Functions of several Variables:

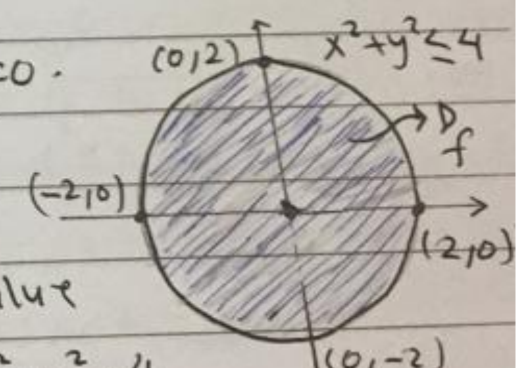
Def^n (1): (Function of two variables)
Let D be a subset of R^2, Then a function of two variables f is a rule which assigns to each point (order pair) (x,y) in D a unique real number which we denote $f(x,y)$. The set D is called the domain of f. The set $\{f(x,y) : (x,y) \in D\}$ which is the set of values the function f takes on is called the range of f.

EX: ① Let $z = f(x,y) = x^2 + 2y^4$

$D_f = \mathbb{R}^2$, since $x^2 + 2y^4 \geq 0$ for every pair (x,y) $\Rightarrow$ $R_f = R^+$ ((the set of nonnegative real no.))

EX: (2) Find the domain and Range of the f^n f given by $f(x,y) = \sqrt{4 - x^2 - y^2}$ and find $f(0,1)$ & $f(1,-1)$

Sol^n: $D_f = \{(x,y) : 4 - x^2 - y^2 \geq 0\} = \{(x,y) : x^2 + y^2 \leq 4\}$



since x^2 & y^2 are nonnegative
$4 - x^2 - y^2$ is largest when $x = y = 0$.
This means that the largest value of $z = \sqrt{4 - x^2 - y^2} = 2$
since $x^2 + y^2 \leq 4$, the smallest value of z is zero, taken when $x^2 + y^2 = 4$ (at all point on the circle $x^2 + y^2 = 4$).
$\Rightarrow$ The range of f is $[0,2]$.

$f(0,1) = \sqrt{4 - 0 - 1} = \sqrt{3}$ and $f(1,-1) = \sqrt{2}$

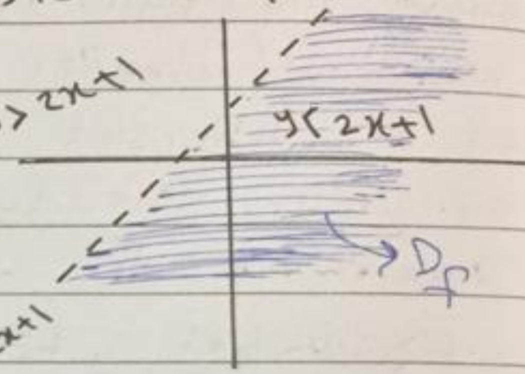
Ex: 3 Let $z = \ln(4x-y+1)$. find D_f & R_f; and sketch the region of D_f.

Soln: $\ln(4x-y+1)$ is defined only when

$4x-y+1>0 \Rightarrow y<4x+1$: this is the equation of the half-plane (below); (

$D_f = \{(x,y) : y<4x+1\}$

$R_f = R$ ($\ln(4x-y+1)$ can take any real value.)



Ex(4): Find D_f & R_f of the f^n

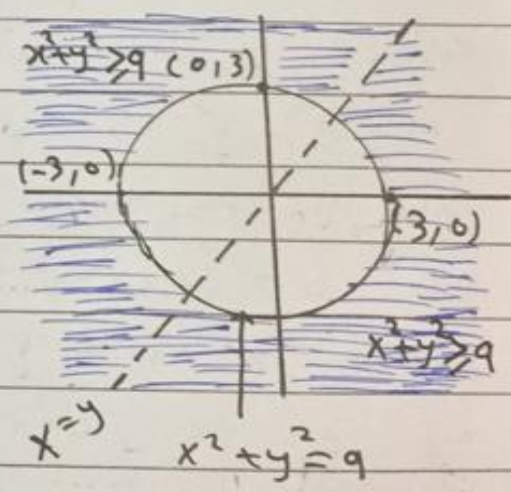
$$z = f(x,y) = \frac{\sqrt{x^2+y^2-9}}{x-y}$$

$f(x,y)$ is not defined when $x-y=0$ $(x=y)$.

and $\sqrt{x^2+y^2-9}$ defined when $x^2+y^2 \geq 9$ (circle $x^2+y^2=9$)

$D_f = \{(x,y) : x^2+y^2 \geq 9$ and $x \neq y\}$

$R_f = R$ (why)?



Ex: Find D_f & R_f of the following f^ns: with sketch.

(i) $z = 3/\sqrt{x^2-y}$

(ii) $z = \tan^{-1}\left(\frac{y}{x}\right)$

(iii) $f(x,y) = \ln(x^2-y^2)$

(iv) $f(x,y) = \tan(x-y)$

(v) $f(x,y) = \sqrt{\frac{(x-y)}{(x+y)}}$

(vi) $z = \cos^{-1}(x-y)$

(vii) $z = e^x + e^y$

Defn: (Function of three variables)

Let D be a subset of $\mathbb{R}^3$. Then a function of three variables f is a rule which assigns to each point (ordered triple) (x,y,z) in D a unique real no. which we denote $f(x,y,z)$. The set D is called the domain of f and the set

$$\{f(x,y,z) : (x,y,z) \in D\}$$

which is the set of values the function f takes on is called the range of f.

