DENSITY, MASS, AND CENTER OF MASS

Let de $p(x.y)$note the density of alpine o object (like a thin lamina, for example). Suppose that the o object occupies a region Ω in the xy plane. Then the mass of a small rectangle of sides $\Delta x\ and\ \Delta y$ centered at the point $(x.y)$is approximated by

$$p(x.y)\Delta x\Delta y = p(x.y)\Delta A \qquad (1)$$

And the total mass of the object is

$$\mu = \iint_{\Omega} p(x.y)\Delta A \qquad (2)$$

Compare this formula for the mass of an object lying along the x- axis with density $p(x)$,we showed how to calculate the first moment and center of mass of an object around the x- and y – axes for example, we defined

$$M_y = \int_a^b x\, p\,(x)dx \qquad (3)$$

To be the first moment about the y –axis when we had a system of masses distributed along the x-axis. Similarly, we calculated x- coordinate of the center of mass of the object to be

$$\bar{x} = \frac{\int_b^a x\, p\,(x)dx}{\int_a^b\, p\,(x)dx} = \frac{first\ moment\ about\ y - axis}{\text{mass}} = \frac{M_y}{\mu} \qquad (4)$$

we found that it was necessary to assume that the region had a constant area density However, by using double integrals, we can get away from this restriction. Consider the plane region whose mass is given by (2). Then we define

$$M_y = first\ moment\ around\ y - axis = \iint_{\Omega} xp(x.y)dA \qquad (5)$$

Look at Figure 1. The first moment about the y-axis of a small rectangle centered at (x, y) is given by: $x_i p(x_i^{*}.y_i^{*})\Delta x\Delta y.$ (6)

And if we add up these moments for all such "sub rectangles" and take a limit, we arrive at equation (5). finally, we define the center of mass of the plane region to be the point $(\bar{x}.\bar{y})$. where

$$\bar{x} = \frac{M_y}{\mu} = \frac{\iint_{\Omega} xp(x.y)dA}{\iint_{\Omega} p(x.y)dA} \qquad (7)$$

and

$$\bar{y} = \frac{M_x}{\mu} = \frac{\iint_{\Omega} yp(x.y)\,dA}{\iint_{\Omega} p(x.y)\,dA} \qquad (8)$$

EXAMPLE 1 :A plane lamina has the shape of the triangle bounded by the lines$y = x . y = 2 - x$. and the x-axis Its density function is given by$\rho(x.y) = 1 + 2x + y$ Distance is measured in meters, and mass is measured in kilograms. Find the mass and center of mass of the lamina.

Solution. The mass is given by

$$\mu = \iint_{\Omega} p(x.y)\,dA = \int_0^1 \int_y^{2-y} (1 + 2x + y)\,dxdy$$

$$= \int_0^1 \{(x + x^2 + xy)|_y^{2-y}\}dy = \int_0^1 (6 - 4y - 2y^2)$$

$$= \left(6y - 2y^2 - \frac{2y^3}{3}\right)|_0^1 = \frac{10}{3}kg$$

Then

$$M_y = \int_0^1 \int_y^{2-y} x(1 + 2x + y)\,dxdy$$

$$= \int_0^1 \int_y^{2-y} (1 + 2x^2 + xy)\,dxdy = \int_0^1 \left\{\left(\frac{x^2}{2} + \frac{2x^3}{3} - \frac{x^2 y}{2}\right)|_y^{2-y}\right\}dy$$

$$= \int_0^1 \left(\frac{22}{3} - 8y + 2y^2 - \frac{4y^3}{3}\right)dy$$

$$= \left(\frac{22}{3}y - 4y^2 + \frac{2y^3}{3} - \frac{y^4}{3}\right)|_0^1 = \frac{11}{3}kg \ \cdot m \cdot$$

$$M_x = \int_0^1 \int_y^{2-y} y(1 + 2x + y)\,dxdy = \int_0^1 \{(xy + x^2 y + xy^2)|_y^{2-y}\}dy$$

$$= \int_0^1 (6y - 4y^2 - 2y^3)dy = \left(3y^2 - \frac{4y^3}{3} - \frac{y^4}{2}\right)|_0^1 = \frac{7}{6}kg \cdot m \cdot$$

