THE TRIPLE INTEGRAL

In this section we discuss the idea behind the triple integral of a function of three variables $f(x, y, z)$ over a region S in $\mathbb{R}^3$. This is really a simple extension of the double integral . For that reason we will omit a number of technical details,We start with a parallelepiped π which $\mathbb{R}^3$, can be written as

$$\pi = \{(x, y, z): a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2\} \tag{1}$$

We construct regular partitions of the three intervals $[a_1, a_2]$, $[b_1, b_2]$ and $[c_1, c_2]$:

$a_1 = x_0 < x_1 < \cdots < x_n = a_2$

$b_1 = y_0 < y_1 < \cdots < y_m = b_2$

$c_1 = z_0 < z_1 < \cdots < z_p = c_2$,

To obtain nmp "boxes" The volume of typical box B_{ijk} is given by

$$\Delta V = \Delta x\, \Delta y\, \Delta z , \tag{2}$$

Where $\Delta x = x_i - x_{i-1}$, $\Delta y = y_j - y_{j-1}$, and $\Delta z = z_k - z_{k-1}$. We then form the sum

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} f(x_i^*, y_j^*, z_k^*)\Delta V, \tag{3}$$

where (x_i^*, y_j^*, z_k^*) is in B_{ijk}

We now define

$$\Delta u = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

Geometrically , Δu is the length of a diagonal of a rectangular solid with sides having lengths Δx , Δy and Δz respectively . We see that as $\Delta u \to 0$, Δx , Δy and Δz approach 0, and the volume of each box tends to zero . We then take the limit as $\Delta u \to 0$.

Definition 1 THE TRIPLE INTEGRAL Let $w = f(x, y, z)$ and let the parallelepiped π be given by (1) . Suppose that

$$\lim_{\Delta u \to 0} \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} f(x_i^*, y_j^*, z_k^*)\ \Delta x\, \Delta y\, \Delta z$$

Exists and is in dependent of the way in which the points (x_i^*, y_j^*, z_k^*) are chosen .Then the triple integral of f over π , written $\iiint_\pi f(x,y,z)dV$, is defined by

$$\iiint_\pi f(x,y,z)dV = \lim_{\Delta u \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{p} f(x_i^*, y_j^*, z_k^*)\ \Delta \tag{4}$$

As with double integrals , we can write triple integrals as iterated (or repeated)integrals . If π is defined by (1) , we have

$$\iiint_\pi f(x,y,z)dV = \int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} f(x,y,z)dz\, dy\, dx \tag{5}$$

EXAMPLE 1 Evaluate $\iiint_\pi xy\cos yz\, dV$, where π is the parallelepiped

$$\left\{(x,y,z): 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le \frac{\pi}{2}\right\}$$

Solution : $\iiint_\pi xy\cos yz\, dV = \int_0^1 \int_0^1 \int_0^{\frac{\pi}{2}} xy\cos yz\, dz\, dy\, dx$

$$= \int_0^1 \int_0^1 \{xy \cdot \frac{1}{y} \sin yz| \begin{matrix}\frac{\pi}{2} \\ 0\end{matrix}\}\ dy\, dx = \int_0^1 \int_0^1 x\sin(\frac{\pi}{2}y)dydx$$

$$= \int_0^1 \{-\frac{2}{\pi} x\cos\frac{\pi}{2} y \left|\begin{matrix}1 \\ 0\end{matrix}\right.\} dx \int_0^1 \frac{2}{\pi} xdx = \frac{x^2}{\pi} |\begin{matrix}1 \\ 0\end{matrix} = \frac{1}{\pi}$$

THE TRIPLE INTEGRAL OVER AMORE GENERAL REGION

We now define the triple integral over a more general region S . We assume that S is bounded . Then we can enclose Sin a parallelepiped π and define a new function F by

$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if}(x,y,z)\text{is in S} \\ 0, & (x,y,z)\text{is in}\pi \text{ but not S} \end{cases}$$

We then define : $\iiint_s f(x,y,z)dV = \iiint_\pi F(x,y,z)dV$ (6)

REMARK 1. If f is continuous over S ,we will discuss Below , then $\iiint_s f(x,y,z)dV$ will exist. The proof of this fact is beyond the scope of this text but can be found in any advanced calculus text .