Ex: ① let $w = f(x,y,z) = \sqrt{1 - x^2 - \frac{y^2}{4} - \frac{z^2}{9}}$

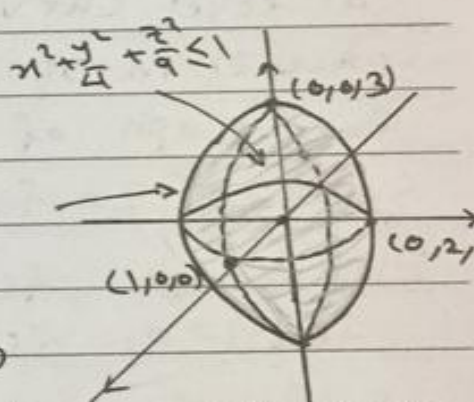
find D_f & R_f, $f(0,1,1)$.

Soln: $f(x,y,z)$ defined if $1 - x^2 - \frac{y^2}{4} - \frac{z^2}{9} \geq 0$

$$\Rightarrow x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1$$

$$D_f = \{(x,y,z) : x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1\}$$

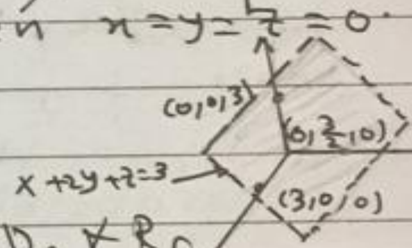
$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1 \quad \text{(elipsoid)}$$



$R_f = [0,1]$, since $x^2 + \frac{y^2}{4} + \frac{z^2}{9} \geq 0$

so that $1 - x^2 - \frac{y^2}{4} - \frac{z^2}{9}$ is max. when $x = y = z = 0$.

$$f(0,1,1) = \sqrt{1 - \frac{1}{4} - \frac{1}{9}} = \frac{\sqrt{23}}{6}$$

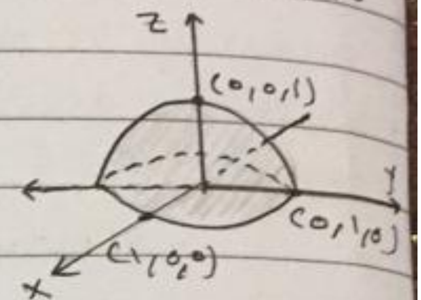


② Let $w = \ln(x + 2y + z - 3)$. find D_f & R_f.

Since $f(x,y,z)$ is defined when $x + 2y + z - 3 > 0$, then $x + 2y + z > 3$. The eqn. $x + 2y + z = 3$ is plane. Thus the domain of f is the half-space above (but not includ[ing]) this plane.

$R_f = \mathbb{R}$.

$\underline{\text{Def}^n}$: The graph of a $f^{\underline{n}}$ of two variables f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$. The graph of a $f^{\underline{n}}$ of two variables is called a Surface in $\mathbb{R}^3$.



$\underline{\text{Ex}}$: Sketch the graph of the $f^{\underline{n}}$

$$z = f(x, y) = \sqrt{1 - x^2 - y^2}.$$

since $z \geq 0$

$$z^2 = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 + z^2 = 1$$ (eqn. of the unit sphere)

the graph of f is hemisphere.

$\underline{\text{Def}^n}$: The set of points in the plane where a $f^{\underline{n}}$ $f(x, y)$ has a constant value $f(x, y) = c$ is called a level curve of f. The set of all points (x, y, z) in space, for (x, y) in the domain of f, is called the graph of f. The graph of f is also called the surface $z = f(x, y)$.

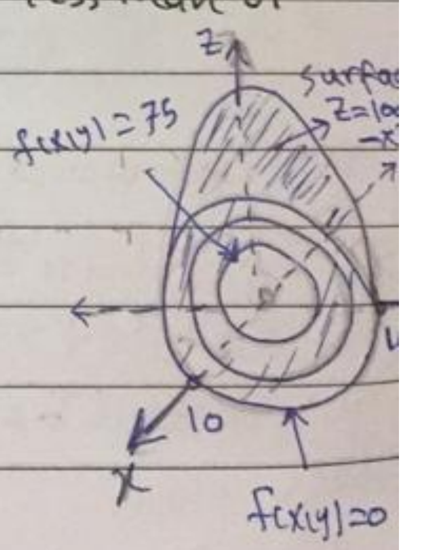
$\underline{\text{Ex}}$: Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, $f(x, y) = 75$ in D_f.

$\underline{\text{Sol}^n}$: The domain is the entire xy-plane, and the range of f is the set of real no.s less than or equal to 100.

(paraboloid) $z = 100 - x^2 - y^2$

$$f(x, y) = 100 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 100$$

$$f(x, y) = 100 - x^2 - y^2 = 75 \Rightarrow x^2 + y^2 = 25$$

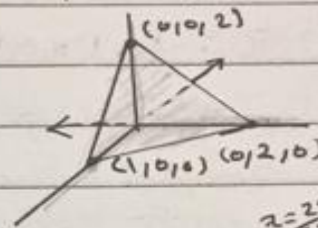


Ex: ① Sketch $z = 2-2x-y$ (plane)

if $x=0,\ y=0 \Rightarrow z=2$

if $x=0,\ z=0 \Rightarrow y=2$

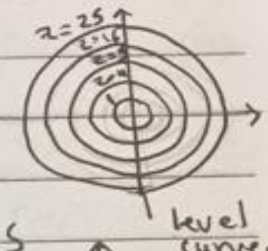
if $y=0,\ z=0 \Rightarrow x=1$

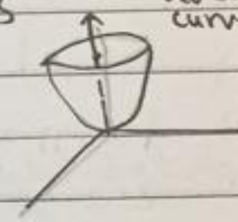


② $z = x^2+y^2$ (circular paraboloid)

in the plane $z=k>0$ circle

in planes $x=k$ & $y=k$ are parabolas

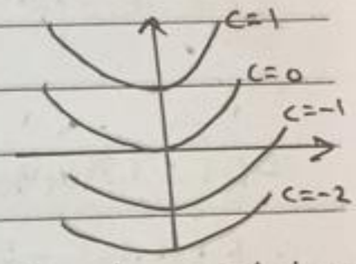




③ $f(x,y) = y-x^2$

level curves are defined by $y-x^2=c$

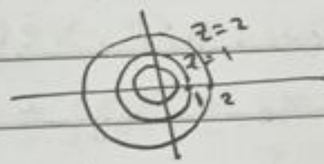
where c is constant.



