BOUNDED AND MONTONIC SEQUENCES

There are certain kinds of sequences that have special properties worthy of mention.

Definition 1 BOUNDEDNESS

(i) The sequence $\{a_n\}$ is bounded above if there a number M_1 such that

$$a_n \leq M_1 \tag{1}$$

For every positive integer n

(ii) It is bounded below if there is number M_2 such that

$$M_2 \leq a_2 \tag{2}$$

For every positive integer n.

(iii) It is bounded below if there is number $M > 0$ such that

$$|a_n| \leq M$$

For every positive integer n

The numbers M_1 , M_2 , and M are called ,respectively , an upper bounded , a lower bounded , and abound for $\{a_n\}$

(iv) If the sequence is not bounded, it is called unbounded

REMARK. If $\{a_n\}$ is bounded above and below, then it is bounded. Simply set $M = \max\{|M_1|.|M_2\}$

EXAMPLE 1 The sequence $\{\sin n\}$ has the upper bound of 1, the lower bound of -1 and the bound of 1 since $-1 \leq \sin n \leq 1$ for every n Of course, any number greater than 1 is also a bound

EXAMPLE 2 The sequence$\{(-1)^n\}$ has the upper bound 1, the lower bound -1 , and the bound 1.

EXAMPLE 3 The sequence$(2)^n$ is bounded below by 2 but has no upper bound and so is un bounded .

EXAMPLE 4 The sequence$\{(-1)^n 2^n\}$ is bounded neither below nor above

It turns out that the following statement is true: Every convergent sequence is abounded.

Theorem 1. If the sequence $\{a_n\}$ is convergent, then it is bounded

Proof. Before giving the technical details, we remark that the idea behind the proof is easy. For if $\lim_{n\to\infty} a_n = L$. then a_n is close to the finite number L if n is large Thus , for example $|a_n| \leq |L| + 1$ if n is large enough . Since a_n is a real number for every n, the first few terms of the sequence are bounded , and these two facts give us abound for the entire sequence

Now to the details Let $\epsilon = 1$ Then there is an $N > 0$ such that

$$|a_n - L| < 1 \qquad \text{if } n \geq N \tag{3}$$

Let

$$K = \max\{|a_1|.|a_2|.\cdots.|a_N|\} \tag{4}$$

Since each is finite, K, being the maximum of a finite number of terms, is also finite. Now let

$$M = \max\{\,|L| + 1. K\} \tag{5}$$

It follows from (4) that if $n \leq N$. then $|a_n| \leq K$ if $n \geq N$ then from(3) , $|a_n| < |L| + 1$; so in either case $|a_n| \leq M$. and the theorem is proved

Sometimes it is difficult to find a bound for a convergent sequence.

EXAMPLE 5 Find an M such that $\frac{5^n}{n!} \leq M$

Solution. We know from Theorem 13.2.3 that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every real number x In particular , $\frac{5^n}{n!}$ is convergent and therefore must be bounded Perhaps the easiest way to find the bound is to tabulate a few values , an in Table 1. It is clear from the table that the maximum value of a_n occurs at $n = 4$ or $n = 5$ and is equal to 26.04. Of course, any number larger than 26.04. is also abound for the sequence.

Since every convergence is bounded, it follows that:

(a) $\{\text{In In } n\}$ (starting at $n = 2$) (b) $\{n \sin n\}$ (c) $\{(-\sqrt{2})^n\}$

The convergent. Of Theorem 1 is not true. That is, it is not true that every bounded sequence is convergent. For example, the sequences $\{(-1)^n\}$ and $\{\sin n\}$ are both bounded and divergent.

