INFINITE SERIES

we defined the geometric series $\sum_{k=0}^{\infty} r^k$ and showed that if $|r| < 1$. series converges to $1/(1-r)$ Let us again look at what we did If S_n denotes the sum of the first $n+1$ terms of the geometric series then

$$S_n = 1 + r + r^2 + \cdots r^n = \frac{1 - r^{n+1}}{1 - r} \;.\; r \neq 1 \tag{1}$$

For each n we obtain the number S_n . and therefore we can define a new sequence $\{S_n\}$ to be the sequence of partial sums of the geometric series If $|r| < 1$ then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}$$

That is , the convergence of the geometric series is implied by the convergence of the sequence of partial sums $\{S_n\}$

We now give a more general definition of these concepts

Definition 1 INFNITE SERIES

Let be a sequence Then the infinite sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{2}$$

Is called an infinite series (or , simply , series) Each a_k in (2) is called a term of the series The partial sums of the series are given by

$$S_n = \sum_{k=1}^{n} a_k$$

The term is called the n th partial sum of the series If the sequence of partial sums $\{S_n\}$ converges to L , then we say that the infinite series $\sum_{k=1}^{\infty} a_k$ converges to L and we write

$$\sum_{k=1}^{\infty} a_k = L \tag{3}$$

Otherwise, we say that the series $\sum_{k=1}^{\infty} a_k$ diverges

REMARK Occasionally a series will be written with the first term other than a_1 for example,$\sum_{k=0}^{\infty}(\frac{1}{2})^k$ *and* $\sum_{k=2}^{\infty} 1/(In\ k)$ are both examples of infinite series. In the second case we must start with $k = 2$ since $1/(In\ 1)$ is not defined

EXAMPLE 1 We can write the number 1/3 as

$$\frac{1}{3} = 0 \cdot 33333 \cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots + \frac{3}{10^n} + \cdots \qquad (4)$$

This expression is an infinite series Here$a_n = \frac{3}{10^n}$ and

$$S_n = \frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n} = \overbrace{0 \cdot 333 \cdots 3}^{n\ placee}$$

we can formally prove that this sum converges by noting that

$$S = \frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right) = \frac{3}{10}\sum_{k=0}^{\infty}\left(\frac{1}{10}\right)^k$$

$\swarrow$ *By Theorem* $14 \cdot 3 \cdot 2$

$$= \frac{3}{10}\left[\frac{1}{1 - \left(\frac{1}{10}\right)}\right] = \frac{3}{10}\left(\frac{1}{\frac{9}{10}}\right) = \frac{3}{10} \cdot \frac{10}{9} = \frac{3}{9} = \frac{1}{3}$$

As a matter of fact, any decimal number x can be thought of as a convergent infinite series, for if $x = 0\ \ a_1 a_2 a_3 \cdots a_n \cdots$. Then

$$x = \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + \cdots + \frac{a_n}{10^n} + \cdots = \sum_{k=1}^{\infty} \frac{a_k}{10^k} \quad \dagger$$

EXAMPLE 2 Express the repeating decimal 0.123123123 …. as a rational number (the quotient of two integers)

Solution : $0 \cdot 123123123 \cdots = 0 \cdot 123 + 0 \cdot 000123 + 0 \cdot 000000123 + \cdots$

$$= \frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \cdots = \frac{123}{10^3}\left[1 + \frac{1}{10^3} + \frac{1}{(10^3)^2} + \cdots\right]$$

$$= \frac{123}{1000}\sum_{k=0}^{\infty}\left(\frac{1}{1000}\right)^k = \frac{123}{1000}\left[\frac{1}{1 - \left(\frac{1}{1000}\right)}\right] = \frac{123}{1000} \cdot \frac{1}{\frac{999}{1000}}$$

†Since $0 \le a_k < 10$.

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} < \sum_{k=1}^{\infty} \frac{10}{10^k} = \sum_{k=1}^{\infty} \frac{1}{10^{k-1}} = 1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \cdots = \frac{1}{1 - \frac{1}{10}} = \frac{10}{9}$$

Once we have this test, the inequality given above implies that $\sum_{k=1}^{\infty} (\frac{a_k}{10^k})$ converges

$$= \frac{123}{1000} \cdot \frac{1000}{999} = \frac{123}{999} = \frac{41}{333}$$

In general, we can use the geometric series to write any repeating decimal in the form of a fraction by using the technique of Example 1 or 2 In fact , the rational numbers are exactly those real numbers that can be written as repeating decimals Repeating decimals include numbers like $3 = 3 \cdot 00000 \cdots and \frac{1}{4} = 0 \cdot 25 = 0 \cdot 25000000 \cdots$

