SEQENCES OF REAL NUMBERS

According to a poplar dictionary, a sequence is " the following of one thing after another. "In mathematics we could define a sequence intuitively as a succession of numbers that never terminates. The numbers in the sequence are called the terms of the sequence .In a sequence there is one term for each positive integer

EXAMPLE 1 Consider the sequence

$$\begin{array}{cccccc} 1st & 2nd & 3rd & 4\,th & 5th & nth \\ \swarrow term & \swarrow term & \swarrow term & \swarrow term & \swarrow term & \swarrow term \\ \frac{1}{2}. & \frac{1}{4}. & \frac{1}{8}. & \frac{1}{16}. & \frac{1}{32}.\cdots. & \frac{1}{2^n}.\cdots \end{array}$$

We see that there is one term for each positive integer. The terms in this sequence form an infinite set of real numbers, which we write as

$$A = \left\{\frac{1}{2}.\ \frac{1}{4}.\ \frac{1}{8}\ .\cdots.\frac{1}{2^n}.\cdots\right\} \qquad (1)$$

That is, the set A consists of all numbers of the form $\frac{1}{2^n}$ where n is a positive integer There is another way to describe this set We define the function *f by* the rule $f\ \ (n) = \frac{1}{2^n}$ where the domain of *f* is the set of positive integers. Then the set A is precisely the set of values taken by the function *f*.

In general, we have the following formal definition.

Definition 1 SEQUENCE A sequence of real numbers is a function whose domain is the set of positive integers. The values taken by the function are called terms of the sequence.

NOTATION. We will often denote the terms of a sequence by a_n Thus if the function given in Definition 1 is *f*, then $a_n = f\ (n)$ with this notation, we can denote the set of values taken by the sequence by $\{a_n\}$ Also, we use *n*, *m*, and so on as integer variables and *x*, *y*, and so on as real variables.

EXAMPLE 2 : The following are sequences of real numbers:

$(a)\quad \{a_n\} = \left\{\frac{1}{n}\right\}$ $\qquad (b)\{\,a_n\} = \{\sqrt{n}\}$ $\qquad (c)\{\,a_n\} = \left\{\frac{1}{n!}\right\}$

$(d)\ \{a_n\} = \{\sin n\}$ $\qquad (e)\{a_n\} = \left\{\frac{e^n}{n!}\right\}$ $\qquad (f)\{a_n\} = \left\{\frac{n-1}{n}\right\}$

We sometimes denote a sequences by writing out the values$\{\,a_1.a_2.a_3.\cdots\}$.

EXAMPLE 3 We write out the values of the sequences in Example 2:

$(a)\left\{\ 1.\frac{1}{2}.\frac{1}{3}.\frac{1}{4}.\cdots.\frac{1}{n}.\cdots\right\}$

$(b)\ \{1.\sqrt{2}.\sqrt{3}.\sqrt{4}.\cdots.\sqrt{n}.\cdots\}$

(c) $\left\{\ 1.\frac{1}{2}.\frac{1}{6}.\frac{1}{24}.\cdots.\frac{1}{n!}.\cdots\right\}$

$(d)\{\sin 1.\sin 2.\sin 3.\sin 4.\cdots.\sin n.\cdots\}$

$(e)\ \left\{e.\frac{e^2}{2}.\frac{e^3}{3}.\frac{e^4}{4}.\cdots.\frac{e^n}{n!}.\cdots\right\}$

$(f)\left\{\ 0.\frac{1}{2}.\frac{2}{3}.\frac{3}{4}.\cdots.\frac{n-1}{n}.\cdots\right\}$

EXAMPLE 4 : Find the general term of the sequence

$$\{-1.1-1.1.-1.1.-1.\cdots\}$$

Solution. We see that$a_1=-1.a_2=1.a_3=-1a_4=1.\cdots$ Hence

$$a_2=\begin{cases}-1. & if\ n\ is\ odd\\ 1. & if\ n\ is\ even\end{cases}$$

A more concise way to write this term is: $a_2=(-1)^n$

It is evident that as n gets large, the numbers $1/n$ get small. We can write $\lim_{n\to\infty}\frac{1}{n}=0$