REMARK 2. Iff ≥ 0 on S, then the triple integral $\iiint_s f(x,y,z)dV$ represents the "volume" in for – dimensional space $\mathbb{R}^4$. of the region bounded above by f and below by S .We carries over to four (and more) dimensions ,ow let S take the form

$$S = \{(x,y,z): a_1 \leq x \leq a_2, g_1(x) \leq y \leq g_2(x), h_1(x,y) \leq z \leq h_2(x,y)\}. \qquad (7)$$

What does such a solid look like ? We first note that the equations $z = h_1(x,y)$ and $z = h_2(x,y)$ are the equations of surfaces in $\mathbb{R}^3$. The equations $y = g_1(x)$ and $y = g_2(x)$ are equations of cylinders in $\mathbb{R}^3$, and the equations are equations of planes in $\mathbb{R}^3$..We assume that g_1 , g_2 , h_1 and h_2 are continuous . If f is continuous , then $\iiint_s f(x,y,z)dV$ will exist and $\iiint_s f(x,y,z)dV = \int_{a_1}^{a_2} \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dzdydx$ (8)

EXAMPLE 2 Evaluate $\iiint_s 2x^3 y^2 z \, dV$, where S is the region

$$\{(x,y,z): 0 \leq x \leq 1, x^2 \leq y \leq x, x-y \leq z \leq x+y\}.$$

Solution: $\iiint_s 2x^3 y^2 z \, dV = \int_0^1 \int_{x^2}^{x} \int_{x-y}^{x+y} 2x^3 y^2 z \, dz \, dy \, dx$

$$= \int_0^1 \int_{x^2}^{x} 2x^3 y^2 \left|_{x-y}^{x+y}\right\} dydx$$

$$= \int_0^1 \int_{x^2}^{x} x^3 y^2 [(x+y)^2 - (x-y)^2] dydx$$

$$= \int_0^1 \int_{x^2}^{x} 4x^4 y^3 dy \, dx = \int_0^1 \left\{x^4 y^4 \left|_{x^2}^{x}\right.\right\} dx$$

$$= \int_0^1 (x^8 - x^{12}) dx = \frac{1}{9} - \frac{1}{13} = \frac{4}{117}$$

Many of the applications we saw for the double integral can be extended to the triple integral . We present three of them below

I .VOLUME

Let the region S be defined by (7) Then , since $\Delta V = \Delta x \Delta y \Delta z$ represents the volume of a "box" in S , when we add up the volumes of these boxes and take a limit , we obtain the total volume of S. That is , volume of $S = \iiint_s dV$ (9)

EXAMPLE 3 Calculate the volume of the region of Example 2 .

$$V=\int_0^1\int_{x^2}^{x}\int_{x-y}^{x+y} dz\,dy\,dx=\int_0^{-1}\int_{x^2}^{x}\left\{z\Big|_{x-y}^{x+y}\right\}dy\,dx$$

$$=\int_0^1\int_{x^2}^{x}2y\,dy\,dx=\int_0^1\left\{y^2\Big|_{x^2}^{x}\right\}dx=\int_0^1(x^2-x^4)dx$$

$$=\frac{1}{3}-\frac{1}{5}=\frac{2}{15}$$

EXAMPLE 4 Find the volume of the tetrahedron formed by the planes $x=0, y=0, z=0,$ and $x+\left(\frac{y}{2}\right)+\left(\frac{z}{4}\right)=1$