Thus $\bar{x} = \frac{M_y}{\mu} = \frac{11/3}{10/3} = \frac{11}{10}m \ \ and \ \bar{y} = \frac{M_x}{\mu} = \frac{7/6}{10/3} = \frac{7}{20}m$

Theorem 1: Suppose that the plane region Ω is revolved about a line L in the xy –plane that does not intersect it Then the volume generated is equal to the product of the area of Ω and the length of the circumference of the circle traced by the centroid of Ω

Proof We construct a coordinate system, placing the y – axis so that it coincides with the line L and Ω is in the first quadrant. The situation is then as depicted in Figure 3. We form a regular partition of the region Ω and choose the point$(x_i^{*}.\, y_i^{*})$ to be appoint in the sub rectangle Ω_{ij} If Ω_{ij} is revolved about L (the y-axis), it forms airing, The volume of the ring is, approximately,

$$V_{ij} \approx (circumference\ of\ circle\ with\ radius\ x_i^{*}) \times (thickness\ of\ ring\ ) \times (\ height\ of\ ring)$$

$= 2\pi x_i^{*}\Delta x\Delta y = 2\pi x_i^{*}\Delta A$. And using a familiar limiting argument, we have

$$V = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{i=1}^{n}\sum_{j=1}^{m} V_{ij} = \iint_{\Omega} 2\pi x dA = 2\pi \iint_{\Omega} x dA \qquad (9)$$

Now we can think of Ω as a thin lamina with constant density $\rho = 1$ Then from (7)(with $\rho = 1$) .

Form (9)

$$\bar{x} = \frac{\iint_{\Omega} x dA}{\iint_{\Omega} dA} = \frac{V/2\pi}{\iint_{\Omega} dA} = \frac{V/2\pi}{area\ of \Omega}.$$

Or $V = (2\pi\bar{x})(area\ of\ \Omega\ )$

$$= (length\ of\ the\ circumference\ of\ the\ circle\ traced\ by\ the\ cen\ troid\ of \Omega\ ) \times (\ area\ of \Omega)$$

EXAMPLE 2 Use the first theorem of pappus to calculate the volume of the torus generated by rotating the circle$(x-a)^2 + y^2 = r^2(r < a)$ about the y – axis.

Solution. The circle and the torus are sketched in Figure 5. The area of the circle is πr^2 The radius of the circle traced by the centroid (a, 0) is a, and the circumference is $2\pi a$ Thus

$V = (2\pi a)\, \pi r^2 = 2\, \pi r^2\, ar^2$

PROBLEMS

In problems 1- 12, find the mass and center of mass of an object that lies in the given region with the given area density function.

$1\cdot\Omega=\{(x.y):1\le x\le 2.-1\le y\le 1\};\rho(x.y)=x^2+y^2$

$2\cdot\Omega=\{(x.y):0\le x\le 4.1\le y\le 3\};\rho(x.y)=2xy$

$3\cdot\Omega=\left\{(x.y):0\le x\le \frac{\pi}{6}.0\le y\le \frac{\pi}{18}\right\};\rho(x.y)=\sin(2x+3y)$

$4\cdot\Omega=\{(x.y):-2\le x\le 2.0\le y\le 1\};\rho(x.y)=(x-y)^2$

$5\cdot\Omega\ is\ the\ region\ of problem\ 2;\rho(x.y)=xe^{x-y}$

$6\cdot\Omega\ is\ the\ quartero\ fthe\ unit\ circle\ lying\ in\ the\ first\ quadrant;\ \rho(x.y)\ x+y^2$

$7\cdot\Omega\ is\ the\ region\ of Problem\ 6;\rho(x.y)=x^2+y$

$8\cdot\Omega\ is\ the\ region\ of Problem\ 6;\rho(x.y)=x^3+y^3$

$9\cdot\Omega\ is\ the\ triangular\ region\ bounded\ by\ the\ lines\ y=x.y=1-x.and\ the\ x-axis;\ \rho(x.y)=x+2y$

