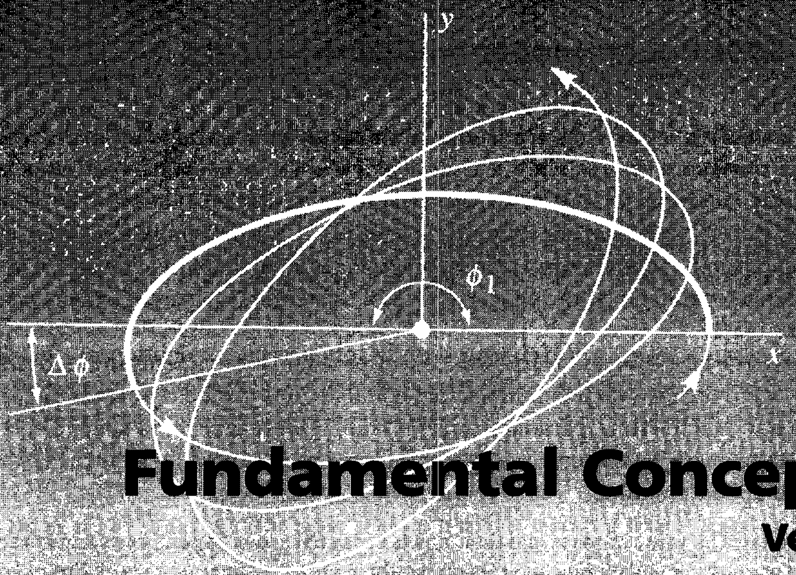

1 Fundamental Concepts: Vectors

"Let no one unversed in geometry enter these portals."
Plato's inscription over his academy in Athens

1.1 | Introduction

The science of classical mechanics deals with the motion of objects through absolute *space* and *time* in the Newtonian sense. Although central to the development of classical mechanics, the concepts of space and time would remain arguable for more than two and a half centuries following the publication of Sir Isaac Newton's *Philosophie naturalis principia mathematica* in 1687. As Newton put it in the first pages of the *Principia*, "Absolute, true and mathematical time, of itself, and from its own nature, flows equably, without relation to anything external, and by another name is called duration. Absolute space, in its own nature, without relation to anything external, remains always similar and immovable."

Ernst Mach (1838–1916), who was to have immeasurable influence on Albert Einstein, questioned the validity of these two Newtonian concepts in *The Science of Mechanics: A Critical and Historical Account of Its Development* (1907). There he claimed that Newton had acted contrary to his expressed intention of "framing no hypotheses," that is, accepting as fundamental premises of a scientific theory nothing that could not be inferred directly from "observable phenomena" or induced from them by argument. Indeed, although Newton was on the verge of overtly expressing this intent in Book III of the *Principia* as the fifth and last rule of his *Regulae Philosophandi* (rules of reasoning in philosophy), it is significant that he refrained from doing so.

Throughout his scientific career he exposed and rejected many hypotheses as false; he tolerated many as merely harmless; he put to use those that were verifiable. But he encountered a class of hypotheses that, neither "demonstrable from the phenomena nor following from them by argument based on induction," proved impossible to avoid. His concepts of space and time fell in this class. The acceptance of such hypotheses as fundamental was an embarrassing necessity; hence, he hesitated to adopt the frame-no-hypotheses rule. Newton certainly could be excused this sin of omission. After all, the adoption of these hypotheses and others of similar ilk (such as the "force" of gravitation) led to an elegant and comprehensive view of the world the likes of which had never been seen.

Not until the late 18th and early 19th centuries would experiments in electricity and magnetism yield observable phenomena that could be understood only from the vantage point of a new space–time paradigm arising from Albert Einstein's special relativity. Hermann Minkowski introduced this new paradigm in a semipopular lecture in Cologne, Germany in 1908 as follows:

> Gentlemen! The views of space and time which I wish to lay before you have sprung from the soil of experimental physics and therein lies their strength. They are radical. From now on, space by itself and time by itself are doomed to fade away into the shadows, and only a kind of union between the two will preserve an independent reality.

Thus, even though his own concepts of space and time were superceded, Newton most certainly would have taken great delight in seeing the emergence of a new space–time concept based upon observed "phenomena," which vindicated his unwritten frame-no-hypotheses rule.

1.2 Measure of Space and Time: Units[1] and Dimensions

We shall assume that space and time are described strictly in the Newtonian sense. Three-dimensional space is Euclidian, and positions of points in that space are specified by a set of three numbers (x,y,z) relative to the origin $(0,0,0)$ of a rectangular Cartesian coordinate system. A length is the spatial separation of two points relative to some standard length.

Time is measured relative to the duration of reoccurrences of a given configuration of a cyclical system—say, a pendulum swinging to and fro, an Earth rotating about its axis, or electromagnetic waves from a cesium atom vibrating inside a metallic cavity. The time of occurrence of any event is specified by a number t, which represents the number of reoccurrences of a given configuration of a chosen cyclical standard. For example, if 1 vibration of a standard physical pendulum is used to define 1 s, then to say that some event occurred at $t = 2.3$ s means that the standard pendulum executed 2.3 vibrations after its "start" at $t = 0$, when the event occurred.

All this sounds simple enough, but a substantial difficulty has been swept under the rug: Just what are the standard units? The choice of standards has usually been made more for political reasons than for scientific ones. For example, to say that a person is 6 feet tall is to say that the distance between the top of his head and the bottom of his foot is six times the length of something, which is taken to be the standard unit of 1 foot.

[1] A delightful account of the history of the standardization of units can be found in H. A. Klein, *The Science of Measurement—A Historical Survey*, Dover Publ., Mineola, 1988, ISBN 0-486-25839-4 (pbk).

In an earlier era that standard might have been the length of an actual human foot or something that approximated that length, as per the writing of Leonardo da Vinci on the views of the Roman architect–engineer Vitruvius Pollio (first century B.C.E.):

> . . . Vitruvius declares that Nature has thus arranged the measurements of a man: four fingers make 1 palm and 4 palms make 1 foot; six palms make 1 cubit; 4 cubits make once a man's height; 4 cubits make a pace, and 24 palms make a man's height . . .

Clearly, the adoption of such a standard does not make for an accurately reproducible measure. An early homemaker might be excused her fit of anger upon being "short-footed" when purchasing a bolt of cloth measured to a length normalized to the foot of the current short-statured king.

The Unit of Length

The French Revolution, which ended with the Napoleanic *coup d'etat* of 1799, gave birth to (among other things) an extremely significant plan for reform in measurement. The product of that reform, the metric system, expanded in 1960 into the Systēme International d'Unitės (SI).

In 1791, toward the end of the first French National Assembly, Charles Maurice de Talleyrand-Perigord (1754–1838) proposed that a task of weight and measure reform be undertaken by a "blue ribbon" panel with members selected from the French Academy of Sciences. This problem was not trivial. Metrologically, as well as politically, France was still absurdly divided, confused, and complicated. A given unit of length recognized in Paris was about 4% longer than that in Bordeaux, 2% longer than that in Marseilles, and 2% shorter than that in Lille. The Academy of Sciences panel was to change all that. Great Britain and the United States refused invitations to take part in the process of unit standardization. Thus was born the antipathy of English-speaking countries toward the metric system.

The panel chose 10 as the numerical base for all measure. The fundamental unit of length was taken to be one ten-millionth of a quadrant, or a quarter of a full meridian. A surveying operation, extending from Dunkirk on the English Channel to a site near Barcelona on the Mediterranean coast of Spain (a length equivalent to 10 degrees of latitude or one ninth of a quadrant), was carried out to determine this fundamental unit of length accurately. Ultimately, this monumental trek, which took from 1792 until 1799, changed the standard meter—estimated from previous, less ambitious surveys—by less than 0.3 mm, or about 3 parts in 10,000. We now know that this result, too, was in error by a similar factor. The length of a standard quadrant of meridian is 10,002,288.3 m, a little over 2 parts in 10,000 greater than the quadrant length established by the Dunkirk–Barcelona expedition.

Interestingly enough, in 1799, the year in which the Dunkirk–Barcelona survey was completed, the national legislature of France ratified new standards, among them the meter. The standard meter was now taken to be the distance between two fine scratches made on a bar of a dense alloy of platinum and iridium shaped in an X-like cross section to minimize sagging and distortion. The United States has two copies of this bar, numbers 21 and 27, stored at the Bureau of Standards in Gaithersburg, MD, just outside Washington, DC. Measurements based on this standard are accurate to about 1 part in 10^6. Thus, an object (a bar of platinum), rather than the concepts that led to it, was established as the standard meter. The Earth might alter its circumference if it so chose, but the standard meter would remain safe forever in a vault in Sevres, just outside Paris, France. This standard persisted until the 1960s.

The 11th General Conference of Weights and Measures, meeting in 1960, chose a reddish-orange radiation produced by atoms of krypton-86 as the next standard of length, with the meter defined in the following way:

> The meter is the length equal to 1,650,763.73 wavelengths in vacuum of the radiation corresponding to the transition between the levels 2 p^{10} and 5 d^5 of the krypton-86 atom.

Krypton is all around us; it makes up about 1 part per million of the Earth's present atmosphere. Atmospheric krypton has an atomic weight of 83.8, being a mixture of six different isotopes that range in weight from 78 to 86. Krypton-86 composes about 60% of these. Thus, the meter was defined in terms of the "majority kind" of krypton. Standard lamps contained no more than 1% of the other isotopes. Measurements based on this standard were accurate to about 1 part in 10^8.

Since 1983 the meter standard has been specified in terms of the velocity of light. A meter is the distance light travels in 1/299,792,458 s in a vacuum. In other words, the velocity of light is defined to be 299,792,458 m/s. Clearly, this makes the standard of length dependent on the standard of time.

The Unit of Time

Astronomical motions provide us with three great "natural" time units: the day, the month, and the year. The day is based on the Earth's spin, the month on the moon's orbital motion about the Earth, and the year on the Earth's orbital motion about the Sun. Why do we have ratios of 60:1 and 24:1 connecting the day, hour, minute, and second? These relationships were born about 6000 years ago on the flat alluvial plains of Mesopotamia (now Iraq), where civilization and city-states first appeared on Earth. The Mesopotamian number system was based on 60, not on 10 as ours is. It seems likely that the ancient Mesopotamians were more influenced by the 360 days in a year, the 30 days in a month, and the 12 months in a year than by the number of fingers on their hands. It was in such an environment that sky watching and measurement of stellar positions first became precise and continuous. The movements of heavenly bodies across the sky were converted to clocks.

The second, the basic unit of time in SI, began as an arbitrary fraction (1/86,400) of a mean solar day ($24 \times 60 \times 60 = 86{,}400$). The trouble with astronomical clocks, though, is that they do not remain constant. The mean solar day is lengthening, and the lunar month, or time between consecutive full phases, is shortening. In 1956 a new second was defined to be 1/31,556,926 of one particular and carefully measured mean solar year, that of 1900. That second would not last for long! In 1967 it was redefined again, in terms of a specified number of oscillations of a cesium atomic clock.

A cesium atomic clock consists of a beam of cesium-133 atoms moving through an evacuated metal cavity and absorbing and emitting microwaves of a characteristic resonant frequency, 9,192,631,770 Hertz (Hz), or about 10^{10} cycles per second. This absorption and emission process occurs when a given cesium atom changes its atomic configuration and, in the process, either gains or loses a specific amount of energy in the form of microwave radiation. The two differing energy configurations correspond to situations in which the spins of the cesium nucleus and that of its single outer-shell electron are either opposed (lowest energy state) or aligned (highest energy state). This kind of a "spin-flip" atomic transition is called a *hyperfine transition*. The energy difference and, hence, the resonant frequency are precisely determined by the invariable structure of the cesium atom. It does not differ from one atom to another. A properly adjusted and maintained cesium clock can keep time with a stability of about 1 part in 10^{12}. Thus, in one year, its deviation from the right time should be no more than about 30 μs (30×10^{-6} s). When two different cesium clocks are compared, it is found that they maintain agreement to about 1 part in 10^{10}.

It was inevitable then that in 1967, because of such stability and reproducibility, the 13th General Conference on Weights and Measures would substitute the cesium-133 atom for any and all of the heavenly bodies as the primary basis for the unit of time. The conference established the new basis with the following historic words:

> The second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between two hyperfine levels of the cesium-133 atom.

So, just as the meter is no longer bound to the surface of the Earth, the second is no longer derived from the "ticking" of the heavens.

The Unit of Mass

This chapter began with the statement that the science of mechanics deals with the motion of objects. *Mass* is the final concept needed to specify completely any physical quantity.[2] The *kilogram* is its basic unit. This primary standard, too, is stored in a vault in Sevres, France, with secondaries owned and kept by most major governments of the world. Note that the units of length and time are based on atomic standards. They are *universally* reproducible and virtually indestructible. Unfortunately, the unit of mass is not yet quite so robust.

A concept involving mass, which we shall have occasion to use throughout this text, is that of the *particle*, or point mass, an entity that possesses mass but no spatial extent. Clearly, the particle is a nonexistent idealization. Nonetheless, the concept serves as a useful approximation of physical objects in a certain context, namely, in a situation where the dimension of the object is small compared to the dimensions of its environment. Examples include a bug on a phonograph record, a baseball in flight, and the Earth in orbit around the Sun.

The units (*kilogram, meter,* and *second*) constitute the basis of the SI system.[3] Other systems are commonly used also, for example, the cgs (*centimeter, gram, second*) and the fps (*foot, pound, second*) systems. These systems may be regarded as secondary because they are defined relative to the SI standard. See Appendix A.

Dimensions

Normally, we think of dimensions as the three mutually orthogonal directions in space along which an object can move. For example, the motion of an airplane can be described in terms of its movement along the directions: east–west, north–south, and up–down. However, in physics, the term has an analogous but more fundamental meaning.

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[2]The concept of mass is treated in Chapter 2.

[3]Other basic and derived units are listed in Appendix A.

