SERIES WITH NONNEGATIVE TERMS

TWO COMPARISON TESTS AND THE INTEGRAL TEST

In this section and the next we consider series of the form $\sum_{k=0}^{\infty} a_k$ where each a_k is nonnegative Such series are often easier to handle then others One fact is easy to prove The sequence $\{S_n\}$ of partial sums is a monotone increasing sequence since $S_{n+1} = S_n + a_{n+1}$ and$a_{n+1} \geq 0$ for every n Then if $\{S_n\}$ is bounded , it is convergent by Theorem 14 . 2.2, and we have the following theorem:

Theorem 1 An infinite series of nonnegative terms is convergent if and only if its sequence of partial sums is bounded

EXAMPLE 1 Show that $\sum_{k=1}^{\infty} 1/k^2$ is convergent .

Solution We group the terms as follows:

$$\sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \overbrace{\frac{1}{2^2} + \frac{1}{3^2}}^{2\ terms} + \overbrace{\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}}^{4\ terms} + \overbrace{\frac{1}{8^2} + \cdots + \frac{1}{15^2}}^{8 terms} + \cdots$$

$$\leq 1 + \overbrace{\frac{1}{2^2} + \frac{1}{2^2}}^{2\ terms} + \overbrace{\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}}^{4\ terms} + \overbrace{\frac{1}{8^2} + \cdots + \frac{1}{8^2}}^{8 terms} + \cdots$$

$$= 1 + \frac{2}{2^2} + \frac{4}{4^2} + \frac{8}{8^2} + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^k} = 2$$

Thus the sequence of partial sums is bounded by 2 and is therefore convergent

With Theorem 1 the convergence or divergence of nonnegative terms depends on whether or not its partial sums are bounded. There are several tests that can be used to determine whether or not the sequence of partial sums of a series is bounded We will deal with these one at a time

Theorem 2 Comparison Test Let$\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0$ for every k

(i) If there exists a convergent series $\sum_{k=1}^{\infty} b_k$ and a number N such that $a_k \leq b_k$ for every $k \geq N$. then$\sum_{k=1}^{\infty} a_k$ converges

(ii) If there exists a convergent series $\sum_{k=1}^{\infty} c_k$ and a number N such that $a_k \geq c_k \geq 0$ for every $k \geq N$. then$\sum_{k=1}^{\infty} a_k$ converges

Proof. In either case the sum of the first N terms is finite, so we need only consider the series since if this is convergent or divergent, then the addition of a finite number of terms does not affect the convergence or divergence

(i) $\sum_{k=N+1}^{\infty} b_k$ Is a nonnegative series (since $b_k \geq a_k \geq 0 \; for \; k > N$) and is convergent Thus the partial sums $T_n = \sum_{k=N+1}^{n} b_k$ are bounded If $S_n = \sum_{k=N+1}^{n} a_k$. then $S_n \leq T_n$. and so the partial sums of $\sum_{k=N+1}^{\infty} a_k$ bounded , implying that $\sum_{k=N+1}^{\infty} a_k$ is convergent

(ii) Let $U_n = \sum_{k=N+1}^{n} c_k$ By Theorem 1 these partial sums are unbounded since $\sum_{k=N+1}^{\infty} c_k$ diverges , Since in this case $S_n \geq U_n$. the partial sums of $\sum_{k=N+1}^{\infty} a_k$ are also unbounded , and the series $\sum_{k=N+1}^{\infty} a_k$ diverges

REMARK. One fact mentioned in the proof of (i) is important enough to state again: if for some positive integer N . $\sum_{k=N+1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges If $\sum_{k=N+1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges That is the addition of a finite number of terms does not affect convergence or divergence

EXAMPLE 2 Determine whether $\sum_{k=1}^{\infty} 1/\sqrt{k}$ converges or diverges

Solution Since $\frac{1}{\sqrt{k}} \geq \frac{1}{k} for \; k \geq 1$. and since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, we see that by the comparison test $\sum_{k=1}^{\infty} 1/\sqrt{k}$ diverges