$10\cdot\Omega\ is\ the\ region\ of Problem\ 9;\rho(x.y)=xe^{x+2y}$

$11\cdot\Omega\ is\ the\ first\ quadrant;\rho(x.y)=e^{-y}/(1+x)^3$

$12\cdot\Omega\ is\ the\ first\ quadrant;\rho(x.y)=(x+y)e^{-(x+y)}$

$13\cdot Use\ the\ first\ theorem\ of\ Pappus\ to\ calculate\ the\ volume\ of\ the\ torus$

$generated\ by\ rotating\ the\ unit\ circl\ about\ the\ line\ y=4-x$

$14\cdot Use\ the\ first\ theorem\ of\ Pappus\ to\ calculate\ the\ volume\ of\ the\ solid$

$generated\ by\ rotating\ the\ triangle\ with\ vertices\ (-1.2).(1.2).and\ (0.4) about\ the\ x-axis$

15. Use the first theorem of Pappus to calculate the volume of the "elliptical torus" generated by rotating the ellipse$\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)=1$ about the line $y=3a$ Assume that $3a>b>0$

DOUBLE INTEGRALS IN POLAR COORDINATES

In this section we will see how to evaluate double integrals of functions in the form$z = f(\ r.\theta)$. where r and θ denote the polar coordinates of appoint in the plane $Let\ z = f(\ r.\theta)\ and\ let\ \Omega$ denote the "polar rectangle"

$$\theta_1 \le \theta \le \theta_2. \qquad r_1 \le r \le r_2 \tag{1}$$

This region is sketched in Figure 1 We will calculate the volume of the solid between the surface $z = f(r.\theta)$ and the region Ω We partition Ω by small "polar rectangle"

And calculate the volume such a region. The volume of the part of the solid over the region Ω_{ij} is given, approximately , by

$$f(r_i\ .\theta_j)A_{ij}\ . \tag{2}$$

WhereA_{ij} is the area of Ω_{ij} Recall Form Section 11. 5 that if $r = f(\theta)$. then the area bounded by the lines $\theta = a.\theta = B.$ and the curve $r = f(\theta)$ is given by

$$A = \int_a^B \frac{1}{2}[(\theta)]^2\, d\theta \tag{3}$$

Thus

$$A_{ij} = \int_{\theta_j}^{\theta_{j+1}} ({r^2}_{i+1} - {r^2}_i) d\theta$$

$$= \frac{1}{2}({r^2}_{i+1} - {r^2}_i)\theta\Big|_{\theta_j}^{\theta_{j+1}} = \frac{1}{2}({r^2}_{i+1} - {r^2}_i)\left(\theta_{j+1} - \theta_j\right)$$

$$= \frac{1}{2}(r_{i+1} - r_i)(r_{i+1} - r_i)\left(\theta_{j+1} - \theta_j\right)$$

$$= \frac{1}{2}(r_{i+1} - r_i)\Delta r\Delta\theta$$

But if Δr is small, then $r_{i+1} \approx r_{i\ .}$ and we have

$$A_{ij} \approx \frac{1}{2}(2r_i)\Delta r\Delta\theta\ = r_i\Delta r\Delta\theta$$

Then

$$V_{ij}\ \approx f\left(r_i\ .\theta_j\right)A_{ij} \approx f\left(r_i\ .\theta_j\right)r_i\Delta r\Delta\theta.$$

So that, adding up the individual volumes and taking a limit, we obtain

$$V = \iint_{\Omega} f(r.\theta) r\, dr\, d\theta \qquad (4)$$

NOTE. Do not forget the extra r in the above formula.

EXAMPLE 1 Find the volume enclosed by the sphere $x^2 + y^2 + z^2 = a^2$

Solution. We will calculate the volume enclosed by the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ and then multiply by two. To do so, we first note that this volume is the volume of the solid under the hemisphere and above the disk$x^2 + y^2 \le a^2$ We use polar coordinates, since in polar coordinates$x^2 + y^2 = (r\, cos\theta)^2 + (r\, cos\theta)^2 = r^2$ On the disk, Then z $\sqrt{a^2 - (x^2 + y^2)} = \sqrt{a^2 - r^2}$, so by (4)

$$V = \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}\, r\, drd\theta = \int_0^{2\pi} \{-\tfrac{1}{3}(a^2 - r^2)^{3/2} |_0^a\} d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} a^3 d\theta = \frac{2\pi a^3}{3}$$