(4) $f(x,y) = x^2+y^2$

$z \geq 0 \Rightarrow z = x^2+y^2$ level curves

contour plots



(5) Let $z = 2x-y+4$. Describe the level curves.

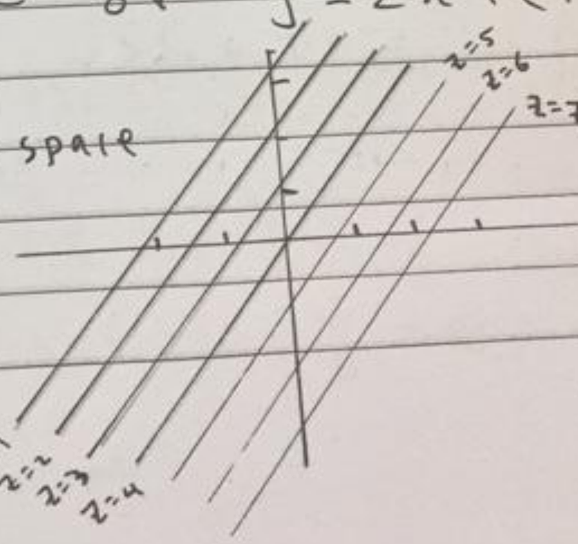
The level curves are the line

$2x-y+4=c$ or $y=2x+(4-c)$

if $x=0,\ y=0 \Rightarrow z=4$

Here the graph of a f^n in space is the plane

$2x-y-z+4=0$



H.W! (1) Describe the level curves each of the following:

(i) $z = \ln(x+y)$ (ii) $z = \frac{1}{xy}$

(iii) $z = x+y-1$, $c = -3, -2, -1, 0, 1, 2, 3$

(iv) $z = xy$, $c = -9, -4, -1, 0, 1, 4, 9$

(v) $z = \sqrt{25 - x^2 - y^2}$, $c = 0, 1, 2, 3, 4$.

(2) find D_f & R_f and sketch the level curves $f(x,y) = c$

(i) $f(x,y) = y - x$ (ii) $z = \sqrt{y-x}$

(iii) $f(x,y) = x^2 - y^2$ (iv) $z = 4x^2 + 9y^2$

(v) $z = \ln(9 - x^2 - y^2)$ (vi) $z = \sin^{-1}(y-x)$

limits and Continuity

Let (x_0, y_0) be a point in $\mathbb{R}^2$.

$|(x,y) - (x_0,y_0)| = r$?

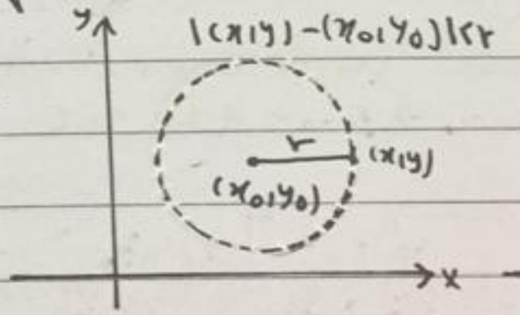
Since (x,y) and (x_0,y_0) are vectors in $\mathbb{R}^2$,

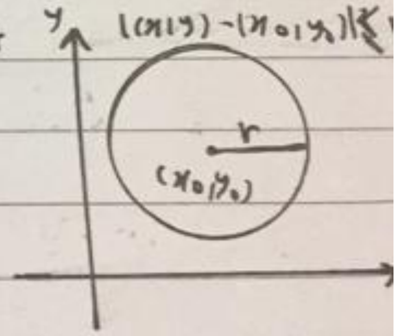
$$|(x,y) - (x_0,y_0)| = |(x-x_0, y-y_0)| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

$|(x,y) - (x_0,y_0)| < r$

$|(x,y) - (x_0,y_0)| \leq r$





Def^n: (1) (i) The open disk D_r centered at (x_0, y_0) with radius r is the subset of $\mathbb{R}^2$ given by $\{(x,y) : |(x,y)-(x_0,y_0)| < r\}$.

(ii) The closed disk centered at (x_0,y_0) with radius r is the subset of $\mathbb{R}^2$ given by $\{(x,y) : |(x,y)-(x_0,y_0)| \leq r\}$

(iii) The boundary of the open or closed disk is the circle $\{(x,y) : |(x,y)-(x_0,y_0)| = r\}$.

(iv) A neighborhood of a point (x_0,y_0) in $\mathbb{R}^2$ is an open disk centered at (x_0,y_0).

Def^n: (2) ((limit))

Let $f(x,y)$ be defined in a neighborhood of (x_0,y_0), but not necessarily at (x_0,y_0) itself. Then the limit of $f(x,y)$ as (x,y) approaches (x_0,y_0) is L, written as

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for every no. $\epsilon > 0$, $\exists$ a $\delta > 0$ s.t. $|f(x,y) - L| < \epsilon$ for every (x,y) (not including (x_0,y_0)) in the open disk centered at (x_0,y_0) with radius δ.

Ex: Show that by def^n $\lim_{(x,y)\to(1,2)} (3x+2y) = 7$.