EXAMPLE 3 Determine whether $\sum_{k=1}^{\infty} 1/k!$ converges or diverges

Solution. If $k \geq 4. k! \geq 2^k$ to see this note that $4! = 24 \; and \; 2^4 = 16$ Then $5! = 5 \cdot 24 \; and \; 2^5 = 2 \cdot 16$ and since $5 > 2 . 5! > 2^5$ .and so on Then since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, we see that $\sum_{k=1}^{\infty} 1/k!$ converges in fact, as we will show in Section 14.9, it converges to $e - 1$ That is,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \qquad (1)$$

Theorem 3 The Integral Test Let f be a function that is continuous, positive, and decreasing for all $x \geq 1$ Then the series

$$\sum_{k=1}^{\infty} f(k) = f(1) + f(2) + f(3) + \cdots + f(n) + \cdots \qquad (2)$$

Converges if $\int_1^n f(x)dx$ converges , and diverges if $\int_1^n f(x)dx \to \infty \; as \; n \to \infty$

Proof The idea behind this proof is fairly easy. Take a look at Figure 1. Com paring areas, we immediately see that

$$f(2)+f(3)+\cdots+f(n) \le \int_1^n f(x)dx \le f\ (1)+f(2)+\cdots+f(n-1)$$

If $\lim_{n\to\infty}\int_1^n f(x)dx$ finite, then the partial sums[$f\ (2)+f\ (3)+\cdots+f(n)$] bounded and the series converges . On the other hand if $\lim_{n\to\infty}\int_1^n f(x)dx=\infty$. then partial sums [$f\ (1)+f(2)+\cdots+f(n-1)$] are unbounded and the series diverges

EXAMPLE 4 Consider the series $\sum_{k=1}^{\infty} 1/k^a$ with $a>0$We have already seen that this series diverges for $a=1$(the harmonic series) and converges for$a=2$ (Example 1). Now let

$$f(x)=\frac{1}{x^a} \quad Then\ \ for\ \ a\neq 1$$

$$\int_1^n f(x)dx=\int_1^n \frac{1}{x^a}dx=\frac{x^{1-a}}{1-a}\int_1^n=\frac{1}{1-a}(n^{1-a}-1)$$

This last expression converges to$\frac{1}{a-1}$ $if\ \ a>1$ and diverges if $a<1$ For $a=1$

$\int_1^n f(x)dx=\int_1^n \frac{1}{x}dx=\ In\ x\int_1^n=In\ n$.

Which diverges (This is another proof that the harmonic series diverges) Hence

$$\sum_{k=1}^{\infty} 1/k^a \ \begin{cases} diverges\ if\ a\ \le 1\ . \\ converges\ if\ a>1 \end{cases}$$

EXAMPLE 5 Determine whether$\sum_{k=1}^{\infty}(In\ \ k)/k^2$ converges or diverges

Solution. We easily see, using L' Hospital's rule, that

$$\lim_{x\to\infty}\frac{In\ x}{\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}=2\lim_{x\to\infty}\frac{\sqrt{x}}{x}=2\lim_{x\to\infty}\frac{1}{\sqrt{x}}=0\ .$$

So that fork sufficiently large , $In\ k\ \le\sqrt{k}$ Thus

$$\frac{In\ k}{k^2}\ \le\ \frac{\sqrt{k}}{k^2}=\frac{1}{k^{\frac{3}{2}}}$$

But $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ converges by the result of Example 4 , and therefor by the comparison test $\sum_{k=1}^{\infty}(In\ k)/k^2$ also converges

NOTE The intergyral test can also be used directly here since $\int_1^n \frac{lnx}{x^2} \quad dx$. can be integrated by parts with $u = In\ x$

EXAMPLE 6 Determine whether $\sum_{k=1}^{\infty} 1/[k\ In\ (k+5)]$ converges or diverges

Solution First , we note that $1/[k\ In\ (k+5)] > 1/[k\ In\ (k+5)In\ (k+5)]$ Also,

$\int_1^n \frac{dx}{(x+5)In\ (x+5)} = In\ In\ (x+5) \int_1^n = InIn\ (n+5) - InIn\ 6.$

Which diverges , so that$\sum_{k=1}^{\infty} 1/[k\ In\ (k+5)]$ also diverges

We now give another test that is an extension of the comparison test

PROBLEMS

In problems 1-33 , determine the convergence or divergence of the given series

1 · $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ 2 · $\sum_{k=10}^{\infty} \frac{1}{k(k-3)}$ 3 · $\sum_{k=4}^{\infty} \frac{1}{5k+50}$