EXAMPLE 1.2.1

Converting Units

What is the length of a light year (LY) in meters?

Solution:

The speed of light is $c = 1$ LY/Y. The distance light travels in $T = 1$ Y is

$$D = cT = (1\ \text{LY}/\cancel{\text{Y}}) \times 1\ \cancel{\text{Y}} = 1\ \text{LY}.$$

If we want to express one light year in terms of meters, we start with the speed of light expressed in those units. It is given by $c = 3.00 \times 10^8$ m/s. However, the time unit used in this value is expressed in seconds, while the interval of time T is expressed in years, so

$$1\ \text{LY} = (3.00 \times 10^8\ \text{m/s}) \times (1\ \text{Y}) = 3.00 \times 10^8\ \text{m} \times (1\ \text{Y}/1\ \text{s})$$

The different times in the result must be expressed in the same unit to obtain a dimensionless ratio, leaving an answer in units of meters only. Converting 1 Y into its equivalent value in seconds achieves this.

$$\begin{aligned}1\ \text{LY} &= (3.00 \times 10^8\ \text{m}) \times (1\ \cancel{\text{Y}}/1\ \text{s}) \times (365\ \cancel{\text{day}}/\cancel{\text{Y}}) \times (24\ \cancel{\text{hr}}/\cancel{\text{day}}) \times (60\ \cancel{\text{min}}/\cancel{\text{hr}}) \times (60\ \text{s}/\cancel{\text{min}}) \\ &= (3.00 \times 10^8\ \text{m}) \times (3.15 \times 10^7\ \cancel{\text{s}}/1\ \cancel{\text{s}}) = 9.46 \times 10^{15}\ \text{m}\end{aligned}$$

We have multiplied 1 year by a succession of ratios whose values each are intrinsically dimensionless and equal to one. For example, 365 days = 1 year, so (365 days/1 year) = (1 year/1 year) = 1. The multiplications have not changed the intrinsic value of the result. They merely convert the value (1 year) into its equivalent value in seconds to "cancel out" the seconds unit, leaving a result expressed in meters.

No more than three fundamental quantities are needed to completely describe or characterize the behavior of any physical system that we encounter in the study of classical mechanics: the space that bodies occupy, the matter of which they consist, and the time during which those bodies move. In other words, classical mechanics deals with the motion of physical objects through space and time. All measurements of that motion ultimately can be broken down into combinations of measurements of mass, length, and time. The acceleration a of a falling apple is measured as a change in speed per change in time and the change in speed is measured as a change in position (length) per change in time. Thus, the measurement of acceleration is completely characterized by measurements of length and time. The concepts of mass, length, and time are far more fundamental than are the arbitrary units we choose to provide a scale for their measurement. Mass, length, and time specify the three primary *dimensions* of all physical quantities. We use the symbols $[M]$, $[L]$, and $[T]$ to characterize these three primary dimensions. The dimension of any physical quantity is defined to be the algebraic combination of $[M]$, $[L]$, and $[T]$ that is needed to fully characterize a measurement of the physical quantity.

In other words, the dimension of any physical quantity can be written as $[M]^{\alpha}[L]^{\beta}[T]^{\gamma}$, where α, β, and γ are powers of their respective dimension. For example, the dimension of acceleration a is

$$[a] = \left[\frac{L/T}{T}\right] = [L][T]^{-2}$$

Be aware! Do not confuse the dimension of a quantity with the units chosen to express it. Acceleration can be expressed in units of feet per second per second, kilometers per hour per hour, or, if you were Galileo investigating a ball rolling down an inclined plane, in units of *punti* per beat per beat! All of these units are consistent with the dimension $[L]\,[T]^{-2}$.

Dimensional Analysis

Dimensional analysis of equations that express relationships between different physical quantities is a powerful tool that can be used to immediately determine whether the result of a calculation has even the possibility of being correct or not. All equations must have consistent dimensions. The dimension of a physical quantity on the left hand side of an equation must have the same dimension as the combination of dimensions of all physical quantities on the right hand side. For example, later on in Example 6.5.3, we calculate the speed of satellite in a circular orbit of radius R_c about the Earth (radius R_e) and obtain the result

$$v_c = \left(\frac{gR_e^2}{R_c}\right)^{1/2}$$

in which g is the acceleration due to gravity, which we introduce in Section 2.2. If this result is correct, the dimensions on both sides of the equation must be identical. Let's see. First, we write down the combination of dimensions on the right side of the equation and reduce them as far as possible

$$\left(\frac{([L][T]^{-2})[L]^2}{[L]}\right)^{1/2} = ([L]^2[T]^{-2})^{1/2} = [L][T]^{-1}$$

The dimensions of the speed v_c are also $[L]\,[T]^{-1}$. The dimensions match; thus, the answer *could* be correct. It could also be incorrect. Dimensional analysis does not tell us unequivocally that it is correct. It can only tell us unequivocally that it is incorrect in those cases in which the dimensions fail to match.

Determining Relationships by Dimensional Analysis

Dimensional analysis can also be used as a way to obtain relationships between physical quantities without going through the labor of a more detailed analysis based on the laws of physics. As an example, consider the simple pendulum, which we analyze in Example 3.2.2. It consists of a small bob of mass m attached to the end of a massless, rigid string of length l. When displaced from its equilibrium configuration, in which it hangs vertically with the mass at its lowest possible position, it swings to and fro because gravity tries to restore the mass to its minimum height above the ground. In the absence of friction, air resistance and all other dissipative forces, it continues to swing to and fro forever! The time it takes to return to any configuration and direction of motion is called its period, or the time τ it takes to execute one complete cycle of its motion. The question before us is: How does its period τ depend on any physical parameters that characterize the pendulum and its environment?

First, we list those parameters that could be relevant. Because we've postulated that the pendulum consists, in part, of an idealized string of zero mass and no flexibility, that it suffers no air resistance and no friction, we eliminate from consideration any factors that are derivable from them. That leaves only three: the mass m of the pendulum bob, the length l of the string, and the acceleration g due to gravity. The period of the pendulum has dimension $[T]$ and the combination of m, l, and g that equates to the period must have dimensions that reduce to $[T]$, also. In other words, the period of the pendulum τ depends on an algebraic combination of m, l, and g of the form

$$\tau \propto m^{\alpha} l^{\beta} g^{\gamma}$$

whose dimensional relationship must be

$$[T] = [M]^{\alpha} [L]^{\beta} ([L]^{\gamma}[T]^{-2\gamma})$$

Because there are no powers of $[M]$ on the left-hand side, $\alpha = 0$ and the mass of the pendulum bob is irrelevant. To match the dimension $[T]$ on both sides of the equation, $\gamma = -\frac{1}{2}$, and to match the dimension $[L]$, $\beta + \gamma = 0$, or $\beta = \frac{1}{2}$. Thus, we conclude that

$$\tau \propto \sqrt{\frac{l}{g}}$$

Dimensional analysis can be taken no further than this. It does not give us the constant of proportionality, but it does tell us how τ likely depends on l and g and it does tell us that the period is independent of the mass m of the bob. Moreover, a single measurement of the period of a pendulum of known length l, would give us the constant of proportionality.

We did leave out one other possible factor, the angle of the pendulum's swing. Could its value affect the period? Maybe, but dimensional analysis alone does not tell us. The angle of swing is a dimensionless quantity, and the period could conceivably depend on it in a myriad of ways. Indeed, we see in Example 3.7.1, that the angle does affect the period if the angular amplitude of the swing is large enough. Yet, what we have learned simply by applying dimensional analysis is quite remarkable. A more detailed analysis based on the laws of physics should yield a result that is consistent with the one obtained from simple dimensional analysis, or we should try to understand why it does not. Whenever we find ourselves faced with such a dilemma, we discover that there is a strong likelihood that we've fouled up the detailed analysis.

Dimensional analysis applied this way is not always so simple. Experience is usually required to zero in on the relevant variables and to make a guess of the relevant functional dependencies. In particular, when trigonometric functions are involved, their lack of dimensionality thwarts dimensional analysis. Be that as it may, it remains a valuable weapon of attack that all students should have in their arsenal.

1.3 | Vectors

The motion of dynamical systems is typically described in terms of two basic quantities: scalars and vectors. A *scalar* is a physical quantity that has magnitude only, such as the mass of an object. It is completely specified by a single number, in appropriate units. Its value is independent of any coordinates chosen to describe the motion of the system. Other familiar examples of scalars include density, volume, temperature, and energy. Mathematically, scalars are treated as real numbers. They obey all the normal algebraic rules of addition, subtraction, multiplication, division, and so on.

A *vector*, however, has both magnitude and direction, such as the displacement from one point in space to another. Unlike a scalar, a vector requires a set of numbers for its complete specification. The values of those numbers are, in general, coordinate system dependent. Besides displacement in space, other examples of vectors include velocity, acceleration, and force. Mathematically, vectors combine with each other according to the parallelogram rule of addition which we soon discuss.[4] The vector concept has led to the emergence of a branch of mathematics that has proved indispensable to the development of the subject of classical mechanics. Vectors provide a compact and elegant way of describing the behavior of even the most complicated physical systems. Furthermore, the use of vectors in the application of physical laws insures that the results we obtain are independent of our choice of coordinate system.

In most written work, a distinguishing mark, such as an arrow, customarily designates a vector, for example, $\vec{A}$. In this text, however, for the sake of simplicity, we denote vector quantities simply by boldface type, for example, **A**. We use ordinary italic type to represent scalars, for example, A.

A given vector **A** is specified by stating its magnitude and its direction relative to some arbitrarily chosen coordinate system. It is represented diagrammatically as a directed line segment, as shown in three-dimensional space in Figure 1.3.1.

A vector can also be specified as the set of its *components*, or projections onto the coordinate axes. For example, the set of three scalars, (A_x, A_y, A_z), shown in Figure 1.3.1, are the components of the vector **A** and are an equivalent representation. Thus, the equation

$$\mathbf{A} = (A_x, A_y, A_z) \tag{1.3.1}$$

implies that either the symbol **A** or the set of three components (A_x, A_y, A_z) referred to a particular coordinate system can be used to specify the vector. For example, if the vector **A** represents a displacement from a point P_1 (x_1, y_1, z_1) to the point P_2 (x_2, y_2, z_2), then its

[4] An example of a directed quantity that does not obey the rule for addition is a finite rotation of an object about a given axis. The reader can readily verify that two successive rotations about different axes do not produce the same result as a single rotation determined by the parallelogram rule. For the present, we shall not be concerned with such nonvector-directed quantities.

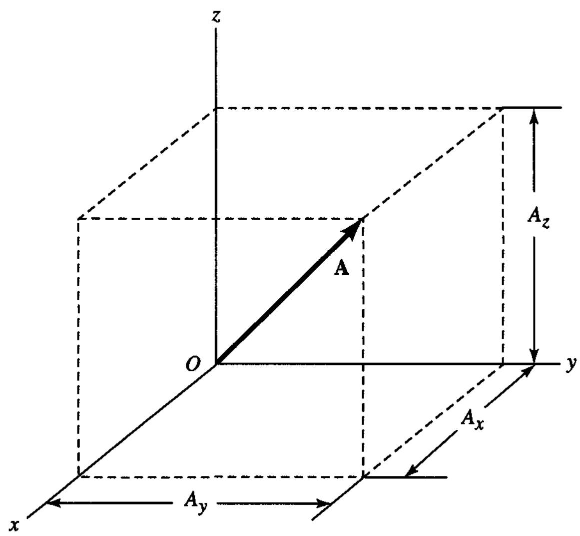

Figure 1.3.1 A vector **A** and its components in Cartesian coordinates.

three components are $A_x = x_2 - x_1$, $A_y = y_2 - y_1$, $A_z = z_2 - z_1$, and the equivalent representation of **A** is its set of three scalar components, $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. If **A** represents a force, then A_x is the x-component of the force, and so on.

If a particular discussion is limited to vectors in a plane, only two components are necessary for their specification. In general, one can define a mathematical space of any number of dimensions n. Thus, the set of n-numbers $(A_1, A_2, A_3, \ldots, A_n)$ represent a vector in an n-dimensional space. In this abstract sense, a vector is an ordered set of numbers.

We begin the study of vector algebra with some formal statements concerning vectors.

I. *Equality of Vectors*

The equation

$$\mathbf{A} = \mathbf{B} \tag{1.3.2}$$

or

$$(A_x, A_y, A_z) = (B_x, B_y, B_z)$$

is equivalent to the three equations

$$A_x = B_x \qquad A_y = B_y \qquad A_z = B_z$$

That is, two vectors are equal if, and only if, their respective components are equal. Geometrically, equal vectors are parallel and have the same length, but they do not necessarily have the same position. Equal vectors are shown in Figure 1.3.2. Though equal, they are physically separate. (Equal vectors are not necessarily equivalent in all respects. Thus, two vectorially equal forces acting at *different* points on an object may produce different mechanical effects.)

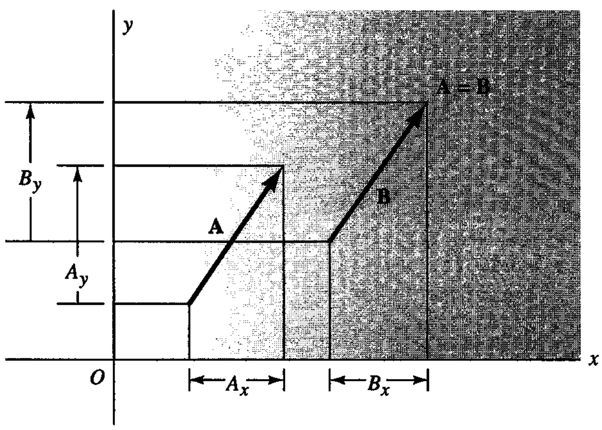

Figure 1.3.2 Illustration of equal vectors.