Solution : We see that z ranges from 0 to the plane This last plane intersects the plane in a line whose equation (obtained by setting z = 0) is given by so that y ranges from 0 to 1 Finally , this line intersects the x – axis at the point (1 , 0, 0) so that x ranges from 0 to 1, and we have

$$V=\int_0^1\int_0^{2(1-x)}\int_0^{4\left(1-x-\frac{y}{2}\right)} dz\,dy\,dx=\int_0^1\int_0^{2(1-x)}4\left(1-x-\frac{y}{2}\right)dy\,dx$$

$$=4\int_0^1\left\{y(1-x)-\frac{y^2}{4}\right)\Big|_0^{2(1-x)}\right\}dx=4\int_0^1[2(1-x)^2-(1-x)^2]dx$$

$$=-\frac{4}{3}(1-x)^3\Big|_0^1=\frac{4}{3}$$

REMARK . It was not necessary to integrate in the order z , then y , then x, We could have written , for example ,

$$0\le x\le 1-\frac{y}{2}-\frac{z}{4}$$

The intersection of this plane with the yz_ plane is the line$0=1-\left(\frac{y}{2}\right)-\left(\frac{z}{4}\right)$, or $z=4\left[1-\left(\frac{y}{4}\right)\right]$ The intersection of this line with the occurs at the point (0, 2, 0) Thus

$$V=\int_0^2\int_0^{4\left(1-\frac{y}{4}\right)}\int_0^{1-\frac{y}{2}-\frac{z}{4}} dx\,dz\,dy$$

$$=\int_0^2\int_0^{4\left(1-\frac{y}{2}\right)}\left(1-\frac{y}{2}-\frac{z}{4}\right)dzdy=\int_0^2\left\{\left[\left(1-\frac{y}{2}\right)z-\frac{z^2}{8}\right]\Bigg|_0^{4\left(1-\frac{y}{2}\right)}\right\}dy$$

$$= \int_0^2 \{4(1 - y/2)^2 - \frac{16[1-(y/2)]^2}{8}\}dy = 2\int_0^2 (1 - \frac{y}{2)^2}dy$$

$$= -\frac{4}{3}(1 - y/2)^3|_0^2 = \frac{4}{3}$$

We could also in any of four other orders (xyz , yzx, yxz, and zxy) to obtain the same result

II. DENSITY AND MASS

Let the function $\rho(x, y, z)$ denote the density (in kilograms per cubic meter , say)of a solid S in Then for a " box" of sides $\Delta x, \Delta y$, and Δz , the approximate mass of the box will be equal to $\rho = (x_i^*, y_j^*, z_k^*)\Delta x\, \Delta y\, \Delta z = \rho(x_i^*, y_j^*, z_k^*)\Delta V$ if $\Delta x, \Delta y$, and Δz small . We then obtain

$$\text{total mass of } S = \mu(S) = \iiint_s \rho(x, y, z)dV \qquad (10)$$

EXAMPLE 5 The density of the solid of Example 2 is given by $\rho(x, y, z) = x + 2y + 4z\ kg/m^3$ Calculate the total mass of the solid .

Solution

$$\mu(S) = \int_0^1\int_{x^2}^x\int_{x-y}^{x+y}(x + 2y + 4z)dzdydx = \int_0^1\int_{x^2}^x\{[(x + 2y)z + 2z^2]\Big|_{x-y}^{x+y}\}dydx$$

$$= \int_0^1\int_{x^2}^x(10xy + 4y)dy\,dx = \int_0^1\{5xy + \frac{4y^3}{3})\Big|_{x^2}^{x}\}dx$$

$$= \int_0^1(5x^3 - 5x^5 + \frac{4}{3}x^3 - \frac{4}{3}x^6)dx = \int_0^1(\frac{19}{3}x^3 - 5x^5 - \frac{4}{3}x^6)dx$$

$$= \frac{19}{12} - \frac{5}{6} - \frac{4}{21} = \frac{47}{84}kg$$

III . FIRST MOMENTS AND CENTER OF MASS

In $\mathbb{R}^3$ we use the symbol M_{yz} to denote the first moment with respect to the yz- plane similarly , M_{xz} denotes the first moment with respect to the xz- plane, and M_{xy} . donates the first moment with respect to the xy – plane Since the distance from appoint

(x , y, z) to the yz – plane is x, and so on , we may use familiar reasoning to obtain

$$M_{yz} = \iiint_s x\rho(x\rho, y, z)dV \qquad (11)$$

$M_{xz} = \iiint_s y\rho(x\rho, y, z)d$ (12)
$M_{xy} = \iiint_s z\rho(x\rho, y, z)d$ (13)

The center of mass of S is then given by

$$(\overline{\overline{x}}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{\mu}, \frac{M_{xz}}{\mu}, \frac{M_{xy}}{\mu}\right), \quad (14)$$

Where μ denotes mass of S . If $\rho(x\rho, y, z)$ is constant, then as in $\mathbb{R}^1$ and $\mathbb{R}^2$ the center of mass is called the centroid

EXAMPLE 6 : Find the center of mass of the solid in Example 5 .