Since boundedness alone does not ensure convergence, we need some other property. We investigate this idea now

Definition 2 MONOTONICITY

(i) The sequence $\{a_n\}$ is monotone increasing if $a_n \leq a_{n+1}$ for every $n \geq 1$

(ii) The sequence $\{a_n\}$ is monotone decreasing if $a_n \geq a_{n+1}$ for every $n \geq 1$

(iii) The sequence $\{a_n\}$ is monotonic if it is either monotone increasing or monotone decreasing

Definition 3 STRICT MONOTONCITY

(i) The sequence $\{a_n\}$ is monotone increasing if $a_n < a_{n+1}$ for every $n \geq 1$

(ii) The sequence $\{a_n\}$ strictly decreasing if $a_n \geq a_{n+1}$ for every $n \geq 1$

(iii) The sequence $\{a_n\}$ strictly monotonic if it is either strictly increasing or strictly decreasing

EXAMPLE 7 The sequence$\{1/2^n\}$ is strictly decreasing since$1/2^n > 1/2^{n+1}$ for every n

EXAMPLE 8 Determine whether the sequence $\{2n/(3n+2)\}$ is increasing, decreasing, or not monotonic

Solution If we write out the first few terms of the sequence, we find that$\left\{\frac{2n}{3n+2}\right\} = \{\frac{2}{5}.\frac{4}{8}.\frac{6}{11}.\frac{8}{14}.\frac{10}{17}.\frac{12}{20}.\cdots\}$ Since these terms are strictly increasing, we suspect that $\{2n/(3n+2)\}$ is an increasing sequence To check this, we try to verify that $a_n < a_{n+1}$ We have

$$a_{n+1} = \frac{2(n+1)}{3(n+1)+2} = \frac{2n+2}{3n+5}$$

Ther $a_n < a_{n+1}$ implies that

$$\frac{2n}{3n+2} < \frac{2n+2}{3n+5}$$

Multiplying both sides of this inequality by $(3n+2)(3n+5)$. we obtain

$$(2n)(3n+5) < (2n+2)(3n+2). \qquad \text{or} \qquad 6n^2 + 10n < 6n^2 + 10n + 4$$

Since this last inequality is obviously true for alln ≥ 1. we can reverse our steps to conclude that $a_n < a_{n+1}$and the sequence is strictly increasing

EXAMPLE 9 Determine whether the sequence$\{(\text{In n})/n\}$ $n > 1$ is increasing, decreasing, or not monotonic

Solution. Let $f(x) = \frac{\text{Inx}}{x}$. Then $f'(x) = \frac{\left[x\left(\frac{1}{x}\right)-(\text{In } x)1\right]}{x^2} = \frac{1-\text{In } x}{x^2}$ If $x > e$. then $\text{In } x > 1$ and$f'(x) < 0$ Thus the sequence $\left\{\frac{\text{In n}}{n}\right\}$ is decreasing for $n \geq 3$However,$\frac{\text{In } 1}{1} = 0 < \frac{(\text{In } 2)}{2} \approx 0 \cdot 35$ so initially , the sequence is increasing Thus the sequence is not monotone It is decreasing if we star with $n = 3$

EXAMPLE 10 The sequence is increasing but not strictly increasing. Here [x] is the "greatest integer" function (see Example 2.2.11) The first twelve terms are 0,0,0,1,1,1,1,2,2,2,2,3 For example , $a_9 = \left[\frac{9}{4}\right] = 2$

EXAMPLE 11 The sequence$\{(-1)^n\}$ is not monotonic since successive terms oscillate between $+1$ and -1

In all the examples we have given, a divergent sequence diverges for one of two reasons: It goes to infinity (it is unbounded) or it oscillates [like $(-1)^n$. which oscillates between-1 and 1] But if a sequence, then it does not oscillate. Thus the following theorem should not be surprising

