causes objects to accelerate, not to move at constant speed.

Galileo's notions of mechanics on Earth were more closely on target with Newton's laws than the conjectures of any of his predecessors had been. He sometimes applied them brilliantly in defense of the Copernican viewpoint, that is, a heliocentric model of the solar system. In particular, even though his notion of the law of inertia was somewhat flawed, he applied it correctly in arguing that terrestrial-based experiments could not be used to demonstrate that the Earth could not be in motion around the Sun. He pointed out that a stone dropped from the mast of a moving ship would not "be left behind" since the stone would share the ship's horizontal speed. By analogy, in contrast to Aristotelian argument, a stone dropped from a tall tower would not be left behind by an Earth in motion. This powerful argument implied that no such observation could be used to demonstrate whether or not the Earth was rotating. The argument contained the seeds of relativity theory.

Unfortunately, as mentioned above, Galileo could not entirely break loose from the Aristotelian dogma of circular motion. In strict contradiction to the law of inertia, he postulated that a body left to itself will continue to move forever, not in a straight line but in a circular orbit. His reasoning was as follows:

> . . . straight motion being by nature infinite (because a straight line is infinite and indeterminate), it is impossible that anything should have by nature the principle of moving in a straight line; or, in other words, towards a place where it is impossible to arrive, there being no finite end. For that which cannot be done, nor endeavors to move whither it is impossible to arrive.

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[1] *Aristotle, Galileo, and the Tower of Pisa*, L. Cooper, Cornell University Press, Ithaca, 1935.

This statement also contradicted his intimate knowledge of centrifugal forces, that is, the tendency of an object moving in a circle to fly off on a tangent in a straight line. He knew that earthbound objects could travel in circles only if this centrifugal force was either balanced or overwhelmed by some other offsetting force. Indeed, one of the Aristotelian arguments against a rotating Earth was that objects on the Earth's surface would be flung off it. Galileo argued that this conclusion was not valid, because the Earth's "gravity" overwhelmed this centrifugal tendency! Yet somehow he failed to make the mental leap that some similar effect must keep the planets in circular orbit about the Sun!

So ultimately it was Newton who pulled together all the fragmentary knowledge that had been accumulated about the motion of earthbound objects into the brilliant synthesis of the three laws and then demonstrated that the motion of heavenly objects obeyed those laws as well.

Newton's laws of motion can be thought of as a prescription for calculating or predicting the subsequent motion of a particle (or system of particles), given a knowledge of its position and velocity at some instant in time. These laws, in and of themselves, say nothing about the reason why a given physical system behaves the way it does. Newton was quite explicit about that shortcoming. He refused to speculate (at least in print) why objects move the way they do. Whatever "mechanism" lay behind the workings of physical systems remained forever hidden from Newton's eyes. He simply stated that, for whatever reason, this is the way things work, as demonstrated by the power of his calculational prescription to predict, with astonishing accuracy, the evolution of physical systems set in motion. Much has been learned since the time of Newton, but a basic fact of physical law persists: the laws of motion are mathematical prescriptions that allow us to predict accurately the future motion of physical systems, given a knowledge of their current state. The laws describe how things work. They do not tell us why.

Newton's First Law: Inertial Reference Systems

The first law describes a common property of matter, namely, *inertia*. Loosely speaking, inertia is the resistance of all matter to having its motion changed. If a particle is at rest, it resists being moved; that is, a force is required to move it. If the particle is in motion, it resists being brought to rest. Again, a force is required to bring it to rest. It almost seems as though matter has been endowed with an innate abhorrence of acceleration. Be that as it may, for whatever reason, it takes a force to accelerate matter; in the absence of applied forces, matter simply persists in its current velocity state—forever.

A mathematical description of the motion of a particle requires the selection of a *frame of reference*, or a set of coordinates in configuration space that can be used to specify the position, velocity, and acceleration of the particle at any instant of time. A frame of reference in which Newton's first law of motion is valid is called an *inertial frame of reference*. This law rules out accelerated frames of reference as inertial, because an object "really" at rest or moving at constant velocity, seen from an accelerated frame of reference, would appear to be accelerated. Moreover, an object seen to be at rest in such a frame would be seen to be accelerated with respect to the inertial frame. So strong is our belief in the concept of inertia and the validity of Newton's laws of motion that we would be forced to invent "fictitious" forces to account for the apparent lack of acceleration of an object at rest in an accelerated frame of reference.

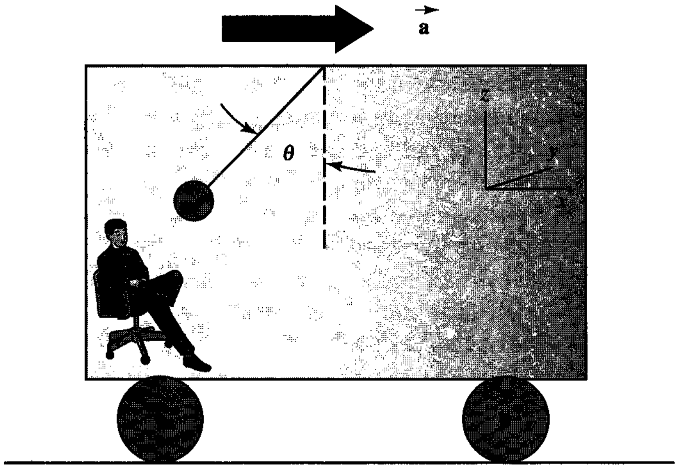

Figure 2.1.1 A plumb bob hangs at an angle θ in an accelerating frame of reference.

A simple example of a noninertial frame of reference should help clarify the situation. Consider an observer inside a railroad boxcar accelerating down the track with an acceleration **a**. Suppose a plumb bob were suspended from the ceiling of the boxcar. How would it appear to the observer? Take a look at Figure 2.1.1. The point here is that the observer in the boxcar is in a noninertial frame of reference and is at rest with respect to it. He sees the plumb bob, also apparently at rest, hanging at an angle θ with respect to the vertical. He knows that, in the absence of any forces other than gravity and tension in the plumb line, such a device should align itself vertically. It does not, and he concludes that some unknown force must be pushing or pulling the plumb bob toward the back of the car. (Indeed, he too feels such a force, as anyone who has ever been in an accelerating vehicle knows from first-hand experience.)

A question that naturally arises is how is it possible to determine whether or not a given frame of reference constitutes an inertial frame? The answer is nontrivial! (For example, if the boxcar were sealed off from the outside world, how would the observer know that the apparent force causing the plumb bob to hang off-vertical was not due to the fact that the whole boxcar was "misaligned" with the direction of gravity—that is, the force due to gravity was actually in the direction indicated by the angle θ?) Observers would have to know that *all* external forces on a body had been eliminated before checking to see whether or not objects in their frame of reference obeyed Newton's first law. It would be necessary to isolate a body completely to eliminate all forces acting upon it. This is impossible, because there would always be some gravitational forces acting unless the body were removed to an infinite distance from all other matter.

Is there a perfect inertial frame of reference? For most practical purposes, a coordinate system attached to the Earth's surface is approximately inertial. For example, a billiard ball seems to move in a straight line with constant speed as long as it does not collide with other balls or hit the cushion. If its motion were measured with very high precision, however, we would see that its path is slightly curved. This is due to the fact that the Earth is rotating and its surface is therefore accelerating toward its axis. Hence, a coordinate system attached to the Earth's surface is not inertial. A better system would be one that uses the center of the Earth as coordinate origin, with the Sun and a star as reference points. But even this system would not be inertial because of the Earth's orbital motion around the Sun.

Suppose, then, we pick a coordinate system whose origin is centered on the Sun. Strictly speaking, this is not a perfect inertial frame either, because the Sun partakes of the general rotational motion of the Milky Way galaxy. So, we try the center of the Milky Way, but to our chagrin, it is part of a local group, or small cluster, of some 20 galaxies that all rotate about their common center of mass. Continuing on, we see that the local group lies on the edge of the Virgo supercluster, which contains dozens of clusters of galaxies centered on the 2000-member-rich Virgo cluster, 60 million light years away, all rotating about their common center of mass! As a final step in this continuing saga of seeming futility, we might attempt to find a frame of reference that is at rest with respect to the observed relative motion of all the matter in the universe; however, we cannot observe all the matter. Some of the potentially visible matter is too dim to be seen, and some matter isn't even potentially visible, the so-called *dark matter*, whose existence we can only infer by indirect means. Furthermore, the universe appears to have a large supply of *dark energy*, also invisible, which nonetheless makes its presence known by accelerating the expansion of the universe.