Sol^n: $\forall\ \epsilon > 0 \quad \exists\ \delta > 0 \ni |3x+2y-7| < \epsilon$ whenever

$0 < \sqrt{(x-1)^2+(y-2)^2} < \delta$.

$$|3x+2y-7| = |(3x-3)+(2y-4)| \leq |3x-3| + |2y-4|$$
$$= 3|x-1| + 2|y-2|$$

$|x-1| \leq \sqrt{(x-1)^2+(y-2)^2}$ & $|y-2| \leq \sqrt{(x-1)^2+(y-2)^2}$

$$3|x-1| + 2|y-2| \leq 5\sqrt{(x-1)^2+(y-2)^2} < 5\delta = 5\frac{\epsilon}{5} = \epsilon$$

Chose $\delta = \frac{\epsilon}{5}$

Ex: let $f(x,y) = \dfrac{y^2 - x^2}{y^2 + x^2}$ for $(x,y) \neq (0,0)$.

Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Sol^n: There are an infinite no. of approaches to the origin. For example, if we approach along the x-axis, then $y=0$ and

$$\lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{y^2+x^2} = \lim_{(x,y)\to(0,0)} \frac{-x^2}{x^2} = -1$$

On the other hand, if we approach along the y-axis, then $x=0$ and

$$\lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{y^2+x^2} = \lim_{(x,y)\to(0,0)} \frac{y^2}{y^2} = 1$$

Hence f cannot have a limit as $(x,y)\to(0,0)$.

Note:

If we get two or more different values for $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ as we approach (x_0,y_0) along different paths, then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Ex: let $f(x,y) = \dfrac{xy^2}{(x^2+y^4)}$. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Sol^n:

(i) let us approach zero along straight line passing through the origin which is not the y-axis. Then $y=mx$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^4)} = \lim_{(x,y)\to(0,0)} \frac{x\,m^2x^2}{x^2+m^4x^4} = \lim_{(x,y)\to(0,0)} \frac{m^2x}{1+m^2x^2} = 0$$

(ii) If $(0,0)$ approaches along the parabola $x=y^2$, then

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^4)} = \lim_{(x,y)\to(0,0)} \frac{y^4}{y^4+y^4} = \frac{1}{2}$$

$\Rightarrow$ The limit does not exist.

Ex: prove that $\lim_{(x,y)\to(0,0)} [xy^2/(x^2+y^2)] = 0$, by def$^{\underline{n}}$.

$\forall \epsilon > 0 \ \exists \delta > 0 \ni \left|\frac{xy^2}{x^2+y^2}\right| < \epsilon$

$|x-0| = |x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$ so that

$$\left|\frac{xy^2}{x^2+y^2}\right| \leq \frac{|xy^2|}{|x^2+y^2|} \leq \frac{\sqrt{x^2+y^2}(x^2+y^2)}{x^2+y^2} = \sqrt{x^2+y^2} < \delta$$

We choose $\delta = \epsilon$.

\# \# \#

H.W: calculate

$$\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2} = \frac{0+0+5}{0+0+2} = \frac{5}{2}$$

$$\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}} = 0$$

$$\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2 = \left[\frac{1}{2}+\left(\frac{1}{-3}\right)\right]^2 = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$$

$$\lim_{(x,y)\to(0,\frac{\pi}{4})} \sec x \tan y = \sec 0 \tan\frac{\pi}{4} = 1\cdot 1 = 1$$

$$\lim_{(x,y)\to(0,\ln 2)} e^{x-y} = e^{0-\ln 2} = e^{-\ln 2} = e^{\ln\frac{1}{2}} = \frac{1}{2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}$$ ~~undefine~~

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{(x-y)}$$

$$= \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = 0$$

Ex: Find $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ if it exists

(i) If $(0,0)$ approaches along the x-axis, then $y=0$ and

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \frac{0}{x^2} = 0$$

(ii) If $(0,0)$ approaches along y-axis, then $x=0$, and

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \frac{0}{y^2} = 0$$

(iii) If $(0,0)$ approaches along the $y=x$,

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{4x^3}{2x^2} = \lim_{(x,y)\to(0,0)} 2x = 0$$

To see if this is true, we apply the def$^{\underline{n}}$ of limit

$\forall \epsilon > 0 \quad \exists \delta > 0 \quad$ s-t

$|f(x,y) - L| < \epsilon$ whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta$$

$$\left|\frac{4xy^2}{x^2+y^2}\right| = \frac{4|x|\,|y^2|}{|x^2+y^2|} = \frac{4|x|\,y^2}{x^2+y^2} < \epsilon$$

since $y^2 \le x^2+y^2$, then

$$\frac{4|x|\,y^2}{x^2+y^2} \le \frac{4|x|\,\cancel{x^2+y^2}}{\cancel{x^2+y^2}} = 4|x| = 4\sqrt{x^2} \le 4\sqrt{x^2+y^2} < 4\delta$$

Choose $\delta = \frac{\epsilon}{4}$, and $0 < \sqrt{x^2+y^2} < \delta$, we get

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| \le 4\sqrt{x^2+y^2} < 4\delta = 4\frac{\epsilon}{4} = \epsilon$$

H.W: (1) Find each of the following limits (If it exist)

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y} \quad ; \quad \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-y^2}{x-y}$$

$$\lim_{\substack{(x,y)\to\\ x\neq 1}} \frac{xy-y-2x+2}{x-1} \quad , \quad \lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y-xy+4x^2-4x}$$

$$\lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} \quad ; \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy} \qquad \lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$$ (not exist)

(2) use definition of limit to verify the indicated limit

$$\lim_{(x,y)\to(1,2)} (3x+y)=5 \quad ; \quad \lim_{(x,y)\to(3,-1)} (x-7y)=10$$

$$\lim_{(x,y)\to(5,-2)} (ax+by)=5a-2b \quad ; \quad \lim_{(x,y)\to(1,1)} \frac{x}{y}=1$$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2}=0 \quad ; \quad \lim_{(x,y)\to(4,1)} (x^2+3y^2)=19.$$