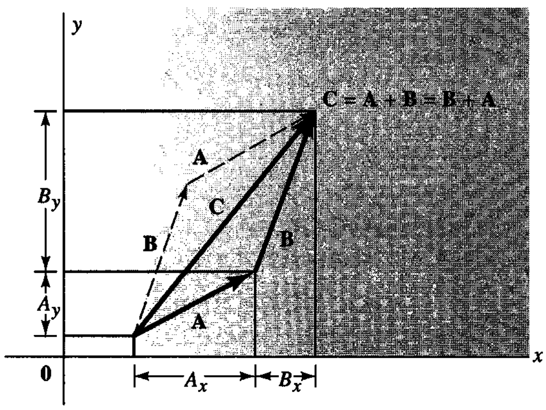

Figure 1.3.3 Addition of two vectors.

II. *Vector Addition*
The addition of two vectors is defined by the equation

$$\mathbf{A} + \mathbf{B} = (A_x, A_y, A_z) + (B_x, B_y, B_z) = (A_x + B_x, A_y + B_y, A_z + B_z) \tag{1.3.3}$$

The sum of two vectors is a vector whose components are sums of the components of the given vectors. The geometric representation of the vector sum of two non-parallel vectors is the third side of a triangle, two sides of which are the given vectors. The vector sum is illustrated in Figure 1.3.3. The sum is also given by the parallelogram rule, as shown in the figure. The vector sum is defined, however, according to the above equation even if the vectors do not have a common point.

III. *Multiplication by a Scalar*
If c is a scalar and $\mathbf{A}$ is a vector,

$$c\mathbf{A} = c(A_x, A_y, A_z) = (cA_x, cA_y, cA_z) = \mathbf{A}c \tag{1.3.4}$$

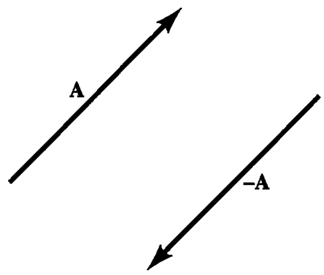

Figure 1.3.4 The negative of a vector.

The product $c\mathbf{A}$ is a vector whose components are c times those of $\mathbf{A}$. Geometrically, the vector $c\mathbf{A}$ is parallel to $\mathbf{A}$ and is c times the length of $\mathbf{A}$. When $c = -1$, the vector $-\mathbf{A}$ is one whose direction is the reverse of that of $\mathbf{A}$, as shown in Figure 13.4.

IV. *Vector Subtraction*
Subtraction is defined as follows:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} = (A_x - B_x, A_y - B_y, A_z - B_z) \tag{1.3.5}$$

That is, subtraction of a given vector $\mathbf{B}$ from the vector $\mathbf{A}$ is equivalent to adding $-\mathbf{B}$ to $\mathbf{A}$.

V. *The Null Vector*
The vector $\mathbf{O} = (0, 0, 0)$ is called the *null* vector. The direction of the null vector is undefined. From (IV) it follows that $\mathbf{A} - \mathbf{A} = \mathbf{O}$. Because there can be no confusion when the null vector is denoted by a zero, we shall hereafter use the notation $\mathbf{O} = 0$.

VI. *The Commutative Law of Addition*
This law holds for vectors; that is,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.3.6}$$

because $A_x + B_x = B_x + A_x$, and similarly for the y and z components.

VII. *The Associative Law*
The associative law is also true, because

$$\begin{aligned}\mathbf{A} + (\mathbf{B} + \mathbf{C}) &= (A_x + (B_x + C_x), A_y + (B_y + C_y), A_z + (B_z + C_z)) \\ &= ((A_x + B_x) + C_x, (A_y + B_y) + C_y, (A_z + B_z) + C_z) \\ &= (\mathbf{A} + \mathbf{B}) + \mathbf{C}\end{aligned} \tag{1.3.7}$$

VIII. *The Distributive Law*
Under multiplication by a scalar, the distributive law is valid because, from (II) and (III),

$$\begin{aligned}c(\mathbf{A} + \mathbf{B}) &= c(A_x + B_x, A_y + B_y, A_z + B_z) \\ &= (c(A_x + B_x), c(A_y + B_y), c(A_z + B_z)) \\ &= (cA_z + cB_x, cA_y + cB_y, cA_z + cB_z) \\ &= c\mathbf{A}_x + c\mathbf{B}\end{aligned} \tag{1.3.8}$$

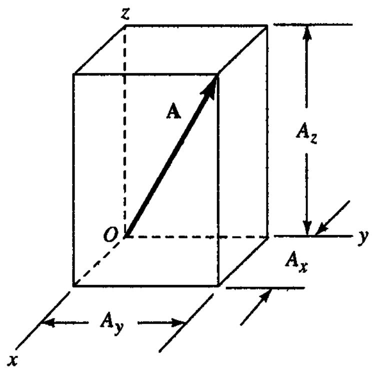

Figure 1.3.5 Magnitude of a vector **A**: $A = (A_x^2 + A_y^2 + A_z^2)^{1/2}$.

Thus, vectors obey the rules of ordinary algebra as far as the above operations are concerned.

IX. *Magnitude of a Vector*
The magnitude of a vector **A**, denoted by $|\mathbf{A}|$ or by A, is defined as the square root of the sum of the squares of the components, namely,

$$A = |\mathbf{A}| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2} \tag{1.3.9}$$

where the positive root is understood. Geometrically, the magnitude of a vector is its length, that is, the length of the diagonal of the rectangular parallelepiped whose sides are A_x, A_y, and A_z, expressed in appropriate units. See Figure 1.3.5.

X. *Unit Coordinate Vectors*
A *unit vector* is a vector whose magnitude is unity. Unit vectors are often designated by the symbol **e**, from the German word *Einheit*. The three unit vectors

$$\mathbf{e}_x = (1,0,0) \qquad \mathbf{e}_y = (0,1,0) \qquad \mathbf{e}_z = (0,0,1) \tag{1.3.10}$$

are called *unit coordinate vectors* or *basis vectors*. In terms of basis vectors, any vector can be expressed as a vector sum of components as follows:

$$\begin{aligned}\mathbf{A} &= (A_x, A_y, A_z) = (A_x,0,0) + (0,A_y,0) + (0,0,A_z)\\ &= A_x(1,0,0) + A_y(0,1,0) + A_z(0,0,1)\\ &= \mathbf{e}_x A_x + \mathbf{e}_y A_y + \mathbf{e}_z A_z\end{aligned} \tag{1.3.11}$$

A widely used notation for Cartesian unit vectors uses the letters **i**, **j**, and **k**, namely,

$$\mathbf{i} = \mathbf{e}_x \qquad \mathbf{j} = \mathbf{e}_y \qquad \mathbf{k} = \mathbf{e}_z \tag{1.3.12}$$

We shall usually employ this notation hereafter.

The directions of the Cartesian unit vectors are defined by the orthogonal coordinate axes, as shown in Figure 1.3.6. They form a right-handed or a left-handed triad,

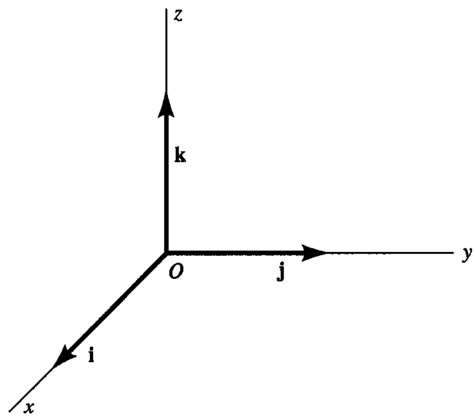

Figure 1.3.6 The unit vectors **ijk**.

depending on which type of coordinate system is used. It is customary to use right-handed coordinate systems. The system shown in Figure 1.3.6 is right-handed. (The handedness of coordinate systems is defined in Section 1.5.)

EXAMPLE 1.3.1

Find the sum and the magnitude of the sum of the two vectors $\mathbf{A} = (1,0,2)$ and $\mathbf{B} = (0,1,1)$.

Solution:

Adding components, we have $\mathbf{A} + \mathbf{B} = (1,0,2) + (0,1,1) = (1,1,3)$.

$$|\mathbf{A} + \mathbf{B}| = (1 + 1 + 9)^{1/2} = \sqrt{11}$$

EXAMPLE 1.3.2

For the above two vectors, express the difference in **ijk** form.

Solution:

Subtracting components, we have

$$\mathbf{A} - \mathbf{B} = (1,-1,1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

EXAMPLE 1.3.3

A helicopter flies 100 m vertically upward, then 500 m horizontally east, then 1000 m horizontally north. How far is it from a second helicopter that started from the same point and flew 200 m upward, 100 m west, and 500 m north?

Solution:

Choosing up, east, and north as basis directions, the final position of the first helicopter is expressed vectorially as $\mathbf{A} = (100,500,1000)$ and the second as $\mathbf{B} = (200,-100,500)$, in meters. Hence, the distance between the final positions is given by the expression

$$\begin{aligned}|\mathbf{A}-\mathbf{B}| &= |((100-200), (500+100), (1000-500))|\ \text{m}\\ &= (100^2+600^2+500^2)^{1/2}\ \text{m}\\ &= 787.4\ \text{m}\end{aligned}$$

1.4 The Scalar Product

Given two vectors **A** and **B**, the scalar product or "dot" product, $\mathbf{A}\cdot\mathbf{B}$, is the scalar defined by the equation

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z \qquad (1.4.1)$$

From the above definition, scalar multiplication is *commutative*,

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \qquad (1.4.2)$$

because $A_xB_x = B_xA_x$, and so on. It is also *distributive*,

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \qquad (1.4.3)$$

because if we apply the definition (1.4.1) in detail,

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= A_x(B_x+C_x) + A_y(B_y+C_y) + A_z(B_z+C_z)\\ &= A_xB_x + A_yB_y + A_zB_z + A_xC_x + A_yC_y + A_zC_z\\ &= \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}\end{aligned} \qquad (1.4.4)$$

The dot product $\mathbf{A}\cdot\mathbf{B}$ has a simple geometrical interpretation and can be used to calculate the angle θ between those two vectors. For example, shown in Figure 1.4.1 are the two vectors **A** and **B** separated by an angle θ, along with an x', y', z' coordinate system arbitrarily chosen as a basis for those vectors. However, because the quantity $\mathbf{A}\cdot\mathbf{B}$ is a scalar, its value is independent of choice of coordinates. With no loss of generality, we can rotate the x', y', z' system into an x, y, z coordinate system, such that the x-axis is aligned with the vector **A** and the z-axis is perpendicular to the plane defined by the two vectors. This coordinate system is also shown in Figure 1.4.1. The components of the vectors, and their dot product, are much simpler to evaluate in this system. The vector **A** is expressed as $(A,0,0)$ and the vector **B** as $(B_x, B_y, 0)$ or $(B\cos\theta, B\sin\theta, 0)$. Thus,

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x = A(B\cos\theta) = |\mathbf{A}|\,|\mathbf{B}|\cos\theta \qquad (1.4.5)$$

Geometrically, $B\cos\theta$ is simply the projection of **B** onto **A**. If we had aligned the x-axis along **B**, we would have obtained the same result but with the geometrical interpretation that $\mathbf{A}\cdot\mathbf{B}$ is now the projection of **A** onto **B** times the length of **B**. Thus, $\mathbf{A}\cdot\mathbf{B}$ can be interpreted as either the projection of **A** onto **B** times the length of **B** or that of **B** onto **A** times the length of **A**. Either interpretation is correct. Perhaps more importantly, we

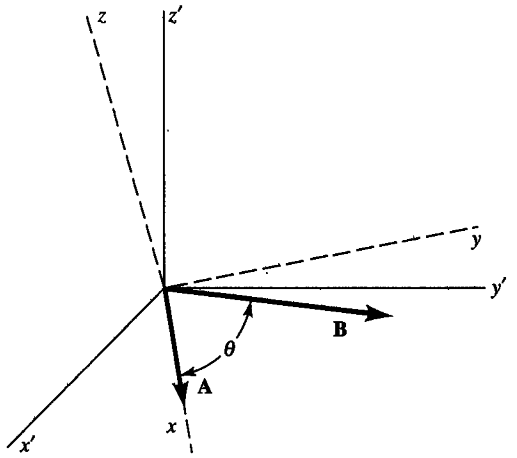

Figure 1.4.1 Evaluating a dot product between two vectors.

can see that we have just proved that the cosine of the angle between two line segments is given by

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} \tag{1.4.6}$$

This last equation may be regarded as an alternative definition of the dot product.

(**Note:** *If* $\mathbf{A}\cdot\mathbf{B}$ *is equal to zero and neither* **A** *nor* **B** *is null, then* $\cos\theta$ *is zero and* **A** *is perpendicular to* **B**.)

The square of the magnitude of a vector **A** is given by the dot product of **A** with itself,

$$A^2 = |\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A} \tag{1.4.7}$$

From the definitions of the unit coordinate vectors **i**, **j**, and **k**, it is clear that the following relations hold:

$$\begin{aligned} \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \\ \mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0 \end{aligned} \tag{1.4.8}$$

Expressing Any Vector as the Product of Its Magnitude by a Unit Vector: Projection

Consider the equation

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z \tag{1.4.9}$$

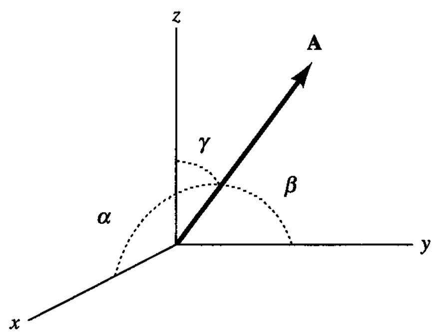

Figure 1.4.2 Direction angles α, β, γ of a vector.