However, all is not lost. The universe began with the *Big Bang* about 12.7 billion years ago and has been expanding ever since. Some of the evidence for this is the observation of the *Cosmic Microwave Background* radiation (CMB), a relic of the primeval fireball that emerged from that singular event.[2] Its existence provides us with a novel means of actually measuring the Earth's "true" velocity through space, without reference to neighboring galaxies, clusters, or superclusters. If we were precisely at rest with respect to the universal expansion,[3] then we would see the CMB as perfectly *isotropic*, that is, the distribution of the radiation would be the same in all directions in the sky. The reason for this is that initially, the universe was extremely hot and the radiation and matter that sprang forth from the Big Bang interacted fairly strongly and were tightly coupled together. But 380,000 years later, the expanding universe cooled down to a temperature of about 3000 K and matter, which up to that point consisted mostly of electrically charged protons and electrons, then combined to form neutral hydrogen atoms and the radiation *decoupled* from it. Since then, the universe has expanded even more, by a factor of about 1000, and has cooled to a temperature of about 2.73 K. The spectral distribution of the left over CMB has changed accordingly. Indeed, the radiation is remarkably, though not perfectly,

[2] For the most up-to-date information about the CMB, dark matter, and dark energy, visit the NASA Goddard Space Flight Center at http://map.gsfc.nasa.gov and look for articles discussing the Wilkinson Microwave Anisotropy Project (WMAP). For a general discussion of the CMB and its implications, the reader is referred to almost any current astronomy text, such as *The Universe*, 6th ed., Kaufmann and Freedman, Wiley Publishing, Indianapolis, 2001.

[3] A common analog of this situation is an inflating balloon on whose surface is attached a random distribution of buttons. Each button is fixed and, therefore, "at rest" relative to the expanding two-dimensional surface. Any frame of reference attached to any button would be a valid inertial frame of reference.

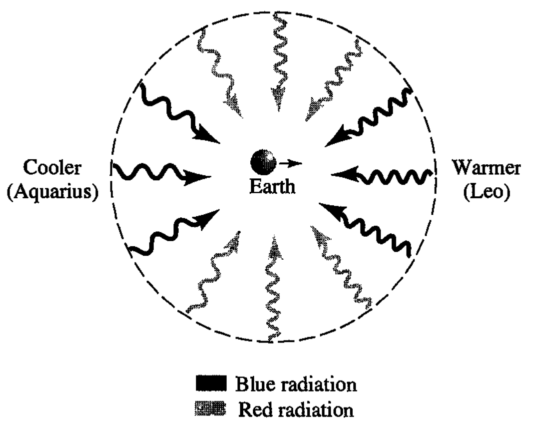

Figure 2.1.2 Motion of Earth through the cosmic microwave background.

isotropic. Radiation arriving at Earth from the direction of the constellation Leo appears to be coming from a slightly warmer region of the universe and, thus, has a slightly shorter or "bluer" wavelength than radiation arriving from the opposite direction in the constellation Aquarius (Figure 2.1.2). This small spectral difference occurs because the Earth moves about 400 km/s towards Leo, which causes a small *Doppler shift* in the observed spectral distribution.[4] Observers in a frame of reference moving from Leo toward Aquarius at 400 km/s *relative to Earth* would see a perfectly isotropic distribution (except for some variations that originated when the radiation decoupled from matter in localized regions of space of slightly different matter densities). These observers would be at rest with respect to the overall expansion of the universe! It is generally agreed that such a frame of reference comes closest to a perfect inertial frame.

However, do not think that we are implying that there is such a thing as an *absolute* inertial frame of reference. In part, the theory of relativity resulted from the failure of attempts to find an absolute frame of reference in which all of the fundamental laws of physics, not just Newton's first law of motion, were supposed to be valid. This led Einstein to the conclusion that the failure to find an absolute frame was because of the simple reason that none exists. Consequently, he proposed as a cornerstone of the theory of relativity that the fundamental laws of physics are the same in all inertial frames of reference and that there is no single preferred inertial frame.

Interestingly, Galileo, who predated Einstein by 300 years, had arrived at a very similar conclusion. Consider the words that one of his characters, Salviati, speaks to another, Sagredo, in his infamous *Dialogue Concerning the Two Chief World Systems*,[5] which poetically expresses the gist of *Galilean relativity*.

[4]Relative motion toward a source of light decreases the observed wavelength of the light. Relative motion away from the source increases the observed wavelength. This change in observed wavelength is called the Doppler Effect. A shortening is called a blueshift and a lengthening is called a redshift.

[5]*Dialogue Concerning the Two Chief World Systems*, Galileo Galilei (1632), *The Second Day*, 2nd printing, p. 186, translated by Stillman Drake, University of California Press, Berkeley, 1970.

"Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though there is no doubt that when the ship is standing still, everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time you are in the air, the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel, without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease toward bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. . . . "

EXAMPLE 2.1.1

Is the Earth a Good Inertial Reference Frame?

Calculate the centripetal acceleration (see Example 1.12.2), relative to the acceleration due to gravity g, of

(a) a point on the surface of the Earth's equator (the radius of the Earth is $R_E = 6.4 \times 10^3$ km)
(b) the Earth in its orbit about the Sun (the radius of the Earth's orbit is $a_E = 150 \times 10^6$ km)
(c) the Sun in its rotation about the center of the galaxy (the radius of the Sun's orbit about the center of the galaxy is $R_G = 2.8 \times 10^4$ LY. Its orbital speed is $v_G = 220$ km/s)

Solution:

The centripetal acceleration of a point rotating in a circle of radius R is given by

$$a_c = \omega^2 R = \left(\frac{2\pi}{T}\right)^2 R = \frac{4\pi^2 R}{T^2}$$

where T is period of one complete rotation. Thus, relative to g we have

$$\frac{a_c}{g} = \frac{4\pi^2 R}{gT^2}$$

(a) $\dfrac{a_c}{g} = \dfrac{4\pi^2(6.4\times 10^6\,\text{m})}{9.8\,\text{m}\cdot\text{s}^{-2}(3.16\times 10^7\,\text{s})^2} = 3.4\times 10^{-3}$

(b) 6×10^{-4}

(c) 1.5×10^{-12}

Question for Discussion

Suppose that you step inside an express elevator on the 120th floor of a tall skyscraper. The elevator starts its descent, but as in your worst nightmare, the support elevator cable snaps and you find yourself suddenly in freefall. Realizing that your goose is cooked—or soon will be—you decide to conduct some physics experiments during the little time you have left on Earth—or above it! First, you take your wallet out of your pocket and remove a dollar bill. You hold it in front of your face and let it go. Wonder of wonders—it does nothing! It just hangs there seemingly suspended in front of your face (Figure 2.1.3)! Being an educated person with a reasonably good understanding of Newton's first law of motion, you conclude that there is no force acting on the dollar bill. Being a skeptical person, however, you decide to subject this conclusion to a second test. You take a piece of string from your pocket, tie one end to a light fixture on the ceiling of the falling elevator, attach your wallet to the other end, having thus fashioned a crude

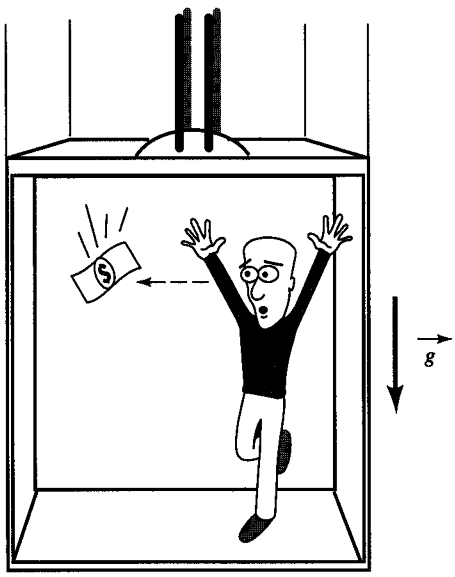

Figure 2.1.3 Person in falling elevator.

plumb bob. You know that a hanging plumb bob aligns itself in the direction of gravity, which you anticipate is perpendicular to the plane of the ceiling. However, you discover that no matter how you initially align the plumb bob relative to the ceiling, it simply hangs in that orientation. There appears to be no gravitational force acting on the plumb bob, either. Indeed, there appears to be no force of any kind acting on any object inside the elevator. You now wonder why your physics instructor had such difficulty trying to find a perfect inertial frame of reference, because you appear to have discovered one quite easily—just get into a freely falling elevator. Unfortunately, you realize that within a few moments, you will not be able to share the joy of your discovery with anyone else.

So—is an elevator in free fall a perfect inertial frame of reference, or not?

Hint: Consider this quotation by Albert Einstein.

> At that moment there came to me the happiest moment of my life . . . for an observer falling freely from the roof of a house no gravitational force exists during his fall—at least not in his immediate vicinity. That is, if the observer releases any objects, they remain in a state of rest or uniform motion relative to him, respectively, independent of their unique chemical and physical nature. Therefore, no observer is entitled to interpret his state as that of "rest."

For a more detailed discussion of inertial frames of reference and their relationship to gravity, read the delightful book, *Spacetime Physics*, 2nd ed., by Taylor and Wheeler, W. H. Freeman & Co., New York, 1992.