Theorem 2 :A bounded monotonic sequence is convergent

Proof. We will prove this theorem for the case in which the sequence $\{a_n\}$ is increasing. The proof of the other case is similar. Since $\{a_n\}$ is bounded, there is a number M such that$a_n \leq M$ for every n Let L be the smallest such upper bound Now let $\in > 0$ be given . Then there is a number $N > 0$ such that $a_n > L - \in$ If this were not true , then we would have $a_n \leq L - \in$ for all $n \geq 1$ Then $L - \in$ would be an upper bound for$\{a_n\}$ and since $L - \in < L$. This would contradict the choice of L as the smallest such upper bound Since $\{a_n\}$ is increasing , we have for $n \geq N$.

$$L - \in \; < a_N \leq a_n \, L < + \in \qquad (6)$$

But the inequalities in (6) imply that $|a_n - L| \leq \in$ for $n \geq N$. which proves , according to the definition of convergence , that $\lim_{n\to\infty} a_n = L$

The number L is called the least upper bound for the sequence$\{a_n\}$ It is an axiom of the real number system that every set of real numbers that is bounded above has a least upper bound and that every set of real numbers that is bounded below has a greatest lower bound This axiom is called the completeness axiom and is of para mount importance in theoretical mathematical analysis We discussed the completeness axiom earlier on page 6 .

We have actually proved a stronger result. Namely, that if the sequence$\{a_n\}$ is bounded above and increasing, then it converges to its least upper bound . Similarly, if $\{a_n\}$is bounded below and decreasing, then it converges to its greatest lower bound.

EXAMPLE 12 In Example 8 we saw that the sequence$\{2n/(3n+2)\}$ is strictly increasing Also , since $2n/(3n+2) < 3n/(3n+2) < 3n/3n = 1$ we see that$\{a_n\}$ is also bounded , so that by Theorem 2 , $\{a_n\}$ is convergent . We easily find that $\lim_{n\to\infty} 2n/(3n+2) = \frac{2}{3}$

PROBLEMS

In problems 1- 12 , determine whether the given sequence is bounded or unbounded If it is bounded , find the smallest bound for $|a_n|$

$1\cdot\left\{\frac{1}{n+1}\right\}$ $2\cdot\{\sin n\pi\}$ $3\cdot\{\cos n\pi\}$

$4\cdot\{\sqrt{n}\sin n\}$ $5\cdot\left\{\frac{2^n}{1+2^n}\right\}$ $6\cdot\left\{\frac{2^n+1}{2^n}\right\}$

$7\cdot\left\{\frac{1}{n!}\right\}$ $8\cdot\left\{\frac{3^n}{n!}\right\}$ $9\cdot\left\{\frac{n^2}{n!}\right\}$

$10\cdot\left\{\frac{2n}{2^n}\right\}$ $11\cdot\left\{\frac{\ln n}{n}\right\}$ $*\,12\cdot\{ne^{-n}\}$

13. Show that for $n > 2^{10}.\frac{n^{10}}{n!} > (n+1)^{10}/(n+1)!$ and use this result to conclude that $\{\frac{n^{10}}{n!}\}$ is bounded

In problems 14 -28, determine whether the given sequence is monotone increasing, strictly increasing, monotone decreasing, or not monotonic.

$14\cdot\{\sin n\pi\}$ $15\cdot\left\{\frac{3^n}{2+3^n}\right\}$ $16\cdot\{\left(\frac{n}{25}\right)^{\frac{1}{3}}\}$

$17\cdot\{n+(-1)^n\sqrt{n}\}$ $18\cdot\left\{\frac{\sqrt{n+1}}{n}\right\}$ $19\cdot\left\{\frac{n!}{n^n}\right\}$

$20\cdot\left\{\frac{n^n}{n!}\right\}$ $21\cdot\left\{\frac{2n!}{1\cdot3\cdot5\cdot7\cdot\cdots\ (2n-1)}\right\}$

$*\,22\cdot\{n+\cos n\}$ $23\cdot\left\{\frac{2^{2n}}{n!}\right\}$ $24\cdot\left\{\frac{\sqrt{n}-1}{n}\right\}$