Multiply and divide on the right by the magnitude of **A**:

$$\mathbf{A} = A\left(\mathbf{i}\frac{A_x}{A} + \mathbf{j}\frac{A_y}{A} + \mathbf{k}\frac{A_z}{A}\right) \tag{1.4.10}$$

Now $A_x/A = \cos\alpha$, $A_y/A = \cos\beta$, and $A_z/A = \cos\gamma$ are the *direction cosines* of the vector **A**, and α, β, and γ are the *direction angles*. Thus, we can write

$$\mathbf{A} = A(\mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma) = A(\cos\alpha, \cos\beta, \cos\gamma) \tag{1.4.11a}$$

or

$$\mathbf{A} = A\mathbf{n} \tag{1.4.11b}$$

where **n** is a unit vector whose components are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. See Figure 1.4.2. Consider any other vector **B**. Clearly, the projection of **B** on **A** is just

$$B\cos\theta = \frac{\mathbf{B}\cdot\mathbf{A}}{A} = \mathbf{B}\cdot\mathbf{n} \tag{1.4.12}$$

where θ is the angle between **A** and **B**.

EXAMPLE 1.4.1

Component of a Vector: Work

As an example of the dot product, suppose that an object under the action of a constant force[5] undergoes a linear displacement $\Delta\mathbf{s}$, as shown in Figure 1.4.3. By definition, the *work* ΔW done by the force is given by the product of the component of the force **F** in the direction of $\Delta\mathbf{s}$, multiplied by the magnitude Δs of the displacement; that is,

$$\Delta W = (F\cos\theta)\,\Delta s$$

[5] The concept of force is discussed in Chapter 2.

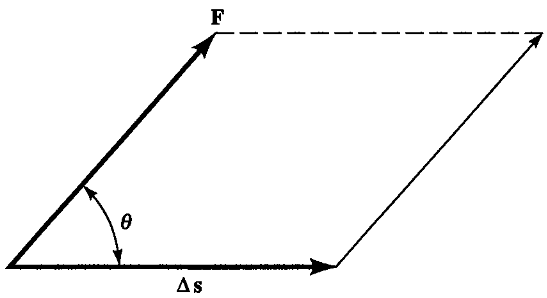

Figure 1.4.3 A force acting on a body undergoing a displacement.

where θ is the angle between **F** and **Δs**. But the expression on the right is just the dot product of **F** and **Δs**, that is,

$$\Delta W = \mathbf{F} \cdot \Delta \mathbf{s}$$

EXAMPLE 1.4.2

Law of Cosines

Consider the triangle whose sides are **A**, **B**, and **C**, as shown in Figure 1.4.4. Then $\mathbf{C} = \mathbf{A} + \mathbf{B}$. Take the dot product of **C** with itself,

$$\begin{aligned} \mathbf{C} \cdot \mathbf{C} &= (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) \\ &= \mathbf{A} \cdot \mathbf{A} + 2\mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{B} \end{aligned}$$

The second step follows from the application of the rules in Equations 1.4.2 and 1.4.3. Replace $\mathbf{A} \cdot \mathbf{B}$ with $AB \cos\theta$ to obtain

$$C^2 = A^2 + 2AB \cos\theta + B^2$$

which is the familiar law of cosines.

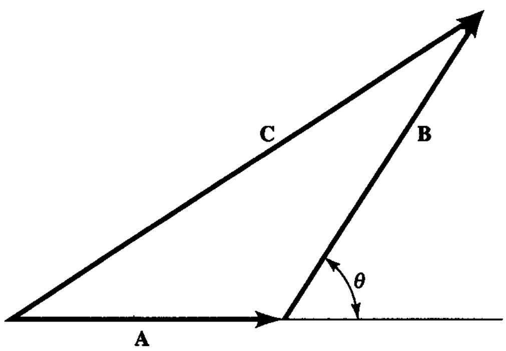

Figure 1.4.4 The law of cosines.

EXAMPLE 1.4.3

Find the cosine of the angle between a long diagonal and an adjacent face diagonal of a cube.

Solution:

We can represent the two diagonals in question by the vectors $\mathbf{A} = (1, 1, 1)$ and $\mathbf{B} = (1, 1, 0)$. Hence, from Equations 1.4.1 and 1.4.6,

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{1+1+0}{\sqrt{3}\sqrt{2}} = \sqrt{\frac{2}{3}} = 0.8165$$

EXAMPLE 1.4.4

The vector $a\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to the vector $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. What is the value of a?

Solution:

If the vectors are perpendicular to each other, their dot product must vanish ($\cos 90° = 0$).

$$(a\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = a + 2 + 3 = a + 5 = 0$$

Therefore,

$$a = -5$$

1.5 | The Vector Product

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, the vector product or cross product, $\mathbf{A}\times\mathbf{B}$, is defined as the vector whose components are given by the equation

$$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x) \tag{1.5.1}$$

It can be shown that the following rules hold for cross multiplication:

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \tag{1.5.2}$$

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C} \tag{1.5.3}$$

$$n(\mathbf{A}\times\mathbf{B}) = (n\mathbf{A})\times\mathbf{B} = \mathbf{A}\times(n\mathbf{B}) \tag{1.5.4}$$

The proofs of these follow directly from the definition and are left as an exercise.

(**Note:** *The first equation states that the cross product is anticommutative.*)

According to the definitions of the unit coordinate vectors (Section 1.3), it follows that

$$\begin{aligned}
&\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0\\
&\mathbf{j}\times\mathbf{k}=\mathbf{i}=-\mathbf{k}\times\mathbf{j}\\
&\mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i}\\
&\mathbf{k}\times\mathbf{i}=\mathbf{j}=-\mathbf{i}\times\mathbf{k}
\end{aligned} \tag{1.5.5}$$

These latter three relations define a right-handed triad. For example,

$$\mathbf{i}\times\mathbf{j}=(0-0,0-0,1-0)=(0,0,1)=\mathbf{k} \tag{1.5.6}$$

The remaining equations are proved in a similar manner.

The cross product expressed in **ijk** form is

$$\mathbf{A}\times\mathbf{B}=\mathbf{i}(A_yB_z-A_zB_y)+\mathbf{j}(A_zB_x-A_xB_z)+\mathbf{k}(A_xB_y-A_yB_x) \tag{1.5.7}$$

Each term in parentheses is equal to a determinant,

$$\mathbf{A}\times\mathbf{B}=\mathbf{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}+\mathbf{j}\begin{vmatrix}A_z & A_x\\ B_z & B_x\end{vmatrix}+\mathbf{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix} \tag{1.5.8}$$

and finally

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix} \tag{1.5.9}$$

which is verified by expansion. The determinant form is a convenient aid for remembering the definition of the cross product. From the properties of determinants, if **A** is parallel to **B**—that is, if $\mathbf{A}=c\mathbf{B}$—then the two lower rows of the determinant are proportional and so the determinant is null. Thus, the cross product of two parallel vectors is null.

Let us calculate the magnitude of the cross product. We have

$$|\mathbf{A}\times\mathbf{B}|^2=(A_yB_z-A_zB_y)^2+(A_zB_x-A_xB_z)^2+(A_xB_y-A_yB_x)^2 \tag{1.5.10}$$

This can be reduced to

$$|\mathbf{A}\times\mathbf{B}|^2=\left(A_x^2+A_y^2+A_z^2\right)\left(B_x^2+B_y^2+B_z^2\right)-(A_xB_x+A_yB_y+A_zB_z)^2 \tag{1.5.11}$$

or, from the definition of the dot product, the above equation may be written in the form

$$|\mathbf{A}\times\mathbf{B}|^2=A^2B^2-(\mathbf{A}\cdot\mathbf{B})^2 \tag{1.5.12}$$

Taking the square root of both sides of Equation 1.15.12 and using Equation 1.4.6, we can express the magnitude of the cross product as

$$|\mathbf{A}\times\mathbf{B}| = AB(1-\cos^2\theta)^{1/2} = AB\sin\theta \tag{1.5.13}$$

where θ is the angle between **A** and **B**.

To interpret the cross product geometrically, we observe that the vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ is perpendicular to both **A** and to **B** because

$$\begin{aligned}\mathbf{A}\cdot\mathbf{C} &= A_xC_x + A_yC_y + A_zC_z \\ &= A_x(A_yB_z - A_zB_y) + A_y(A_zB_x - A_xB_z) + A_z(A_xB_y - A_yB_x) \\ &= 0\end{aligned} \tag{1.5.14}$$

Similarly, $\mathbf{B}\cdot\mathbf{C} = 0$; thus, the vector **C** is perpendicular to the plane containing the vectors **A** and **B**.

The sense of the vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$ is determined from the requirement that the three vectors **A**, **B**, and **C** form a right-handed triad, as shown in Figure 1.5.1. (This is consistent with the previously established result that in the right-handed triad **ijk** we have $\mathbf{i}\times\mathbf{j} = \mathbf{k}$.) Therefore, from Equation 1.5.13 we see that we can write

$$\mathbf{A}\times\mathbf{B} = (AB\sin\theta)\mathbf{n} \tag{1.5.15}$$

where **n** is a unit vector normal to the plane of the two vectors **A** and **B**. The sense of **n** is given by the *right-hand rule*, that is, the direction of advancement of a right-handed screw rotated from the positive direction of **A** to that of **B** through the smallest angle between them, as illustrated in Figure 1.5.1. Equation 1.5.15 may be regarded as an alternative definition of the cross product in a right-handed coordinate system.

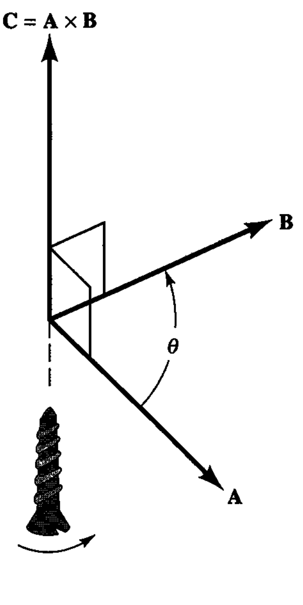

Figure 1.5.1 The cross product of two vectors.

EXAMPLE 1.5.1

Given the two vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{B} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find $\mathbf{A} \times \mathbf{B}$.

Solution:

In this case it is convenient to use the determinant form

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i}(2-1) + \mathbf{j}(-1-4) + \mathbf{k}(-2-1)$$
$$= \mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$$

EXAMPLE 1.5.2

Find a unit vector normal to the plane containing the two vectors **A** and **B** above.

Solution:

$$\mathbf{n} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}}{[1^2 + 5^2 + 3^2]^{1/2}}$$
$$= \frac{\mathbf{i}}{\sqrt{35}} - \frac{5\mathbf{j}}{\sqrt{35}} - \frac{3\mathbf{k}}{\sqrt{35}}$$

EXAMPLE 1.5.3

Show by direct evaluation that $\mathbf{A} \times \mathbf{B}$ is a vector with direction perpendicular to **A** and **B** and magnitude $AB \sin\theta$.

Solution:

Use the frame of reference discussed for Figure 1.4.1 in which the vectors **A** and **B** are defined to be in the x, y plane; **A** is given by $(A, 0, 0)$ and **B** is given by $(B \cos\theta, B \sin\theta, 0)$. Then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A & 0 & 0 \\ B\cos\theta & B\sin\theta & 0 \end{vmatrix} = \mathbf{k}AB\sin\theta$$

1.6 | An Example of the Cross Product: Moment of a Force

Moments of force, or *torques*, are represented by cross products. Let a force **F** act at a point $P(x, y, z)$, as shown in Figure 1.6.1, and let the vector **OP** be designated by **r**; that is,

$$\mathbf{OP} = \mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{1.6.1}$$

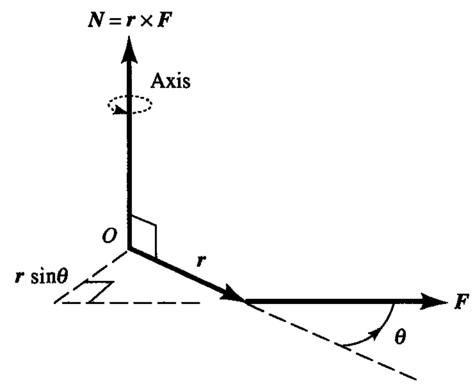

Figure 1.6.1 Illustration of the moment of a force about a point O.

The moment **N** of force, or the *torque* **N**, about a given point O is defined as the cross product

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} \tag{1.6.2}$$

Thus, the moment of a force about a point is a vector quantity having a magnitude and a direction. If a single force is applied at a point P on a body that is initially at rest and is free to turn about a fixed point O as a pivot, then the body tends to rotate. The axis of this rotation is perpendicular to the force **F**, and it is also perpendicular to the line OP; therefore, the direction of the torque vector **N** is along the axis of rotation.

The magnitude of the torque is given by

$$|\mathbf{N}| = |\mathbf{r} \times \mathbf{F}| = rF \sin\theta \tag{1.6.3}$$

in which θ is the angle between **r** and **F**. Thus, $|\mathbf{N}|$ can be regarded as the product of the magnitude of the force and the quantity $r \sin\theta$, which is just the perpendicular distance from the line of action of the force to the point O.

When several forces are applied to a single body at different points, the moments add vectorially. This follows from the distributive law of vector multiplication. The condition for rotational equilibrium is that the vector sum of all the moments is zero:

$$\sum_i (r_i \times \mathbf{F}_i) = \sum_i \mathbf{N}_i = 0 \tag{1.6.4}$$

A more complete discussion of force moments is given in Chapters 8 and 9.

1.7 | Triple Products

The expression

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

is called the *scalar triple product* of **A**, **B**, and **C**. It is a scalar because it is the dot product of two vectors. Referring to the determinant expressions for the cross product, Equations 1.5.8 and 1.5.9, we see that the scalar triple product may be written

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{1.7.1}$$

Because the exchange of the terms of two rows or of two columns of a determinant changes its sign but not its absolute value, we can derive the following useful equation:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \tag{1.7.2}$$

Thus, the dot and the cross may be interchanged in the scalar triple product.

The expression

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$

is called the *vector triple product*. It is left for the student to prove that the following equation holds for the vector triple product:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{1.7.3}$$

This last result can be remembered simply as the "back minus cab" rule.