Mass and Force: Newton's Second and Third Laws

The quantitative measure of inertia is called *mass*. We are all familiar with the notion that the more massive an object is, the more resistive it is to acceleration. Go push a bike to get it rolling, and then try the same thing with a car. Compare the efforts. The car is much more massive and a much larger force is required to accelerate it than the bike. A more quantitative definition may be constructed by considering two masses, m_1 and m_2, attached by a spring and initially at rest in an inertial frame of reference. For example, we could imagine the two masses to be on a frictionless surface, almost achieved in practice by two carts on an air track, commonly seen in elementary physics class demonstrations. Now imagine someone pushing the two masses together, compressing the spring, and then suddenly releasing them so that they fly apart, attaining speeds v_1 and v_2. We *define* the ratio of the two masses to be

$$\frac{m_2}{m_1} = \left|\frac{\mathbf{v}_1}{\mathbf{v}_2}\right| \tag{2.1.1}$$

If we let m_1 be the standard of mass, then all other masses can be operationally defined in the above way relative to the standard. This operational definition of mass is consistent with Newton's second and third laws of motion, as we shall soon see. Equation 2.1.1 is equivalent to

$$\Delta(m_1\mathbf{v}_1) = -\Delta(m_2\mathbf{v}_2) \tag{2.1.2}$$

because the initial velocities of each mass are zero and the final velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are in opposite directions. If we divide by Δt and take limits as $\Delta t \rightarrow 0$, we obtain

$$\frac{d}{dt}(m_1\mathbf{v}_1) = -\frac{d}{dt}(m_2\mathbf{v}_2) \tag{2.1.3}$$

The product of mass and velocity, $m\mathbf{v}$, is called *linear momentum*. The "change of motion" stated in the second law of motion was rigorously defined by Newton to be the time rate of change of the linear momentum of an object, and so the second law can be rephrased as follows: *The time rate of change of an object's linear momentum is proportional to the impressed force,* **F**. Thus, the second law can be written as

$$\mathbf{F} = k\frac{d(m\mathbf{v})}{dt} \tag{2.1.4}$$

where k is a constant of proportionality. Considering the mass to be a constant, independent of velocity (which is not true of objects moving at "relativistic" speeds or speeds approaching the speed of light, 3×10^8 m/s, a situation that we do not consider in this book), we can write

$$\mathbf{F} = km\frac{d\mathbf{v}}{dt} = km\mathbf{a} \tag{2.1.5}$$

where **a** is the resultant acceleration of a mass m subjected to a force **F**. The constant of proportionality can be taken to be $k = 1$ by defining the unit of force in the SI system to be that which causes a 1-kg mass to be accelerated 1 m/s^2. This force unit is called 1 newton.

Thus, we finally express Newton's second law in the familiar form

$$\mathbf{F} = \frac{d(m\mathbf{v})}{dt} = m\mathbf{a} \tag{2.1.6}$$

The force **F** on the left side of Equation 2.1.6 is the *net* force acting upon the mass m; that is, it is the vector sum of all of the individual forces acting upon m.

We note that Equation 2.1.3 is equivalent to

$$\mathbf{F}_1 = -\mathbf{F}_2 \tag{2.1.7}$$

or Newton's third law, namely, that two interacting bodies exert equal and opposite forces upon one another. Thus, our definition of mass is consistent with both Newton's second and third laws.

Linear Momentum

Linear momentum proves to be such a useful notion that it is given its own symbol:

$$\mathbf{p} = m\mathbf{v} \tag{2.1.8}$$

Newton's second law may be written as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{2.1.9}$$

Thus, Equation 2.1.3, which describes the behavior of two mutually interacting masses, is equivalent to

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0 \tag{2.1.10}$$

or

$$\mathbf{p}_1 + \mathbf{p}_2 = \text{constant} \tag{2.1.11}$$

In other words, Newton's third law implies that the total momentum of two mutually interacting bodies is a constant. This constancy is a special case of the more general situation in which the total linear momentum of an isolated system (a system subject to no net externally applied forces) is a conserved quantity. The law of linear momentum conservation is one of the most fundamental laws of physics and is valid even in situations in which Newtonian mechanics fails.

EXAMPLE 2.1.2

A spaceship of mass M is traveling in deep space with velocity $v_i = 20$ km/s relative to the Sun. It ejects a rear stage of mass $0.2\,M$ with a relative speed $u = 5$ km/s (Figure 2.1.4). What then is the velocity of the spaceship?

Solution:

The system of spaceship plus rear stage is a closed system upon which no external forces act (neglecting the gravitational force of the Sun); therefore, the total linear momentum is conserved. Thus

$$\mathbf{P}_f = \mathbf{P}_i$$

where the subscripts i and f refer to initial and final values respectively. Taking velocities in the direction of the spaceship's travel to be positive, before ejection of the rear stage, we have

$$P_i = Mv_i$$

Let U be the velocity of the ejected rear stage and v_f be the velocity of the ship after ejection. The total momentum of the system after ejection is then

$$P_f = 0.20\,MU + 0.80\,Mv_f$$

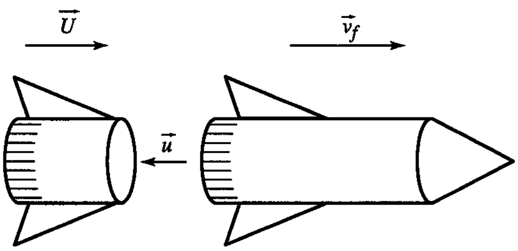

Figure 2.1.4 Spaceship ejecting a rear stage.

The speed u of the ejected stage relative to the spaceship is the difference in velocities of the spaceship and stage

$$u = v_f - U$$

or

$$U = v_f - u$$

Substituting this latter expression into the equation above and using the conservation of momentum condition, we find

$$0.20\, M(v_f - u) + 0.8\, Mv_f = Mv_i$$

which gives us

$$v_f = v_i + 0.2u = 20 \text{ km/s} + 0.20\ (5 \text{ km/s}) = 21 \text{ km/s}$$

Motion of a Particle

Equation 2.1.6 is the fundamental equation of motion for a particle subject to the influence of a *net* force, **F**. We emphasize this point by writing **F** as $\mathbf{F}_{net}$, the vector sum of all the forces acting on the particle.

$$\mathbf{F}_{\text{net}} = \Sigma \mathbf{F}_i = m\frac{d^2\mathbf{r}}{dt^2} = m\mathbf{a} \tag{2.1.12}$$

The usual problem of dynamics can be expressed in the following way: Given a knowledge of the forces acting on a particle (or system of particles), calculate the acceleration of the particle. Knowing the acceleration, calculate the velocity and position as functions of time. This process involves solving the second-order differential equation of motion represented by Equation 2.1.12. A complete solution requires a knowledge of the initial conditions of the problem, such as the values of the position and velocity of the particle at time $t = 0$. The initial conditions plus the dynamics dictated by the differential equation of motion of Newton's second law completely determine the subsequent motion of the particle. In some cases this procedure cannot be carried to completion in an analytic way. The solution of a complex problem will, in general, have to be carried out using numerical approximation techniques on a digital computer.

2.2 | Rectilinear Motion: Uniform Acceleration Under a Constant Force

When a moving particle remains on a single straight line, the motion is said to be *rectilinear*. In this case, without loss of generality we can choose the x-axis as the line of motion. The general equation of motion is then

$$F_x(x, \dot{x}, t) = m\ddot{x} \tag{2.2.1}$$

(**Note:** *In the rest of this chapter, we usually use the single variable* x *to represent the position of a particle. To avoid excessive and unnecessary use of subscripts, we often use the symbols* v *and* a *for* $\dot{x}$ *and* $\ddot{x}$, *respectively, rather than* v_x *and* a_x, *and* F *rather than* F_x.)

The simplest situation is that in which the force is constant. In this case we have constant acceleration

$$\ddot{x} = \frac{dv}{dt} = \frac{F}{m} = \text{constant} = a \tag{2.2.2a}$$

and the solution is readily obtained by direct integration with respect to time:

$$\dot{x} = v = at + v_0 \tag{2.2.2b}$$

$$x = \tfrac{1}{2}at^2 + v_0 t + x_0 \tag{2.2.2c}$$

where v_0 is the velocity and x_0 is the position at $t = 0$. By eliminating the time t between Equations 2.2.2b and 2.2.2c, we obtain

$$2a(x - x_0) = v^2 - v_0^2 \tag{2.2.2d}$$

The student will recall the above familiar equations of uniformly accelerated motion. There are a number of fundamental applications. For example, in the case of a body falling freely near the surface of the Earth, neglecting air resistance, the acceleration is very nearly constant. We denote the acceleration of a freely falling body with **g**. Its magnitude is $g = 9.8\ \text{m/s}^2$. The downward force of gravity (the *weight*) is, accordingly, equal to $m\mathbf{g}$. The gravitational force is always present, regardless of the motion of the body, and is independent of any other forces that may be acting.[6] We henceforth call it $m\mathbf{g}$.

EXAMPLE 2.2.1

Consider a block that is free to slide down a smooth, frictionless plane that is inclined at an angle θ to the horizontal, as shown in Figure 2.2.1(a). If the height of the plane is h and the block is released from rest at the top, what will be its speed when it reaches the bottom?

Solution:

We choose a coordinate system whose positive x-axis points down the plane and whose y-axis points "upward," perpendicular to the plane, as shown in the figure. The only force along the x direction is the component of gravitational force, $mg \sin\theta$, as shown in Figure 2.2.1(b). It is constant. Thus, Equations 2.2.2a–d are the equations of motion, where

$$\ddot{x} = a = \frac{F_x}{m} = g \sin\theta$$

[6] Effects of the Earth's rotation are studied in Chapter 5.