This is also suggested by the graph in Figure 1a Similarly, it is not hard to show that as n gets large, $\frac{(n-1)}{n}$ gets close to 1. We write : $\lim_{n\to\infty}\frac{n-1}{n}=1$

For the remainder of this section we will be concerned with calculating the limit of a sequence as $n\to\infty$ Since a sequence is a special type of function, our formal definition of the limit of a sequence is going to be very similar to the definition of $\lim_{n\to\infty} f(x)$

Definition 2 FINITE LIMIT OF ASEQUENCE A sequence $\{a_2\}$ † has limit L if for every $\epsilon>0$ there exists an integer $N>0$ such that if $n\geq N$ then$|a_2-L|<\in$ We

$$\operatorname*{Lim}_{n\to\infty} a_2=L \tag{2}$$

Intuitively, this definition states that if as n increases without bound, gets arbitrarily close to L

Definition 2 FINITE LIMIT OF A SEQUENCE The sequence $\{a_n\}$ has the limit ∞ if for every positive number M there is an integer $N > 0$ such that if $n > N$. then $a_n > M$ In this case we write $\lim_{n\to\infty} a_n = \infty$

Intuitively, $\lim_{n\to\infty} a_n = \infty$ means that as n increases without bound, also increases without bound.

The theorem below gives us a very useful result.

Theorem 1: Let r be a real number Then

$\operatorname{Lim}_{n\to\infty} r^n = 0 \qquad if \ |r| < 1$

And $\lim_{n\to\infty} |r^n| = \infty \qquad if \ |r| > 1$

Proof

Case 1 $r = 0$ Then $r^n = 0$. and the sequence has the limit 0

Case 2 $0 < |r| < 1$ For a given $\in > 0$. choose N such that $N > \frac{In \in}{In\,|r|}$

Note that since $|r| < 1$. $In\ |r| < 0\ Now\ if\ n > N$.

$$n > \frac{In\ \in}{In\,|r|} \qquad and \qquad n\,In\ |r| < \in$$

The second inequality follows from the fact that In |r | is negative, and multiplying both sides of an inequality by a negative number reverses the inequality. Thus

$$In\ |r^n - 0| = In\,|r^n| = In\ |r|^n = n\,In\ |r| < In\ \in$$

Since In$|r^n - 0| <$ In $\in$ and In x is an increasing function, we conclude that$|r^n - 0| < \in$ Thus according to the definition of a finite limit of a sequence , $\operatorname{Lim}_{n\to\infty} |r^n| = 0$

Case 3 $|r| >$ 1. Let $M > 0$ be given Choose$N > \frac{InM}{In|r|}$ Then if $n > N$.

$\swarrow n > N$

$$In\,|r^n| = n\ In\ |r| > \left(\frac{In\ M}{In\,|r|}\right)(In\,|r|) = In\ M\,.$$

So that $|r^n| > \mathrm{M}$ if $\mathrm{n} > \mathrm{N}$

Form the definition of an infinite limit of a sequence, we see that $\lim_{n\to\infty} |r^n| = \infty$

Theorem 2 Suppose that $\lim_{n\to\infty} a_n and \lim_{n\to\infty} b_n$ both exist and are finite.

$$(i)\lim_{n\to\infty} aa_n = a \lim_{n\to\infty} a_n \; for \; any \; number \; a \qquad (3)$$

$$(ii)\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \qquad (4)$$

$$(iii)\ \lim_{n\to\infty} a_n\, b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) \qquad (5)$$

$(iv) If \ \lim_{n\to\infty} b_n \neq 0 .\ then$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \qquad (6)$$

Theorem 3 Continuity Theorem Suppose that L is finite and $\lim_{n\to\infty} a_n = L \ \ if \ f \ is$ continuous in an open interval containing L , then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L) \qquad (7)$$

Theorem 4 Squeezing Theorem Suppose that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$ and that $\{ c_n\}$ is a sequence having the property that for $n > N$ (appositive integer), $a_n \leq c_n \leq b_n$ Then

$$\lim_{n\to\infty} c_n = L \qquad (8)$$

We now give a central definition in the theory of sequences

Definition 4 CONVERGENCE AND DIVERGENCE OF A SEQUENCE If the limit in (2) exists and if L is finite, we say that the sequence converges or is convergent Otherwise, we say that the sequence diverges or is divergent