EXAMPLE 3 Telescoping Series Consider the infinite series$\sum_{k=1}^{\infty} \frac{1}{k}(1+1)$ We write the first three partial sums:

$$S_1 = \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = 1 - \frac{1}{2}.$$

$$S_2 = \sum_{k=1}^{2} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = 1 - \frac{1}{3}.$$

$$S_3 = \sum_{k=1}^{3} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = 1 - \frac{1}{4}.$$

We can use partial fractions to rewrite the general term as

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

From which we can get a better view of the n th partial as

$$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}.$$

Because all other terms cancel. Since $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left\{1-\left[\frac{1}{n-1}\right]\right\} = 1$. we see that

$\sum_{k=1}^{\infty}\frac{1}{k}(1+1) = 1$

When, as here a terms cancel, we say that the series is a telescoping series

REMARK. Often, it is not possible to calculate the exact sum of an infinite series even if it can be shown that series converges

EXAMPLE 4 Consider the series

$$\sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots \qquad (5)$$

This series is called the harmonic series Although$a_n = \frac{1}{n} \to 0 as\ n \to \infty$ it is not difficult to show that the harmonic series diverges To see this, we write

$$\sum_{k=1}^{z}\frac{1}{k} = 1+\frac{1}{2}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>}^{2terms}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>}^{4terms}+\underbrace{\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)}_{>}^{8terms}+\cdots$$

Here we have written the terms in groups containing 2^n numbers Note that$\frac{1}{3}+\frac{1}{4} > \frac{2}{4} = \frac{1}{2}.\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}.$ and so on Thus $\sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{2}+\cdots.$ and the series diverges

WARNING. Example 4 clearly shows that even though the sequence $\{a_n\}$ converges to 0, the series may, in fact, diverge. That is, if $a_n \to 0$. then$\sum_{k=1}^{\infty} a_k$ may or may not converge. Some additional test is needed to determine convergence or divergence

Theorem 1 Let c be constant. Suppose that$\sum_{k=1}^{\infty} a_k$ and$\sum_{k=1}^{\infty} b_k$ both converge Then$\sum_{k=1}^{\infty}(a_k + b_k)\ and\ \sum_{k=1}^{\infty} ca_k$ converge, and

$$(i)\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \qquad (6)$$

$$(ii) \ \sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k \qquad (7)$$

This theorem should not be surprising Since the sum in a series is the limit of a sequence (the sequence of partial sum) the first part, for example, simply restates the fact that the limit of the sum is the sum of the limits.

Proof

(i) Let $S = \sum_{k=1}^{\infty} a_k$ and $T = \sum_{k=1}^{\infty} b_k$ The partial sums are given by $S_n = \sum_{k=1}^{n} a_k$ and $T_n = \sum_{k=1}^{n} b_k$ Then

$\sum_{k=1}^{\infty}(a_k + b_k) = \lim_{n\to\infty} \sum_{k=1}^{\infty}(a_k + b_k) = \lim_{n\to\infty} (\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k) = \lim_{n\to\infty}(S_n + T_n)$

$= \lim_{n\to\infty} S_n + \lim_{n\to\infty} T_n = S + T = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{n} b_k$

$$(ii) \sum_{k=1}^{\infty} ca_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} ca_k = \lim_{n\to\infty} c \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} c\ S_n$$

$$= c \lim_{n\to\infty} S_n = cS = c\sum_{k=1}^{\infty} a_k$$

EXAMPLE 5 Show that $\sum_{k=1}^{\infty}\{ \left[\frac{1}{k(k+1)}\right] + (\frac{5}{6})^k\}$ converges

Solution This follows since $\sum_{k=1}^{\infty} \frac{1}{k}(1+1)$ converges (Example 3) and $\sum_{k=1}^{\infty}(\frac{5}{6})^k$ converges because $\sum_{k=1}^{\infty}(\frac{5}{6})^k = \sum_{k=0}^{\infty}(\frac{5}{6})^k - (\frac{5}{6})^0$ [we added and subtracted the term $\left(\frac{5}{6}\right)^0 = 1$] $= 1/(1-\frac{5}{6}) - 1 = 5$

EXAMPLE 6 Does $\sum_{k=1}^{\infty} \frac{1}{50} k$ converge or diverge ?