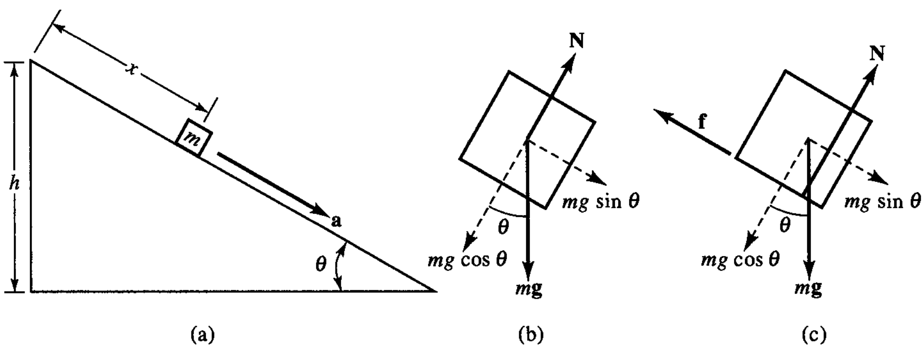

Figure 2.2.1 (a) A block sliding down an inclined plane. (b) Force diagram (no friction). (c) Force diagram (friction $f = \mu_K N$).

and

$$x - x_0 = \frac{h}{\sin\theta}$$

Thus,

$$v^2 = 2(g\sin\theta)\left(\frac{h}{\sin\theta}\right) = 2gh$$

Suppose that, instead of being smooth, the plane is rough; that is, it exerts a frictional force **f** on the particle. Then the net force in the x direction, (see Figure 2.2.1(c)), is equal to $mg\sin\theta - f$. Now, for sliding contact it is found that the magnitude of the frictional force is proportional to the magnitude of the normal force N; that is,

$$f = \mu_K N$$

where the constant of proportionality μ_K is known as the *coefficient of sliding* or *kinetic friction*.[7] In the example under discussion, the normal force, as shown in the figure, is equal to $mg\cos\theta$; hence,

$$f = \mu_K mg\cos\theta$$

Consequently, the net force in the x direction is equal to

$$mg\sin\theta - \mu_K mg\cos\theta$$

Again the force is constant, and Equations 2.2.2a–d apply where

$$\ddot{x} = \frac{F_x}{m} = g(\sin\theta - \mu_k\cos\theta)$$

[7]There is another coefficient of friction called the *static* coefficient μ_s, which, when multiplied by the normal force, gives the maximum frictional force under static contact, that is, the force required to barely start an object to move when it is initially at rest. In general, $\mu_s > \mu_K$.

The speed of the particle increases if the expression in parentheses is positive—that is, if $\theta > \tan^{-1}\mu_K$. The angle, $\tan^{-1}\mu_K$, usually denoted by ϵ, is called the *angle of kinetic friction*. If $\theta = \epsilon$, then $a = 0$, and the particle slides down the plane with constant speed. If $\theta < \epsilon$, a is negative, and so the particle eventually comes to rest. For motion *up* the plane, the direction of the frictional force is reversed; that is, it is in the positive x direction. The acceleration (actually deceleration) is then $\ddot{x} = g(\sin\theta + \mu_K \cos\theta)$.

2.3 Forces that Depend on Position: The Concepts of Kinetic and Potential Energy

It is often true that the force a particle experiences depends on the particle's position with respect to other bodies. This is the case, for example, with electrostatic and gravitational forces. It also applies to forces of elastic tension or compression. If the force is independent of velocity or time, then the differential equation for rectilinear motion is simply

$$F(x) = m\ddot{x} \tag{2.3.1}$$

It is usually possible to solve this type of differential equation by one of several methods, such as using the chain rule to write the acceleration in the following way:

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = v\frac{dv}{dx} \tag{2.3.2}$$

so the differential equation of motion may be written

$$F(x) = mv\frac{dv}{dx} = \frac{m}{2}\frac{d(v^2)}{dx} = \frac{dT}{dx} \tag{2.3.3}$$

The quantity $T = \frac{1}{2}mv^2$ is called the *kinetic energy* of the particle. We can now express Equation 2.3.3 in integral form:

$$W = \int_{x_0}^{x} F(x)\,dx = T - T_0 \tag{2.3.4}$$

The integral $\int F(x)\,dx$ is the *work* W done on the particle by the impressed force $F(x)$. *The work is equal to the change in the kinetic energy of the particle.* Let us *define* a function $V(x)$ such that

$$-\frac{dV(x)}{dx} = F(x) \tag{2.3.5}$$

The function $V(x)$ is called the *potential energy*; it is defined only to within an arbitrary additive constant. In terms of $V(x)$, the work integral is

$$W = \int_{x_0}^{x} F(x)\,dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0 \tag{2.3.6}$$

Notice that Equation 2.3.6 remains unaltered if $V(x)$ is changed by adding *any* constant C, because

$$-[V(x) + C] + [V(x_0) + C] = -V(x) + V(x_0) \tag{2.3.7}$$

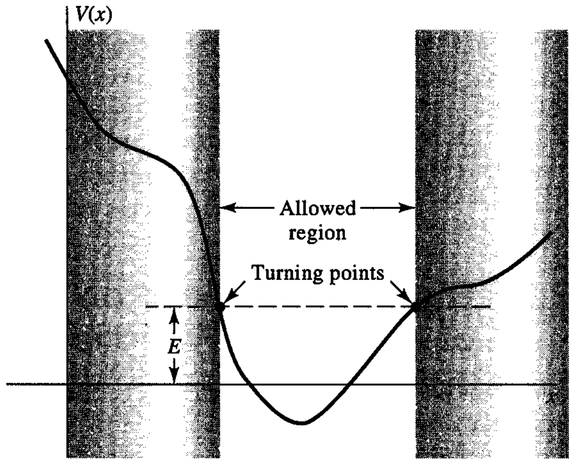

Figure 2.3.1 Graph of a one-dimensional potential energy function $V(x)$ showing the allowed region of motion and the turning points for a given value of the total energy E.

We now transpose terms and write Equation 2.3.6 in the following form:

$$T_0 + V(x_0) = \text{constant} = T + V(x) \equiv E \tag{2.3.8}$$

This is the *energy equation*. E is defined to be the *total energy* of the particle (technically, it's the total *mechanical* energy). It is equal to the sum of the kinetic and potential energies and is constant throughout the motion of the particle. This constancy results from the fact that the impressed force is a function only of the position x (of the particle and consequently can be derived from a corresponding potential energy) function $V(x)$. Such a force is said to be *conservative*.[8] Nonconservative forces—that is, those for which no potential energy function exists—are usually of a dissipational nature, such as friction.

The motion of the particle can be obtained by solving the energy equation (Equation 2.3.8) for v,

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - V(x)]} \tag{2.3.9}$$

which can be written in integral form,

$$\int_{x_0}^{x} \frac{dx}{\pm\sqrt{\frac{2}{m}[E - V(x)]}} = t - t_0 \tag{2.3.10}$$

thus giving t as a function of x.

In view of Equation 2.3.9, we see that the expression for v is real only for those values of x such that $V(x)$ is less than or equal to the total energy E. Physically, this means that the particle is confined to the region or regions for which the condition $V(x) \leq E$ is satisfied. Furthermore, v goes to zero when $V(x) = E$. This means that the particle must come to rest and reverse its motion at points for which the equality holds. These points are called the *turning points* of the motion. The above facts are illustrated in Figure 2.3.1.

[8] A more complete discussion of conservative forces is found in Chapter 4.

EXAMPLE 2.3.1

Free Fall

The motion of a freely falling body (discussed above under the case of constant acceleration) is an example of conservative motion. If we choose the x direction to be positive upward, then the gravitational force is equal to $-mg$. Therefore, $-dV/dx = -mg$, and $V = mgx + C$. The constant of integration C is arbitrary and merely depends on the choice of the reference level for measuring V. We can choose $C = 0$, which means that $V = 0$ when $x = 0$. The energy equation is then

$$\tfrac{1}{2}mv^2 + mgx = E$$

The energy constant E is determined from the initial conditions. For instance, let the body be projected upward with initial speed v_0 from the origin $x = 0$. These values give $E = mv_0^2/2 = mv^2/2 + mgx$, so

$$v^2 = v_0^2 - 2gx$$

The turning point of the motion, which is in this case the maximum height, is given by setting $v = 0$. This gives $0 = v_0^2 - 2gx_{max}$, or

$$h = x_{max} = \frac{v_0^2}{2g}$$

EXAMPLE 2.3.2

Variation of Gravity with Height

In Example 2.3.1 we assumed that g was constant. Actually, the force of gravity between two particles is inversely proportional to the square of the distance between them (Newton's law of gravity).[9] Thus, the gravitational force that the Earth exerts on a body of mass m is given by

$$F_r = -\frac{GMm}{r^2}$$

in which G is Newton's constant of gravitation, M is the mass of the Earth, and r is the distance from the center of the Earth to the body. By definition, this force is equal to the quantity $-mg$ when the body is at the surface of the Earth, so $mg = GMm/r_e^2$. Thus, $g = GM/r_e^2$ is the acceleration of gravity at the Earth's surface. Here r_e is the radius of the Earth (assumed to be spherical). Let x be the distance above the surface, so that $r = r_e + x$. Then, neglecting any other forces such as air resistance, we can write

$$F(x) = -mg\frac{r_e^2}{(r_e + x)^2} = m\ddot{x}$$

[9]We study Newton's law of gravity in more detail in Chapter 6.

for the differential equation of motion of a vertically falling (or rising) body with the variation of gravity taken into account. To integrate, we set $\ddot{x} = v dv/dx$. Then

$$-mgr_e^2 \int_{x_0}^{x} \frac{dx}{(r_e + x)^2} = \int_{v_0}^{v} mv\, dv$$

$$mgr_e^2 \left(\frac{1}{r_e + x} - \frac{1}{r_e + x_0} \right) = \tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_0^2$$

This is just the *energy equation* in the form of Equation 2.3.6. The potential energy is $V(x) = -mg[r_e^2/(r_e + x)]$ rather than mgx.