Thus the volume of the sphere is 2($2\pi a^{3/3}$)=(4/3) πa^3

EXAMPLE 2 : Find the volume of the solid bounded above by the surface$z = 3 + r$ and blow by the region enclosed by the cardioid r = 1 sin θ

Solution : The cardioid can be described by

$$\Omega = \{(r.\theta): 0 \le \theta \le 2\pi \; and \; 0 \le r \le 1 + \sin\theta \}$$

Then from (4)

$$V = \int_0^{2\pi} \int_0^{1+\sin\theta} (3 + r) r\, drd\theta = \int_0^{2\pi} \left\{ \left(\frac{3r^2}{2} + \frac{r^2}{3} \right) |_0^{1+sin\theta} \right\} d\theta$$

$$= \int_0^{2\pi} \{\tfrac{3}{2}(1 + 2\sin\theta + sin^2\theta) + (1 + 3\sin\theta + 3sin^2\theta + sin^3\theta)\} d\theta$$

$$= \int_0^{2\pi} (\frac{11}{6} + 4\sin\theta + \frac{5}{2} sin^2\theta + \frac{sin^3\theta}{3}) d\theta$$

$$= \int_0^{2\pi} \{\frac{11}{6} + 4\sin\theta + \frac{5}{4}(1 - \cos 2\theta) + \frac{sin\theta}{3}(1 - cos^2\theta)\} d\theta$$

$$= \left(\frac{11}{6}\theta - 4\cos\theta + \frac{5}{4}\theta - \frac{5}{8} sin2\theta - \frac{cos\theta}{3} + \frac{cos^3\theta}{9} \right) |_0^{2\pi} = \frac{37}{6}\pi$$

REMARK. We can also use this technique to calculate areas If $(r.\theta) = 1$ $then$

$$\iint_{\Omega} r\,drd\theta = area\ of\Omega \quad (5)$$

EXAMPLE 3 Calculate the area enclosed by the cardioid $r = 1 + \sin\theta$

Solution

$$A = \int_0^{2\pi}\int_0^{1+\sin\theta} rdrd\theta = \int_0^{2\pi}\left\{\frac{r^2}{2}\Big|_0^{1+\sin\theta}\right\}d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(1 + 2sin\theta + sin^2\theta)d\theta = \frac{1}{2}\int_0^{2\pi} 1 + sin\theta + \frac{1 - cos2\theta}{2})d\theta$$

$$= \frac{1}{2}\left(\frac{3\theta}{2} - 2cos\theta - \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi} = \frac{3\pi}{2}$$

As we will see, it is often very useful to write a double integral in terms of polar coordinates Let $z = f(x.y)$ be a function defined over a region Ω Then using polar coordinates, we can write

$$z = f(rcos\,\theta . r\sin\theta). \quad (6)$$

And we can also describe in terms of polar coordinates. The volume of the solid under f and over is the same whether we use rectangular or polar coordinates. Thus writing the volume in both rectangular and polar coordinates, we obtain the useful change – of - variables formula

$$\iint_{\Omega} f(x.y)dA = \iint_{\Omega} f(r\cos\theta . r\sin\theta)rd\theta \quad (7)$$

EXAMPLE 4 The density at any point on a semicircular plane lamina is proportional to the square of the distance from the point to the center of the circle. find the mass of the lamina

Solution. We have $\rho(x.y) = a\,(x^2 + y^2)ar^2 and\ \Omega = \{(r.\theta): 0 \le r \le a\ and\ 0 \le \theta \le \pi\}$. where a is the radius of the circle. Then

$$\mu = \int_0^{\pi}\int_0^{a}(ar^2)r\,dr\,d\theta = \int_0^{\pi}\left\{\frac{ar^4}{4}\Big|_0^a\right\}d\theta = \frac{aa^4}{4}\int_0^{\pi}d\theta = \frac{a\pi a^4}{4}$$

This double integral can be computed without using polar coordinates, but the com-potation is much more tedious. Try it!