25 $\cdot\left\{\frac{n-1}{n+1}\right\}$ 26 $\cdot\left\{\text{In }\left(\frac{3n}{n+1}\right)\right\}$ 27 $\cdot\{\text{In } n-\text{In }(n+2)\}$

28 $\cdot\{\left(1+\frac{3}{n}\right)^{\frac{1}{n}}\}$

*29. Show that the sequence $\{(2^n+3^n)^{1/n}\}$ is convergent

*30. Show that $\{\left(a^n+b^n\right)^{\frac{1}{n}}\}$ is convergent for any positive real numbers a and b [Hint : First do Problem 29 Then treat the cases $a=b$ and $a\neq b$ separately]

31. Show that the sequence $\left\{\frac{n!}{n^n}\right\}$ is bounded [Hint : Show that $\left\{\frac{n!}{n^n}\right\}>(n+1)!/(n+1)^n$ sufficiently large n]

32. Prove that the sequence $\left\{\frac{n!}{n^n}\right\}$ converges [hint: Use the result of Problem 31.]

33. Use Theorem 2 to show that $\{\text{In } n-\text{In }(n+4)\}$ converges.

GEOMETRIC SERIES

Consider the sum $S_7=1+2+4+8+16+32+64+128$

This can be written as $S_7=1+2+2^2+2^3+2^4+2^5+2^6+2^7=\sum_{k=0}^{7}2^k$

GEOMETRIC PROGRESSION

In general, the sum of a geometric progression is a sum of the form

$$S_n=1+r+r^2+r^3+\cdots+r^{n-1}+r^n=\sum_{k=0}^{7}r^k\ . \qquad (1)$$

Where r is a real number and n is affixed positive integer

We now obtain a formula for the sum in (1)

Theorem 1 If $r\neq 1$. the sum of a geometric progression (1) is given by

$$S_n=\frac{1-r^{n+1}}{1-r} \qquad (2)$$

Proof. We write

$$S_n=1+r+r^2+r^3+\cdots+r^{n-1}+r^n \qquad (3)$$

And then multiply both sides of (3) by r:

$$rS_n = r + r^2 + r^3 + r^4 + \cdots + r^n + r^{n+1} \tag{4}$$

We now subtract (4) from (3) and note that all terms except the first and the last cancel:

$$S_n - rS_n = 1 - r^{n+1} .$$

Or

$$(1 - r)S_n = 1 - r^{n+1} \tag{5}$$

Finally, we divide both sides of (5) by $1 - r$ (which is nonzero) to obtain equation (2)

NOTE If $r = 1$. we obtain

$$\overbrace{S_n = 1 + 1 + \cdots + 1 = n + 1}^{\text{n+1 terms}}$$

EXAMPLE 1 Calculate $S_7 = 1 + 2 + 4 + 8 + 16 + 32 + 128$. using formula (2)

Solution Here $r = 2$ and $n = 7$. so that

$$S_7 = \frac{1 - 2^8}{1 - 2} = 2^8 - 1 = 256 - 1 = 255$$

EXAMPLE 2 Calculate $\sum_{k=0}^{10} \left(\frac{1}{2}\right)^k$

Solution Here $r = \frac{1}{2}$ and $n = 10$. so that

$$S_{10} = \frac{1 - \left(\frac{1}{2}\right)^{11}}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{2048}}{\frac{1}{2}} = 2\left(\frac{2047}{2048}\right) = \frac{2047}{1048}$$

EXAMPLE 3 Calculate

$$S_6 = 1 - \frac{2}{3} + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 - \left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^6 = \sum_{k=0}^{6} \left(-\frac{2}{3}\right)^k$$

Solution Here $r = -\frac{2}{3}$ and $n = 6$. so that

$$S_6 = \frac{1-\left(-\frac{2}{3}\right)^7}{1-\left(-\frac{2}{3}\right)} = \frac{1+\frac{128}{2187}}{\frac{5}{3}} = \frac{3}{5}\left(\frac{2315}{2187}\right) = \frac{463}{729}$$