Maximum Height: Escape Speed

Suppose a body is projected upward with initial speed v_0 at the surface of the Earth, $x_0 = 0$. The energy equation then yields, upon solving for v^2, the following result:

$$v^2 = v_0^2 - 2gx \left(1 + \frac{x}{r_e} \right)^{-1}$$

This reduces to the result for a uniform gravitational field of Example 2.2.1, if x is very small compared to r_e so that the term x/r_e can be neglected. The turning point (maximum height) is found by setting $v = 0$ and solving for x. The result is

$$x_{max} = h = \frac{v_0^2}{2g} \left(1 - \frac{v_0^2}{2gr_e} \right)^{-1}$$

Again we get the formula of Example 2.2.1 if the second term in the parentheses can be ignored, that is, if v_0^2 is much smaller than $2gr_e$.

Using this last, exact expression, we solve for the value of v_0 that gives an infinite value for h. This is called the *escape speed*, and it is found by setting the quantity in parentheses equal to zero. The result is

$$v_e = (2gr_e)^{1/2}$$

This gives, for $g = 9.8$ m/s^2 and $r_e = 6.4 \times 10^6$ m,

$$v_e \simeq 11 \text{ km/s} \simeq 7 \text{ mi/s}$$

for the numerical value of the escape speed from the surface of the Earth.

In the Earth's atmosphere, the average speed of air molecules (O_2 and N_2) is about 0.5 km/s, which is considerably less than the escape speed, so the Earth retains its atmosphere. The moon, on the other hand, has no atmosphere; because the escape speed at the moon's surface, owing to the moon's small mass, is considerably smaller than that at the Earth's surface, any oxygen or nitrogen would eventually disappear. The Earth's atmosphere, however, contains no significant amount of hydrogen, even though hydrogen is the most abundant element in the universe as a whole. A hydrogen atmosphere would have escaped from the Earth long ago, because the molecular speed of hydrogen is large enough (owing to the small mass of the hydrogen molecule) that at any instant a significant number of hydrogen molecules would have speeds exceeding the escape speed.

EXAMPLE 2.3.3

The *Morse function* $V(x)$ approximates the potential energy of a vibrating diatomic molecule as a function of x, the distance of separation of its constituent atoms, and is given by

$$V(x) = V_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - V_0$$

where V_0, x_0, and δ are parameters chosen to describe the observed behavior of a particular pair of atoms. The force that each atom exerts on the other is given by the derivative of this function with respect to x. Show that x_0 is the separation of the two atoms when the potential energy function is a minimum and that its value for that distance of separation is $V(x_0) = -V_0$. (When the molecule is in this configuration, we say that it is in equilibrium.)

Solution:

The potential energy of the diatomic molecule is a minimum when its derivative with respect to x, the distance of separation, is zero. Thus,

$$F(x) = -\frac{dV(x)}{dx} = 0 =$$

$$2\frac{V_0}{\delta}\left(1 - e^{-(x-x_0)/\delta}\right)\left(e^{-(x-x_0)/\delta}\right) = 0$$

$$1 - e^{-(x-x_0)/\delta} = 0$$

$$\ln(1) = -(x - x_0)/\delta = 0$$

$$\therefore x = x_0$$

The value of the potential energy at the minimum can be found by setting $x = x_0$ in the expression for $V(x)$. This gives $V(x_0) = -V_0$.

EXAMPLE 2.3.4

Shown in Figure 2.3.2 is the potential energy function for a diatomic molecule. Show that, for separation distances x close to x_0, the potential energy function is parabolic and the resultant force on each atom of the pair is linear, always directed toward the equilibrium position.

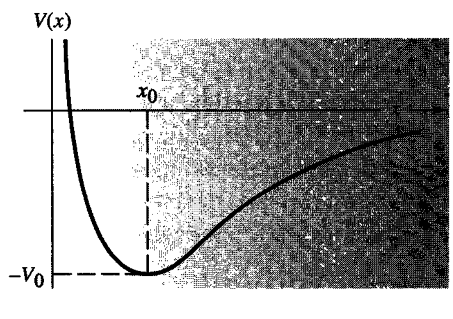

Figure 2.3.2 Potential energy function for a diatomic molecule.

Solution:

All we need do here is expand the potential energy function near the equilibrium position.

$$V(x) \approx V_0\left[1-\left(1-\left(\frac{x-x_0}{\delta}\right)\right)\right]^2 - V_0$$

$$\approx \frac{V_0}{\delta^2}(x-x_0)^2 - V_0$$

$$F(x) = -\frac{dV(x)}{dx} = -\frac{2V_0}{\delta^2}(x-x_0)$$

(**Note:** The force is linear and is directed in such a way as to restore the diatomic molecule to its equilibrium position.)

EXAMPLE 2.3.5

The binding energy ($-V_0$) of the diatomic hydrogen molecule H_2 is -4.52 eV (1 eV = 1.6×10^{-19} joules; 1 joule = 1 N · m). The values of the constants x_0 and δ are .074 and .036 nm, respectively (1 nm = 10^{-9} m). Assume that at room temperature the total energy of the hydrogen molecule is about ΔE = 1/40 eV higher than its binding energy. Calculate the maximum separation of the two atoms in the diatomic hydrogen molecule.

Solution:

Because the molecule has a little more energy than its minimum possible value, the two atoms will vibrate between two values of x, where their kinetic energy is zero. At these turning points, all the energy is potential; hence,

$$V(x) = -V_0 + \Delta E \approx \frac{V_0}{\delta^2}(x-x_0)^2 - V_0$$

$$x = x_0 \pm \delta\sqrt{\frac{\Delta E}{V_0}}$$

Putting in numbers, we see that the hydrogen molecule vibrates at room temperature a distance of about ±4% of its equilibrium separation.

For this situation where the oscillation is small, the two atoms undergo a symmetrical displacement about their equilibrium position. This arises from approximating the potential function as a parabola near equilibrium. Note from Figure 2.3.2 that, farther away from the equilibrium position, the potential energy function is not symmetrical, being steeper at smaller distances of separation. Thus, as the diatomic molecule is "heated up," on the average it spends an increasingly greater fraction of its time separated by a distance greater than their separation at equilibrium. This is why most substances tend to expand when heated.

2.4 Velocity-Dependent Forces: Fluid Resistance and Terminal Velocity

It often happens that the force that acts on a body is a function of the velocity of the body. This is true, for example, in the case of viscous resistance exerted on a body moving through a fluid. If the force can be expressed as a function of v only, the differential equation of motion may be written in either of the two forms

$$F_0 + F(v) = m\frac{dv}{dt} \tag{2.4.1}$$

$$F_0 + F(v) = mv\frac{dv}{dx} \tag{2.4.2}$$

Here F_0 is any constant force that does not depend on v. Upon separating variables, integration yields either t or x as a function of v. A second integration can then yield a functional relationship between x and t.

For normal fluid resistance, including air resistance, $F(v)$ is not a simple function and generally must be found through experimental measurements. However, a fair approximation for many cases is given by the equation

$$F(v) = -c_1 v - c_2 v\,|v| = -v\,(c_1 + c_2\,|v|) \tag{2.4.3}$$

in which c_1 and c_2 are constants whose values depend on the size and shape of the body. (The absolute-value sign is necessary on the last term because the force of fluid resistance is always opposite to the direction of v.) If the above form for $F(v)$ is used to find the motion by solving Equation 2.4.1 or 2.4.2, the resulting integrals are somewhat messy. But for the limiting cases of small v and large v, respectively, the linear or the quadratic term in $F(v)$ dominates, and the differential equations become somewhat more manageable.

For spheres in air, approximate values for the constants in the equation for $F(v)$ are, in SI units,

$$c_1 = 1.55 \times 10^{-4} D$$
$$c_2 = 0.22\,D^2$$

where D is the diameter of the sphere in meters. The ratio of the quadratic term $c_2 v|v|$ to the linear term $c_1 v$ is, thus,

$$\frac{0.22v\,|v|\,D^2}{1.55\times10^{-4}\,vD} = 1.4\times10^3\,|v|\,D$$

This means that, for instance, with objects of baseball size ($D \sim 0.07$ m), the quadratic term dominates for speeds in excess of 0.01 m/s (1 cm/s), and the linear term dominates for speeds less than this value. For speeds around this value, *both* terms must be taken into account. (See Problem 2.15.)