Solution : We have already found that $\mu = 47/64$ We next calculate the moments .

$$M_{yz} = \int_0^1 \int_{x^2}^x \int_{x-y}^{x+y} x(x - 2y + 4z)dz\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^x \{((x^2 + 2xy)z + 2xz^2 \Big|_{x-y}^{x+y}\} dx$$

$$= \int_0^1 \int_{x^2}^x (10x^2y + 4xy^2)dydx = \int_0^1 \{(5x^2 y^2 + \frac{4xy^3}{3} \Big|_{x^2}^{x}\} dx$$

$$= \int_0^1 (5x^4 - 5x^6 + \frac{4}{3}x^4 - \frac{4}{3}x^7)dx = 1 - \frac{5}{7} + \frac{4}{15} - \frac{1}{6} = \frac{27}{70}$$

$$M_{xz} = \int_0^1 \int_{x^2}^x \int_{x-y}^{x+y} y(x + 2y + 4z)dzdydx$$

$$= \int_0^1 \int_{x^2}^x \left\{[(xy + 2y^2)z + 2yz^2]\Big|_{x-y}^{x+y}\right\} dydx$$

$$= \int_0^1 \int_{x^2}^x 10xy^2 + 4y^3)dydx = \int_0^1 \{(\frac{10}{3}xy^3 + y^4) \Big|_{x^2}^{x}\} dx$$

$$= \int_0^1 \left(\frac{10}{3}x^4 - \frac{10}{3}x^7 + x^4 - x^8\right) dx = \frac{2}{3} - \frac{5}{12} + \frac{1}{5} - \frac{1}{9} = \frac{61}{180}$$

$$M_{xy} = \int_0^1 \int_{x^2}^x \int_{x-y}^{x+y} z(x + 2y + 4z)dzdydx$$

$$= \int_0^1 \int_{x^2}^x \{[x + 2y)\frac{z^2}{2} + \frac{4z^3}{3}] \Big|_{x-y}^{x+y}\} dydz$$

$$=\int_0^1\int_{x^2}^{x}[(x+2y)(2xy)+\frac{4}{3}(6x^2\,y+2y^3)]dydx$$

$$=\int_0^1\int_{x^2}^{x}(10x^2y+4x^2+\frac{8}{3}y^3)\;dy\;dx$$

$$=\int_0^1\{(5x^2y^2+\frac{4}{3}xy^3+\frac{2}{3}y^4)\Big|_{x^2}^{x}\}\,dx$$

$$=\int_0^1(5x^4-5x^6+\frac{4}{3}x^4-\frac{4}{3}x^7+\frac{2}{3}x^4-\frac{2}{3}x^8)dx$$

$$=\int_0^1(7x^4-5x^6-\frac{4}{3}x^7-\frac{2}{3}x^8)dx=\frac{7}{5}-\frac{5}{7}-\frac{1}{6}-\frac{2}{27}=\frac{841}{1890}$$

Thus

$$(\bar{x}\,,\bar{y},\bar{z}\,)=\left(\frac{M_{yz}}{\mu},\frac{M_{xz}}{\mu}\,,\frac{M_{xy}}{\mu}\right)=\left(\frac{\frac{27}{70}}{\frac{47}{84}}\,,\frac{\frac{61}{180}}{\frac{47}{84}}\,,\frac{\frac{841}{1890}}{\frac{47}{84}}\right)$$

$$=\left(\frac{162}{235},\frac{427}{705}\,,\frac{1682}{2115}\right)\approx(0.689,0.606,0.795\,)$$

PROBLEMS

In problems 1-7 evaluate the repeated triple integral .