(3) Show that the indicated limit exists and calculat it

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{\sqrt{x^2+y^2}} \quad ; \quad \lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^4+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} \quad ; \quad \lim_{(x,y)\to(0,0,0)} \frac{yx^2+z^3}{x^2+y^2+z^2}$$

1- $$\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(x-y)+2(\sqrt{x}-\sqrt{y})}{(\sqrt{x}-\sqrt{y})}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})+2(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y}+2)}{(\sqrt{x}-\sqrt{y})}$$

$$= \lim_{(x,y)\to(0,0)} (\sqrt{x}+\sqrt{y}+2) = 2.$$

2- $$\lim_{\substack{(x,y)\to(4,3)\\ (x-y)\neq 1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{x-(y+1)} = \lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{(\sqrt{x}-\sqrt{y+1})(\sqrt{x}+\sqrt{y+1})}$$

$$= \lim_{(x,y)\to(4,3)} \frac{1}{\sqrt{x}+\sqrt{y+1}} = \frac{1}{4}$$

3- $$\lim_{(x,y)\to(0,0)} \frac{e^{y}\sin x}{x} = \lim_{(x,y)\to(0,0)} e^{y} \cdot \lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$$

$$= e^{0} \cdot 1 = 1 \cdot 1 = 1$$

4- $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin r^2}{r^2} = \lim_{r\to 0} \frac{2r\cos r^2}{2r} = \lim_{r\to 0} \frac{2r\cos r^2}{2r}$$

$$= \lim_{r\to 0} \cos r^2 = 1$$

5- $$\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy} = \lim_{u\to 0} \frac{1-\cos u}{u} = \lim_{u\to 0} \frac{\sin u}{1} = \frac{0}{1} = 0$$

* Sandwich Theorem: If $g(x,y) \le f(x,y) \le h(x,y)$ for all $(x,y) \neq (x_0,y_0)$ in a disk centered at (x_0,y_0) and if g and h have the same finite limit L as $(x,y)\to(x_0,y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Ex: ① find $\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy}$ if $1-\frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$

$$\lim_{(x,y)\to(0,0)} \left(1-\frac{x^2y^2}{3}\right) = 1 \text{ and } \lim_{(x,y)\to(0,0)} 1 = 1 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy} = 1$$

② find $\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}$

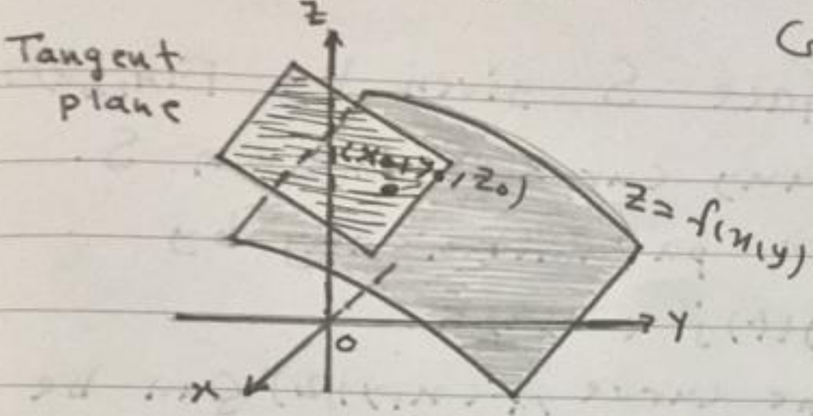
since $\left|\sin\frac{1}{x}\right| \le 1 \Rightarrow -1 \le \sin\frac{1}{x} \le 1 \Rightarrow -y \le y\sin\frac{1}{x} \le y$ for $y \ge 0$

and $-y \ge y\sin\frac{1}{x} \ge y$ for $y \le 0$. As $(x,y)\to(0,0)$, both y and $-y$ approache 0

$$\lim y\sin\frac{1}{x} = 0.$$

Application of Partial Derivatives

Tangent planes, Normal Lines and Gradients



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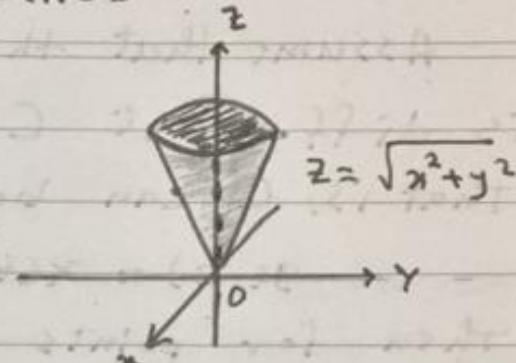


-2-

Let $z = f(x,y)$ be a function of two variables. As we have seen, the graph of f is a surface in R^3. More generally, the graph of the equation $F(x,y,z)=0$ is a surface in R^3. The 'surface' $F(x,y,z)=0$ is called differentiable at a point (x_0, y_0, z_0) if $\partial F/\partial x$, $\partial F/\partial y$ and $\partial F/\partial z$ all exist and are continuous at (x_0, y_0, z_0). Just as a differentiable curve in R^2 has a unique tangent line at each point, a differentiable surface in R^3 has a unique tangent plane at each point. We shall formally define what we mean by a tangent plane to a surface after a bit, although it should be easy enough to visualize (see 1). We note here that not every surface has a tangent plane at every point. For example, the cone $\boxed{z = \sqrt{x^2+y^2}}$ clearly has no tangent plane at the origin (see 2).