EXAMPLE 2.4.1

Horizontal Motion with Linear Resistance

Suppose a block is projected with initial velocity v_0 on a smooth horizontal surface and that there is air resistance such that the linear term dominates. Then, in the direction of the motion, $F_0 = 0$ in Equations 2.4.1 and 2.4.2, and $F(v) = -c_1 v$. The differential equation of motion is then

$$-c_1 v = m\frac{dv}{dt}$$

which gives, upon integrating,

$$t = \int_{v_0}^{v} -\frac{m\,dv}{c_1 v} = -\frac{m}{c_1}\ln\left(\frac{v}{v_0}\right)$$

Solution:

We can easily solve for v as a function of t by multiplying by $-c_1/m$ and taking the exponential of both sides. The result is

$$v = v_0 e^{-c_1 t/m}$$

Thus, the velocity decreases exponentially with time. A second integration gives

$$x = \int_0^t v_0 e^{-c_1 t/m}\,dt$$
$$= \frac{mv_0}{c_1}(1 - e^{-c_1 t/m})$$

showing that the block approaches a limiting position given by $x_{lim} = mv_0/c_1$.

EXAMPLE 2.4.2

Horizontal Motion with Quadratic Resistance

If the parameters are such that the quadratic term dominates, then for positive v we can write

$$-c_2 v^2 = m\frac{dv}{dt}$$

which gives

$$t = \int_{v_n}^{v} \frac{-m\,dv}{c_2 v^2} = \frac{m}{c_2}\left(\frac{1}{v} - \frac{1}{v_0}\right)$$

Solution:

Solving for v, we get

$$v = \frac{v_0}{1 + kt}$$

where $k = c_2v_0/m$. A second integration gives us the position as a function of time:

$$x(t) = \int_0^t \frac{v_0 dt}{1+kt} = \frac{v_0}{k}\ln(1+kt)$$

Thus, as $t \to \infty$, v decreases as $1/t$, but the position does not approach a limit as was obtained in the case of a linear retarding force. Why might this be? You might guess that a quadratic retardation should be more effective in stopping the block than is a linear one. This is certainly true at large velocities, but as the velocity approaches zero, the quadratic retarding force goes to zero much faster than the linear one—enough to allow the block to continue on its merry way, albeit at a very slow speed.

Vertical Fall Through a Fluid: Terminal Velocity

(a) *Linear case*. For an object falling vertically in a resisting fluid, the force F_0 in Equations 2.4.1 and 2.4.2 is the weight of the object, namely, $-mg$ for the x-axis positive in the upward direction. For the linear case of fluid resistance, we then have for the differential equation of motion

$$-mg - c_1v = m\frac{dv}{dt} \tag{2.4.4}$$

Separating variables and integrating, we find

$$t = \int_{v_0}^{v} \frac{m\,dv}{-mg - c_1v} = -\frac{m}{c_1}\ln\frac{mg + c_1v}{mg + c_1v_0} \tag{2.4.5}$$

in which v_0 is the initial velocity at $t = 0$. Upon multiplying by $-c_1/m$ and taking the exponential, we can solve for v:

$$v = -\frac{mg}{c_1} + \left(\frac{mg}{c_1} + v_0\right)e^{-c_1t/m} \tag{2.4.6}$$

The exponential term drops to a negligible value after a sufficient time ($t \gg m/c_1$), and the velocity approaches the limiting value $-mg/c_1$. The limiting velocity of a falling body is called the *terminal velocity*; it is that velocity at which the force of resistance is just equal and opposite to the weight of the body so that the total force is zero, and so the acceleration is zero. The magnitude of the terminal velocity is the *terminal speed*.

Let us designate the terminal speed mg/c_1 by v_t, and let us write τ (which we may call the *characteristic time*) for m/c_1. Equation 2.4.6 may then be written in the more significant form

$$v = -v_t(1 - e^{-t/\tau}) + v_0e^{-t/\tau} \tag{2.4.7}$$

These two terms represent two velocities: the terminal velocity v_t, which exponentially "fades in," and the initial velocity v_0, which exponentially "fades out" due to the action of the viscous drag force.

In particular, for an object dropped from rest at time $t = 0$, $v_0 = 0$, we find

$$v = -v_t(1 - e^{-t/\tau}) \tag{2.4.8}$$

Thus, after one characteristic time the speed is $1 - e^{-1}$ times the terminal speed, after two characteristic times it is the factor $1 - e^{-2}$ of v_t, and so on. After an interval of 5τ, the speed is within 1% of the terminal value, namely, $(1 - e^{-5})v_t = 0.993\, v_t$.

(b) *Quadratic case.* In this case, the magnitude of $F(v)$ is proportional to v^2. To ensure that the force remains resistive, we must remember that the sign preceding the $F(v)$ term depends on whether or not the motion of the object is upward or downward. This is the case for any resistive force proportional to an *even* power of velocity. A general solution usually involves treating the upward and downward motions separately. Here, we simplify things somewhat by considering only the situation in which the body is either dropped from rest or projected downward with an initial velocity v_0. We leave it as an exercise for the student to treat the upward-going case. We take the downward direction to be the positive y direction. The differential equation of motion is

$$\begin{aligned} m\frac{dv}{dt} &= mg - c_2 v^2 = mg\left(1 - \frac{c_2}{mg}v^2\right) \\ &= mg\left(1 - \frac{v^2}{v_t^2}\right) \\ \frac{dv}{dt} &= g\left(1 - \frac{v^2}{v_t^2}\right) \end{aligned} \tag{2.4.9}$$

where

$$v_t = \sqrt{\frac{mg}{c_2}} \qquad (\textit{terminal speed}) \tag{2.4.10}$$

Integrating Equation 2.4.9 gives t as a function of v,

$$t - t_0 = \int_{v_0}^{v} \frac{dv}{g\left(1 - \frac{v^2}{v_t^2}\right)} = \tau\left(\tanh^{-1}\frac{v}{v_t} - \tanh^{-1}\frac{v_0}{v_t}\right) \tag{2.4.11}$$

where

$$\tau = \frac{v_t}{g} = \sqrt{\frac{m}{c_2 g}} \qquad (\textit{characteristic time}) \tag{2.4.12}$$

Solving for v, we obtain

$$v = v_t \tanh\left(\frac{t - t_0}{\tau} - \tanh^{-1}\frac{v_0}{v_t}\right) \tag{2.4.13}$$

If the body is released from rest at time $t = 0$,

$$v = v_t \tanh\frac{t}{\tau} = v_t\left(\frac{e^{2t/\tau} - 1}{e^{2t/\tau} + 1}\right) \tag{2.4.14}$$

The terminal speed is attained after the lapse of a few characteristic times; for example, at $t = 5\tau$, the speed is $0.99991\, v_t$. Graphs of speed versus time of fall for the linear and quadratic cases are shown in Figure 2.4.1.

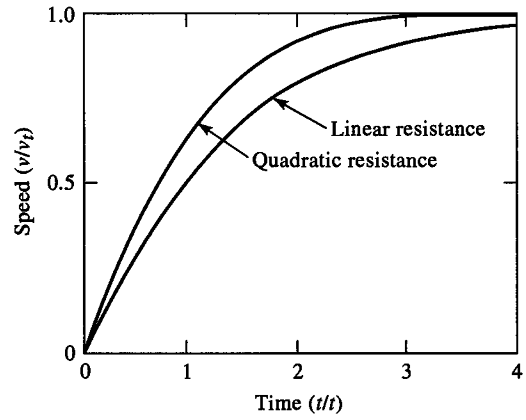

Figure 2.4.1 Graphs of speed (units of terminal speed) versus time (units of time constant τ) for a falling body.

In many instances we would like to know the speed attained upon falling a given distance. We could find this out by integrating Equation 2.4.13, obtaining y as a function of time, and then eliminating the time parameter to find speed versus distance. A more direct solution can be obtained by direct modification of the fundamental differential equation of motion so that the independent variable is distance instead of time. For example, because

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{1}{2}\frac{dv^2}{dy} \tag{2.4.15}$$

Equation 2.4.9 can be rewritten with y as the independent variable:

$$\frac{dv^2}{dy} = 2g\left(1 - \frac{v^2}{v_t^2}\right) \tag{2.4.16}$$

We solve this equation as follows:

$$u = 1 - \frac{v^2}{v_t^2} \qquad \text{so} \qquad \frac{du}{dy} = -\frac{1}{v_t^2}\frac{dv^2}{dy} = -\left(\frac{2g}{v_t^2}\right)u$$

$$u = u(y=0)e^{-2gy/v_t^2} \qquad \text{but} \qquad u(y=0) = 1 - \frac{v_0^2}{v_t^2}$$

$$u = \left(1 - \frac{v_0^2}{v_t^2}\right)e^{-2gy/v_t^2} = 1 - \frac{v^2}{v_t^2}$$

$$\therefore v^2 = v_t^2\left(1 - e^{-2gy/v_t^2}\right) + v_0^2 e^{-2gy/v_t^2} \tag{2.4.17}$$

Thus, we see that the squares of the initial velocity and terminal velocity exponentially fade in and out within a characteristic length of $v_t^2/2g$.

EXAMPLE 2.4.3

Falling Raindrops and Basketballs

Calculate the terminal speed in air and the characteristic time for (a) a very tiny spherical raindrop of diameter 0.1 mm = 10^{-4} m and (b) a basketball of diameter 0.25 m and mass 0.6 kg.

Solution:

To decide which type of force law to use, quadratic or linear, we recall the expression that gives the ratio of the quadratic to the linear force for air resistance, namely, $1.4 \times 10^3 |v| D$. For the raindrop this is $0.14v$, and for the basketball it is $350v$, numerically, where v is in meters per second. Thus, for the raindrop, v must exceed 1/0.14 = 7.1 m/s for the quadratic force to dominate. In the case of the basketball, v must exceed only 1/350 = 0.0029 m/s for the quadratic force to dominate. We conclude that the linear case should hold for the falling raindrop, while the quadratic case should be correct for the basketball. (See also Problem 2.15.)