Vector triple products are particularly useful in the study of rotating coordinate systems and rotations of rigid bodies, which we take up in later chapters. A geometric application is given in Problem 1.12 at the end of this chapter.

EXAMPLE 1.7.1

Given the three vectors $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{i} - \mathbf{j}$, and $\mathbf{C} = \mathbf{k}$, find $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$.

Solution:

Using the determinant expression, Equation 1.7.1, we have

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(-1+0) = -1$$

EXAMPLE 1.7.2

Find $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ above.

Solution:

From Equation 1.7.3 we have

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{i}-\mathbf{j})0 - \mathbf{k}(1-0) = -\mathbf{k}$$

EXAMPLE 1.7.3

Show that the vector triple product is nonassociative.

Solution:

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = -\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = -\mathbf{a}(\mathbf{c}\cdot\mathbf{b}) + \mathbf{b}(\mathbf{c}\cdot\mathbf{a})$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) - (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}(\mathbf{c}\cdot\mathbf{b}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

which is not necessarily zero.

1.8 Change of Coordinate System: The Transformation Matrix

In this section we show how to represent a vector in different coordinate systems. Consider the vector **A** expressed relative to the triad **ijk**:

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z \tag{1.8.1}$$

Relative to a new triad **i′j′k′** having a different orientation from that of **ijk**, the same vector **A** is expressed as

$$\mathbf{A} = \mathbf{i}'A_{x'} + \mathbf{j}'A_{y'} + \mathbf{k}'A_{z'} \tag{1.8.2}$$

Now the dot product $\mathbf{A}\cdot\mathbf{i}'$ is just $A_{x'}$, that is, the projection of **A** on the unit vector **i′**. Thus, we may write

$$\begin{aligned}
A_{x'} &= \mathbf{A}\cdot\mathbf{i}' = (\mathbf{i}\cdot\mathbf{i}')A_x + (\mathbf{j}\cdot\mathbf{i}')A_y + (\mathbf{k}\cdot\mathbf{i}')A_z \\
A_{y'} &= \mathbf{A}\cdot\mathbf{j}' = (\mathbf{i}\cdot\mathbf{j}')A_x + (\mathbf{j}\cdot\mathbf{j}')A_y + (\mathbf{k}\cdot\mathbf{j}')A_z \\
A_{z'} &= \mathbf{A}\cdot\mathbf{k}' = (\mathbf{i}\cdot\mathbf{k}')A_x + (\mathbf{j}\cdot\mathbf{k}')A_y + (\mathbf{k}\cdot\mathbf{k}')A_z
\end{aligned} \tag{1.8.3}$$

The scalar products $(\mathbf{i}\cdot\mathbf{i}')$, $(\mathbf{i}\cdot\mathbf{j}')$, and so on are called the *coefficients of transformation*. They are equal to the direction cosines of the axes of the primed coordinate system relative to the unprimed system. The unprimed components are similarly expressed as

$$\begin{aligned}
A_x &= \mathbf{A}\cdot\mathbf{i} = (\mathbf{i}'\cdot\mathbf{i})A_{x'} + (\mathbf{j}'\cdot\mathbf{i})A_{y'} + (\mathbf{k}'\cdot\mathbf{i})A_{z'} \\
A_y &= \mathbf{A}\cdot\mathbf{j} = (\mathbf{i}'\cdot\mathbf{j})A_{x'} + (\mathbf{j}'\cdot\mathbf{j})A_{y'} + (\mathbf{k}'\cdot\mathbf{j})A_{z'} \\
A_z &= \mathbf{A}\cdot\mathbf{k} = (\mathbf{i}'\cdot\mathbf{k})A_{x'} + (\mathbf{j}'\cdot\mathbf{k})A_{y'} + (\mathbf{k}'\cdot\mathbf{k})A_{z'}
\end{aligned} \tag{1.8.4}$$

All the coefficients of transformation in Equation 1.8.4 also appear in Equation 1.8.3, because $\mathbf{i}\cdot\mathbf{i}' = \mathbf{i}'\cdot\mathbf{i}$ and so on, but those in the rows (equations) of Equation 1.8.4 appear in the columns of terms in Equation 1.8.3, and conversely. The transformation rules expressed in these two sets of equations are a general property of vectors. As a matter of fact, they constitute an alternative way of defining vectors.[6]

[6] See, for example, J. B. Marion and S. T. Thornton, *Classical Dynamics*, 5th ed., Brooks/Cole—Thomson Learning, Belmont, CA, 2004.

The equations of transformation are conveniently expressed in matrix notation.[7] Thus, Equation 1.8.3 is written

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \mathbf{i}\cdot\mathbf{i}' & \mathbf{j}\cdot\mathbf{i}' & \mathbf{k}\cdot\mathbf{i}' \\ \mathbf{i}\cdot\mathbf{j}' & \mathbf{j}\cdot\mathbf{j}' & \mathbf{k}\cdot\mathbf{j}' \\ \mathbf{i}\cdot\mathbf{k}' & \mathbf{j}\cdot\mathbf{k}' & \mathbf{k}\cdot\mathbf{k}' \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{1.8.5}$$

The 3-by-3 matrix in Equation 1.8.5 is called the *transformation matrix*. One advantage of the matrix notation is that successive transformations are readily handled by means of matrix multiplication.

The application of a given transformation matrix to some vector **A** is also formally equivalent to rotating that vector within the unprimed (fixed) coordinate system, the components of the rotated vector being given by Equation 1.8.5. Thus, finite rotations can be represented by matrices. (Note that the sense of rotation of the vector in this context is opposite that of the rotation of the coordinate system in the previous context.)

From Example 1.8.2 the transformation matrix for a rotation about a different coordinate axis—say, the y-axis through an angle θ—is given by the matrix

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Consequently, the matrix for the combination of two rotations, the first being about the z-axis (angle ϕ) and the second being about the new y'-axis (angle θ), is given by the matrix product

$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \tag{1.8.6}$$

Now matrix multiplication is, in general, noncommutative; therefore, we might expect that the result would be different if the order of the rotations, and, therefore, the order of the matrix multiplication, were reversed. This turns out to be the case, which the reader can verify. This is in keeping with a remark made earlier, namely, that finite rotations do not obey the law of vector addition and, hence, are not vectors even though a single rotation has a direction (the axis) and a magnitude (the angle of rotation). However, we show later that infinitesimal rotations do obey the law of vector addition and can be represented by vectors.

[7] A brief review of matrices is given in Appendix H.

EXAMPLE 1.8.1

Express the vector $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ in terms of the triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$, where the $x'y'$–axes are rotated 45° around the z-axis, with the z- and z'-axes coinciding, as shown in Figure 1.8.1. Referring to the figure, we have for the coefficients of transformation $\mathbf{i} \cdot \mathbf{i}' = \cos 45°$ and so on; hence,

$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i}' = 1/\sqrt{2} & \mathbf{j}\cdot\mathbf{i}' = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{i}' = 0 \\ \mathbf{i}\cdot\mathbf{j}' = -1/\sqrt{2} & \mathbf{j}\cdot\mathbf{j}' = 1/\sqrt{2} & \mathbf{k}\cdot\mathbf{j}' = 0 \\ \mathbf{i}\cdot\mathbf{k}' = 0 & \mathbf{j}\cdot\mathbf{k}' = 0 & \mathbf{k}\cdot\mathbf{k}' = 1 \end{array}$$

These give

$$A_{x'} = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}} \qquad A_{y'} = \frac{-3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad A_{z'} = 1$$

so that, in the primed system, the vector $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{5}{\sqrt{2}}\mathbf{i}' - \frac{1}{\sqrt{2}}\mathbf{j}' + \mathbf{k}'$$

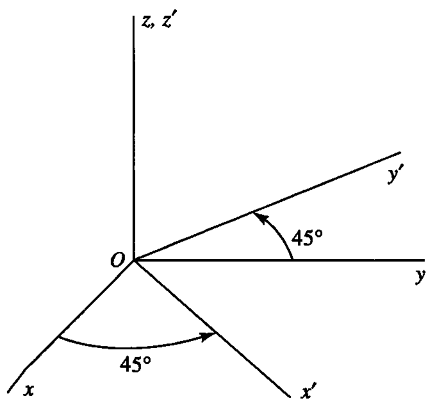

Figure 1.8.1 Rotated axes.

EXAMPLE 1.8.2

Find the transformation matrix for a rotation of the primed coordinate system through an angle ϕ about the z-axis. (Example 1.8.1 is a special case of this.) We have

$$\mathbf{i}\cdot\mathbf{i}' = \mathbf{j}\cdot\mathbf{j}' = \cos\phi$$
$$\mathbf{j}\cdot\mathbf{i}' = -\mathbf{i}\cdot\mathbf{j}' = \sin\phi$$
$$\mathbf{k}\cdot\mathbf{k}' = 1$$

and all other dot products are zero; hence, the transformation matrix is

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

EXAMPLE 1.8.3

Orthogonal Transformations

In more advanced texts, vectors are defined as quantities whose components change according to the rules of *orthogonal* transformations. The development of this subject lies outside the scope of this text; however, we give a simple example of such a transformation that the student may gain some appreciation for the elegance of this more abstract definition of vectors. The rotation of a Cartesian coordinate system is an example of an orthogonal transformation. Here we show how the components of a vector transform when the Cartesian coordinate system in which its components are expressed is rotated through some angle θ and then back again.

Let us take the velocity **v** of a projectile of mass m traveling through space along a parabolic trajectory as an example of the vector.[8] In Figure 1.8.2, we show the position and velocity of the projectile at some instant of time t. The direction of **v** is tangent to the trajectory of the projectile and designates its instantaneous direction of travel. Because the motion takes place in two dimensions only, we can specify the velocity in terms of its components along the x- and y-axes of a two-dimensional Cartesian coordinate system. We can also specify the velocity of the projectile in terms of components referred to an $x'y'$ coordinate system obtained by rotating the xy system through the angle θ. We choose an angle of rotation θ that aligns the x'-axis with the direction of the velocity vector.

We express the coordinate rotation in terms of the transformation matrix, defined in Equation 1.8.5. We write all vectors as column matrices; thus, the vector $\mathbf{v} = (v_x, v_y)$ is

$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$$

Given the components in one coordinate system, we can calculate them in the other using the transformation matrix of Equation 1.8.5. We represent this matrix by the symbol **R**.[9]

$$\mathbf{R} = \begin{pmatrix} i\cdot i' & j\cdot i' \\ i\cdot j' & j\cdot j' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

[8] Galileo demonstrated back in 1609 that the trajectory of such a projectile is a parabola. See for example: (1) Stillman Drake, *Galileo at Work—His Scientific Biography*, Dover Publications, New York 1978. (2) *Galileo Manuscripts*, Folio 116v, vol. 72, Biblioteca Nationale Centrale, Florence, Italy.

[9] We also denote matrices in this text with boldface type symbols. Whether the symbol represents a vector or a matrix should be clear from the context.

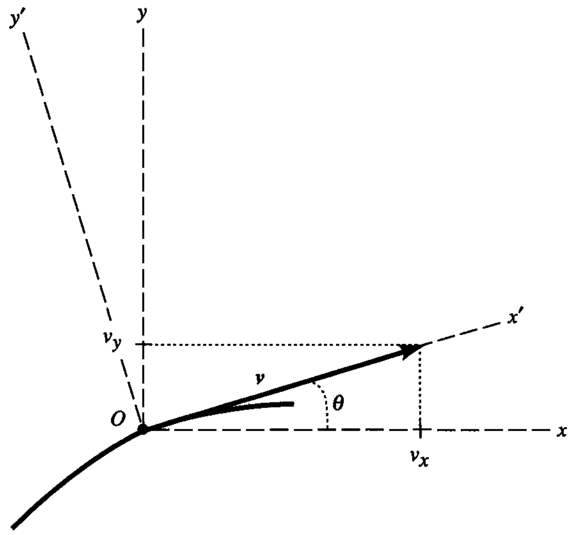

Figure 1.8.2 Velocity of a moving particle referred to two different two-dimensional coordinate systems.

The components of $\mathbf{v}'$ in the $x'y'$ coordinate system are

$$\mathbf{v}' = \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$$

or symbolically, $\mathbf{v}' = \mathbf{R}\mathbf{v}$. Here we have denoted the vector in the primed coordinate system by $\mathbf{v}'$. Bear in mind, though, that $\mathbf{v}$ and $\mathbf{v}'$ represent the same vector. The velocity vector points along the direction of the x'-axis in the rotated $x'y'$ coordinate system and, consistent with the figure, $v_{x'} = v$ and $v_{y'} = 0$. The components of a vector change values when we express the vector in coordinate systems rotated with respect to each other.

The square of the magnitude of $\mathbf{v}$ is

$$(\mathbf{v}\cdot\mathbf{v}) = \tilde{\mathbf{v}}\mathbf{v} = (v\cos\theta \quad v\sin\theta)\begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix} = v^2\cos^2\theta + v^2\sin^2\theta = v^2$$

($\tilde{\mathbf{v}}$ is the transpose of the column vector $\mathbf{v}$—the transpose $\tilde{\mathbf{A}}$ of any matrix $\mathbf{A}$ is obtained by interchanging its columns with its rows.)

Similarly, the square of the magnitude of $\mathbf{v}'$ is

$$(\mathbf{v}'\cdot\mathbf{v}') = \tilde{\mathbf{v}}'\mathbf{v}' = (v \quad 0)\begin{pmatrix} v \\ 0 \end{pmatrix} = v^2 + 0^2 = v^2$$

In each case, the magnitude of the vector is a scalar v whose value is independent of our choice of coordinate system. The same is true of the mass of the projectile. If its mass is one kilogram in the xy coordinate system, then its mass is one kilogram in the $x'y'$ coordinate system. Scalar quantities are *invariant* under a rotation of coordinates.