EXAMPLE 5 Find the volume of the solid bounded by thexy plane, the cylinder$x^2 + y^2 = 4$ and the paraboloid $z = 2(x^2 + y^2)$

Solution : The volume requested is the volume under the surface $z = 2(x^2 + y^2) = 2r^2$and above the circle $x^2 + y^2 = 4\ Thus$

$$V = \int_0^{2\pi}\int_0^{2} 2r^2 \cdot rdr\, d\theta = \int_0^{2\pi}\left\{\frac{r^4}{2}\Big|_0^2\right\} d\theta = 16\pi$$

EXAMPLE 6: In probability theory one of the most important integrals and polar that is encountered is the integral

$\int_{-\infty}^{\infty} e^{-x^2} dx$

We now show how a combination of double integrals and polar coordinates can be used to evaluate it. Let

$I = \int_0^{\infty} e^{-x^2} dx$

Then by symmetry $\int_{-\infty}^{\infty} e^{-x^2} dx$ Thus we need only to evaluate I. But since any dummy variable can be used in a definite integral, we also have

$$I = \int_0^{\infty} e^{-y^2} dy$$

Thus

$$I^2 = \left(\int_0^{\infty} e^{-x^2} dx\right)\left(\int_0^{\infty} e^{-y^2} dy\right).$$

$$I^2 = \int_0^{\infty}\int_0^{\infty} e^{-x^2} dx e^{-y^2} dy = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dxdy = \iint_{\Omega} e^{-(x^2+y^2)} dA$$

Where denotes the first quadrant. In polar coordinates the first quadrant can be written as

$$\Omega = \left\{(r.\theta) : 0 \le r \le \infty \; and \; \le 0 \le \theta \le \frac{\pi}{2}\right\}$$

Thus since $x^2 + y^2 = r^2$ we obtain

$$I^2 = \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} rdrd\theta = \int_0^{\frac{\pi}{2}}(\lim_{N\to\infty}\int_0^{N} e^{-r2} rdr)d\theta$$

$$= \int_0^{\pi/2}(\lim_{N\to\infty} -\frac{1}{2}e^{-r^2}|_0^N)d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

Hence $I^2 = \frac{\pi}{4}.\; so\;\; I = \sqrt{\frac{\pi}{2}}\; .and$

$\int_{-\infty}^{\infty} e^{-x^2} dx = 2I = \sqrt{\pi}$

By making the substitution $u = \frac{x}{\sqrt{2}}$. it is easy to show that

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

The function $\rho(x) = (1\sqrt{2\pi})e^{-x^2/2}$ is called the density function for the unit normal distribution. We have just shown that $\int_{-\infty}^{\infty} \rho(x)\, dx = 1$

EXAMPLE 7 Find the volume of the solid bounded by the circular paraboloid$x = y^2 + z^2$ and the plane x = 1 .

Solution. :$V = \iint_{\Omega} (1 - y^2 - z^2) dy\, dz$.

Where Ω is the circle$y^2 + z^2$ =1 But this integral is easily evaluated by using polar coordinates (with y and z in place of x and y in the polar coordinate formulas). We have

$$V = \int_0^{2\pi} \int_0^1 (1 - r^2) r\, dr d\theta$$

$$= \int_0^{2\pi} \left\{ \left(\frac{r^2}{2} - \frac{r^4}{4} \right) |_0^1 \right\} d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \frac{\pi}{2}$$

PROBLMS

In problem 1-5, calculate the volume under the given surface that lies over the given region Ω

1 · $z = r$; Ω is the circle of radius a

2 · $z = r^n; n\ is\ a\ positive\ integer\ ; \Omega\ is\ the\ circle\ of\ radius\ a$

3 · $z = 3 - r;\ \Omega\ is\ the\ circle\ r = 2\cos\theta$

4 · $z = r^2$; Ω is the cardioid $r = 4(1 - \cos\theta)$

5. $z = r^3; \Omega$ *is the region enclosed by the spiral of Archimedes* $r = a\theta$ *and the polar axis for* θ*between 0 and* 2π

In problems 6-15, calculate the region by the given curve or curves.