1. $\int_0^1\int_0^y\int_0^x y\;dzdxdy$

2. $\int_0^2\int_{-z}^{z}\int_{y-z}^{y+z}2xzdx\;dy\;dz$

3. $\int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2}dy\;\;dx\;dz$

4. $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{z}\sin\left(\frac{x}{z}\right)\;dx\;dz\;dy$

5. $\int_1^2\int_{1-y}^{1+y}\int_0^{yz}6xyz\;dx\;dz\;dy$

6. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{x}yz\;dz\;dy\;dx$

7. $\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}z^2\;dz\;dy\;dx$

8. Change the order of integration in problem 1 and writ the integral in the form
(a) $\int_?^? \int_?^? \int_?^? y\, dx \quad dy\, dz$; (b) $\int_?^? \int_?^? \int_?^? y\, dy\, dz\, dx$ [Hint : Sketch the region in Problem 1 from the given limits and then find the new limits directly from the figure.]

9. Write the integral of Problem 6 in the form
(a) $\int_?^? \int_?^? \int_?^? yz\, dx\, dy\, dz$; (b) $\int_?^? \int_?^? \int_?^? yz\, dy\, dz\, dx$

10. Write the integral of problem 7 in the from $\int_?^? \int_?^? \int_?^? z^2 dx\, dz\, dy$

In problems 11-18, find the volume of the given solid

11. The tetrahedron with vertices at the points (0, 0,0), (1, 0,0), (0, 1,0) and (0, 0,1).

12 . The tetrahedron with vertices at the points (0, 0,0), (a, 0,0), (0, b,0) and (0, 0,c)

13. The solid in the first octant bounded by the cylinder $x^2 + z^2 = 9$, the plane

$x + y = 4$, and the three coordinate planes

14. The solid bounded by planes $x - 2y + 4z = 4, -2x + 3y - z = 6, x = 0$ and $y = 0$

15 .The solid bounded above by the sphere $x^2 + y^2 + z^2 = 16$, and below by the plane z = 2

16. The solid in the first octant bounded by the cylinder $z = 5 - x^2$ and the planes $z = y$ and $z = 2y$ that lies in the half space $y \geq 0$

17. The solid bounded by the ellipsoid $\left(x^2/a^2\right) + \left(y^2/b^2\right) + \left(z^2/c^2\right) = 1$

18. The solid bounded by the elliptic cylinder $9x^2 + y^2 = 9$ and the planes

$z = 0$ and $x + y + 9z = 9$

THE TRIPLE INTEGRAL IN CYLINDRICAL AND SPHERICAL COORDINATES

In this section we show how triple integrals can be written by using cylindrical and spherical coordinates .

I. CYLINDRICAL COORDINATES

Recall the cylindrical coordinates of a point in$\mathbb{R}^2$ are (r, θ, z) where r and θ are the polar coordinates of the projection of the point onto the xy- plane and z is the usual z- coordinates , In order to calculate an integral of a region given in cylindrical coordinates we go through a procedure very similar to the one we used to write double integrals in polar coordinates

Consider the " cylindrical parallelepiped" given by

$$\pi_c = \{(r, \theta, z): r_1 \leq r \leq r_2; \theta_1 \leq \theta \leq \theta_2 \,; z_1 \leq z \leq z_2\}$$

This solid is sketched in Figure 1. If we partition the z- axis for z in $[z_1 \,, z_2]$, we obtain "slices "For affixed z the area of the face of a slice of π_c is , given by

$$A_i = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r \, dr \, d\theta \qquad (1)$$

The volume of a slice is , by (1), given by

$$V_i = A_i \Delta z = \left\{ \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r \, dr \, d\theta \right\} \Delta z$$

Then adding these volumes and taking the limit as before , we obtain

$$\text{volume of } \pi_c = \int_{z_1}^{z_2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r \, dr \, d\theta \; dz \qquad (2)$$

In general , let region S be given in cylindrical coordinates by

$$S = \{(r, \theta, z): \theta_1 \leq \theta \leq \theta_2 \,, 0 \; \leq g_1(\theta) \leq r \leq g_2(\theta), h_1(r, \theta) \leq \; z \leq h_2(r, \theta)\}, \qquad (3)$$