Assume that the surface S given by $F(x,y,z)=0$ is diff. Let C be any curve lying on S. That is, C can be given parametrically by

$$g(t) = x(t)i + y(t)j + z(t)k$$

Then for points on the curve, $F(x,y,z)$ can be written as a fn. of t and, from the vector form of the chain rule, we have

$$F'(t) = \frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt}$$

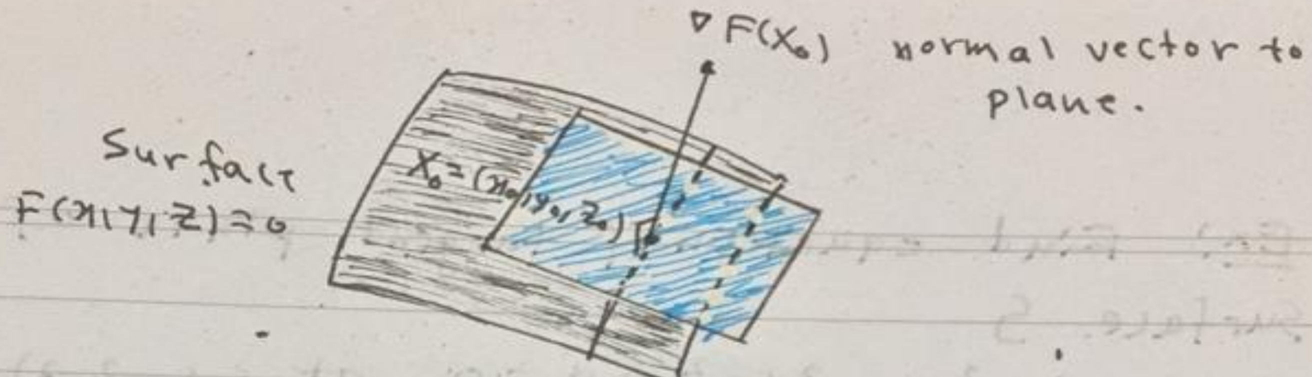
but

$$F'(t) = \nabla F \cdot g'(t) \quad \text{———} \quad (1)$$

$$= \left(\frac{\partial F}{\partial x}i + \frac{\partial F}{\partial y}j + \frac{\partial F}{\partial z}k\right)\cdot\left(\frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k\right)$$

$$= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt}$$

where ∇F is called a gradient of F.
But since $F(x(t), y(t), z(t)) = 0$ for all t (since $(x(t), y(t), z(t))$ is on S), we see that $F'(t) = 0$ for all t. But $g'(t)$ is tangent to the curve C for every no. t.
Thus (1) implies that:
The gradient of F at a point $X_0 = (x_0, y_0, z_0)$ in S is orthogonal to the tangent vector to any curve C remaining on S and passing through.



Def^n: Let F be diff. at $X_0=(x_0,y_0,z_0)$, and Let the surface S be defined by $F(x,y,z)=0$

(i) Then the tangent plane to S is the plane passing through the point $X_0=(x_0,y_0,z_0)$ with normal vector $\nabla F(X_0)$.

(ii) The normal line to S at X_0 is the line passing through X_0 having the same direction as $\nabla F(X_0)$.

Ex: Find the eqn. of the tangent plane of $z=f(x,y)$ at (x_0,y_0,z_0)

Sol^n: let $F(x,y,z)=z-f(x,y)=0$

the eqn. of tangent plane is

$$(x-x_0)F_x+(y-y_0)F_y+(z-z_0)F_z=0 \qquad (1)$$

s.t.

$$F_z=1\ ,\ F_y=-f_y\ ,\ F_x=-f_x$$

(1) becomes,

$$-(x-x_0)f_x-(y-y_0)f_y+(z-z_0)=0$$

$$z-z_0=(x-x_0)f_x+(y-y_0)f_y$$

Ex: Find equation of tangent plane of the surface S

$$4x^2+9y^2+z^2-49=0 \quad \text{at } (1,-2,3)$$

Soln: $F(x,y,z)=4x^2+9y^2+z^2-49=0$

$F_x=8x, \quad F_y=18y, \quad F_z=2z$

$F_x\big|_{(1,-2,3)}=8, \quad F_y\big|_{(1,-2,3)}=-36, \quad F_z\big|_{(1,-2,3)}=6$

$(x-x_0)F_x+(y-y_0)F_y+(z-z_0)F_z=0$

$8(x-1)-36(y+2)+6(z-3)=0$

$8x-36y+6z-8-72-18=0$

$8x-36y+6z-98=0$

$4x-18y+3z-49=0$

Def^n: (Normal line)

The normal line is define as

$$\frac{(x-x_0)}{f_x\big|_{(x_0,y_0)}}=\frac{(y-y_0)}{f_y\big|_{(x_0,y_0)}}=\frac{(z-z_0)}{-1}$$

and the normal vector N to the tangent plane is

$$N=f_x(x_0,y_0)i+f_y(x_0,y_0)j-k$$

Note!

If $z=f(x,y)$ and, if $\nabla f(x_0,y_0)=0$, then the tangent plane to the surface at (x_0,y_0,z_0) is parallel to the xy-plane (i.e it is horizontal).

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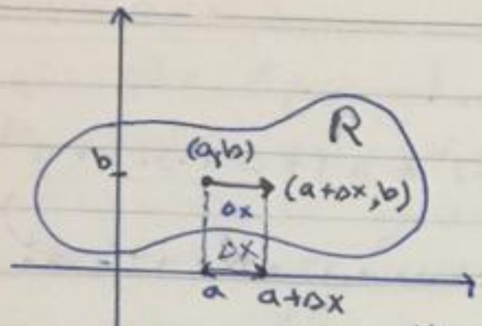
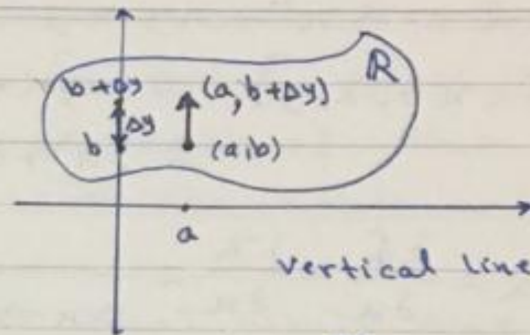
Partial Derivatives



Horizontal line

I

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x}$$



Vertical line

II

$$\lim_{\Delta y \to 0} \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y}$$

<u>Def^n</u> (1) : Let $z = f(x,y)$. then

(i) The partial derivative of f with respect to x is the f^n:

(1) ——— $$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$\frac{\partial f}{\partial x}$ is defined at every point (x,y) in D_f s.t. the limit (1) exists.