4 · $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+2k}}$ 5 · $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+1}}$ 6 · $\sum_{k=1}^{\infty} \frac{In\ k}{k^3}$

7 · $\sum_{k=2}^{\infty} \frac{1}{k^2+1}$ 8 · $\sum_{k=2}^{\infty} \frac{4}{k\ In\ k}$ 9 · $\sum_{k=0}^{\infty} ke^{-k}$

10 · $\sum_{k=3}^{\infty} k^2e^{-k}$ 11 · $\sum_{k=5}^{\infty} \frac{1}{k(In\ k)^3}$ 12 · $\sum_{k=4}^{\infty} \frac{1}{k^2\sqrt{In\ k}}$

13 · $\sum_{k=2}^{\infty} \frac{1}{(3k\ -1\ )^{\frac{3}{2}}+1}$ 14 · $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+3}}$ 15 · $\sum_{k=2}^{\infty} \frac{1}{k\sqrt{In\ k}}$

16 · $\sum_{k=1}^{\infty} \frac{1}{50+\sqrt{k}}$ 17 · $\sum_{k=2}^{\infty} (\frac{k}{k+1})^k$ 18 · $\sum_{k=1}^{\infty} (\frac{k}{k+1})^{\frac{1}{k}}$

19 · $\sum_{k=4}^{\infty} \frac{1}{k\ In\ \ In\ k}$ 20 · $\sum_{k=1}^{\infty} \sin\frac{1}{k}$ 21 · $\sum_{k=1}^{\infty} \frac{1}{(k+2)\sqrt{In(k+1)}}$

$22 \cdot \sum_{k=1}^{\infty} \frac{e^{\frac{1}{k}}}{k^2}$ $\quad 23 \cdot \sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$ $\quad 24 \cdot \sum_{k=1}^{\infty} \operatorname{sech} k$

$25 \cdot \sum_{k=10}^{\infty} \frac{1}{k(Ink)(InIn\,k)}$ $\quad 26 \cdot \sum_{k=1}^{\infty} \frac{k^2}{50k}$ $\quad 27 \cdot \sum_{k=1}^{\infty} \frac{1}{cosh^2 k}$

$28 \cdot \sum_{k=1}^{\infty} \tan^{-1} k$ $\quad *29 \cdot \sum_{k=1}^{\infty} \frac{k^2}{50k}$ $\quad 30 \cdot \sum_{k=1}^{\infty} \frac{\sqrt{k}}{3k^2+2k+20}$

$31 \cdot \sum_{k=1}^{\infty} \frac{(k+1)^{\frac{7}{8}}}{k^3+k^2+3}$ $\quad 32 \cdot \sum_{k=1}^{\infty} \frac{1}{(k+1)In(k+1)}$ $\quad 33 \cdot \sum_{k=1}^{\infty} \frac{k^5+2k^4+3k+7}{k^6+3k^4+2k^2+1}$

THE RATIO AND ROOT TESTS

In this section we discuss two more tests that can be used to determine whether an infinite series converges or diverges. The first of these, the ratio test, is useful in a wide variety of applications

Theorem 1 The Ratio Test Let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k > 0$ for every k, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \qquad (1)$$

$(i) If\ \ L < 1 . \sum_{k=1}^{\infty} a_k \ \ converges$

$(ii) If\ \ L > 1 . \sum_{k=1}^{\infty} a_k \ \ diverges$

$(iii) If\ \ L = 1 . \sum_{k=1}^{\infty} a_k\ may\ \ converge\ or\ diverge\ and\ \ the\ ratio\ \ test\ is\ inconclusive;$

Some other test must be used

Proof

(i) Pick $\in > 0$ such that L+ $\in < 1$ By the definition of the limit in (1), there is a number $N > 0$ such that if $n \geq N$ we have

$\frac{a_{n+1}}{a_n} < L + \in$

Then $a_{n+1} < a_n(L{+}\in)$. $\quad a_{n+2} < a_{n+1}\ (L{+}\in) < a_n\ (L{+}\in)^2$