And let f be a function of r , and z Then the triple integral of f over S is given by

$$\iiint_s f = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r, \theta, z) r \, dz \, dr \, d\theta \qquad (4)$$

EXAMPLE 1 Find the mass of a solid bounded by the cylinder $r = \sin \theta$, the planes $z = 0, \theta = 0, = \theta = \frac{\pi}{3}$, and the cone $z = r$, if the density is given by $\rho \,(r, \theta, z) = 4r$

Solution . We first note that the solid may be written as

$$S = \{(r, \theta, z): 0 \le \theta \le \frac{\pi}{3}, 0 \le r \le \sin\theta, 0 \le z \le r\}.$$

Then

$$\mu = \int_0^{\frac{\pi}{3}} \int_0^{\sin\theta,} \int_0^{r} (4r) r \, dz \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \int_0^{\sin\theta,} \{4r^2 z \Big|_0^r\} dr d\theta = \int_0^{\frac{\pi}{3}} \int_0^{\sin\theta,} 4r^3 \, dr d\theta$$

$$= \int_0^{\pi/3} \left\{r^4 \Big|_0^{\sin\theta}\right\} d\theta = \int_0^{\pi/3} \sin^4\theta d\theta = \int_0^{\pi/3} (\frac{1 - \cos 2\theta}{2})^2 d\theta$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{3}} (1 - 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}) \, d\theta$$

$$= \frac{1}{4} (\frac{3\theta}{2} - \sin 2\theta + \frac{\sin 4\theta}{8}) \Big|_0^{\frac{\pi}{3}} = \frac{1}{4} (\frac{\pi}{2} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{16})$$

$$= \frac{\pi}{8} - \frac{9\sqrt{3}}{64}$$

Let S be a region in $\mathbb{R}^3$ Sincex $= r\cos x = r\cos\theta.$ $y = r\sin\theta.$ and z=z , we have

$$\iiint_S f(x.y.z) dV = \iiint_S f\ (r\cos\theta.\ r\sin\theta\ .z\ ) r\ dz\ dr\ d\theta \qquad (5)$$

Given in rectangular Coordinates — Given in cylindrical coordinates

REMARK. A gain, do not forget the extra r when you convert to cylindrical coordinates.

EXAMPLE 2 Find the mass of the solid bounded by paraboloid $z = x^2 + y^2$ and the plane $z = 4$ if the density at any point is proportional to the distance from the point to the z-axis

Solution:The density is given by$\rho(x.y.z) = a\sqrt{x^2 + y^2}$. where a is a constant of proportionality. Thus since the solid may be written as

$$S = \left\{(x.y.z): -2 \le x \le 2\ .-\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}. x^2 + y^2 \le z \le 4\right\}.$$

We have

$$\mu = \iiint_s a\sqrt{x^2+y^2}dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} a\sqrt{x^2+y^2}\, dz\, dydx$$

We write this expression in cylindrical , using the fact that a $\sqrt{x^2+y^2} = ar$ We note that the largest value of r is 2 , since at the "top" of the solid $r^2 = x^2 + y^2 = 4$ Then

$$\mu = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4}(ar)r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{2} ar^2(4-x^2)dr\, d\theta$$

$$= a\int_0^{2\pi}\{(\frac{4r^3}{3}-\frac{r^2}{5})|_0^2\}d\theta = \frac{64a}{15}\int_0^{2\pi} d\theta = \frac{128}{15}\pi a$$

II. SPHERICAL COORDINATES

Recall that a point p in $\mathbb{R}^3$ can be written in the spherical coordinates $(\rho . \theta . \varphi)$where $\rho \geq 0$, $0 \leq \theta < 2\pi . 0 \leq \varphi \leq \pi$ Here is distance between the point and the origin , θ is the same as in cylindrical coordinates ,and φ is the angle between$\overrightarrow{0P}$ and the positive z- axis Consider the "parallelepiped"

$$\pi_s = \{(\rho . \theta . \varphi) : \rho_1 \leq \rho \leq \rho_2 . \theta_1 \leq \theta \leq \theta_2 . \varphi_1 \leq \varphi \leq \varphi_2\} \quad (6)$$