Suppose we transform back to the xy coordinate system. We should obtain the original components of $\mathbf{v}$. The transformation back is obtained by rotating the $x'y'$ coordinate system through the angle $-\theta$. The transformation matrix that accomplishes this can be obtained by changing the sign of θ in the matrix $\mathbf{R}$.

$$\mathbf{R}(-\theta) = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \equiv \tilde{\mathbf{R}}$$

We see that the rotation back is generated by the *transpose* of the matrix $\mathbf{R}$, or $\tilde{\mathbf{R}}$.

If we now operate on $\mathbf{v}'$ with $\tilde{\mathbf{R}}$, we obtain $\tilde{\mathbf{R}}\mathbf{v}' = \tilde{\mathbf{R}}\mathbf{R}\mathbf{v} = \mathbf{v}$ or in matrix notation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$$

In other words, $\tilde{\mathbf{R}}\mathbf{R} = \mathbf{I}$, the *identity* operator, or $\tilde{\mathbf{R}} = \mathbf{R}^{-1}$, the inverse of $\mathbf{R}$. Transformations that exhibit this characteristic are called orthogonal transformations. Rotations of coordinate systems are examples of such a transformation.

1.9 Derivative of a Vector

Up to this point we have been concerned mainly with vector algebra. We now begin the study of the calculus of vectors and its use in the description of the motion of particles.

Consider a vector $\mathbf{A}$, whose components are functions of a single variable u. The vector may represent position, velocity, and so on. The parameter u is usually the time t, but it can be any quantity that determines the components of $\mathbf{A}$:

$$\mathbf{A}(u) = \mathbf{i}A_x(u) + \mathbf{j}A_y(u) + \mathbf{k}A_z(u) \tag{1.9.1}$$

The derivative of $\mathbf{A}$ with respect to u is defined, quite analogously to the ordinary derivative of a scalar function, by the limit

$$\frac{d\mathbf{A}}{du} = \lim_{\Delta u \to 0} \frac{\Delta\mathbf{A}}{\Delta u} = \lim_{\Delta u \to 0}\left(\mathbf{i}\frac{\Delta A_x}{\Delta u} + \mathbf{j}\frac{\Delta A_y}{\Delta u} + \mathbf{k}\frac{\Delta A_z}{\Delta u}\right)$$

where $\Delta A_x = A_x(u + \Delta u) - A_x(u)$ and so on. Hence,

$$\frac{d\mathbf{A}}{du} = \mathbf{i}\frac{dA_x}{du} + \mathbf{j}\frac{dA_y}{du} + \mathbf{k}\frac{dA_z}{du} \tag{1.9.2}$$

The derivative of a vector is a vector whose Cartesian components are ordinary derivatives.

It follows from Equation 1.9.2 that the derivative of the sum of two vectors is equal to the sum of the derivatives, namely,

$$\frac{d}{du}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du} \tag{1.9.3}$$

The rules for differentiating vector products obey similar rules of vector calculus. For example,

$$\frac{d(n\mathbf{A})}{du} = \frac{dn}{du}\mathbf{A} + n\frac{d\mathbf{A}}{du} \tag{1.9.4}$$

$$\frac{d(\mathbf{A}\cdot\mathbf{B})}{du} = \frac{d\mathbf{A}}{du}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{du} \tag{1.9.5}$$

$$\frac{d(\mathbf{A}\times\mathbf{B})}{du} = \frac{d\mathbf{A}}{du}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{du} \tag{1.9.6}$$

Notice that it is necessary to preserve the order of the terms in the derivative of the cross product. The proofs are left as an exercise for the student.

1.10 Position Vector of a Particle: Velocity and Acceleration in Rectangular Coordinates

In a given reference system, the position of a particle can be specified by a single vector, namely, the displacement of the particle relative to the origin of the coordinate system. This vector is called the *position vector* of the particle. In rectangular coordinates (Figure 1.10.1), the position vector is simply

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{1.10.1}$$

The components of the position vector of a moving particle are functions of the time, namely,

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \tag{1.10.2}$$

In Equation 1.9.2 we gave the formal definition of the derivative of any vector with respect to some parameter. In particular, if the vector is the position vector **r** of a moving

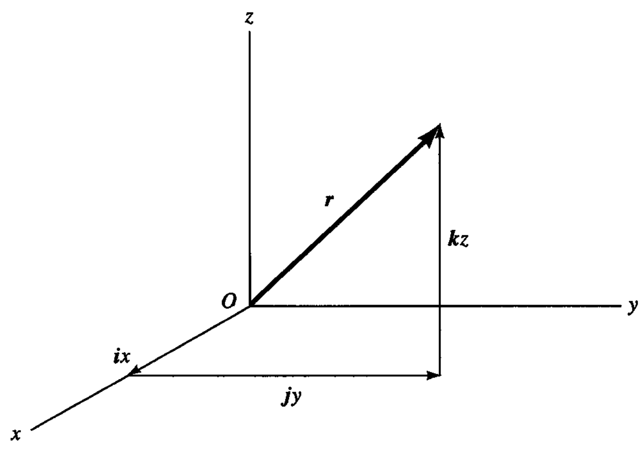

Figure 1.10.1 The position vector **r** and its components in a Cartesian coordinate system.

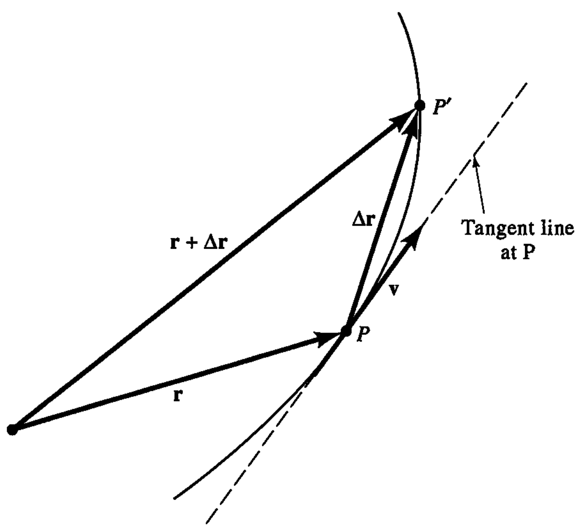

Figure 1.10.2 The velocity vector of a moving particle as the limit of the ratio $\Delta\mathbf{r}/\Delta t$.

particle and the parameter is the time t, the derivative of $\mathbf{r}$ with respect to t is called the *velocity*, which we shall denote by $\mathbf{v}$:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}\dot{x} + \mathbf{j}\dot{y} + \mathbf{k}\dot{z} \tag{1.10.3}$$

where the dots indicate differentiation with respect to t. (This convention is standard and is used throughout the book.) Let us examine the geometric significance of the velocity vector. Suppose a particle is at a certain position at time t. At a time Δt later, the particle will have moved from the position $\mathbf{r}(t)$ to the position $\mathbf{r}(t + \Delta t)$. The vector displacement during the time interval Δt is

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \tag{1.10.4}$$

so the quotient $\Delta\mathbf{r}/\Delta t$ is a *vector* that is parallel to the displacement. As we consider smaller and smaller time intervals, the quotient $\Delta\mathbf{r}/\Delta t$ approaches a limit $d\mathbf{r}/dt$, which we call the *velocity*. The vector $d\mathbf{r}/dt$ expresses both the direction of motion and the rate. This is shown graphically in Figure 1.10.2. In the time interval Δt, the particle moves along the path from P to P'. As Δt approaches zero, the point P' approaches P, and the direction of the vector $\Delta\mathbf{r}/\Delta t$ approaches the direction of the tangent to the path at P. The velocity vector, therefore, is always tangent to the path of motion.

The magnitude of the velocity is called the *speed*. In rectangular components the speed is just

$$v = |\mathbf{v}| = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \tag{1.10.5}$$

If we denote the cumulative scalar distance along the path with s, then we can express the speed alternatively as

$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{1/2}}{\Delta t} \tag{1.10.6}$$

which reduces to the expression on the right of Equation 1.10.5.

The time derivative of the velocity is called the *acceleration*. Denoting the acceleration with **a**, we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \tag{1.10.7}$$

In rectangular components,

$$\mathbf{a} = \mathbf{i}\ddot{x} + \mathbf{j}\ddot{y} + \mathbf{k}\ddot{z} \tag{1.10.8}$$

Thus, acceleration is a vector quantity whose components, in rectangular coordinates, are the second derivatives of the positional coordinates of a moving particle.

EXAMPLE 1.10.1

Projectile Motion

Let us examine the motion represented by the equation

$$\mathbf{r}(t) = \mathbf{i}bt + \mathbf{j}\left(ct - \frac{gt^2}{2}\right) + \mathbf{k}0$$

This represents motion in the xy plane, because the z component is constant and equal to zero. The velocity **v** is obtained by differentiating with respect to t, namely,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b + \mathbf{j}(c - gt)$$

The acceleration, likewise, is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{j}g$$

Thus, **a** is in the negative y direction and has the constant magnitude g. The path of motion is a parabola, as shown in Figure 1.10.3. The speed v varies with t according to the equation

$$v = [b^2 + (c - gt)^2]^{1/2}$$

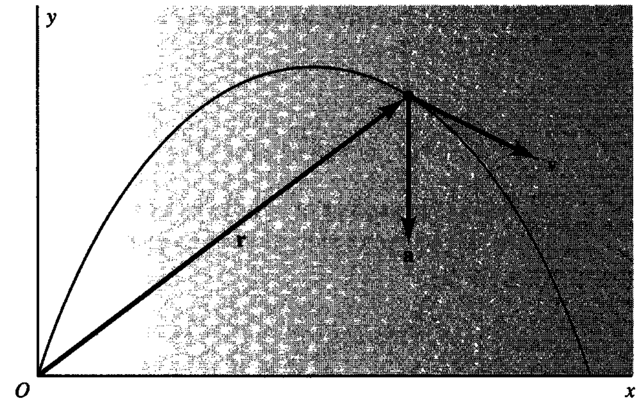

Figure 1.10.3 Position, velocity, and acceleration vectors of a particle (projectile) moving in a parabolic path.

EXAMPLE 1.10.2

Circular Motion

Suppose the position vector of a particle is given by

$$\mathbf{r} = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t$$

where ω is a constant.

Let us analyze the motion. The distance from the origin remains constant:

$$|\mathbf{r}| = r = (b^2 \sin^2 \omega t + b^2 \cos^2 \omega t)^{1/2} = b$$

So the path is a circle of radius b centered at the origin. Differentiating $\mathbf{r}$, we find the velocity vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i}b\omega \cos \omega t - \mathbf{j}b\omega \sin \omega t$$

The particle traverses its path with constant speed:

$$v = |\mathbf{v}| = (b^2\omega^2 \cos^2 \omega t + b^2\omega^2 \sin^2 \omega t)^{1/2} = b\omega$$

The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\mathbf{i}b\omega^2 \sin \omega t - \mathbf{j}b\omega^2 \cos \omega t$$

In this case the acceleration is perpendicular to the velocity, because the dot product of $\mathbf{v}$ and $\mathbf{a}$ vanishes:

$$\mathbf{v} \cdot \mathbf{a} = (b\omega \cos \omega t)(-b\omega^2 \sin \omega t) + (-b\omega \sin \omega t)(-b\omega^2 \cos \omega t) = 0$$

Comparing the two expressions for $\mathbf{a}$ and $\mathbf{r}$, we see that we can write

$$\mathbf{a} = -\omega^2 \mathbf{r}$$

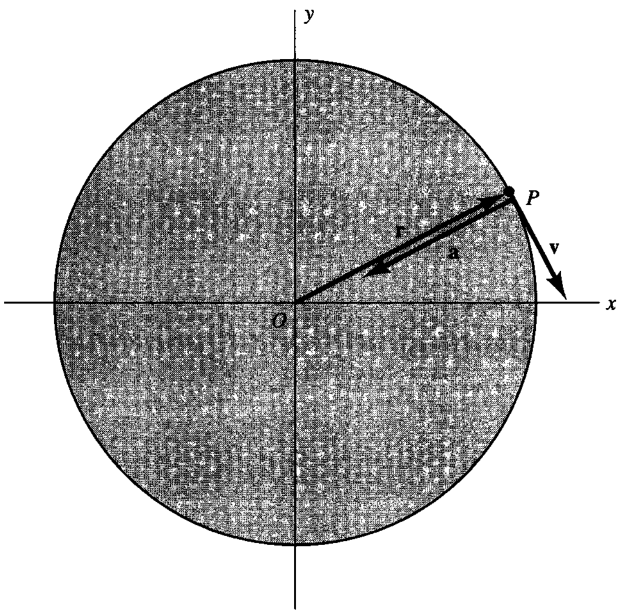

Figure 1.10.4 A particle moving in a circular path with constant speed.

so **a** and **r** are oppositely directed; that is, **a** always points toward the center of the circular path (Fig. 1.10.4).

EXAMPLE 1.10.3

Rolling Wheel

Let us consider the following position vector of a particle P:

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$$

in which

$$\mathbf{r}_1 = \mathbf{i}b\omega t + \mathbf{j}b$$
$$\mathbf{r}_2 = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t$$

Now $\mathbf{r}_1$ by itself represents a point moving along the line $y = b$ at constant velocity, provided ω is constant; namely,

$$\mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt} = \mathbf{i}b\omega$$

The second part, $\mathbf{r}_2$, is just the position vector for circular motion, as discussed in Example 1.10.2. Hence, the vector sum $\mathbf{r}_1 + \mathbf{r}_2$ represents a point that describes a circle of radius b about a moving center. This is precisely what occurs for a particle on the rim of a rolling wheel, $\mathbf{r}_1$ being the position vector of the center of the wheel and $\mathbf{r}_2$ being

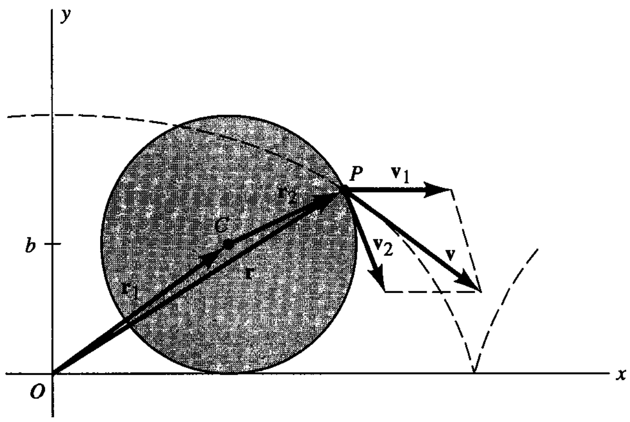

Figure 1.10.5 The cycloidal path of a particle on a rolling wheel.