EXAMPLE: The sequence $\left\{\frac{1}{2^n}\right\}$ is convergent since, by Theorem 1, $\lim_{n\to\infty} \frac{1}{2^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

EXAMPLE 6 The sequence $\{r^n\}$ is divergent for r since $\lim_{n\to\infty} r^n = \infty \, if \ r > 1$

EXAMPLE 7 The sequence $\{(-1)^n\}$ is divergent since the values a_n alternate between -1 and +1 but do not stay close to any fixed number as n becomes large

Theorem 5: Suppose that $\lim\limits_{x\to\infty} f(x) = L.$ a finite number, $\infty . or - \infty$ If f is defined for every positive integer, then the limit of the sequence $\{a_n\} = \{f\ (n)\}$ is also equal to L. That is $\lim\limits_{n\to\infty} f(x) = \lim\limits_{n\to\infty}\ a_n = L$

EXAMPLE 8 Calculate $\lim\limits_{n\to\infty} 1/n^2$

Solution Since $\lim\limits_{x\to\infty} \frac{1}{x^2} = 0.$ we have $\lim\limits_{n\to\infty} 1/n^2$ (by Theorem 5)

EXAMPLE 9 Does the sequence $\{\frac{e^n}{n}\}$converge or diverge ?

Solution Since $\lim\limits_{n\to\infty} \frac{e^x}{x} = \lim\limits_{n\to\infty} \frac{e^x}{1}$ (by L 'Hospital's rule)$= \infty$. we find that the sequence diverges

REMARK. It should be emphasized that Theorem 5 does not say that if does not exist, then diverges for example, let $f(x) = \sin \pi x$

Then $\lim\limits_{x\to\infty} f(x)$ does not exist, but $\lim\limits_{n\to\infty} f(n) = \lim\limits_{n\to\infty} \sin \pi n = 0$ since $\sin \pi n = 0$ for every integer n

EXAMPLE 10 Let$\{\ a_n\} = \{[1 + \left(\frac{1}{n}\right)]^n\}$ Does this sequence converge or diverge ? $y = \left(1 + \frac{1}{x}\right) x\ lny = x1n\left(1 + \frac{1}{x}\right)$

Solution Since $\lim\limits_{x\to\infty} [\ 1 + (\frac{1}{x})]^x = e.$ we see that a_n converges to the limit e

EXAMPLE 11 Determine the convergence or divergence of the sequence $\left\{\frac{In\ n}{n}\right\}$

Solution $\lim\limits_{x\to\infty} \left[\frac{In\ x}{x}\right] = \lim\limits_{x\to\infty} \left[(\frac{1}{x})/1\right] = 0$ by L' Hospital's rule, so that the sequence converges to 0 .

EXAMPLE 12 Let$p(x) = c_0 + c_1 x + \cdots + c_m x^m\ and\ \ q(x) = d_0 + d_1 x + \cdots + d_2 x^2$ In Problem 2.4.39 we showed that if the rational function $r(x) = \frac{p(x)}{q(x)}.\ then\ .if\ c_m d_r \neq 0.$

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} \begin{cases} 0. & if\ \ m < r \\ \pm\infty. & if\ m > r \\ \dfrac{c_m}{d_r}\ . & if\ m = r \end{cases}$$

Thus the sequence $\left\{\frac{p(n)}{q(n)}\right\}$ converges to 0 if $m < r$. converges to $\frac{c_m}{d_r}$ $if\ m = r\ .and$ diverges if $m > r$

EXAMPLE 13 Does the sequence$\{(5n^3 + 2n^2 + 1)(2n^2 + 3n + 4)$ converge or diverge ?

Solution Here $m = r = 3$. that by the result of Example 12 the sequence converges to $\frac{c_3}{d_3} = \frac{5}{2}$

EXAMPLE 14 Does the sequence$\{n^{\frac{1}{n}}\}$ converge or diverge?