6 · $r = 1 - \cos\theta$

7 · $r = 4(1 + \cos\theta)$

8 · $r = 1 + 2\cos\theta (outer\ loop)$

9 · $r = 3 - 2\sin\theta$

10 · $r^2 = \cos 2\theta$

11 · $r^2 = 4\sin 2\theta$

$12 \cdot \mathrm{r} = \mathrm{a} + \mathrm{b}\sin\theta . a > b > 0$ $13 \cdot \mathrm{r} = \tan\theta$ *and the lin* $\theta = \frac{\pi}{4}$

$14 \cdot$ *Outside the circle* $r = 6$ *and inside the cardioid* $r = 4(1 + \sin\theta)$

$15 \cdot$ *Inside the cardioid* $r = 2(1 + \cos\theta)$ *but outside the circle* $r = 2$

16. Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 4a^2$ below by the xy - plane , and on the sides by the cylinder $x^2 + y^2 = a^2$

17. Find the area of the region interior to the curve $(x^2 + y^2)^3 = 9y^2$

18. Find the volume of the solid bounded by the cone $x^2 + y^2 = z^2$ and the cylinder $x^2 + y^2 = 4y$

19. Find the volume of the solid bounded by the cone $z^2 = x^2 + y^2$ and the paraboloid

$2z = x^2 + y^2$

20. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 9$ and the paraboloid $x^2 + y^2 - z^2 = 1$

21. Find the volume of the solid centered at the origin that is bounded above by the surface $z = e^{-(x^2+y^2)}$ and below by the unit circle

22. Find the centroid of the region bounded by $r = \cos\theta + 2\sin\theta$

23. Find the centroid of the region bounded by the limacon $r = 3 + \sin\theta$

24. Find the centroid of the region bounded by the limacon $r = a + b\cos\theta . a > b > 0$

25. Show that $\iint_\Omega 1/(1 + x^2 + y^2)d\Lambda = \pi In\, 2$. where Ω is the unit disk .

SURFACE AREA

we found a formula for the length of a plane curve[$given\ by\ y = f(x)$] by showing that the length of a small " piece" of the curve was approximately equal to $\sqrt{1+[f'(x)]^2}\ \ \Delta x$We now define the area of a surface $z = f\ (x.y\ )$ that lies over a region in the plane We define it by analogy with the are length formula

Definition 1 LATERAL SURFACEA Let f be continuous with continuous partial

derivatives in the region Ω in the xy – plane. Then the lateral surface area σ of the graph of f over Ω is defined by

$$\sigma = \iint_{\Omega}\sqrt{1 + f_{x^2}(x.y\ ) + f_{y^2}(x.y\ )dA} \qquad (1)$$

REMARK 1. The assumption that f is continuously differentiable over Ω ensures that the integral in (1) exists

REMARK 2. We will show why formula (1) makes intuitive sense at the end of this section.

REMARK 3. We emphasize that formula (1) is a definition. A definition is not some – thing that we have to prove, of course, but it is something that we have to live with It will be comforting, therefore, to use our definition to evaluate areas where we know what we want the answer to be

EXAMPLE 1 : Calculate the lateral surface area cut from the cylinder $x^2 + y^2 = 9$ by the planes $x = 0\ and\ x = 4$

Solution. The surface is right circular cylinder with height 4 and radius 3 whose axis lies along the x – axis Thus $S = 2\pi rh = 24\pi$ ·

We now solve the problem by using formula (1) The surface area consists of two equal parts, one for $z > 0\ and\ \ one\ \ for\ z < 0$ We calculate the surface area for $z > 0$ and multiply the result by 2. We have, for $z > 0. z = f\ (x.y\ ) = \sqrt{9-y^2}. so\ f_x = 0. f_y = -y/\surd(9 - y\text{^}2\ ). and f^2{}_y = y^2/(9-y^2)$ Thus since$0 \le x \le 4\ and - 3 \le y\ \le 3\ .we$ have

$$S = 2\int_0^4\int_{-3}^{3}\sqrt{1+\frac{y^2}{9-y^2}}\ \ dydx = 4\int_0^4\int_0^3\sqrt{\frac{9}{9-y^2}}\ \ dy\,dx$$

$$= 12\int_0^4 (\int_0^{\frac{\pi}{2}}\frac{dy}{\sqrt{9-y^2}}\ dx\ \cdot$$

Setting $y = 3\sin\theta$. we have

$S = 12 \int_0^4 (\int_0^{\pi/2} \frac{3\cos\theta}{3\cos\theta}\, d\theta)\, dx = 12 \int_0^4 \frac{\pi}{2} dx = 12\,(4)\left(\frac{\pi}{2}\right) = 24\,\pi$

We now compute surface area in problems where we don't know the answer in advance.