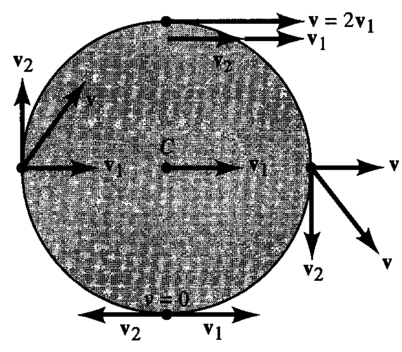

Figure 1.10.6 Velocity vectors for various points on a rolling wheel.

the position vector of the particle P *relative* to the moving center. The actual path is a *cycloid*, as shown in Figure 1.10.5. The velocity of P is

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{i}(b\omega + b\omega \cos \omega t) - \mathbf{j}b\omega \sin \omega t$$

In particular, for $\omega t = 0, 2\pi, 4\pi, \ldots$, we find that $\mathbf{v} = \mathbf{i}2b\omega$, which is just twice the velocity of the center C. At these points the particle is at the uppermost part of its path. Furthermore, for $\omega t = \pi, 3\pi, 5\pi, \ldots$, we obtain $\mathbf{v} = 0$. At these points the particle is at its lowest point and is instantaneously in contact with the ground. See Figure 1.10.6.

1.11 | Velocity and Acceleration in Plane Polar Coordinates

It is often convenient to employ polar coordinates r, θ to express the position of a particle moving in a plane. Vectorially, the position of the particle can be written as the product of the radial distance r by a unit radial vector $\mathbf{e}_r$:

$$\mathbf{r} = r\mathbf{e}_r \tag{1.11.1}$$

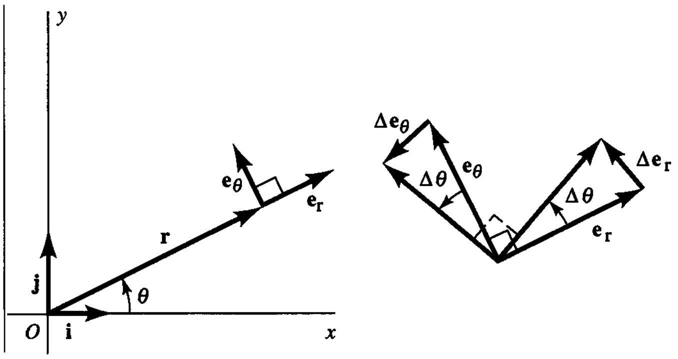

Figure 1.11.1 Unit vectors for plane polar coordinates.

As the particle moves, both r and $\mathbf{e}_r$ vary; thus, they are both functions of the time. Hence, if we differentiate with respect to t, we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} \tag{1.11.2}$$

To calculate the derivative $d\mathbf{e}_r/dt$, let us consider the vector diagram shown in Figure 1.11.1. A study of the figure shows that when the direction of $\mathbf{r}$ changes by an amount $\Delta\theta$, the corresponding change $\Delta\mathbf{e}_r$ of the unit radial vector is as follows: The magnitude $|\Delta\mathbf{e}_r|$ is approximately equal to $\Delta\theta$ and the direction of $\Delta\mathbf{e}_r$ is very nearly perpendicular to $\mathbf{e}_r$. Let us introduce another unit vector, $\mathbf{e}_\theta$, whose direction is perpendicular to $\mathbf{e}_r$. Then we have

$$\Delta\mathbf{e}_r \simeq \mathbf{e}_\theta\Delta\theta \tag{1.11.3}$$

If we divide by Δt and take the limit, we get

$$\frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\theta\frac{d\theta}{dt} \tag{1.11.4}$$

for the time derivative of the unit radial vector. In a precisely similar way, we can argue that the change in the unit vector $\mathbf{e}_\theta$ is given by the approximation

$$\Delta\mathbf{e}_\theta \simeq -\mathbf{e}_r\Delta\theta \tag{1.11.5}$$

Here the minus sign is inserted to indicate that the direction of the change $\Delta\mathbf{e}_\theta$ is opposite to the direction of $\mathbf{e}_r$, as can be seen from Figure 1.11.1. Consequently, the time derivative is given by

$$\frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r\frac{d\theta}{dt} \tag{1.11.6}$$

By using Equation 1.11.4 for the derivative of the unit radial vector, we can finally write the equation for the velocity as

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{1.11.7}$$

Thus, $\dot{r}$ is the radial component of the velocity vector, and $r\dot{\theta}$ is the transverse component.

To find the acceleration vector, we take the derivative of the velocity with respect to time. This gives

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{e}_r + \dot{r}\frac{d\mathbf{e}_r}{dt} + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt} \tag{1.11.8}$$

The values of $d\mathbf{e}_r/dt$ and $d\mathbf{e}_\theta/dt$ are given by Equations 1.11.4 and 1.11.6 and yield the following equation for the acceleration vector in plane polar coordinates:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \tag{1.11.9}$$

Thus, the radial component of the acceleration vector is

$$a_r = \ddot{r} - r\dot{\theta}^2 \tag{1.11.10}$$

and the transverse component is

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \tag{1.11.11}$$

The above results show, for instance, that if a particle moves on a circle of constant radius b, so that $\dot{r} = 0$, then the radial component of the acceleration is of magnitude $b\dot{\theta}^2$ and is directed inward toward the center of the circular path. The transverse component in this case is $b\ddot{\theta}$. On the other hand, if the particle moves along a fixed radial line—that is, if θ is constant—then the radial component is just $\ddot{r}$ and the transverse component is zero. If r and θ both vary, then the general expression (1.11.9) gives the acceleration.

EXAMPLE 1.11.1

A honeybee hones in on its hive in a spiral path in such a way that the radial distance decreases at a constant rate, $r = b - ct$, while the angular speed increases at a constant rate, $\dot{\theta} = kt$. Find the speed as a function of time.

Solution:

We have $\dot{r} = -c$ and $\ddot{r} = 0$. Thus, from Equation 1.11.7,

$$\mathbf{v} = -c\mathbf{e}_r + (b - ct)kt\mathbf{e}_\theta$$

so

$$v = [c^2 + (b - ct)^2 k^2 t^2]^{1/2}$$

which is valid for $t \le b/c$. Note that $v = c$ both for $t = 0$, $r = b$ and for $t = b/c$, $r = 0$.

EXAMPLE 1.11.2

On a horizontal turntable that is rotating at constant angular speed, a bug is crawling outward on a radial line such that its distance from the center increases quadratically with time: $r = bt^2$, $\theta = \omega t$, where b and ω are constants. Find the acceleration of the bug.

Solution:

We have $\dot{r} = 2bt$, $\ddot{r} = 2b$, $\dot{\theta} = \omega$, $\ddot{\theta} = 0$. Substituting into Equation 1.11.9, we find

$$\begin{aligned}\mathbf{a} &= \mathbf{e}_r(2b - bt^2\omega^2) + \mathbf{e}_\theta[0 + 2(2bt)\omega] \\ &= b(2 - t^2\omega^2)\mathbf{e}_r + 4b\omega t\mathbf{e}_\theta\end{aligned}$$

Note that the radial component of the acceleration becomes negative for large t in this example, although the radius is always increasing monotonically with time.

1.12 | Velocity and Acceleration in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

In the case of three-dimensional motion, the position of a particle can be described in cylindrical coordinates R, ϕ, z. The position vector is then written as

$$\mathbf{r} = R\mathbf{e}_R + z\mathbf{e}_z \tag{1.12.1}$$

where $\mathbf{e}_R$ is a unit radial vector in the xy plane and $\mathbf{e}_z$ is the unit vector in the z direction. A third unit vector $\mathbf{e}_\phi$ is needed so that the three vectors $\mathbf{e}_R\mathbf{e}_\phi\mathbf{e}_z$ constitute a right-handed triad, as illustrated in Figure 1.12.1. We note that $\mathbf{k} = \mathbf{e}_z$.

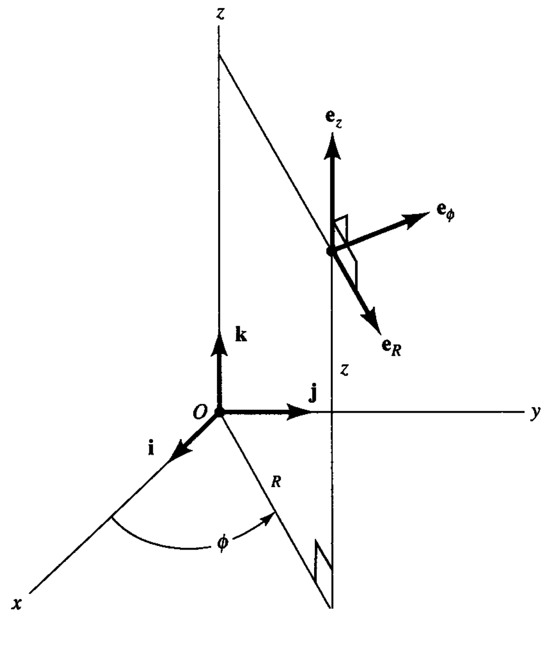

Figure 1.12.1 Unit vectors for cylindrical coordinates.

The velocity and acceleration vectors are found by differentiating, as before. This again involves derivatives of the unit vectors. An argument similar to that used for the plane case shows that $d\mathbf{e}_R/dt = \mathbf{e}_\phi \dot{\phi}$ and $d\mathbf{e}_\phi/dt = -\mathbf{e}_R \dot{\phi}$. The unit vector $\mathbf{e}_z$ does not change in direction, so its time derivative is zero.

In view of these facts, the velocity and acceleration vectors are easily seen to be given by the following equations:

$$\mathbf{v} = \dot{R}\,\mathbf{e}_R + R\dot{\phi}\,\mathbf{e}_\phi + \dot{z}\,\mathbf{e}_z \tag{1.12.2}$$

$$\mathbf{a} = (\ddot{R} - R\dot{\phi}^2)\mathbf{e}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z \tag{1.12.3}$$

These give the values of **v** and **a** in terms of their components in the *rotated* triad $\mathbf{e}_R\,\mathbf{e}_\phi\,\mathbf{e}_z$.

An alternative way of obtaining the derivatives of the unit vectors is to differentiate the following equations, which are the relationships between the fixed unit triad **ijk** and the rotated triad:

$$\begin{aligned} \mathbf{e}_R &= \mathbf{i}\cos\phi + \mathbf{j}\sin\phi \\ \mathbf{e}_\phi &= -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi \\ \mathbf{e}_z &= \mathbf{k} \end{aligned} \tag{1.12.4}$$

The steps are left as an exercise. The result can also be found by use of the rotation matrix, as given in Example 1.8.2.

Spherical Coordinates

When spherical coordinates r, θ, ϕ are employed to describe the position of a particle, the position vector is written as the product of the radial distance r and the unit radial vector $\mathbf{e}_r$, as with plane polar coordinates. Thus,

$$\mathbf{r} = r\mathbf{e}_r \tag{1.12.5}$$

The direction of $\mathbf{e}_r$ is now specified by the two angles ϕ and θ. We introduce two more unit vectors, $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$, as shown in Figure 1.12.2.

The velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} \tag{1.12.6}$$

Our next problem is how to express the derivative $d\mathbf{e}_r/dt$ in terms of the unit vectors in the rotated triad.

Referring to Figure 1.12.2, we can derive relationships between the **ijk** and $\mathbf{e}_r\mathbf{e}_\theta\mathbf{e}_\phi$ triads. For example, because any vector can be expressed in terms of its projections on to the x, y, z, coordinate axes

$$\mathbf{e}_r = \mathbf{i}(\mathbf{e}_r \cdot \mathbf{i}) + \mathbf{j}(\mathbf{e}_r \cdot \mathbf{j}) + \mathbf{k}(\mathbf{e}_r \cdot \mathbf{k}) \tag{1.12.7}$$

$\mathbf{e}_r \cdot \mathbf{i}$ is the projection of the unit vector $\mathbf{e}_r$ directly onto the unit vector **i**. According to Equation 1.4.11a, it is equal to $\cos\alpha$, the cosine of the angle between those two unit vectors. We need to express this dot product in terms of θ and ϕ, not α. We can obtain the

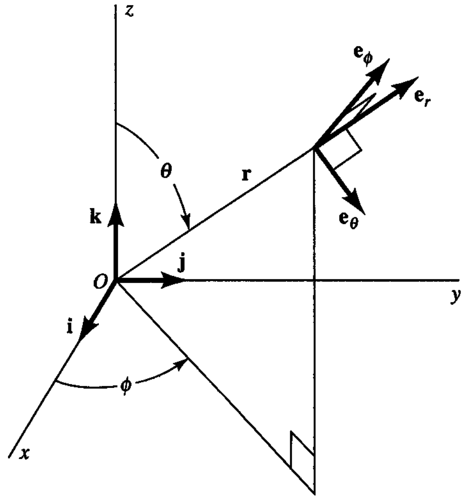

Figure 1.12.2 Unit vectors for spherical coordinates.

desired relation by making two successive projections to get to the x-axis. First project $\mathbf{e}_r$ onto the xy plane, and then project from there onto the x-axis. The first projection gives us a factor of sin θ, while the second yields a factor of cos ϕ. The magnitude of the projection obtained in this way is the desired dot product:

$$\mathbf{e}_r \cdot \mathbf{i} = \sin\theta\cos\phi \tag{1.12.8a}$$

The remaining dot products can be evaluated in a similar way,

$$\mathbf{e}_r \cdot \mathbf{j} = \sin\theta\sin\phi \qquad \text{and} \qquad \mathbf{e}_r \cdot \mathbf{k} = \cos\theta \tag{1.12.8b}$$