EXAMPLE 4 Calculate the sum $1 + b^2 + b^4 + b^6 + \cdots + b^{20} = \sum_{k=0}^{10} b^{2k}$ for $b \neq \pm 1$

Solution Not that the sum can be written $1 + b^2 + (b^2)^2 + (b^2)^3 + \cdots + (b^2)^{10}$ Here $r = b^2 \neq 1$ and $n = 10$. so that

$$S_{10} = \frac{1-(b^2)^{11}}{1-b^2} = \frac{b^{22}-1}{b^2-1}$$

The sum of a geometric progression is the sum of a finite number of terms We now see what happens if the number of terms is infinite Consider the sum

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \qquad (6)$$

What can such a sum mean? We will give a formal definition in a moment For now let us show why it is reasonable to say that $S = 2$ Let $S_n = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^n$ Then

$$S_n = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2\left[1-\left(\frac{1}{2}\right)^{n+1}\right]$$

Thus for any n (no matter how large), $1 \leq S_n < 2$ Hence the numbers S_n are bounded Also, since $S_{n+1} = S_n + \left(\frac{1}{2}\right)^{n+1} > S_n$ the numbers S_n are monotone increasing. Thus the sequence $\{S_n\}$ converges But

$$S = \lim_{n\to\infty} S_n$$

Thus S has a finite sum To compute it, we note that

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left[1-\left(\frac{1}{2}\right)^{n+1}\right] = 2\lim\left[1-\left(\frac{1}{2}\right)^{n+1}\right] = 2$$

Since $\lim_{n\to\infty} \left(\frac{1}{2}\right)^{n+1} = 0$

GEOMETRIC SERIES

The infinite sum is called a geometric series In general, a geometric series is an infinite sum of the form

$$S = \sum_{k=0}^{\infty} (r)^k = 1 + r + r^2 + r^3 + \cdots \qquad (7)$$

CONVERGENCE AND DIVERGENCE OF A GEOMETRIC SERIES

Let $S_n = \sum_{k=0}^{n} r^k$ Then we say that the geometric series converges exists and is finite O otherwise, the series is said to diverge

EXAMPLE 5 Let $r = 1$ Then

$S_n = \sum_{k=0}^{n} 1^k = \sum_{k=0}^{n} 1 = \underbrace{1 + 1 + \cdots + 1 = n + 1}_{1+n}$

Since $\lim_{n\to\infty}(n + 1) = \infty$. the series $\sum_{k=0}^{\infty} 1^k$ diverges

EXAMPLE 6 Let $r = -2$ Then

$$S_n = \sum_{k=0}^{n} r^k = \frac{1 - (-2)^{n+1}}{1 - (-2)} = \frac{1}{3}\left[1 - (-2)^{n+1}\right]$$

But $(-2)^{n+1} = (-1)^{n+1}(2^{n+1}) = \pm 2^{n+1}$ As $n \to \infty$. $2^{n+1} \to \infty$ Thus the series $\sum_{k=0}^{\infty} (-2)^{k+1}$ diverges

Theorem

Let $S = \sum_{k=0}^{\infty} r^k$ be a geometric series

(i) The series converges to

$\frac{1}{1-r}$ if $|r| < 1$

(ii) The series diverges if $|r| \geq 1$

Proof (i)if $|r| < 1$. then $\lim_{n\to\infty} r^{n+1} = 0$ Thus

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r} \lim_{n\to\infty}(1 - r^{n+1})$$

$$= \frac{1}{1-r}\ (1-0) = \frac{1}{1-r}$$

(ii) If $|r| \ > 1$. then $\lim_{n\to\infty} |\,r|^{n+1} = \infty$ Thus $1 - r^{n+1}$ does not have a finite limit and the series diverges Finally , if $r = 1$. then series diverges , by Example 5, and if $r = -1$ then S_n alternates between the numbers 0 and 1, so that the series diverges