The volume of the raindrop is $\pi D^3/6 = 0.52 \times 10^{-12}$ m^3, so, multiplying by the density of water, 10^3 kg/m^3, gives the mass $m = 0.52 \times 10^{-9}$ kg. For the drag coefficient we get $c_1 = 1.55 \times 10^{-4} D = 1.55 \times 10^{-8}$ N · s/m. This gives a terminal speed

$$v_t = \frac{mg}{c_1} = \frac{0.52 \times 10^{-9} \times 9.8}{1.55 \times 10^{-8}} \text{ m/s} = 0.33 \text{ m/s}$$

The characteristic time is

$$\tau = \frac{v_t}{g} = \frac{0.33 \text{ m/s}}{9.8 \text{ m/s}^2} = 0.034 \text{ s}$$

For the basketball the drag constant is $c_2 = 0.22D^2 = 0.22 \times (0.25)^2 = 0.0138$ N · s^2/m^3, and so the terminal speed is

$$v_t = \left(\frac{mg}{c_2}\right)^{1/2} = \left(\frac{0.6 \times 9.8}{0.0138}\right)^{1/2} \text{ m/s} = 20.6 \text{ m/s}$$

and the characteristic time is

$$\tau = \frac{v_t}{g} = \frac{20.6 \text{ m/s}}{9.8 \text{ m/s}^2} = 2.1 \text{ s}$$

Thus, the raindrop practically attains its terminal speed in less than 1 s when starting from rest, whereas it takes several seconds for the basketball to come to within 1% of the terminal value.

For more information on aerodynamic drag, the reader is referred to an article by C. Frohlich in *Am. J. Phys.*, **52**, 325 (1984) and the extensive list of references cited therein.

*2.5 Vertical Fall Through a Fluid: Numerical Solution

Many problems in classical mechanics are described by fairly complicated equations of motion that cannot be solved analytically in closed form. When one encounters such a problem, the only available alternative is to try to solve the problem numerically. Once one decides that such a course of action is necessary, many alternatives open up. The widespread use of personal computers (PCs) with large amounts of memory and hard-disk storage capacity has made it possible to implement a wide variety of problem-solving tools in high-level languages without the tedium of programming. The tools in most widespread use among physicists include the software packages *Mathcad, Mathematica* (see Appendix I), and *Maple*, which are designed specifically to solve mathematical problems numerically (and symbolically).

As we proceed through the remaining chapters in this text, we use one or another of these tools, usually at the end of the chapter, to solve a problem for which no closed-form solution exists. Here we have used *Mathcad* to solve the problem of an object falling vertically through a fluid. The problem was solved analytically in the preceding section, and we use the solution we obtained there as a check on the numerical result we obtain here, in hopes of illustrating the power and ease of the numerical problem-solving technique.

Linear and quadratic cases revisited. The first-order differential equation of motion for an object falling vertically through a fluid in which the retarding force is linear was given by Equation 2.4.4:

$$mg - c_1 v = m\frac{dv}{dt} \tag{2.5.1a}$$

Here, though, we have chosen the downward y direction to be positive, because we consider only the situation in which the object is dropped from rest. The equation can be put into a much simpler form by expressing it in terms of the characteristic time $\tau = m/c_1$ and terminal velocity $v_t = mg/c_1$.

$$\frac{dv/v_t}{dt/\tau} = 1 - \frac{v}{v_t} \tag{2.5.1b}$$

Now, in the above equation, we "scale" the velocity v and the time of fall t in units of v_t and τ, respectively; that is, we let $u = v/v_t$ and $T = t/\tau$. The preceding equation becomes

$$\textit{Linear:}\ \frac{du}{dT} = u' = 1 - u \tag{2.5.1c}$$

where we denote the first derivative of u by u'.

*Sections in the text marked with * may be skipped with impunity.

An analysis similar to the one above leads to the following "scaled" first-order differential equation of motion for the case in which the retarding force is quadratic (see Equation 2.4.9).

$$\textit{Quadratic: } \frac{du}{dT} = u' = 1 - u^2 \tag{2.5.2}$$

The *Mathcad* software package comes with the *rkfixed* function, a general-purpose Runge–Kutta solver that can be used on nth-order differential equations or on systems of differential equations whose initial conditions are known. This is the situation that faces us in both of the preceding cases. All we need do, it turns out, to solve these two differential equations is to "supply" them to the *rkfixed* function in *Mathcad*. This function uses the fourth-order Runge–Kutta method[10] to solve the equations. When called in *Mathcad*, it returns a two-column matrix in which

- the left-hand (or 0th) column contains the data points at which the solution to the differential equation is evaluated (in the case here, the data points are the times T_i);
- the right-hand (or first) column contains the corresponding values of the solution (the values u_i).

The syntax of the call to the function and the arguments of the function is:
rkfixed($\mathbf{y}$, x_0, x_f, *npoints*, $\mathbf{D}$)

$\mathbf{y}$	=	a vector of n initial values, where n is the order of the differential equation or the size of the system of equations you're solving. For a single first-order differential equation, like the one in this case, the vector degenerates to a single initial value, $y(0) = y(x_0)$.
x_0, x_f	=	the endpoints of the interval within which the solutions to the differential equation are to be evaluated. The initial values of $\mathbf{y}$ are the values at x_0.
npoints	=	the number of points beyond the initial point at which the solution is to be evaluated. This value sets the number of rows to (1 + *npoints*) in the matrix *rkfixed*.
$\mathbf{D}(x, \mathbf{y})$	=	an n-element vector function containing the first derivatives of the unknown functions $\mathbf{y}$. Again, for a single first-order differential equation, this vector function degenerates to a single function equal to the first derivative of the single function y.

We show on the next two pages an example of a *Mathcad* worksheet in which we obtained a numerical solution for the above first-order differential equations (2.5.1c and 2.5.2). The worksheet was imported to this text directly from *Mathcad*. What is shown there should be self-explanatory, but exactly how to implement the solution might not be. We discuss the details of how to do it in Appendix I. The important thing here is to note the simplicity of the solution (as evidenced by the brevity of the worksheet) and its accuracy (as can be seen by comparing the numerical solutions shown in Figure 2.5.1 with the analytic solutions shown in Figure 2.4.1). The accuracy is further detailed in Figure 2.5.2, where we have

[10] See, for example, R. L. Burden and J. Douglas Faires, *Numerical Analysis*, 6th ed, Brooks/Cole, Pacific Grove, ITP, 1977.

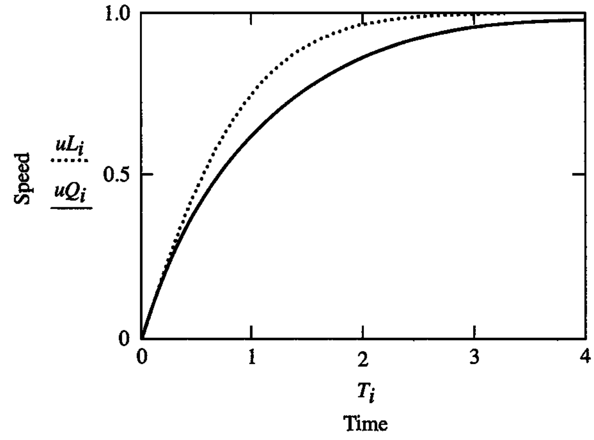

Figure 2.5.1 Numerical solution of speed versus time for a falling body. uL, linear case; uQ, quadratic case.

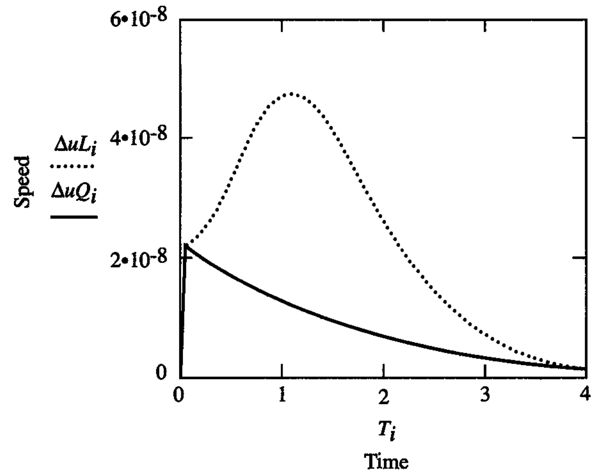

Figure 2.5.2 Difference between analytic and numerical solutions for the speed of a falling object. ΔuL, linear case; ΔuQ, quadratic case.

plotted the percent difference between the numerical and analytic solutions. The worst error, about 5×10^{-8}, occurs in the quadratic solution. Even greater accuracy could be achieved by dividing the time interval (0–4) into even more data points than the 100 chosen here.

Mathcad Solution for Speed of Falling Object: Linear Retarding Force.

$u_0 := 0$ ← Define initial value (use [to make the subscript)

$D(T, u) := 1 - u$ ← Define function for first derivative u'

$Y := rkfixed(u, 0, 4, 100, D)$ ← Evaluates solution at 100 points between 0 and 4 using fourth-order Runge–Kutta.