To approximate the volume of π_ϑ we partition the intervals$[\rho_1 . \rho_2] . [\theta_1 . \theta_2]$ and $[\varphi_1 . \varphi_2]$This partition gives us a number of "spherical boxes ", one , The length of an arc of a circle is given by

$$L = r\theta . \quad (7)$$

Where r is the radius of the circle and θ is the angle that "cuts off" ,the one side of the spherical box $\Delta\rho$is. Since $\rho = \rho_i$the equation of a sphere, we find, from (7), that the length of a second side is $\rho_i \sin\varphi_k \Delta\theta$, approximately, by

$$V_i \approx (\Delta\rho)(\rho_i\Delta\varphi)(\rho_i \sin\varphi_k \Delta\theta).$$

And using a familiar argument, we have

$$\text{volume of } \pi_s = \iiint_{\pi_s} \rho^2 \sin\varphi\, d\rho\, d\vartheta\, d\theta = \int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_{\rho_1}^{\rho_2} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \quad (8)$$

If f is a function of the variables $\rho . \theta$ and φ . we have

$$\iiint_{\pi_s} f = \iiint_{\pi_s} f(\rho . \theta . \varphi)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \quad (9)$$

More generally, let the region S be defined in spherical coordinates by

$$S = \{(\rho . \theta . \varphi) : \theta_1 \leq \theta \leq \theta_2 . g_1(\theta) \leq \varphi \leq g_2(\theta) . h_1(\theta . \varphi) \leq p \leq h_2(\theta . \varphi)\} \quad (10)$$

$$\iiint_s f = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\theta.\varphi)}^{h_2(\theta.\varphi)} f(\rho\,.\,\theta.\,\varphi)\rho^2 \sin\varphi \; d\rho \; d\varphi \; d\theta \qquad (11)$$

Recall that to convert from rectangular to spherical coordinates, we have that formulas

$x = \rho \sin\varphi \cos\theta\,.$ $\quad$ $y = \rho \sin\varphi \sin\theta\,.$ $\quad$ $z = \rho \cos\varphi$

Thus to convert from a triple integral in rectangular coordinates to a triple integral in spherical coordinates, we have

$$\iiint_s f(x.y.z)dV = \iiint_s f(\rho \sin\varphi \cos\theta \;\; \rho \sin\varphi \sin\theta \,.\; \rho \cos\varphi\,) \quad \rho^2 \sin\varphi \; d\rho \; d\varphi \; d\theta$$

Given in rectangular coordinates $\qquad$ *Given in spherical coordinates*

EXAMPLE 3 Calculate the volume enclosed by the sphere $x^2 + y^2 + z^2 = a^2$

Solution. In rectangular coordinates, since

$$S = \left\{(x.\; y.z): -a \le x \le a\,. -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2} - \sqrt{a^2 - x^2 - y^2} \le z \le \sqrt{a^2 - x^2 - y^2}\right\}$$

We can write the volume as

$$V = \int_{-a}^{a} \int_{\sqrt{a^2-x^2}-}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\; dy\; dx$$

This expression is very tedious to calculate. So instead, we note that the region enclosed by a sphere can be represented in spherical coordinates as

$$S = \{(\rho\,.\,\theta.\,\varphi): 0 \le \rho \le a.\, 0 \le \theta \le 2\pi.\, 0 \le \varphi \le \pi\}$$

Thus

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\varphi \; d\rho \; d\varphi \; d\theta \;\; = \int_0^{2\pi} \int_0^{\pi} \frac{a^3}{3} \sin\varphi \;\; d\,\varphi \; d\theta$$

$$\frac{a^3}{3} \int_0^{2\pi} \left\{-\cos\varphi\Big|_0^{\pi}\right\} d\theta = \frac{a^3}{3} \int_0^{2\pi} d\theta \frac{a^3}{3}$$

EXAMPLE 4 Find the mass of the sphere in Example 3 if its density at a proportional to the distance from the point to the origin