EXAMPLE 2 Calculate the lateral surface area cut form the cylinder $y^2 + z^2 = 9$ by the planes $x = 0. x = 1. y = 0$ *and* $y = 2$

Solution It consists of two equal parts: one for $z > 0$ *and one for* $z < 0$ We calculate the surface area for $z > 0$ multiply the result by 2 . For $z > 0$. We have $z = \sqrt{9 - y^2}$ Ω is the rectangle$\{(x. y): 0 \le x \le 1 \text{ and } 0 \le y \le 2\}$ · Then

$$f_x = 0 \qquad and \qquad f_y = -\frac{y}{\sqrt{9 - y^2}}.$$

So that $\sigma = \int_0^2 \int_0^1 \sqrt{1 + 0^2 + (\frac{y}{\sqrt{9-y^2}})^2}\; dxdy = \int_0^2 \int_0^1 \sqrt{1 + \frac{y^2}{\sqrt{9-y^2}}}\, dxdy$

$$= 3\int_0^2 \int_0^1 \frac{1}{\sqrt{9 - y^2}} dxdy = 3\int_0^2 (\frac{1}{\sqrt{9 - y^2}} |_0^1) dy = 3\int_0^2 (\frac{dy}{\sqrt{9 - y^2}}$$

$$= 3\sin^{-1}\frac{y}{3}|_0^2 = \sin^{-1}\frac{2}{3}$$

Thus the total surface area is $6\sin^{-1}\frac{2}{3} \approx 4 \cdot 38$

EXAMPLE 3 : Find the lateral surface area of the circular paraboloid $z = x^2 + y^2$ between the $xy\_$ plane and the plane $z = 9$

Solution. The region Ω is the disk $x^2 + y^2 \le 9$,We have

$$f_x = 2x \qquad and \qquad f_y = 2y\,.$$

So $\sigma = \iint_\Omega \sqrt{1 + 4x^2 + 4y^2}\; dA$

Clearly, this problem calls for the use of polar coordinates. We have

$$\sigma = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, rdrd\theta = \int_0^{2\pi} \{\frac{1}{12}(1 + 4r^2)^{3/2}|_0^3\} d\theta$$

$$= \frac{\pi}{6}(37^{3/2} - 1) \approx 117 \cdot 3$$

EXAMPLE 4 Calculate the area of the part of the surface $z = x^3 + y^4$ that lies over the square $\{(x.y): 0 \le x \le 1.0 \le y \le 1\}$

Solution $f_x = 3x^2$ *and* $f_y = 4y^3$.*so*

$$\sigma = \int_0^1 \int_0^1 \sqrt{1 + 9x^4 + 16y^6}\ dxdy$$

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Derivation of Formula for Surface Area. We begin by calculating the surface area over a rectangle with sides and ,we assume that has continuous partial derivatives over are small, then the region PQSR in space has, approximately, the shape of a parallelogram.

$$\Delta\sigma \approx area\ of \text{parallelogram} = \left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| \quad (2)$$

Now $\overrightarrow{PQ} = \big(x + \Delta x. y . f(x + \Delta x. y)\big) - \big(x. y . f(x. y)\big)$

$= \big(\Delta x. 0.\ f(x + \Delta x. y) - f(x. y)\big)$

But if Δx small , then

$$\frac{f(x + \Delta x. y) - f(x. y)}{\Delta x} \approx f_x(x. y).$$

So $f(x + \Delta x. y) - f(x. y) \approx f_x(x. y)\Delta x$

And $\overrightarrow{PQ} \approx (\Delta x. 0 f_x(x. y)\Delta x)$ (3)

Similarly,

$\overrightarrow{PQ} = \big(x. y + \Delta y . f(x. y + \Delta y)\big) - \big(x. y . f(x. y)\big)$

$= \big(0. \Delta y . f(x. y + \Delta y) - f(x. y)\big).$

And if Δy *is small* .