Solution. Since $\lim_{x\to\infty} x^{1/x} = 1$ (see Example 12.3.7), the sequence converges to 1

EXAMPLE 15 Determine the convergence or divergence of the sequence$\left\{\sin\frac{an}{n^B}\right\}$ where a is a real number and $B > 0$

Solution. Since $-1 \leq \sin ax \leq 1$. we see that.

$$-\frac{1}{x^B} \leq \frac{\sin ax}{x^B} \leq \frac{1}{x^B} \quad for\ any\ \ x > 0$$

But$\pm \lim_{x\to\infty} \frac{1}{x^B} = 0$. and so by the squeezing theorem (Theorem 2.4.1), $\lim_{x\to\infty}[(\sin ax)/x^B] = 0$ Therefore the sequence$\{(\sin an)/n^B\}$ converges to 0

As in Example 15, the squeezing theorem can often be used to calculate the limit of a sequence.

PROBLMS

(I)find the first five terms of the given sequence.

$1 \cdot \left\{\frac{1}{3^n}\right\}$ $2 \cdot \left\{\frac{n+1}{n}\right\}$ $3 \cdot \left\{1 - \frac{1}{4^n}\right\}$

$4 \cdot \{\sqrt[3]{n}\}$ $5 \cdot \left\{e^{\frac{1}{n}}\right\}$ $6 \cdot \{n \cos n\}$

$7 \cdot \{\sin n\pi\}$ $8 \cdot \{\cos n\pi\}$ $9 \cdot \left\{\sin\frac{n\pi}{2}\right\}$

In problems 10 -27, determine whether the given sequence is convergent or divergent. If it is convergent, find its limit.

$10 \cdot \left\{\frac{3}{n}\right\}$ $11 \cdot \left\{\frac{1}{\sqrt{n}}\right\}$ $12 \cdot \left\{\frac{n+1}{n^{\frac{5}{2}}}\right\}$

$13 \cdot \{\sin n\}$ $14 \cdot \{\sin n\pi\}$ $15 \cdot \left\{\cos\left(\mathrm{n}+\frac{\pi}{2}\right)\right\}$

$16 \cdot \left\{\frac{n^5+3n^2+1}{n^6+4n}\right\}$ $17 \cdot \left\{\frac{4n^5-3}{7n^5+n^2+2}\right\}$ $18 \cdot \{\left(1+\frac{4}{n}\right)^n\}$

$19 \cdot \{\left(1+\frac{1}{4n}\right)^n\}$ $20 \cdot \left\{\frac{\sqrt{n}}{In\, n}\right\}$

$21 \cdot \{\sqrt{n+3}-\sqrt{n}\}$ $[Hint\!: Multiply\ \ and\ divide\ by\ \ \sqrt{n+3}+\sqrt{n}]$

$22 \cdot \left\{\frac{2^n}{n!}\right\}$ $23 \cdot \left\{\frac{a^n}{n!}\right\}(a\ real)$ $24 \cdot \left\{\frac{4}{\sqrt{n^2+3-n}}\right\}$

$25 \cdot \left\{\frac{(-1)^n\, n^3}{n^3+1}\right\}$ $26 \cdot \{(-1)^n \cos n\pi\}$ $27 \cdot \left\{\frac{(-1)^n}{\sqrt{n}}\right\}$

In Problems 28 -33, find the general term a_n of the given sequence.

$28 \cdot \{1.-2.3\,.-4.5.-6.\cdots\}$ $29 \cdot \{1.2\cdot 5.\ 3\cdot 5^2.4\cdot 5^3.5\cdot 5^4.\cdots\}$

$30 \cdot \left\{\frac{1}{2}.\frac{2}{3}.\frac{3}{4}.\frac{4}{5}.\frac{5}{6}.\cdots\right\}$ $31 \cdot \left\{\frac{1}{2}.\frac{3}{4}.\frac{7}{8}.\frac{15}{16}.\frac{31}{32}.\cdots\right\}$

$*\,32 \cdot \left\{\frac{1}{3}.\frac{2}{5}.\frac{3}{7}.\frac{4}{9}.\frac{5}{11}.\cdots\right\}$ $33 \cdot \left\{1-\frac{1}{3}.\frac{1}{9}.-\frac{1}{27}.\cdots\right\}$

34. Show that $a_n = [1+(\frac{a}{n})]^n$ converges to e^a