Solution. We have density = $a\sqrt{a^2 + x^2 + y^2} = a\rho$ Thus

$\mu = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} a\rho(\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{aa^4}{4}\int_0^{2\pi}\int_0^{\pi} \sin\varphi \; d\,\varphi \, d\theta$

$= \frac{aa^4}{4}(4\pi) = \pi a^4 \, a$

EXAMPLE 5 Find the centroid of the "ice-cream- cone –shaped" region below the sphere $x^2 + y^2 + z^2 = z$ and above the cone $z^2 = x^2 + y^2$

Solution. The region is sketched in Figure 5. Since $x^2 + y^2 + z^2 = \rho^2$ and$z = \rho \cos\varphi.$ the sphere$\{(x.y.z):\ x^2 + y^2 + z^2 = z\}$ can be written in spherical coordinates as

Since $x^2 + y^2 = \rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta\rho^2 \sin^2\varphi$ the equation of the cone is

$\rho^2 \cos^2\varphi = \rho^2\sin^2\varphi.$ or $\cos^2\varphi = \sin^2\varphi.$

And since the equation is

$\varphi = \frac{\pi}{4}$

Thus, assuming that the density of the region is the constant k, we have

$$\mu = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos^2\varphi} k\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= k\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\varphi\{\frac{\rho^3}{3}\Big|_0^{\cos\varphi}\} d\varphi d\theta = k\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\cos^3\varphi}{3}\sin\varphi d\varphi \, d\theta$$

$$= k\int_0^{2\pi}\{-\frac{\cos^4\varphi}{12}\Big|_0^{\frac{\pi}{4}}\} d\theta = \frac{1}{12}k\int_0^{2\pi}\frac{3}{4}d\theta = \pi k/8$$

Since $x = \rho \sin\varphi \cos\theta$. we have

$$M_{yz} = k\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi} (\rho \sin\varphi \cos\varphi)\rho^2 \; \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= k\int_0^{2\pi}\cos\theta d\theta \int_0^{\pi/4}\int_0^{\cos\varphi} \rho^2 \sin^2\varphi \, d\rho \, d\varphi = 0$$

Since $\int_0^{2\pi}\cos\theta d\theta = 0$

(an unsurprising result because of symmetry) Similarly , since $y = \rho \sin\varphi \sin\theta$.

$$M_{xz} = k\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi}(\rho\sin\varphi\sin\varphi)\rho^2\ \sin\varphi\ d\rho\ d\varphi\ d\theta = 0$$

(again because of symmetry) Finally, since $z = \rho\cos\varphi$.

$$M_{xy} = k\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi}(\rho\cos\varphi)\rho^2\ \sin\varphi\ d\rho\ d\varphi\ d\theta$$

$$= \frac{k}{4}\int_0^{2\pi}\int_0^{\pi/4}\cos^5\varphi\sin\varphi\ d\varphi\ d\theta = \frac{k}{4}\int_0^{2\pi}\left\{-\frac{\cos^6\varphi}{6}\bigg|\begin{matrix}\pi/4\\0\end{matrix}\right\}d\theta$$

$$\frac{1}{24}\cdot\frac{7}{8}\ k\int_0^{2\pi}d\theta = \frac{7\pi k}{96}$$

PROBLEMS

Solve problems 1-11 by using cylindrical coordinates

1. Find the volume of the region inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder$(x - 1)^2 + y^2 = 1$

2. Find the centroid of the region of problem 1

3. Suppose the density of the region of problem 1 is proportional to the square of the distance to the xy-plane and is measured in kilograms per cubic meter. Find the center of mass of the region

4 .Find the volume of the solid bounded above by the paraboloid$z = 4 - x^2 - y^2$ and below by the xy- plane

5. Find the center of mass of the solid of Problem 4 if the density is proportional to the distance to the xy- plane

6. Find the volume of the solid bounded by the plane $z = y$ and the paraboloid $z = x^2 + y^2$

7. Find the centroid of the region of Problem 6.

8. Find the volume of the solid bounded by the two cones $z^2 = x^2 + y^2$ and $z^2 = 16x^2 + 16y^2$ between $z = 0$ and $z = 2$

9. Find the center of mass of the solid in problem 8 if the density at any point is proportional to the distance to the z- axis