$\overrightarrow{PQ} \approx \big(0. \Delta y. f_y(x. y)\Delta y\big)$ (4)

Thus from (3) and (4),

$$\overrightarrow{PQ} \times \overrightarrow{PQ} \approx \begin{vmatrix} i & j & k \\ \Delta x & 0 & f_x(x. y)\Delta x \\ 0 & \Delta y & f_y(x. y)\Delta y \end{vmatrix}$$

$= -f_x(x.y)\Delta x\Delta y i - f_y(x.y)\Delta x\,\Delta y j + \Delta x\Delta y k$

$= \left(-f_x(x.y)i - f_y(x.y)j + k\right)\Delta x\Delta y$.

So that from (2)

$$\Delta\sigma \approx \sqrt{f_x{}^2(x.y) + f^2{}_y(x.y) + 1}\,\overbrace{\Delta x\Delta y}^{=\Delta A} \qquad (5)$$

Finally , adding up the surface area over rectangles that partition Ω and taking a limit yields

$$\sigma = \iint_{\Omega}\sqrt{1 + f^2{}_x(x,y) + f^2{}_y(x,y)}\,dA \qquad (1)$$

REMARK. We caution that equation (1) has not been proved. However, if f has continuous partial derivatives , it certainly is plausible .

PROBLEMS

In problems 1-9 ,find the area of the part of the surface that lies over the given region

1. $z = x + 2y;\ \Omega = \{(x, y): 0 \le x \le y, 0 \le y \le 2\}$

2 . $z = x + 7y;\ \Omega = region\ between\ y = x^2\ and\ y = x^5$

3 . $z = ax + by; \Omega = upper\ half\ of\ unit\ circle$

4 . $z = y^2;\ \Omega = \{(x, y): 0 \le x \le 2, 0 \le y \le 4\}$

5. $z = 3 + x^{\frac{2}{3}};\ \Omega = \{(x, y): -1 \le x \le 1, 1 \le y \le 2\}$

6. $z = \left(x^{\frac{4}{4}}\right) + \left(\frac{1}{8}x^2\right);\ \Omega = \{(x, y): 1 \le x \le 2, 0 \le y \le 5\}$

7. $z = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}};\ \Omega = \{(x, y): -4 \le x \le 7, 0 \le y \le 3\}$

8. $z = 2In\ (1 + y);\ \Omega = \{(x, y): 0 \le x \le 2, 0 \le y \le 1\}$

9. $(z + 1)^2 = 4x^3;\ \Omega = \{(x, y): 0 \le x \le 1, 0 \le y \le 2\}$

10. Calculate the lateral surface area of the cylinder $y^{2/3} + z^{2/3} = 1$ for x in the interval [0,2]

11. Find the surface area of the hemisphere $x^2 + y^2 + z^2 = a^2, z \ge 0$

12. Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that is also inside the cylinder $x^2 + y^2 = ay$

13. Find the area of the surface in the first octant cut from the cylinder $x^2 + y^2 = 16z$ by the plane y = z

14. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 16z$ lying within the circular paraboloid $z = x^2 + y^2$

15. Find the area of the surface cut from the hyperbolic paraboloid $4z = x^2 - y^2$ by the cylinder $x^2 + y^2 = 16$

16. Let $z = f(x, y)$ be the equation of a plane $(i.e, z = ax + by + c)$ Show that over the region Ω, the area of the plane is given by $\sigma = \iint_\Omega \sec\gamma\, dA$,

Where γ is the angle between the normal vector N to the plane and the positive z – axis [Hint: Show, using the dot product , that $\cos\gamma = \frac{N.K}{|N|} = \frac{1}{\sqrt{1+a^2+b^2}}$]

In Problems 17- 20 , find a double integral that represents the area of the given surface over the given region Do not try to evaluate the integral

17 .$z = x^3 + y^3; \Omega$ is the unit circle

18 . $z = In\,(x + 2y);\ \Omega = \{(x\,, y\,): 0 \le x \le 1, 0 \le y \le 4\}$

19 . $z = \sqrt{1 + x + y}\,; \Omega\ is\ the\ triangle\ bounded\ by\ y = x, y = 4 - x, and\ the\ y\_axis$.

20 . $z = e^{x-y}; \Omega\ is\ the\ ellipse\ 4x^2 + 9y^2 = 36$

21.Find a double integral that represents the surface area of the ellipsoid $\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) + \left(\frac{z^2}{c^2}\right) = 1$