(ii) the partial derivative of f w.r.to y is the f^n

(2) ——— $$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_x = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$\frac{\partial f}{\partial y}$ is defined at every point (x,y) in D_f s.t. the limit (2) exists.

<u>EX</u> : Computing Partial Derivatives of $f(x,y) = 3x^2 + xy + y^2$, $f_x(1,1)$, $f_y(0,2)$ by def^n

$$z = 3x^2 + xy + y^2$$

$$f(x+\Delta x, y) = 3(x+\Delta x)^2 + (x+\Delta x)y + y^2$$
$$= 3x^2 + 6x(\Delta x) + 3(\Delta x)^2 + xy + \Delta x(y) + y^2$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{\cancel{3x^2} + 6x(\Delta x) + 3(\Delta x)^2 + xy + y\Delta x + \cancel{y^2} - \cancel{3x^2} - xy - \cancel{y^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x(\Delta x) + 3(\Delta x)^2 + y\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 6x + 3\Delta x + y = 6x + y.$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{3x^2 + x(y+\Delta y) + (y+\Delta y)^2 - 3x^2 - xy - y^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\cancel{3x^2} + \cancel{xy} + x\Delta y + \cancel{y^2} + 2y\Delta y + (\Delta y)^2 - \cancel{3x^2} - \cancel{xy} - \cancel{y^2}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x\Delta y + 2y\Delta y + (\Delta y)^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} x + 2y + \Delta y = x + 2y$$

$$f_x(1,1) = 7, \quad f_y(0,2) = 4$$

Remark:

(i) The partial derivatives f_x, and, f_y gives us the rate of change of f as each of the variables x and y change with the other one held fixed.

(ii) It should be emphasized that while the f^{n}'s f_x and f_y are computed with one of the variables held constant, each is a function of both variables

Ex(1) Let $f(x,y) = \sqrt{x+y^2}$. Calculate $\partial f/\partial x$, $\partial f/\partial y$

$$z = (x+y^2)^{1/2}$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{1}{2}(x+y^2)^{-1/2} \cdot (1) = \frac{1}{2\sqrt{x+y^2}}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{1}{2}(x+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x+y^2}}$$

Ex: (2)

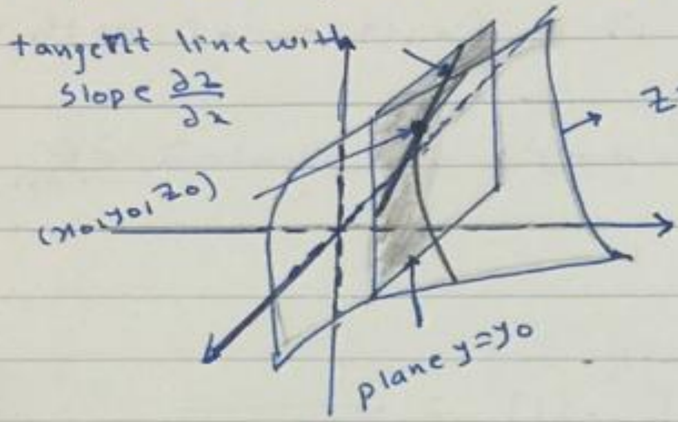
Let $z = \frac{x}{y}\sin(x^2y^3)$. Calculate f_x, f_y

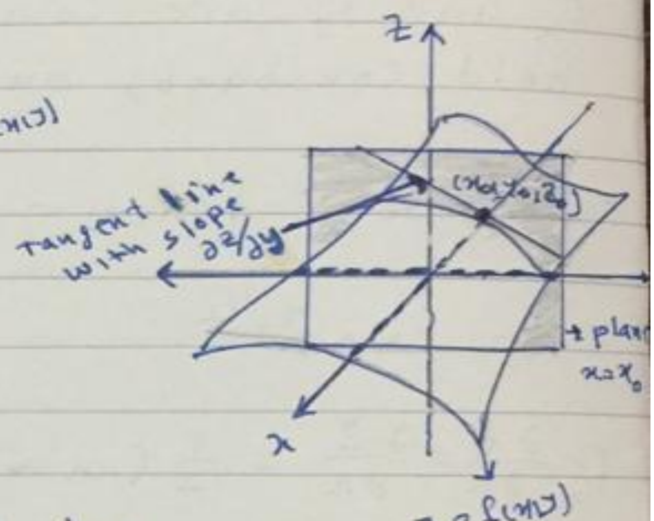
$$\frac{\partial f}{\partial x} = \left(\frac{x}{y}\right)\cos(x^2y^3)(2xy^3) + \frac{1}{y}\sin(x^2y^3)$$
$$= 2x^2y^2\cos(x^2y^3) + \frac{1}{y}\sin(x^2y^3)$$

$$\frac{\partial f}{\partial y} = \left(\frac{x}{y}\right)\cos(x^2y^3)(3y^2x^2) + \left(-\frac{x}{y^2}\right)\sin(x^2y^3)$$

$$= \frac{x}{y}\cos(x^2y^3)(3y^2x^2) - \frac{x}{y^2}\sin(x^2y^3)$$
$$= 3x^3y\cos(x^2y^3) - \frac{x}{y^2}\sin(x^2y^3)$$

Remark: If (x_0, y_0, z_0) is a point on the surface $z = f(x,y)$, then $\partial z/\partial x$ evaluated at (x_0, y_0) is the slope of the line tangent to the surface at (x_0, y_0, z_0) which lies in the plane $y = y_0$.
Also, $\partial z/\partial y$ evaluated at (x_0, y_0) is the slope of the line tangent to the surface at (x_0, y_0, z_0) which lies in the plane $x = x_0$





Def^n (2) Let $w = f(x,y,z)$ be defined in a neighborhood of the point (x,y,z). Then

(i) The partial derivative of f w.r.to x is the function

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} = f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x} \quad (3)$$

f_x in defined at every point (x,y,z) in D_f at which the limit in (3) exists.