$i := 0 .. \text{rows}(Y) - 1$ ← i denotes each element pair in the matrix Y (a 101×2 matrix). First column contains data points (time T) where solution (velocity u) is evaluated. Second column contains u values.

$uL_i := (Y^{\langle 1 \rangle})_i$ ← Rename normalized velocity, linear case

Mathcad Solution for Speed of Falling Object: Quadratic Retarding Force.

$u_0 := 0$ ← Define initial value (use [to make the subscript)

$D(T, u) := 1 - u^2$ ← Define function for first derivative u'

$Z := rkfixed(u, 0, 4, 100, D)$ ← Evaluates solution at 100 points between 0 and 4 using fourth-order Runge–Kutta.

$T_i := 0.04i$ ← Define time in terms of array element

$uQ_i := (Z^{\langle 1 \rangle})_i$ ← Rename normalized velocity, quadratic case

Difference Between Analytic and Numerical Solutions.

$v_i := 1 - e^{-T_i}$ ← Analytic solution for linear retarding force

$u_i := \dfrac{(e^{2 \cdot T_i} - 1)}{(e^{2 \cdot T_i} + 1)}$ ← Analytic solution for quadratic retarding force

$\Delta uL_i := \dfrac{(v_i - uL_i)}{v_i}$ ← Difference, linear case

$\Delta uQ_i := \dfrac{(u_i - uQ_i)}{u_i}$ ← Difference, quadratic case

Problems

2.1 Find the velocity $\dot{x}$ and the position x as functions of the time t for a particle of mass m, which starts from rest at $x = 0$ and $t = 0$, subject to the following force functions:
(a) $F_x = F_0 + ct$
(b) $F_x = F_0 \sin ct$
(c) $F_x = F_0 e^{ct}$
where F_0 and c are positive constants.

2.2 Find the velocity $\dot{x}$ as a function of the displacement x for a particle of mass m, which starts from rest at $x = 0$, subject to the following force functions:
(a) $F_x = F_0 + cx$
(b) $F_x = F_0 e^{-cx}$
(c) $F_x = F_0 \cos cx$
where F_0 and c are positive constants.

2.3 Find the potential energy function $V(x)$ for each of the forces in Problem 2.2.

2.4 A particle of mass m is constrained to lie along a frictionless, horizontal plane subject to a force given by the expression $F(x) = -kx$. It is projected from $x = 0$ to the right along the positive x direction with initial kinetic energy $T_0 = 1/2\, kA^2$. k and A are positive constants. Find (**a**) the potential energy function $V(x)$ for this force; (**b**) the kinetic energy, and (**c**) the total energy of the particle as a function of its position. (**d**) Find the turning points of the motion. (**e**) Sketch the potential, kinetic, and total energy functions. (*Optional*: Use *Mathcad* or *Mathematica* to plot these functions. Set k and A each equal to 1.)

2.5 As in the problem above, the particle is projected to the right with initial kinetic energy T_0 but subject to a force $F(x) = -kx + kx^3/A^2$, where k and A are positive constants. Find (**a**) the potential energy function $V(x)$ for this force; (**b**) the kinetic energy, and (**c**) the total energy of the particle as a function of its position. (**d**) Find the turning points of the motion and the condition the total energy of the particle must satisfy if its motion is to exhibit turning points. (**e**) Sketch the potential, kinetic, and total energy functions. (*Optional:* Use *Mathcad* or *Mathematica* to plot these functions. Set k and A each equal to 1.)

2.6 A particle of mass m moves along a frictionless, horizontal plane with a speed given by $v(x) = \alpha/x$, where x is its distance from the origin and α is a positive constant. Find the force $F(x)$ to which the particle is subject.

2.7 A block of mass M has a string of mass m attached to it. A force **F** is applied to the string, and it pulls the block up a frictionless plane that is inclined at an angle θ to the horizontal. Find the force that the string exerts on the block.

2.8 Given that the velocity of a particle in rectilinear motion varies with the displacement x according to the equation

$$\dot{x} = bx^{-3}$$

where b is a positive constant, find the force acting on the particle as a function of x. (*Hint:* $F = m\ddot{x} = m\dot{x}\, d\dot{x}/dx$.)

2.9 A baseball (radius = .0366 m, mass = .145 kg) is dropped from rest at the top of the Empire State Building (height = 1250 ft). Calculate (**a**) the initial potential energy of the baseball, (**b**) its final kinetic energy, and (**c**) the total energy dissipated by the falling baseball by computing the line integral of the force of air resistance along the baseball's total distance of fall. Compare this last result to the difference between the baseball's initial potential energy and its final kinetic energy. *(Hint: In part (c) make approximations when evaluating the hyperbolic functions obtained in carrying out the line integral.)*

2.10 A block of wood is projected up an inclined plane with initial speed v_0. If the inclination of the plane is 30° and the coefficient of sliding friction $\mu_K = 0.1$, find the total time for the block to return to the point of projection.

2.11 A metal block of mass m slides on a horizontal surface that has been lubricated with a heavy oil so that the block suffers a viscous resistance that varies as the $\frac{3}{2}$ power of the speed:

$$F(v) = -cv^{3/2}$$

If the initial speed of the block is v_0 at $x = 0$, show that the block cannot travel farther than $2mv_0^{1/2}/c$.

2.12 A gun is fired straight up. Assuming that the air drag on the bullet varies quadratically with speed, show that the speed varies with height according to the equations

$$v^2 = Ae^{-2kx} - \frac{g}{k} \quad (\textit{upward motion})$$

$$v^2 = \frac{g}{k} - Be^{2kx} \quad (\textit{downward motion})$$

in which A and B are constants of integration, g is the acceleration of gravity, and $k = c_2/m$ where c_2 is the drag constant and m is the mass of the bullet. (*Note:* x is measured positive upward, and the gravitational force is assumed to be constant.)

2.13 Use the above result to show that, when the bullet hits the ground on its return, the speed is equal to the expression

$$\frac{v_0 v_t}{\left(v_0^2 + v_t^2\right)^{1/2}}$$

in which v_0 is the initial upward speed and

$$v_t = (mg/c_2)^{1/2} = \text{terminal speed} = (g/k)^{1/2}$$

(This result allows one to find the fraction of the initial kinetic energy lost through air friction.)

2.14 A particle of mass m is released from rest a distance b from a fixed origin of force that attracts the particle according to the inverse square law:

$$F(x) = -kx^{-2}$$

Show that the time required for the particle to reach the origin is

$$\pi\left(\frac{mb^3}{8k}\right)^{1/2}$$

2.15 Show that the terminal speed of a falling spherical object is given by

$$v_t = [(mg/c_2) + (c_1/2c_2)^2]^{1/2} - (c_1/2c_2)$$

when *both* the linear and the quadratic terms in the drag force are taken into account.

2.16 Use the above result to calculate the terminal speed of a soap bubble of mass 10^{-7} kg and diameter 10^{-2} m. Compare your value with the value obtained by using Equation 2.4.10.

2.17 Given: The force acting on a particle is the product of a function of the distance and a function of the velocity: $F(x, v) = f(x)g(v)$. Show that the differential equation of motion can be solved by integration. If the force is a product of a function of distance and a function of time, can the equation of motion be solved by simple integration? Can it be solved if the force is a product of a function of time and a function of velocity?

2.18 The force acting on a particle of mass m is given by

$$F = kvx$$

in which k is a positive constant. The particle passes through the origin with speed v_0 at time $t = 0$. Find x as a function of t.

2.19 A surface-going projectile is launched horizontally on the ocean from a stationary warship, with initial speed v_0. Assume that its propulsion system has failed and it is slowed by a retarding force given by $F(v) = -Ae^{\alpha v}$. **(a)** Find its speed as a function of time, $v(t)$. Find **(b)** the time elapsed and **(c)** the distance traveled when the projectile finally comes to rest. A and α are positive constants.

2.20 Assume that a water droplet falling though a humid atmosphere gathers up mass at a rate that is proportional to its cross-sectional area A. Assume that the droplet starts from rest and that its initial radius R_0 is so small that it suffers no resistive force. Show that **(a)** its radius and **(b)** its speed increase linearly with time.

Computer Problems

C 2.1 A parachutist of mass 70 kg jumps from a plane at an altitude of 32 km above the surface of the Earth. Unfortunately, the parachute fails to open. (In the following parts, neglect horizontal motion and assume that the initial velocity is zero.)

(a) Calculate the time of fall (accurate to 1 s) until ground impact, given no air resistance and a constant value of g.

(b) Calculate the time of fall (accurate to 1 s) until ground impact, given constant g and a force of air resistance given by

$$F(v) = -c_2 v|v|$$

where c_2 is 0.5 in SI units for a falling man and is constant.

(c) Calculate the time of fall (accurate to 1 s) until ground impact, given c_2 scales with atmospheric density as

$$c_2 = 0.5e^{-y/H}$$

where $H = 8$ km is the scale height of the atmosphere and y is the height above ground. Furthermore, assume that g is no longer constant but is given by

$$g = \frac{9.8}{\left(1+\frac{y}{R_e}\right)^2}\ \text{ms}^{-2}$$

where R_e is the radius of the Earth and is 6370 km.

(d) For case (c), plot the acceleration, velocity, and altitude of the parachutist as a function of time. Explain why the acceleration becomes positive as the parachutist falls.