And $a_{n+k} < a_n\ (L{+}\in)^k \qquad (2)$

For each $k \geq 1$ and each $n \geq N$ In particular, for $k \geq N$ we use (2) to obtain

$$a_k = a_{(k-N)+N} \leq a_N \,(L{+}\in)^{k-N}$$

Then $S_n = \sum_{k=N}^{n} a_k \leq \sum_{k=N}^{n} a_N \,(L{+}\in)^{k-N} = \frac{a_N}{(L+\in)^N} \sum_{k=N}^{n} (L{+}\in)^k$

But since $L{+}\in < 1 . \sum_{k=0}^{n} (L{+}\in)^k = 1/[1-(L{+}\in)]$ (since this last sum is the sum of a geometric series)Thus

$S_n \leq \frac{a_N}{(L+\in)^N} \cdot \frac{1}{1-(L+\in)}$

And so the partial sums of $\sum_{k=N}^{\infty} a_k$ are bounded , implying that $\sum_{k=N}^{\infty} a_k$ converges
Thus $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{\infty} a_k$ also converges

(ii) If $1 < L < \infty$ pick such that $L -> 1$ Then for $n \geq N.$ the same proof as before (with the inequalities reversed) shows that

$a_k \geq a_N \,(L{+}\in)^{k-N}$

And that $S_n = \sum_{k=N}^{\infty} a_k > \frac{a_N}{(L+\in)^N} \sum_{k=N}^{n} (L{-}\in)^k$

But since $L - \in > \sum_{k=N}^{n} (L{+}\in)^k$ diverges, so that the partial sums are unbounded and $\sum_{k=N}^{\infty} a_k$ diverges The proof in the case $L = \infty$ is suggested in problem 33

(iii) To illustrate (iii), we show that $L = 1$ can occur for converging or diverging series

(a) The harmonic series $\sum_{k=1}^{\infty} 1/k$ diverges But

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

(b) The series $\sum_{k=1}^{\infty} 1/k^2$ converges Here

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \left(\frac{n}{n+1}\right)^2 = 1$$

REMARK. The ratio test is very useful But in those cases where$L = 1$. we must try another test , to determine whether the series converges or diverges

EXAMPLE 1 We have used the comparison test to show that converges Using the ratio test, we find that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

So that the series converges

EXAMPLE 2 Determine whether the series $\sum_{k=0}^{\infty}(100)^k /k!$ converges or diverges

Solution Here

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(100)^{n+1}}{(n+1)!}}{\frac{(100)^n}{n!}} = \lim_{n\to\infty} \frac{100}{n+1} = 0$$

So that the series diverges

EXAMPLE 3 Determine whether the series $\sum_{k=1}^{\infty} k^k /k!$ converges or diverges

Solution

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\lim_{n\to\infty}\left(\left[\frac{(n+1)^{n+1}}{(n+1)!}\right]\right)}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1\,.$$

So that the series diverges

EXAMPLE 4 Determine whether the series $\sum_{k=1}^{\infty}(k+1)/[k(k+2)]$ converges or diverges

Solution Here

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+2}{(n+1)(n+3)}}{\frac{n+1}{n(n+2)}} = 1$$

Thus the ratio test fails. However $\lim_{n\to\infty} [(k+1)/k(k+2)/(1/k) = 1 \; so \;\; that \; \sum_{k=1}^{\infty}((k+1)/k(k+2))/(1/k)]$ diverges by the limit comparison test

Theorem 2 The Root Test Let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k > 0$ and suppose that $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = R$

$(i) If \ R < 1. \sum_{k=1}^{\infty} a_k \ converges$

$(ii) If \ R > 1. \sum_{k=1}^{\infty} a_k \ diverges$

$(i) If \ R = 1. \sum_{k=1}^{\infty} a_k \ the \ series \ either \ convergesor \ diverges \ . and \ no \ conclusions$

Can be drawn from this test

EXAMPLE 5 Determine whether $\sum_{k=2}^{\infty} 1/(In\ k)^k$ converges or diverges

Solution Note first that we start at $k = 2$ since $1/(In1)^1$ is not defined

$$\lim_{n\to\infty} \left[\frac{1}{(In\ n)^n}\right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{In\ n} = 0$$

So that the series converges

EXAMPLE 6 Determine whether the series $\sum_{k=1}^{\infty} (\frac{k^k}{3^{4k+5}})$ converges or diverges

Solution $\lim_{n\to\infty} (n^n/3^{4+5})^{\frac{1}{n}} = \lim_{n\to\infty} (n/3^{4+5}) = \infty. since \lim_{n\to\infty} 3^{4+5} = 3^4 = 81$