The relationships for $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ can be obtained as above, yielding the desired relations

$$\begin{aligned}
\mathbf{e}_r &= \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta \\
\mathbf{e}_\theta &= \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta \\
\mathbf{e}_\phi &= -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi
\end{aligned} \tag{1.12.9}$$

which express the unit vectors of the rotated triad in terms of the fixed triad **ijk.** We note the similarity between this transformation and that of the second part of Example 1.8.2. The two are, in fact, identical if the correct identification of rotations is made. Let us differentiate the first equation with respect to time. The result is

$$\frac{d\mathbf{e}_r}{dt} = \mathbf{i}(\dot\theta\cos\theta\cos\phi - \dot\phi\sin\theta\sin\phi) + \mathbf{j}(\dot\theta\cos\theta\sin\phi + \dot\phi\sin\theta\cos\phi) - \mathbf{k}\dot\theta\sin\theta \tag{1.12.10}$$

Next, by using the expressions for $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ in Equation 1.12.9, we find that the above equation reduces to

$$\frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\phi\dot\phi\sin\theta + \mathbf{e}_\theta\dot\theta \tag{1.12.11a}$$

The other two derivatives are found through a similar procedure. The results are

$$\frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r\dot{\theta} + \mathbf{e}_\phi\dot{\phi}\cos\theta \tag{1.12.11b}$$

$$\frac{d\mathbf{e}_\phi}{dt} = -\mathbf{e}_r\dot{\phi}\sin\theta - \mathbf{e}_\theta\dot{\phi}\cos\theta \tag{1.12.11c}$$

The steps are left as an exercise. Returning now to the problem of finding **v**, we insert the expression for $d\mathbf{e}_r/dt$ given by Equation 1.12.11a into Equation 1.12.6. The final result is

$$\mathbf{v} = \mathbf{e}_r\dot{r} + \mathbf{e}_\phi r\dot{\phi}\sin\theta + \mathbf{e}_\theta r\dot{\theta} \tag{1.12.12}$$

giving the velocity vector in terms of its components in the rotated triad.

To find the acceleration, we differentiate the above expression with respect to time. This gives

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \mathbf{e}_r\ddot{r} + \dot{r}\frac{d\mathbf{e}_r}{dt} + \mathbf{e}_\phi\frac{d(r\dot{\phi}\sin\theta)}{dt} + r\dot{\phi}\sin\theta\frac{d\mathbf{e}_\phi}{dt} + \mathbf{e}_\theta\frac{d(r\dot{\theta})}{dt} + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt} \end{aligned} \tag{1.12.13}$$

Upon using the previous formulas for the derivatives of the unit vectors, the above expression for the acceleration reduces to

$$\begin{aligned} \mathbf{a} &= (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\mathbf{e}_\theta \\ &\quad + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\mathbf{e}_\phi \end{aligned} \tag{1.12.14}$$

giving the acceleration vector in terms of its components in the triad $\mathbf{e}_r\mathbf{e}_\theta\mathbf{e}_\phi$.

EXAMPLE 1.12.1

A bead slides on a wire bent into the form of a helix, the motion of the bead being given in cylindrical coordinates by $R = b$, $\phi = \omega t$, $z = ct$. Find the velocity and acceleration vectors as functions of time.

Solution:

Differentiating, we find $\dot{R} = \ddot{R} = 0$, $\dot{\phi} = \omega$, $\ddot{\phi} = 0$, $\dot{z} = c$, $\ddot{z} = 0$. So, from Equations 1.12.2 and 1.12.3, we have

$$\mathbf{v} = b\omega\mathbf{e}_\phi + c\mathbf{e}_z$$
$$\mathbf{a} = -b\omega^2\mathbf{e}_R$$

Thus, in this case both velocity and acceleration are constant in magnitude, but they vary in direction because both $\mathbf{e}_\phi$ and $\mathbf{e}_r$ change with time as the bead moves.

EXAMPLE 1.12.2

A wheel of radius b is placed in a gimbal mount and is made to rotate as follows. The wheel spins with constant angular speed ω_1 about its own axis, which in turn rotates with constant angular speed ω_2 about a vertical axis in such a way that the axis of the wheel stays in a horizontal plane and the center of the wheel is motionless. Use spherical coordinates to find the acceleration of any point on the rim of the wheel. In particular, find the acceleration of the highest point on the wheel.

Solution:

We can use the fact that spherical coordinates can be chosen such that $r = b$, $\theta = \omega_1 t$, and $\phi = \omega_2 t$ (Fig. 1.12.3). Then we have $\dot{r} = \quad = 0$, $\dot{\theta} = \omega_1$, $\ddot{\theta} = 0$, $\dot{\phi} = \omega_2$, $\ddot{\phi} = 0$. Equation 1.12.14 gives directly

$$\mathbf{a} = (-b\omega_2^2 \sin^2\theta - b\omega_1^2)\mathbf{e}_r - b\omega_2^2 \sin\theta\cos\theta\,\mathbf{e}_\theta + 2b\omega_1\omega_2\cos\theta\,\mathbf{e}_\phi$$

The point at the top has coordinate $\theta = 0$, so at that point

$$\mathbf{a} = -b\omega_1^2\mathbf{e}_r + 2b\omega_1\omega_2\mathbf{e}_\phi$$

The first term on the right is the centripetal acceleration, and the last term is a transverse acceleration normal to the plane of the wheel.

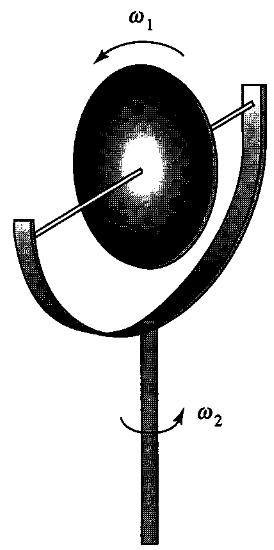

Figure 1.12.3 A rotating wheel on a rotating mount.

Problems

1.1 Given the two vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and $\mathbf{B} = \mathbf{j} + \mathbf{k}$, find the following:
(a) $\mathbf{A} + \mathbf{B}$ and $|\mathbf{A} + \mathbf{B}|$
(b) $3\mathbf{A} - 2\mathbf{B}$
(c) $\mathbf{A} \cdot \mathbf{B}$
(d) $\mathbf{A} \times \mathbf{B}$ and $|\mathbf{A} \times \mathbf{B}|$

1.2 Given the three vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{i} + \mathbf{k}$, and $\mathbf{C} = 4\mathbf{j}$, find the following:
(a) $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C})$ and $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C}$
(b) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
(c) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

1.3 Find the angle between the vectors $\mathbf{A} = a\mathbf{i} + 2a\mathbf{j}$ and $\mathbf{B} = a\mathbf{i} + 2a\mathbf{j} + 3a\mathbf{k}$. (*Note:* These two vectors define a face diagonal and a body diagonal of a rectangular block of sides a, $2a$, and $3a$.)

1.4 Consider a cube whose edges are each of unit length. One corner coincides with the origin of an xyz Cartesian coordinate system. Three of the cube's edges extend from the origin along the positive direction of each coordinate axis. Find the vector that begins at the origin and extends
(a) along a major diagonal of the cube;
(b) along the diagonal of the lower face of the cube.
(c) Calling these vectors $\mathbf{A}$ and $\mathbf{B}$, find $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.
(d) Find the angle between $\mathbf{A}$ and $\mathbf{B}$.

1.5 Assume that two vectors $\mathbf{A}$ and $\mathbf{B}$ are known. Let $\mathbf{C}$ be an unknown vector such that $\mathbf{A} \cdot \mathbf{C} = u$ is a known quantity and $\mathbf{A} \times \mathbf{C} = \mathbf{B}$. Find $\mathbf{C}$ in terms of $\mathbf{A}$, $\mathbf{B}$, u, and the magnitude of $\mathbf{A}$.

1.6 Given the time-varying vector

$$\mathbf{A} = \mathbf{i}\alpha t + \mathbf{j}\beta t^2 + \mathbf{k}\gamma t^3$$

where α, β, and γ are constants, find the first and second time derivatives $d\mathbf{A}/dt$ and $d^2\mathbf{A}/dt^2$.

1.7 For what value (or values) of q is the vector $\mathbf{A} = \mathbf{i}q + 3\mathbf{j} + \mathbf{k}$ perpendicular to the vector $\mathbf{B} = \mathbf{i}q - q\mathbf{j} + 2\mathbf{k}$?

1.8 Give an algebraic proof and a geometric proof of the following relations:

$$|\mathbf{A} + \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$$
$$|\mathbf{A} \cdot \mathbf{B}| \le |\mathbf{A}|\,|\mathbf{B}|$$

1.9 Prove the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

1.10 Two vectors $\mathbf{A}$ and $\mathbf{B}$ represent concurrent sides of a parallelogram. Show that the area of the parallelogram is equal to $|\mathbf{A} \times \mathbf{B}|$.

1.11 Show that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is not equal to $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$.

1.12 Three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ represent three concurrent edges of a parallelepiped. Show that the volume of the parallelepiped is equal to $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$.

1.13 Verify the transformation matrix for a rotation about the z-axis through an angle ϕ followed by a rotation about the y'-axis through an angle θ, as given in Example 1.8.2.

1.14 Express the vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ in the primed triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$ in which the $x'y'$-axes are rotated about the z-axis (which coincides with the z'-axis) through an angle of 30°.

1.15 Consider two Cartesian coordinate systems xyz and $x'\ y'\ z'$ that initially coincide. The $x'\ y'\ z'$ undergoes three successive counterclockwise 45° rotations about the following axes: first, about the fixed z-axis; second, about its own x'-axis (which has now been rotated); finally, about its own z'-axis (which has also been rotated). Find the components of a unit vector $\mathbf{X}$ in the xyz coordinate system that points along the direction of the x'-axis in the rotated $x'\ y'\ z'$ system. (*Hint: It would be useful to find three transformation matrices that depict each of the above rotations. The resulting transformation matrix is simply their product.*)

1.16 A racing car moves on a circle of constant radius b. If the speed of the car varies with time t according to the equation $v = ct$, where c is a positive constant, show that the angle between the velocity vector and the acceleration vector is 45° at time $t = \sqrt{b/c}$. (*Hint: At this time the tangential and normal components of the acceleration are equal in magnitude.*)

1.17 A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in an elliptical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b\cos\omega t + \mathbf{j}2b\sin\omega t$$

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at $t = 0$ and at $t = \pi/2\omega$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

1.18 A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b\sin\omega t + \mathbf{j}b\cos\omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

1.19 A bee goes out from its hive in a spiral path given in plane polar coordinates by

$$r = be^{kt} \qquad \theta = ct$$

where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (*Hint: Find* $\mathbf{v}\cdot\mathbf{a}/va$.)

1.20 Work Problem 1.18 using cylindrical coordinates where $R = b$, $\phi = \omega t$, and $z = ct^2$.

1.21 The position of a particle as a function of time is given by

$$\mathbf{r}(t) = \mathbf{i}(1 - e^{-kt}) + \mathbf{j}e^{kt}$$

where k is a positive constant. Find the velocity and acceleration of the particle. Sketch its trajectory.

1.22 An ant crawls on the surface of a ball of radius b in such a manner that the ant's motion is given in spherical coordinates by the equations

$$r = b \qquad \phi = \omega t \qquad \theta = \frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]$$

Find the speed of the ant as a function of the time t. What sort of path is represented by the above equations?

1.23 Prove that $\mathbf{v} \cdot \mathbf{a} = v\dot{v}$ and, hence, that for a moving particle $\mathbf{v}$ and $\mathbf{a}$ are perpendicular to each other if the speed v is constant. (*Hint: Differentiate both sides of the equation* $\mathbf{v} \cdot \mathbf{v} = v^2$ *with respect to t. Note,* $\dot{v}$ *is not the same as* $|\mathbf{a}|$. *It is the magnitude of the acceleration of the particle along its instantaneous direction of motion.*)

1.24 Prove that

$$\frac{d}{dt}[\mathbf{r} \cdot (\mathbf{v} \times \mathbf{a})] = \mathbf{r} \cdot (\mathbf{v} \times \dot{\mathbf{a}})$$

1.25 Show that the tangential component of the acceleration of a moving particle is given by the expression

$$a_\tau = \frac{\mathbf{v} \cdot \mathbf{a}}{v}$$

and the normal component is therefore

$$a_n = \left(a^2 - a_\tau^2\right)^{1/2} = \left[a^2 - \frac{(\mathbf{v} \cdot \mathbf{a})^2}{v^2}\right]^{1/2}$$

1.26 Use the above result to find the tangential and normal components of the acceleration as functions of time in Problems 1.18 and 1.19.

1.27 Prove that $|\mathbf{v} \times \mathbf{a}| = v^3/\rho$, where ρ is the radius of curvature of the path of a moving particle.

1.28 A wheel of radius b rolls along the ground with constant forward acceleration a_0. Show that, at any given instant, the magnitude of the acceleration of any point on the wheel is $(a_0^2 + v^4/b^2)^{1/2}$ relative to the center of the wheel and is also $a_0[2 + 2\cos\theta + v^4/a_0^2b^2 - (2v^2/a_0b)\sin\theta]^{1/2}$ relative to the ground. Here v is the instantaneous forward speed, and θ defines the location of the point on the wheel, measured forward from the highest point. Which point has the greatest acceleration relative to the ground?

1.29 What is the value of x that makes of following transformation $\mathbf{R}$ orthogonal?

$$\mathbf{R} = \begin{pmatrix} x & x & 0 \\ -x & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What transformation is represented by $\mathbf{R}$?

1.30 Use vector algebra to derive the following trigonometric identities
(a) $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$
(b) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$