(ii) The partial derivative of f w.r.to y is

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} = f_y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y} \quad (4)$$

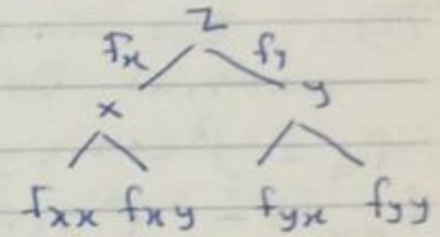
f_y is defined at every point (x,y,z) in D_f at which the limit in (4) exists.

Higher Order Partial Derivatives

If $z = f(x,y)$, then we can differentiate each of the two "first" partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ w.r. to both x and y to obtain four second partial derivatives as:

(i) Differentiate twice with respect to x:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$



(ii) Diff. first w.r.to x, and, then w.r.to y

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

(iii) Diff. first w.r. to y, and, then w.r.to x

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

(iv) Diff. twice w.r.to y

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

<u>Remark</u> The derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are called the mixed second partials.

Ex: ① Let $z = f(x,y) = x^3y^2 + xy^5$. Calculate the four second partial derivatives

$f_x = 3x^2y^2 + y^5$, $f_y = 2x^3y + 5xy^4$

$f_{xx} = 6xy^2$; $\boxed{f_{xy} = 6x^2y + 5y^4}$ → Mixed second partials.

$f_{yy} = 2x^3 + 20xy^3$; $\boxed{f_{yx} = 6x^2y + 5y^4}$

Ex ② $z = \sin xy^3$. Calculate the four second partial Derivatives

$f_x = y^3\cos xy^3$, $f_y = 3xy^2\cos xy^3$

$f_{xx} = -y^6\sin xy^3$

$f_{xy} = -y^3\cdot\sin xy^3\,(3xy^2) + 3y^2\cos xy^3$
$= -3xy^5\sin xy^3 + 3y^2\cos xy^3$

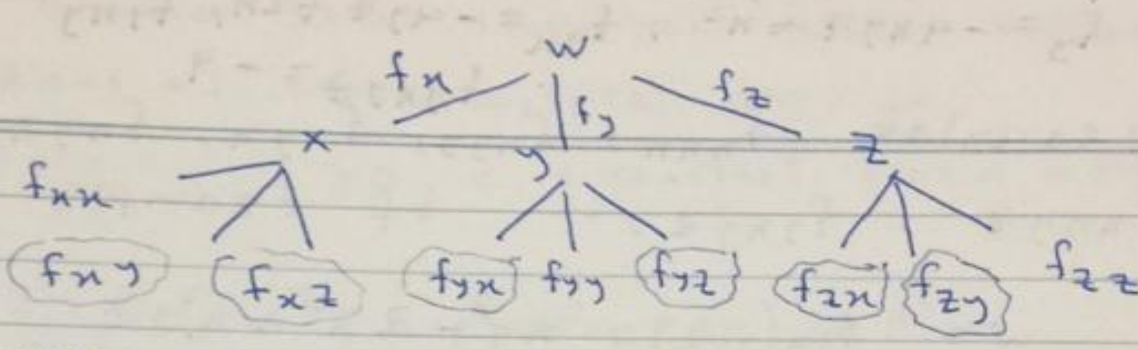
$f_{yx} = 3xy^2(-\sin xy^3)(y^3) + \cos xy^3\,(3y^2)$
$= -3xy^5\sin xy^3 + 3y^2\cos xy^3$

$f_{yy} = 3xy^2(-\sin xy^3)(3xy^2) + \cos xy^3\,(6xy)$
$= -9x^2y^4\sin xy^3 + 6xy\cos xy^3$

Theorem 1: Suppose that f, f_x, f_y, f_{xy} and f_{yx} are all continuous at (x_0, y_0). Then

$$f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$$

Remark: If $w = f(x,y,z)$, then we have the nine second partial derivatives



Theorem(2): If f, f_x, f_y, f_z and all six Mixed partials are continuous at apoint (x_0, y_0, z_0). Then at that point

$$f_{xy} = f_{yx} \quad ; \quad f_{xz} = f_{zx} \quad ; \quad f_{yz} = f_{zy}$$

Ex(3) Let $w = xy^3 - zx^5 + x^2yz$. calculate all second partial derivatives and show that all three pairs of mixed partials are equal.

Sol^n:

$f_x = y^3 - 5zx^4 + 2xyz$.

$f_y = 3xy^2 + x^2z$. ; $f_z = -x^5 + x^2y$

$f_{xx} = -20zx^3 + 2yz$; $f_{yy} = 6xy$; $f_{zz} = 0$

$f_{xy} = 3y^2 + 2xz$; $f_{yx} = 3y^2 + 2xz$; $f_{zx} = -5x^4 + 2xy$

$f_{xz} = -5x^4 + 2xy$; $f_{yz} = x^2$; $f_{zy} = x^2$

$f_{zx} = -5x^4 + 2xy$

we can define partial derivatives of orders higher than two, For example

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial x}(f_{xx})$$

$$f_{xyz} = \frac{\partial^3 f}{\partial z \partial y \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial}{\partial z}(f_{xy})$$

$$f_{xyy} = \frac{\partial^3 f}{\partial y \partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right), \cdots$$