Thus the series diverges

EXAMPLE 7 Determine whether the series $\sum_{k=1}^{\infty} (\frac{1}{2} + \frac{1}{k})^k$ converges or diverges

Solution : $\lim_{n\to\infty} \left[\left(\frac{1}{2} + \frac{1}{n}\right)^n\right]^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n}\right) = \frac{1}{2} < 1.$

So that series converges

PROBLEMS

In problems 1- 25, determine whether the given series converges or diverges

$1\cdot\sum_{k=1}^{\infty}\frac{2^k}{k^2}$ $2\cdot\sum_{k=1}^{\infty}\frac{5^k}{k^5}$ $3\cdot\sum_{k=1}^{\infty}\frac{r^k}{k^r}\ 0<r<1$

$4\cdot\sum_{k=1}^{\infty}\frac{r^k}{k^r}\ r>1$ $5\cdot\sum_{k=1}^{\infty}\frac{k!}{k^k}$ $6\cdot\sum_{k=1}^{\infty}\frac{k^k}{(2k)!}$

$7\cdot\sum_{k=1}^{\infty}\frac{e^k}{k^5}$ $8\cdot\sum_{k=1}^{\infty}\frac{e^k}{k!}$ $9\cdot\sum_{k=1}^{\infty}\frac{k^{\frac{2}{3}}}{10^k}$

$10\cdot\sum_{k=1}^{\infty}\frac{3^k+k}{k!+2}$ $11\cdot\sum_{k=1}^{\infty}\frac{k}{(In\ k)^k}$ $12\cdot\sum_{k=1}^{\infty}\frac{4^k}{k^3}$

$13\cdot\sum_{k=2}^{\infty}\left(1+\frac{1}{k}\right)^k$ $14\cdot\sum_{k=1}^{\infty}\frac{\sqrt{k}In\ k}{k^3+1}$ $15\cdot\sum_{k=1}^{\infty}\frac{3^{4k+5}}{k^k}$

$16\cdot\sum_{k=1}^{\infty}\frac{a^{mk+b}}{k^k}\ \ a>1\ .b\ real$ $17\cdot\sum_{k=1}^{\infty}\frac{k^k}{a^{mk+b}}\ \ a>1\ .b\ real$ $18\cdot\sum_{k=1}^{\infty}\frac{k^6 5^k}{(k+1)!}$

$19\cdot\sum_{k=1}^{\infty}\frac{k^2 k!}{(2k)!}$ $20\cdot\sum_{k=1}^{\infty}\frac{(2k)!}{k^2 k!}$ $*\,21\cdot\sum_{k=1}^{\infty}\left(\frac{k!}{k^k}\right)^k$

$22\cdot\sum_{k=1}^{\infty}\left(\frac{k^k}{k!}\right)^k$ $23\cdot\sum_{k=2}^{\infty}\frac{e^k}{(In\ k)^k}$ $24\cdot\sum_{k=1}^{\infty}\frac{(In\ k)^k}{k^2}$

$25\cdot\sum_{k=1}^{\infty}\left(\frac{k}{3k+2}\right)^k$

26. Show that $\sum_{k=0}^{\infty}\frac{x^k}{k!}$ converges for every real number x

27. Show that if $a_n^{\frac{1}{n}}\to 1$. then $\sum_{k=1}^{\infty}a_k$ may converge or diverge [Hint : Consider $\sum 1/k$ and $\sum 1/k^2$]

28. Prove that $\frac{k!}{k^k}\to 0\ \ as\ k\to\infty$

29.Let $a_k=3/k^2$ if even and $a_k=1/k^2$ is odd Show that $\lim_{n\to\infty}(a_{n+1}/a_n)$ does not exist , but $\sum_{k=1}^{\infty}a_k$ converges

30. Construct a series of positive terms for which $\lim_{n\to\infty}(a_{n+1}/a_n)$ does not exist but for which $\sum_{k=1}^{\infty} a_k$ diverges

31. Prove that if $a_n > 0$ and $\lim_{n\to\infty}(a_{n+1}/a_n) = \infty$. then $\sum_{k=1}^{\infty} a_k$ diverges [Hint : Show that for N sufficiently large . $a_k \geq 2^{k-N} a_n$]