EXAMPLE 7 $1 - \frac{2}{3} + \left(\frac{2}{3}\right)^2 - \cdots \ = \sum_{k=0}^{\infty} \ (-\frac{2}{3})^k = 1/[1 - \left(-\frac{2}{3}\right) = 1/(\frac{5}{3}) = \frac{3}{5}$

EXAMPLE 8 $1 + \frac{\pi}{4} + \left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{4}\right)^3 + \cdots \sum_{k=0}^{\infty} \ \left(\frac{\pi}{4}\right)^k \ = \frac{1}{1-\left(\frac{\pi}{4}\right)}$

$$= \frac{4}{4-\pi} \approx 4\cdot 66$$

PROBLEMS

In Problems 1-11, calculate the sum of the given geometric progression

$1 \cdot 1 + 3 + 9 + 27 + 81 + 243$

$2 \cdot 1 + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^8}$

$3 \cdot 1 - 5 + 25 - 125 + 625 - 3125$

$4 \cdot 0 \cdot 2 - 0 \cdot 2^2 + 0 \cdot 2^3 + \cdots 0 \cdot 2^9$

$5 \cdot 0 \cdot 3^2 - 0 \cdot 3^3 + 0 \cdot 3^4 - 0 \cdot 3^5 + 0 \cdot 3^6 - 0 \cdot 3^7 + 0 \cdot 3^8$

$6 \cdot 1 + b^3 + b^6 + b^9 + b^{12} + b^{15} + b^{18} + b^{21}$

$7 \cdot 1 - \frac{1}{b^2} + \frac{1}{b^4} - \frac{1}{b^6} + \frac{1}{b^8} + \frac{1}{b^{10}} + \frac{1}{b^{12}} - \frac{1}{b^{14}}$

$8 \cdot \pi - \pi^3 + \pi^5 - \pi^7 + \pi^9 - \pi^{11} + \pi^{13}$

$9 \cdot 1 + \sqrt{2} + 2 + 2^{\frac{3}{2}} + 4 + 2^{\frac{5}{2}} + 8 + 2^{\frac{7}{2}} + 16$

$10 \cdot 1 - \frac{1}{\sqrt{3}} + \frac{1}{3} - \frac{1}{3\sqrt{3}} + \frac{1}{9} - \frac{1}{9\sqrt{3}} + \frac{1}{27} - \frac{1}{27\sqrt{3}} + \frac{1}{81}$

$11 \cdot 1 - 16 + 64 - 256 + 1024 - 4096$

In Problems 13 – 22, calculate the sum of the given geometric series

$13 \cdot 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots$

$14 \cdot 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

$15 \cdot 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$

$16 \cdot \; 1 - \frac{1}{10} + \frac{1}{100} - \frac{1}{1000} + \cdots$

$17 \cdot 1 + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3} + \cdots$

$18 \cdot 1 + 0 \cdot 7 + 0 \cdot 7^2 + 0 \cdot 7^3 + \cdots$

$19 \cdot 1 - 0 \cdot 62 + 0 \cdot 62^2 + 0 \cdot 62^3 + 0 \cdot 62^4 - \cdots$

$20 \cdot \frac{1}{4} + \frac{1}{16} + \frac{1}{46} + \cdots \left[\text{Hint Factor out the term } \frac{1}{4}\right]$

$21 \cdot \frac{3}{5} - \frac{3}{25} + \frac{3}{125} - \cdots$

$22 \cdot \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

$23 \cdot$ How large must n be in order that $\left(\frac{1}{2}\right)^n < 0 \cdot 01$?

$24 \cdot$ How large must n be in order that $(0 \cdot 8)^n < 0 \cdot 01$?

$25 \cdot$ How large must n be in order that $(0 \cdot 99)^n < 0 \cdot 01$?

$26 \cdot$ If $x > 1$. show that

$$1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots = \frac{x}{x - 1}$$