Solution. We show that the series diverges by assuming that it converges to obtain a contradiction If Does $\sum_{k=1}^{\infty} \frac{1}{50} k$ did converge , then Does $50 \ \sum_{k=1}^{\infty} \frac{1}{50} k$ would also converge by Theorem 1. But then $50 \ \sum_{k=1}^{\infty} \frac{1}{50} k = \sum_{k=1}^{\infty} 50 \cdot \frac{1}{50} k = \sum_{k=1}^{\infty} 1/k$ and this series is the harmonic series , which we know diverges Hence $\sum_{k=1}^{\infty} \frac{1}{50} k$ diverges

Another useful test is given by the following theorem and corollary

Theorem 2 If $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{n\to\infty} a_n = 0$

Proof. Let $S = \sum_{k=1}^{\infty} a_k$ Then the partial sums S_n and S_{n-1} are given by

$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_{n-1} + a_n$

And $S_{n-1} = \sum_{k=1}^{n-1} a_k = a_1 + a_2 + \cdots + a_{n-1}$

So that $S_n - S_{n-1} = a_n$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$

We have already seen that the converse of this theorem is false. The convergence of $\{a_n\}$ to 0 does not imply that $\sum_{k=1}^{\infty} a_k$ converges For example, the harmonic series does not converge, but the sequence $\{\frac{1}{n}\}$ does converge to zero

Corollary

If $\{a_n\}$ does note converge to 0, then $\sum_{k=1}^{\infty} a_k$ diverges

EXAMPLE 7 $\sum_{k=1}^{\infty} (-1)^k$ diverges since the sequence does not converge to zero

EXAMPLE 8 $\sum_{k=1}^{\infty} k(k+100)$ diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+100} = 1 \neq 0$

PROBLEMS

In problems 1-15, a convergent infinite series is given Find its sum

1 · $\sum_{k=0}^{\infty} \frac{1}{4^k}$ 2 · $\sum_{k=0}^{\infty} \left(-\frac{2}{3}\right)^k$ 3 · $\sum_{k=2}^{\infty} \frac{1}{2^k}$

4 · $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ 5 · $\sum_{k=-3}^{\infty} \frac{1}{2^{k+3}}$ 6 · $\sum_{k=3}^{\infty} \left(\frac{2}{3}\right)^k$

7 · $\sum_{k=0}^{\infty} \frac{100}{5^k}$ 8 · $\sum_{k=0}^{\infty} \frac{5}{100^k}$ 9 · $\sum_{k=2}^{\infty} \frac{1}{k(k+1)}$

10 · $\sum_{k=3}^{\infty} \frac{1}{k(k-1)}$ 11 · $\sum_{k=0}^{\infty} \frac{1}{k(k+1)(k+2)}$ 12 · $\sum_{k=-1}^{\infty} \frac{1}{k(k+3)(k+4)}$

13 · $\sum_{k=2}^{\infty} \frac{2^{k+3}}{3^k}$ 14 · $\sum_{k=2}^{\infty} \frac{2^{k+4}}{3^{k-1}}$ 15 · $\sum_{k=4}^{\infty} \frac{5^{k-2}}{6^{k+1}}$

In Problems 16-24, write the repeating decimals as rational numbers

$16 \cdot 0 \cdot 666 \cdots$ $17 \cdot 0 \cdot 353535 \cdots$ $18 \cdot 0 \cdot 282828 \cdots$

$19 \cdot 0 \cdot 717171 \cdots$ $20 \cdot 0 \cdot 214214214 \cdots$ $21 \cdot 0 \cdot 501501501 \cdots$

$22 \cdot 0 \cdot 124242424 \cdots$ $23 \cdot 0 \cdot 11362362362 \cdots$ $24 \cdot 0 \cdot 513651365136 \cdots$

25. Give a new proof , using the corollary to Theorem 2 , that the geometric series diverges if $|r| \geq 1$

In problems 26 – 30, use theorem 1 to calculate the sum of the convergent series

$26 \cdot \sum_{k=0}^{\infty}[\frac{1}{2^k} + \frac{1}{5^k}]$ $27 \cdot \sum_{k=1}^{\infty}\left[\frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}\right]$

$28 \cdot \sum_{k=0}^{\infty}[\frac{3}{5^k} + \frac{7}{5^k}]$ $29 \cdot \sum_{k=1}^{\infty}[\frac{8}{5^k} + \frac{7}{(k+3)(k+4)}]$

$30 \cdot \sum_{k=0}^{\infty}[\frac{12 \cdot 2^{k+1}}{3^{k-2}} + \frac{15 \cdot 3^{k+1}}{4^{k+2}}]$

31. Show that for any nonzero real numbers a andb_1 $\sum_{k=1}^{\infty} a/bk$ diverges

32. Show that if the sequences$\{a_k\}$ $and\{b_k\}$ differ only for a finite number of terms, then$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ and either both converge or both diverge

38. Show that $\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k\ x^{2k}\ if\ |x| < 1$

39. Picka_0 anda_1 For$n \geq 2$. compute a_n recursively so that $n(n-1)\ a_n = (n-2)\ a_{n-1} - (n-3)\ a_{n-2}$ Evaluate$\sum_{n=0}^{\infty} a_n$